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Abstract

This paper investigates the nexus between foreign aid (in the form of loans), poverty trap, and economic development in a recipient country by using a Solow model with two new ingredients: a development loan and a fixed cost in the production process. The presence of this fixed cost generates a poverty trap. Development loans may help the country to escape from the poverty trap and converge to a stable steady-state in the long run, but only if (i) the country's characteristics, such as saving rate, initial capital, governance quality, and in particular productivity, are good enough, (ii) the fixed cost is relatively low, and (iii) loans rule is generous enough. We also show that there is room for endogenous cycles in our model, unlike the standard Solow model.

Keywords: Economic dynamics, economic growth, foreign aid, development loan, poverty trap.
JEL Classification: O11, O19, O41

1 Introduction

In a context of continual increase of foreign aid allocated by the donors in the OECD-Development Assistance Committee (DAC), the evaluation of aid effects is ever more necessary for determining an efficient allocation. Constituting a great source of revenue for

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numerous recipient countries, foreign aid composes of two main forms: grant and loan (with low interest rate). Even that donors generally adopt a mixture of two types of aid, their determinants and their effectiveness are pretty different (Collier, 2006; Cohen et al., 2007; Gaibulloev and Younas, 2018). These two transfer types, both accounted in foreign aid category of the OECD-DAC,[1] are two complementary instruments having usually different motives from donors. In general, loans are allocated for infrastructure and development projects with obligations for repayment in the future and grants are for social and humanitarian purposes with free transfers. For instance, one of the recommendations of the Meltzer Commission (2000) for the development banks is to "use grants instead of loans to improve the quality of life in the poorest countries".[2] However, a development loan is a sustainable instrument for ODA. It represents a higher share in total bilateral than multilateral aid. As shown in Table 1, bilateral loans of members DAC in 2020 are amounted to 12.586 US$ billion (2019) over a total of 111.507 US$ billion (2019) of bilateral aid, representing 11.28 % of bilateral aid, while multilateral loans represent only 2.5 % of multilateral aid.

Table 1: Aid flows of DAC members in grant equivalent (in 2019 US$ million). Source: OECD.

Types of aid	2018	2019	2020
ODA, bilateral total	107 749.84	108 723.10	111 507.10
ODA, bilateral grants	96 041.42	95 055.58	95 321.44
ODA, bileteral loans	9 055.78	9 812.36	12 586.06
ODA, multilateral total	43 013.39	42 959.50	45 518.85
ODA, multilateral grants	41 513.74	42 116.42	44 377.86
ODA, multilateral loans	1 530.94	843.09	1 140.98

Both theoretical and empirical studies analyze the aid effectiveness by investigating the effect of the total aid or the grant component of aid. The aid effectiveness literature pays little attention to development loans while this component of aid is on a rising proportion of total aid flows. As a matter of fact, the increase in ODA gross disbursements to the least developed countries since 2011 is due to an increase in ODA loans, whereas grants have remained essentially stagnant.[3] Table 1 indicates that bilateral loans in 2020 by DAC members on a grant equivalent basis increased by 28.27% compared to 2019 while multilateral loans, even with a lower amount, increased by 26.1%. For bilateral aid, as shown Figure 1, loans made up almost two thirds of Japan's bilateral ODA (61.2% in 2017) while France disburses almost half of its bilateral ODA as loans.

Motivated by the fact that the literature of aid effectiveness has paid little attention to development loans, our paper aims to explore the effects of development loans on the economic dynamics of the recipient countries.[4] Precisely, we address the following questions:

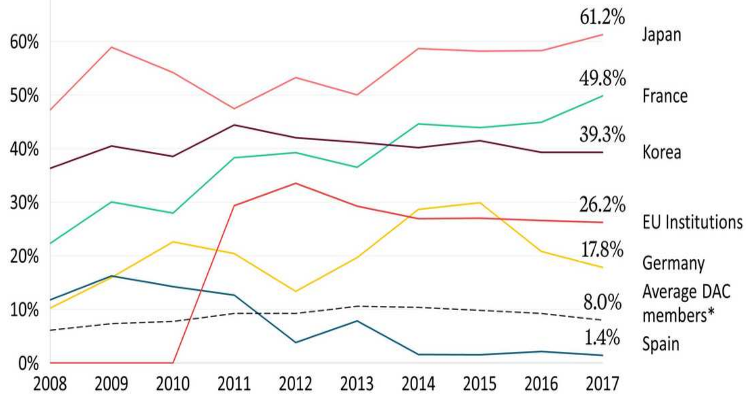
[1]From 2018, the ODA grant-equivalent methodology is used, whereby only the amount given by lending below market rates, is counted as ODA (https://data.oecd.org/oda/net-oda.htm).

[2]The Meltzer Commission: The future of the IMF and World Bank (page 7).

[3]For more details, see "Least developed countries Report 2019"-UNCTAD.

[4]A great number of studies examine the aid effectiveness in terms of economic growth, poverty reduction, income distribution, or fiscal behavior, etc. See for instance, Burnside and Dollar (2000), Guillaumont and Chauvet (2001), Ouattara (2006), Guillaumont and Wagner (2014), Morissey (2015), Maruta et al. (2020)

Figure 1: Share of bilateral ODA disbursed as loans, from "the OECD's new way of counting ODA loans - what's the impact?" (https://donortracker.org/. *: Average includes DAC donor countries and the EU institutions)



How do development loans (usually different from the exogenous grant) and the rule of loan imposed by donors affect the recipient's development process? What are the determinants of the recipient country's optimal choice of loans? How does the country's optimal choice change over time?

To answer these questions, we introduce two new ingredients (fixed cost and development loan) in a growth model à la Solow. The fixed costs in developing countries may be due to the lack of essential infrastructure such as road, rail, access to electricity, etc. (See Le Van et al. (2016) for more details.) Concerning the development loans, the country can promote its investment by borrowing (with low interest rates) from foreign organizations. However, the loans rule imposes two constraints: (i) the country can access development loans only if its income is not high (in the sense that it is lower than the so-called *eligibility threshold*), and (ii) the loan amount cannot exceed an exogenous limit imposed by lenders.

Our tractable framework allows us to explore the global dynamics of the economy.

We point out that in the absence of aid, the presence of the fixed cost generates a poverty trap which is a threshold such that the economy cannot overcome this threshold (resp., converges to a steady state which is higher than this threshold) if its initial capital stock is lower (resp., higher) than this threshold. Then, we investigate conditions under which development loans may enable the recipient country to escape from its poverty trap.

Given a low income country that will collapse if this is no foreign aid, we figure out different scenario depending on the loan rule and the country's characteristic.

(i) If the fixed cost is too high and the maximum amount of loan is low (so that even if the country borrows the maximum level to promote its investment, its capital stock cannot overcome the fixed cost), then the country will collapse whatever the level of the productivity. Moreover, although the country is eligible to access to the development loans, it does not borrow at any period of time. An implication of this finding is that,

for empirical studies and Charterjee et al. (2003), Charteerjee and Tursnovky (2007), Dalgaard (2008), Pham and Pham (2020) for theoretical growth models with foreign aid (in form of grant).

when the maximum amount of loan is very limited and donors want to take care of the recipient country, they should offer a grant rather than a loan because a permanent grant allows the country to avoid a collapse while loans cannot. This means that aid composition matters for the economic growth of the recipient country.[5]

(ii) If the fixed cost is not so high and the rule of aid is quite generous, the country can avoid the collapse by borrowing the maximum amount of loan to promote their investment. This can happen if the productivity and the governance quality have a middle level. In this scenario, although there are two different steady states, the economy converges to a high steady state and this is also higher than that in the economy without aid. Notice that the country is dependent on the loan (in the sense that it borrows at any period) because the capital stock is always lower than the eligibility threshold.

(iii) If the fixed cost is not so high, the rule of aid is quite generous, and more importantly, the productivity and the governance quality are good, then the economy converges to a steady state. Moreover, this steady state is higher than the eligibility threshold and does not depend on loans. In this scenario, the country does not borrow development loans in the long run. It just needs development loans for its first stage of development process.

(iv) Finally, we show that a cycle may appear. This happens if the rate of capital depreciation is high, the productivity is low and the rule of loans is generous (i.e., with a high borrowing bound and a low interest rate). The idea is simple: when the country has a low income, it can borrow from a development loan having a low interest rate to promote their investment and the income goes up. When the income is higher than the eligibility threshold, the country can no longer borrow, and then, the income goes down because of a low productivity and a high depreciation rate. Hence, a cycle arises.

It should be noticed that our above analyses still apply to the case of grants because grants can be considered as a particular case of loans (with zero interest rate). To the best of our knowledge, our paper is the first one focusing on the loan component of foreign aid and its impacts on the dynamics of the recipient country, in particular the possibility to escape the poverty trap.

Our paper is in line with theoretical studies examining the effectiveness of foreign aid (Charterjee et al., 2003; Charteerjee and Tursnovky, 2007; Dalgaard, 2008; Pham and Pham, 2020).[6] However, these papers focus on the case of grants. By contrast, we focus on the

[5]Gupta et al. (2004) suggest that aid composition matters for the relationship between the aid and the fiscal revenue. Indeed, Gupta et al. (2004) examine the effects of grants and loans on the recipient revenue effort using a data with 107 recipient countries during 1970-2000. They shows that an increase in loans is associated with higher tax effort whereas an increase in grants is associated with weaker tax effort in recipient countries. These effects vary across countries and depend on their quality of institutions.

[6]For instance, Charterjee et al. (2003), Charteerjee and Tursnovky (2007) use continuous-time growth models to analyze the economic dynamics around the steady-state and show the contrasting effects of untied and tied foreign aid on the recipient macroeconomic performance. Dalgaard (2008) considers an OLG growth model where households live for two periods while Pham and Pham (2020) uses a discrete-time general equilibrium framework with AK technology. In both Dalgaard (2008) and Pham and Pham (2020), the capital path can be recursively described with a capital stock at period $t+1$ depending on that at period

development loans with obligations for repayment in the future and its nexus with poverty trap, economic dynamics in a recipient country. We also note that the amount of loans in our paper is endogenous and chosen by the recipient country but not by donors as in the above studies. In our paper, donors impose the limit of loan, the interest rate and the eligibility threshold.

Our paper is also related to the literature on poverty trap. Le Van et al. (2016) use an optimal growth model without foreign aid to study how the country can avoid the poverty trap. In their paper, the existence of a poverty trap is due to high fixed costs or lack of infrastructure, being unfavorable for economic development. Under mild conditions on the efficiency of the infrastructure technology and the initial level of fixed costs, Le Van et al. (2016) shows that a poverty trap can be avoided. We complement Le Van et al. (2016) by pointing out that foreign aid (in form of loans or grants) may contribute to prevent such a poverty trap. Moreover, unlike Le Van et al. (2016) where the capital stock converges, an endogenous cycle may arise in our model due to the introducing of development loan.

If we interpret the agent in our growth model as a micro-entrepreneur, our paper has a link with the literature of poverty dynamics (see Diwakar and Shepherp (2021) and references therein). Diwakar and Shepherp (2021) uses the q-squared method to investigate why some households' escapes from poverty are sustained, while others escape only to fall back into poverty (a transitory escape) in several countries (Bangladesh, Cambodia, Ethiopia, Kenya, Nepal, the Philippines, Rwanda, Tanzania, and Uganda). Diwakar and Shepherp (2021) point out that the ability to manage risks to livelihoods and (tangible and intangible) assets makes the difference between a sustained and a temporary escape from poverty. They then argue it is critical to enable environment (economic and social policies and norms, education and health systems, financial infrastructure, ...).

Unlike Diwakar and Shepherp (2021), we use growth models and prove that fixed costs (lack of essential infrastructure such as road, rail, electricity, ...), productivity (including human capital, quality of machines, ...), and credit markets play an important role in getting sustained poverty escapes.

The remainder of the paper is organized as follows. Section 2 describes our framework, and characterizes its economic dynamics when there is no development loan. Section 3 focuses on the endogenous choice of loans and its impacts in such an economy with poverty trap. Section 4 concludes. Technical proofs are presented in Appendix A.

2 A Solow growth model with development loan and fixed cost

For the sake of tractability, we consider a model à la Solow.[7] The population size is constant over time and normalized to unity. At each date t, the wealth W_t (after having reimbursed

t. Pham and Pham (2020) underline, however, the role of the recipient fiscal policy and government effort in financing public investment and the efficiency of aid use.

[7]Adopting a model à la Solow with exogenous saving rate allows us to examine the global dynamics and the cycles, contrary to optimal growth models à la Ramsey, where we can only explicitly compute the optimal allocation in some specific utility and production functions. Moreover, in Ramsey models, it is difficult to study the global dynamics without the monotonicity of the capital path.

loans) is divided between consumption and saving.

$$c_t + S_t = W_t \tag{1a}$$
$$S_t = sW_t \tag{1b}$$
$$k_{t+1} = (1-\delta)k_t + I_t \tag{1c}$$

where c_t, S_t, I_t are consumption, saving, and investment at the period t ($t = 0, 1, \dots, +\infty$), $s \in (0, 1)$ is the exogenous saving rate. k_t represents the physical capital stock at date t ($k_0 > 0$ is given) while $\delta \in [0, 1]$ is the capital depreciation rate.

We introduce two new ingredients with respect to the standard Solow model: (1) development loan and (2) fixed cost in the production function.

2.1 Development loans

We consider that the total investment of the recipient country at t equals the sum of its savings and a fraction of foreign capital flow:

$$I_t = S_t + \lambda a_t \tag{2}$$

where a_t represents the foreign capital flow while $\lambda \in [0, 1]$ is an exogenous parameter. Notice that $(1-\lambda)a_t$ measures the amount of foreign aid, which is wasted due to, for example, a corruption problem or an inefficient use. So, λ may represent the governance quality of the recipient country.

The model in Chenery and Strout (1966) corresponds to the case where $\lambda = 1$ and a_t equals a fraction of the recipient country's GNP.[8]

We now describe how the amount a_t is endogenously determined.

1. First, a_t equals zero if the capital stock of the recipient country at date t exceeds an exogenous threshold b_1: $k_t \geq b_1$. This assumption may be illustrated for instance by the case of countries changing their income category from one period to another such as South Korea. South Korea was a recipient during 1960-1990, after the Korean war (1950-1953) and experienced a high economic growth during this period. As growing quickly during the mid-1960s, South Korea became increasingly integrated into the international capital market. Its investment was enssentially financed by the competitive international market and increasling from domestic saving. South Korea is now a developed country and becomes one of the thirty members of the OECD-DAC since 2010. Other countries such as China and Argentina see their net ODA (in % of GNI) becoming null in the recent years (2016 for Argentina and 2010 for China).

2. Second, when $k_t < b_1$, the recipient country can borrow from foreign organizations but it cannot borrow more than a bound $\bar{a}$ which is exogenous. The country amount a_t is

[8] Charterjee et al. (2003), Charteerjee and Tursnovky (2007) also assume that $\lambda = 1$ and a_t equals a fraction of the recipient country's output. See Dalgaard (2008), Carter (2014), Pham and Pham (2020) for other rules.

chosen by the recipient country so that it maximizes the country's wealth in the next period. Formally, a_t equals the solution of the following problem:

$$(P_t): \quad W_{t+1} \equiv \max_{0\leq x\leq \bar{a}} \Big\{F(k_{t+1}) - Rx\Big\} = \max_{0\leq x\leq \bar{a}} \Big\{F[(1-\delta)k_t + S_t + \lambda x] - Rx\Big\}$$

where k_t and S_t are taken as given, $R = 1 + r$ and r is the exogenous net interest rate imposed by the donor while.[9] As an example, Japan applies fixed interest rates of 0.10%, 0.15%, 0.20% or 0.25% to low-income least developed countries (i.e., income per capita below US$ 1036) for a 15, 20, 25, or 30-year repayment period, respectively.[10] In other words, when the capital stock at date t is below the threshold b_1, the country is considered as in need of international help and benefits from a loan with interest rate r below private rates practiced at international capital markets. Several lower-middle income countries, such as Vietnam, Tunisia and Morocco, always benefit from the ODA loan with a low-interest rate.[11]

To sum up, the foreign capital flow a_t is defined by the following rule:

Assumption 1 (loans rule)**.** *We consider the following rule for the loans with the interest rate R*

$$a_t = \begin{cases} x_t & \text{if } k_t < b_1 \\ 0 & \text{if } k_t \geq b_1 \end{cases} \tag{3}$$

where x_t is the solution of the problem (P_t). The loan rule is represented by three exogenous parameters $(b_1, \bar{a}, R)$.

2.2 Dynamical system

According to the setup above, the capital accumulation becomes:

$$k_{t+1} = (1-\delta)k_t + sW_t + \lambda a_t = \begin{cases} (1-\delta)k_t + sW_t + \lambda x_t & \text{if } k_t < b_1 \\ (1-\delta)k_t + sW_t & \text{if } k_t \geq b_1 \end{cases} \tag{4}$$

while the recipient wealth at date $t+1$ is determined by:

$$W_{t+1} = \begin{cases} F[(1-\delta)k_t + sW_t + \lambda x_t] - Rx_t & \text{if } k_t < b_1 \\ F[(1-\delta)k_t + sW_t] & \text{if } k_t \geq b_1 \end{cases} \tag{5}$$

When there is no loan (i.e., $a_t = 0, \forall t$), the wealth becomes the production (i.e., $W_t = F(k_t)$, $\forall t$), and hence we recover the standard capital accumulation: $k_{t+1} = (1-\delta)k_t + sF(k_t)$.

[9] Here, we do not focus on the case where the recipient country can lend.

[10] Interest rates are higher for the lower and upper-middle income countries.

[11] For instance, Viet Nam's gross loan is amounted at 901,77 US$ million, representing around 42% of 2376 US$ million of received ODA in 2019.

Remark 1 (grants versus loans)**.** *When $R = 0, x_t = \bar{a}\ \forall t$, we recover the case of grants. The foreign aid amount verifies the following rule:*

$$\textit{(Rule of grant): } a_t = \begin{cases} \bar{a} & \textit{if } k_t < b_1 \\ 0 & \textit{if } k_t \geq b_1 \end{cases} \tag{6}$$

and the dynamics of capital follows a one-dimension dynamical system

$$k_{t+1} = \begin{cases} (1-\delta)k_t + sF(k_t) + \lambda\bar{a} & \textit{if } k_t < b_1 \\ (1-\delta)k_t + sF(k_t) & \textit{if } k_t \geq b_1 \end{cases} \tag{7}$$

We now describe the form of production function F. Following Le Van et al. (2016), we introduce a fixed cost in the production process.

Assumption 2 (production function with fixed cost)**.** *Assume that*

$$F(k_t) = \begin{cases} 0 & \textit{if } k_t < b_0 \\ Af(k_t - b_0) & \textit{if } k_t \geq b_0 \end{cases} \tag{8}$$

where A represents the exogenous total factor productivity, and $b_0 > 0$ the fixed cost. The function f is strictly increasing, concave. Moreover, $f'(0) = \infty$, $f'(\infty) = 0$, and $f(0) = 0$.

The presence of b_0 in Assumption 2 represents the fact that the economy has to pay a fixed cost before the production process takes place. The fixed costs in developing countries may be due to the lack of essential infrastructure such as road, rail, access to electricity, etc (see Le Van et al. (2016) for more details).

Our main objective is to explore the role of loans rule $(b_1, \bar{a}, R)$ as well as the threshold b_0 on the economic dynamics of the recipient. Before doing so, we adopt the standard definition of poverty trap as follows:

Definition 1 (collapse and poverty trap)**.**

1. *The economy collapses if $\lim_{t\to\infty} k_t = 0$.*
2. *A value $\bar{k}$ is called a poverty trap if, for any initial capital stock $k_0 \leq \bar{k}$, and we have $k_t \leq \bar{k}$ for any t high enough.*

Our formal definition of poverty trap means that a poor country ($k_0 \leq \bar{k}$) continues to be poor. It is in line with the notion of poverty trap in Azariadis and Stachurski (2005): A poverty trap is a self-reinforcing mechanism which causes poverty to persist.

2.3 Economic dynamics without development loans

The following proposition presents the transitional dynamics and the existence of a poverty trap in this economy in the absence of development loans.

Proposition 1 (poverty trap without development loans)**.** *In the absence of foreign capital flow, the dynamics of capital become*

$$k_{t+1} = \begin{cases} (1-\delta)k_t & \text{if } k_t < b_0 \\ (1-\delta)k_t + sAf(k_t - b_0) & \text{if } k_t \geq b_0 \end{cases}. \tag{9}$$

Denote $M_f \equiv \max_{k \geq 0} \left\{ sAf((k-b_0)^+) - \delta k \right\}$.

1. *If* $M_f < 0$, *then the economy collapses, i.e.,* $\lim_{t\to\infty} k_t = 0$ *for any* $k_0 > 0$.
2. *If* $M_f > 0$, *then* $k = (1-\delta)k + sAf\big(k-b_0)^+\big)$ *has two positive solutions,* $k^0_{low} < k^0_{high}$, *and these solutions are higher than* b_0.
   (a) *If the initial capital* k_0 *is strictly lower than* k^0_{low}, *then* k_t *converges to zero.*
   (b) *If the initial capital* k_0 *is strictly higher than* k^0_{low}, *then* k_t *converges to* k^0_{high}.
3. *If* $M_f = 0$, *then* $k = (1-\delta)k + sAf\big(k-b_0)^+\big)$ *has a unique positive solution, denoted by* k_s.
   (a) *If* $k_0 < k_s$, *then* k_t *converges to zero.*
   (b) *If* $k_0 \geq k_s$, *then* k_t *converges to* k_s.

Proof. See Appendix A. □

Figure 2 illustrates the transitional dynamics of capital in different cases. First, in the absence of fixed cost, we recover the standard Solow model (see the graph in the left side of Figure 2). In this case, the capital stock converges to a unique steady-state.

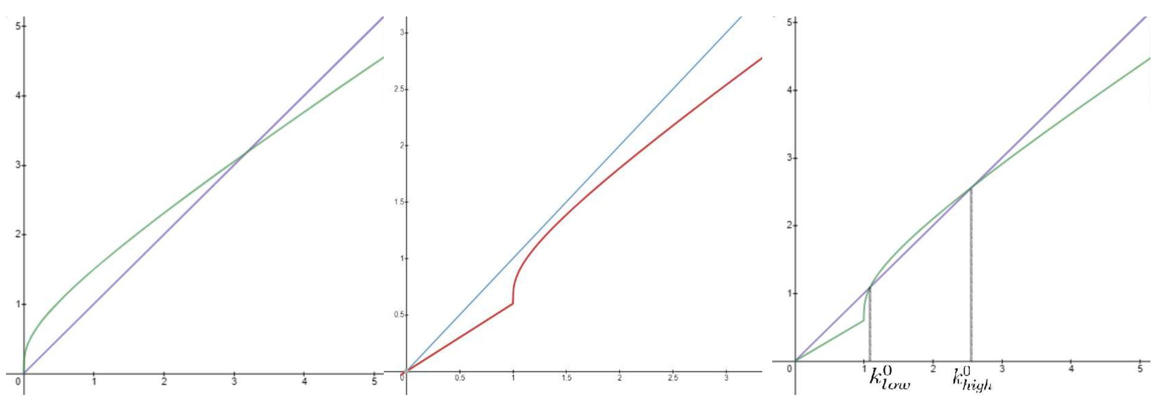


Figure 2: Dynamics of capital. LHS: without fixed cost. Middle: with fixed cost and low investment (no steady-state). RHS: with fixed cost and high investment (two steady-states).

Second, the presence of the fixed cost b_0 generates rich dynamics. Point 1 of Proposition 1 provides a necessary and sufficient condition under which the economy collapses whatever its initial capital level. This scenario, corresponding to the middle graph in Figure 2, happens when $M_f < 0$, i.e., the recipient country's characteristics are not favorable for the development process: technology level A and saving rate are low, the lack of infrastructure is important (i.e., fixed costs b_0 very high), the depreciation rate of capital δ is high.

When conditions in technology and investment are good enough (given δ) so that $M_f > 0$, point 2 of Proposition 1 indicates the existence of two steady-states k^0_{low}, k^0_{high}, which are shown in the graph in the right hand side. According to Proposition 1 and Definition 1, the low steady-state k^0_{low} is considered as a poverty trap for this economy. More precisely, for all initial capital k_0 higher than k^0_{low}, the economy will be on a convergent path towards the high steady-state k^0_{high} while the economy will collapse for all initial capital k_0 lower than k^0_{low}, even k_0 is higher than the fixed cost b_0.

3 Development loans and global dynamics

3.1 Static analysis: optimal choice of loans

In this section, given a period of time, we study the optimal choice of the recipient country regarding foreign loans. We obtain the following result.

Proposition 2 (optimal borrowing)**.** *Assume that the technology verifies Assumption 2. We consider the problem (P_t).*

$$(P_t): \quad \max_{0\leq x\leq \bar{a}} \Big\{ Af\Big(\big[(1-\delta)k_t + S_t + \lambda x - b_0\big]^+\Big) - Rx \Big\}$$

The optimal borrowed amount x_t is determined as follows:

1. $x_t = 0$ *if and only if one of the two following conditions holds:*
   - *(a)* $(1-\delta)k_t + S_t + \lambda \bar{a} \leq b_0$.
   - *(b)* $(1-\delta)k_t + S_t \geq b_0$ *and* $\lambda A f'\big[(1-\delta)k_t + S_t - b_0\big] < R$.
2. $x_t = \bar{a}$ *if and only if* $(1-\delta)k_t + S_t + \lambda\bar{a} > b_0$ *and* $\lambda A f'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] \geq R$.
3. $x_t = a^*_t$*, where* a^*_t *is uniquely determined by* $(1-\delta)k_t + S_t + \lambda a^*_t - b_0 > 0$ *and* $\lambda A f'\big((1-\delta)k_t + S_t + \lambda a^*_t - b_0\big) = R$*, if and only if one of two following conditions holds:*
   - *(a)* $(1-\delta)k_t + S_t \geq b_0$ *and* $\lambda A f'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] < R < \lambda A f'\big[(1-\delta)k_t + S_t - b_0\big]$.
   - *(b)* $(1-\delta)k_t + S_t < b_0 < (1-\delta)k_t + S_t + \lambda\bar{a}$ *and* $\lambda A f'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] < R$.

Proof. See Appendix A. □

Proposition 2 provides several insights. First, given the loans rule $(b_1, \bar{a}, R)$, a country does not necessarily choose the maximum amount $\bar{a}$ proposed by the donors. Indeed, point 1 shows us that the country does not borrow if either (i) the fixed costs are too high so that even if the country borrows the maximum level, the capital stock cannot overcome the fixed costs and/or (ii) the TFP A and the governance quality are low given the repayment rate R.

Second, the country borrows the maximum level if the TFP and the governance quality are relatively high, and the capital stock augmented by loans exceeds the fixed costs so that the problem related to the lack of infrastructure is solved and the production process (fonction (8)) takes place. In other words, we can state that given the loans rule $(b_1, \bar{a}, R)$, the recipient's choice of loans strongly depend on lack of infrastructure, its institutional and macroeconomique performance (TFP, capital productivity and governance quality), as well as on domestic investment rate and depreciation of capital.

3.2 Global dynamics: role of loans

This section focuses on the global dynamics of the economy in the presence of loans. We note that the dynamical system (4) and (5) is of two-dimension, non-linear, non-monotonic, non-continuous. Providing a full analysis is far from trivial. However, we obtain important and insightful properties. First, we look at the steady-states.

Definition 2. *A steady-state is a couple (k, a) such that a sequence of capital stock and foreign capital flow $(k_{t+1}, a_t)_{t\geq 0}$ determined by $k_t = k, a_t = a$, $\forall t \geq 0$ satisfies the dynamical system (4) and (5).*

We focus here on the steady-state with $a = \bar{a}$ because the steady-state with $a = 0$ has been already described in Proposition 1.

Lemma 1. *Denote $H(k) \equiv (1-\delta)k + s\big(Af(k-b_0) - R\bar{a}\big) + \lambda\bar{a}$.*

1. *(k, a) is a steady state with $a = \bar{a}$ if and only if*

$$k = H(k) \tag{10}$$
$$b_0 < k < b_1 \tag{11}$$
$$\lambda A f'(k-b_0) \geq R \tag{12}$$

2. *The equation $k = H(k)$ has at most two solutions on $[b_0, \infty)$. By consequence, there are at most two steady states with $a = \bar{a}$.*

3. *If $(\lambda - sR)\bar{a} + \max_{k\geq b_0}\big\{sAf((k-b_0)^+) - \delta k\big\} > 0$ and $(\lambda - sR)\bar{a} - \delta b_0 < 0$, then the equation $k = H(k)$ has two solutions (denoted by k^a_{low}, k^a_{high}, with $k^a_{low} < k^a_{high}$) on $[b_0, \infty)$. Moreover,*

   (a) For $k \in [k^a_{low}, k^a_{high}]$, we have $H(k) \geq k$.

   (b) For $k \in [b_0, k^a_{low}]$ or $[k^a_{high}, \infty)$, we have $H(k) \leq k$.

Notice that the existence of the steady-state $(k, \bar{a})$ requires that $b_0 < k < b_1$. Condition $k > b_0$ ensures a positive output while $k < b_1$ implies that this country is always eligible for receiving foreign loans at the steady-state. In addition, condition $\lambda A f'(k-b_0) \geq R$ means that the marginal productivity is higher or equal to its marginal cost, justifying a positive amount of loan.

Look at point 3 of Lemma 1. Conditions $(\lambda - sR)\bar{a} + \max_{k\geq b_0}\big\{sAf((k-b_0)^+) - \delta k\big\} \geq 0$ and $(\lambda - sR)\bar{a} - \delta b_0 < 0$ ensure the existence of two steady-states k^a_{low}, k^a_{high} in the presence of loans. Notice that k^a_{low}, k^a_{high} depend on $s, A, \delta, b_0, \lambda$ and the loans rule $(b_1, \bar{a}, R)$. Moreover, k^a_{low} is decreasing but k^a_{high} is increasing in $\bar{a}$. Observe also that

$$k^a_{low} < k^0_{low} < k^0_{high} < k^a_{high}, \tag{13}$$

where recall that k^0_{low} and k^0_{high}, defined in Proposition 1, correspond to the two steady-states in the absence of foreign aid. When $\bar{a} = 0$, k^a_{low} becomes k^0_{low} while k^a_{high} becomes k^0_{high}. This observation seems obvious as the level of poverty trap becomes lower with a positive amount of loan (i.e., we decrease from k^0_{low} to k^a_{low}), implying a higher opportunity for this economy to escape from its poverty trap, all things remain unchanged.

We are now ready to state our main result in this section.

Proposition 3 (role of development loans)**.** *Let Assumptions 1 and 2 be satisfied.*

1. *(Collapse with too high fixed cost and limited loans). Assume that* $R > 0$, $\lambda\bar{a} < \delta b_0$, *and* $(1-\delta)k_0 + sW_0 + \lambda\bar{a} \leq b_0$.

   (a) *The country never borrow:* $a_t = 0$, $\forall t \geq 0$.

   (b) $k_t < b_0$, $k_{t+1} = (1-\delta)k_t$, $\forall t \geq 1$, *and hence* k_t *converges to zero.*

2. *(Avoiding a collapse with loans at every period). Assume that* $(\lambda - sR)\bar{a} + \max_{k \geq b_0}\big\{sAf((k-b_0)^+) - \delta k\big\} > 0$ *and* $(\lambda - sR)\bar{a} - \delta b_0 < 0$. *Assume also that* $b_0 < k^a_{low} < k^a_{high} < b_1 < k^{bind}$.

   (a) *If* $(1-\delta)k_0 + sW_0 < k^0_{low}$ *and there is no aid, then* k_t *converges to zero.*

   (b) *If*

   $$k^a_{low} < (1-\delta)k_0 + sW_0 + \lambda\bar{a} < k^a_{high} \tag{14}$$

   then the country always borrows the maximum level ($a_t = \bar{a}$, $\forall t \geq 0$), *and* k_t *increasingly converges to* k^a_{high}.

3. *(Escaping poverty trap with loan only at the first period). Assume that* $\max_{k \geq 0} sAf((k-b_0)^+) - \delta k > 0$ *and that*

$$k_0 < b_1 \tag{15a}$$

$$\lambda Af'\big((1-\delta)k_0 + sW_0 + \lambda\bar{a} - b_0\big) \geq R \tag{15b}$$

$$(1-\delta)k_0 + sW_0 + \lambda\bar{a} > \max(k_0, b_1, b_0, k^0_{low}) \tag{15c}$$

$$s\Big(Af\big((1-\delta)k_0 + sW_0 + \lambda\bar{a} - b_0\big) - R\bar{a}\Big) > \delta\big((1-\delta)k_0 + sW_0 + \lambda\bar{a}\big). \tag{15d}$$

   Then k_t *converges to* k^0_{high}. *Moreover, the foreign aid at date 0 is* $a_0 = \bar{a}$ *but becomes zero from date 1:* $a_t = 0\ \forall t \geq 1$.

Proof. See Appendix A. □

Proposition 3 shows the role of development loans and figures out different economic mechanisms. Let us provide interpretations and link our theoretical findings to empirical evidences. The first part in Proposition 3 highlights a situation where a country is eligible for receiving foreign loans, but chooses a null amount. This scenario happens when the fixed costs b_0 and capital depreciation δ are relatively high, and other macroeconomic indicators (saving rate, TFP, governance quality) are bad. Meanwhile, the maximum loan allocated by the lenders could be relatively low so that $\lambda\bar{a} < \delta b_0$ and $(1-\delta)k_0 + sW_0 + \lambda\bar{a} \leq b_0$.

A null amount of loans may be chosen by countries having a considerable lack of infrastructure and a low macroeconomic and institutional performance. In this case, if choosing a positive amount of loans, these countries may not respect the rule of loans with obligation for repayment. Indeed, even the country borrows the maximum amount $\bar{a}$, its capital stock is not enough to overcome the fixed costs b_0. Hence, it is better not to borrow. Therefore, the capital stocks decreasingly converge to zero, as domestic saving is not sufficiently dynamic to cover capital depreciation in the absence of a grant or other external sources.

By consequence, if the amount of aid $\bar{a}$ is very limited and donors take care of the recipient countries' development, then they should offer a grant (which corresponds to the case $R = 0$) rather than a loan (i.e., $R > 0$) because a permanent grant helps the country to prevent a collapse while loans cannot.[12] Our point is in line with the practical policies of the French Development Agency (AFD) - a development bank and an implementing agency. Indeed, the AFD differentiates recipient countries based on their income-level: it focuses loans on emerging economies and grants on low-income countries.[13]

Our insight helps us to explain why there is no loan but only free grants in some very low-income countries such as Burundi, Central African Republic, Malawi, or Afghanistan, etc.[14] Actually, these countries are grant-dependent. For decades, their received foreign aid (in form of grant) always represents a considerable ratio in their GNI. For instance, in 2019, received net ODA of Afghanistan, Burundi, Central African Republic, and Malawi represent 22.9%, 19.5 %, 31.6%, and 16.1% of their GNI, respectively. Notice that these recipients are in the groupe of world's poorest countries. The GNI per capita is 540, 250, 520, and 380 Atlas US$ for Afghanistan, Burundi, Central African Republic, and Malawi, respectively. Foreign grants increasing their domestic revenues and investment, might prevent them a collapse.

Now, look at the second part of Proposition 3. Conditions $(\lambda - sR)\bar{a} - \delta b_0 < 0$ and $(\lambda - sR)\bar{a} + \max_{k \geq b_0} \big\{sAf((k-b_0)^+) - \delta k\big\} > 0$ (as stated in Lemma 1) guarantee that equation $k = H(k) \equiv (1-\delta)k + s\big(Af(k-b_0) - Rx\big) + \lambda x$ has two solutions k^a_{low} and k^a_{high}. Condition $k^a_{high} < b_1$ implies that the country is always eligible for receiving loans in the long run.

Part 2 highlights the role of permanent foreign aid (loan) in the dynamics of a recipient country where the initial situation is bad, and in the worse case, if $k_1 = (1-\delta)k_0 + sW_0 < k^0_{low}$

[12]Indeed, in the case of grant ($R = 0$, $a_t = \bar{a}\ \forall t$), our model becomes (6), (7). According to (7), we can see that the capital stock cannot converge to zero whatever the level of fixed cost b_0. In other words, a collapse is avoided thanks to permanent grants.

[13]AFD disburses 95% of France's ODA loans (2017) (The OECD's new way of counting ODA loans-what's the impac?-https://donortracker.org/)

[14]https://www.oecd.org/dac/financing-sustainable-development/development-finance-data/aid-at-a-glance.htm and https://stats.oecd.org/

(given $k_0 < k^0_{low}$), then the economy may collapse. In such a situation, a judicious choice of loans combined with mild initial conditions enables the economy to reduce its poverty trap. Consequently, the economy can escape from its poverty trap and converges to the high steady-state k^a_{high}. It should be noticed that even there is no risk for collapse in the absence of loans (i.e., $k_1 = (1-\delta)k_0 + sW_0 > k^0_{low}$), a permanent loan satisfying condition (14) allows this country to converge to the steady-state k^a_{high}.

Our theoretical points concerning development loans contribute to explain the economic dynamics of several recipient countries, in particular that in the lower-middle income group such as Vietnam, Tunisia, and Morocco where loan represents a high share in their received total ODA. For instance, in the case of Vietnam, gross loan is amounted at 901770 US$ million, representing around 42% of 2376 US$ million of received ODA in 2019. Gross loan represents 34.6% of Tunisia's received ODA while it represents around 28.8% of Morocco's received ODA in 2019. Figure 3 illustrates the evolution of loans in US$ million of these three lower-middle income countries over the period 2010-2019. We can imagine that the choice of permanent loans fits these dynamic economies, always in need of foreign aid, but having possibilities to repay in the future.

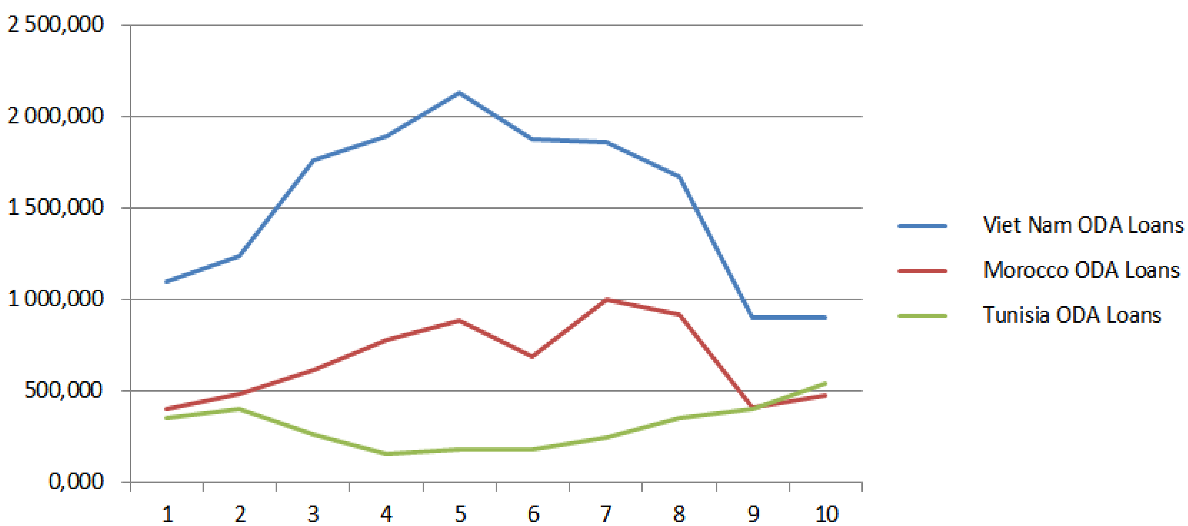


Figure 3: Evolution of loans of three lower-middle income countries over the period 2010-2019 in US$ million. Dataset: Creditor Reporting System-OECD (https://stats.oecd.org/).

The last part in Proposition 3 shows that development loan combined with a good macroeconomic and institutional performance can stimulate a strong dynamics of capital, which enables a recipient country to escape from the poverty trap. Here, the country needs to borrow only in the first period. A strong dynamics of capital is characterized not only by a high amount of loan but also by the recipient's high level of TFP A and high quality of governance λ as well as its low fixed costs b_0, given other variables.[15] In other words, if the TFP A, the governance quality λ and the maximum level of loan $\bar{a}$ are relatively high and the

[15]We do not introduce in our analysis variables representing macroeconomic performance such as debt policy and management, trade and financial sector, etc., which strongly explain an economy's dynamics. These factors are assumed to be exogenous and may be included in TFP A and fixed costs level b_0.

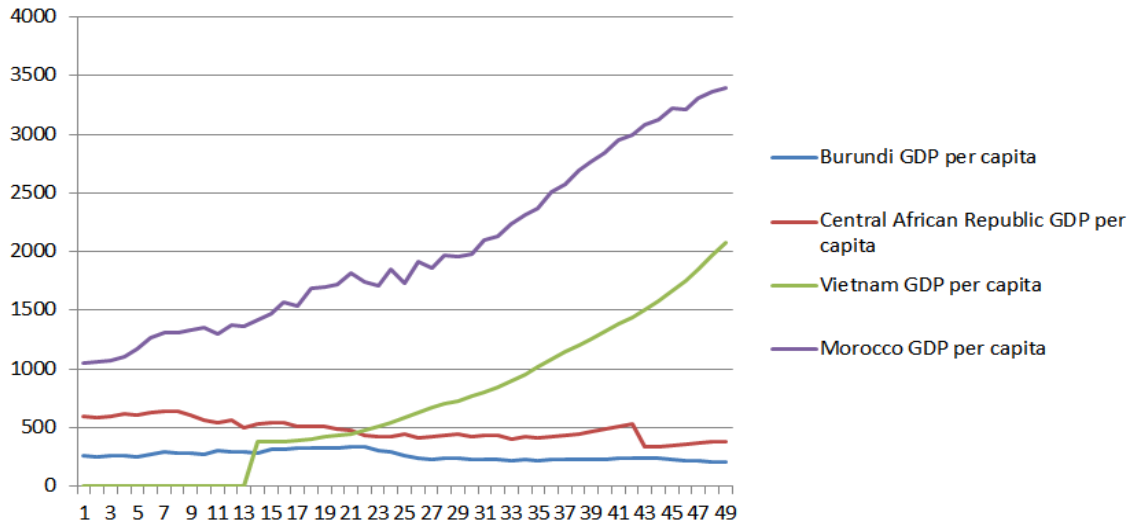


Figure 4: GDP per capita in 2010 US$ over the period 1971-2019 of 4 aid recipient countries (Viet Nam's GDP per capita is available since 1984). Source: the World Development Indicators of the World Bank.

interest rate R is low enough at a given period, then the country can overcome its fixed costs b_0 and the poverty trap can be avoided. A permanent development loan is not necessary in this situation as the capital stock converges autonomously to a high steady-state due to a strong domestic saving. Besides, the country is no longer eligible for concessional loan from the following period because its capital stock exceeds the critical threshold b_1. South Korea constitutes an interesting illustration for this result. Indeed, after the Korean war in the 1950s, foreign economic assistance was essential to the South Korea's economy with about US$ 3 billion of grant aid received before 1968, forming an average of 60% of investment of the country. Between 1964 and 1974, loans at concessionary rates of interest averaged around 6.5% of all foreign borrowing. South Korea's economy grew rapidly and cannot benefit from the concessional loans, but the country became increasingly integrated into the international capital market. From the late 1960s to the mid-1980s, its capital investment was financed from the competitive international market, and increasingly from domestic saving.[16]

The above discussions underline the role of the recipient's initial conditions and efficient use of foreign aid in the latter's impact on the recipient's outputs. Our discussion may be completed by Figure 4 which shows the evolution of GDP per capita of four aid recipient countries: Vietnam and Morocco as actual lower-middle-income countries, Burundi and the Central African Republic in the group of low-income and aid-dependent countries. These four countries were in the low-income group before the 1990s, but Burundi and the Central African Republic have a GDP per capita almost unchanged over a haft-century. Figure 4 is not a sufficient illustration for the efficient use of aid in Morocco and Vietnam. Still, we cannot ignore the correlation between foreign aid and their economic dynamics as they receive an essential amount of grants and loans during several years, which complete their capital formation. Foreign aid attained its highest share in 1990 in Morocco's capital formation

[16]Since 2010, South Korea has become a member of the OECD-DAC. For more information, see "South Korea: Aid, Loans, and Investment" (http://www.country-data.com)

(around 13.2%) and 1992 in Viet Nam's capital formation (around 32.3%). Even that the shares of ODA in GNI and in capital formation are decreasing, this international assistance should be an important factor, among others, to explain how Viet Nam and Morocco escape from their low-income threshold. Since the early 1990s, Morroco has a status of lower-middle income country while Viet Nam has transformed from one of the world's poorest economies to one of the fastest-growing economies. It attained lower-middle income status in 2010, whereas Figure 4 indicates that in the 1990s, Vietnam's GDP per capita was not pretty different from Burundi and the Central African Republic.[17]

3.3 Endogenous cycle

Since the dynamical system (4-5) is non-monotonic and non-continuous, we may expect a room for endogenous fluctuation. The following result formalizes our idea.

Proposition 4 (endogenous fluctuation and cycle)**.** *Assume that* $F(k) = A(k - b_0)^+$ *(i.e.,* $f(x) = x^+$*). Denote*

$$\gamma_1 \equiv \frac{\bar{a}\big[\lambda - sR(1-\delta)\big] - sAb_0(1-\delta)}{1-(1-\delta)(1-\delta+sA)}, \quad \gamma_0 \equiv \frac{\bar{a}\Big(\lambda(sA+1-\delta) - sR\Big) - sAb_0}{1-(1-\delta)(1-\delta+sA)} \tag{16}$$

Assume that $\gamma_1 > b_0 > b_1 > \gamma_0 > 0$, $1 > (1-\delta)(1-\delta+sA)$, $\lambda A > R$, $\lambda \bar{a} > b_0$.[18]

1. *If the initial capital stock* k_0 *is in the interval* $(0, b_1)$*, then we have, for any* $t \geq 0$*,*

$$a_{2t} = \bar{a}, \ a_{2t+1} = 0 \tag{17a}$$
$$k_{2t+1} > b_0 > b_1 > k_{2t} \tag{17b}$$
$$k_{2t+1} = (1-\delta)k_{2t} + \lambda\bar{a} \tag{17c}$$
$$k_{2t+2} = (1-\delta)k_{2t+1} + s\Big(A(k_{2t+1} - b_0) - R\bar{a}\Big) \tag{17d}$$
$$\lim_{t\to\infty} k_{2t+1} = \gamma_1 > b_0 > b_1 > \gamma_0 = \lim_{t\to\infty} k_{2t}. \tag{17e}$$

2. *If* $k_0 > b_0$ *and* $(1-\delta)k_0 + sW_0 < b_1$*, then we have, for any* $t \geq 0$*,*

$$a_{2t+1} = \bar{a}, \ a_{2t} = 0 \tag{18a}$$
$$k_{2t} > b_0 > b_1 > k_{2t+1} \tag{18b}$$
$$k_{2t+2} = (1-\delta)k_{2t+1} + \lambda\bar{a} \tag{18c}$$
$$k_{2t+1} = (1-\delta)k_{2t} + s\Big(A(k_{2t} - b_0) - R\bar{a}\Big) \tag{18d}$$
$$\lim_{t\to\infty} k_{2t} = \gamma_1 > b_0 > b_1 > \gamma_0 = \lim_{t\to\infty} k_{2t+1}. \tag{18e}$$

Proof. See Appendix A. □

[17] Data for Vietnam's gdp per capita is available from 1984.
[18] We can easily check that these conditions are not empty.

The key factors generating a two-period cycle and an endogenous fluctuation in Proposition 4 are: (i) a high depreciation rate δ, (ii) a low productivity A, and (iii) a generous rule of loans (a high borrowing bound $\bar{a}$ and a low interest rate R). Let us explain the mechanism of the cycle. At the period 0, the country's capital stock k_0 is lower than the threshold b_1. So, the country can borrow. Moreover, it borrows the maximum amount $\bar{a}$ because the interest rate R is low (i.e., $R < \lambda A$). This in turn implies that the capital stock at the first period k_1 becomes higher than the threshold b_1, which prevents the country from borrowing. Since the depreciation rate is high and the productivity is low, the capital stock in the second period k_2 goes down. Therefore, a two-period cycle arises. Figure 5 illustrates our mechanism.[19] Here, we simulate by setting $b_0 = 0.85, b_1 = 0.73, \delta = 0.4, s = 0.3, A = 0.8, \lambda = 0.9, R = 0.79, \bar{a} = 1$, and $k_0 = 0.5$

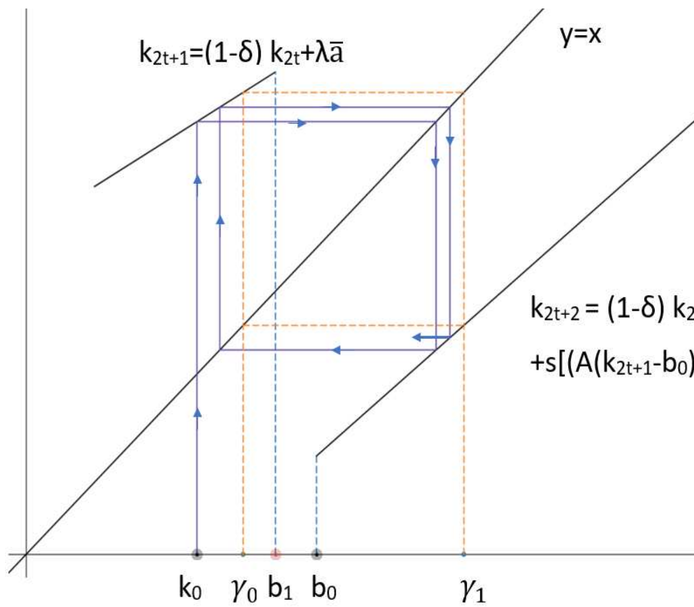


Figure 5: Dynamics of capital. For any $k_0 \in (0, b_1)$, we have $\lim_{t\to\infty} k_{2t+1} = \gamma_1, \lim_{t\to\infty} k_{2t} = \gamma_0$.

Recall that the dynamics of capital is given by the system (17c-17d) which is neither continuous nor increasing, as shown in Figure (5). This is the main source of the cycle in our model. In the literature, endogenous fluctuations have several sources such as complex dynamics, sunspot equilibria, learning (see Guesnerie and Woodford (1993) for an excellent survey on the literature of endogenous cycle in economics). Our cycle concerns the first one which suggests that the dynamics of a (simple) system can have very complex dynamics.

Classically, in two-sector optimal growth models, two-period cycles endogenously arise when the factor intensity of the capital good sector is larger than the factor intensity of the consumption sector (see, for instance, Le Van and Dana (2003)). Proposition 4 contributes to the literature by showing that introducing a development loan in a Solow model can generate a two-period cycle. The cycle we obtain comes from, one the one hand the rule of loans given by (3) which is a source of non-monotonicity of our dynamical system, and on

[19] In the general case, however, we have a two-dimensional dynamical system (see (4) and (5)), and hence, we cannot use Figure (2) to represent the dynamics of capital.

the other hand, the fundamentals of the economy such as its initial endowment, its TFP and the quality of its governance.[20]

Remark 2 (poverty dynamics)**.** *Proposition 4 is related to an important notion in the literature of poverty dynamics: transitory poverty escape, i.e., a situation where people used to live in poverty, succeeded in escaping poverty and then subsequently fell back into poverty (see Diwakar and Shepherp (2021) and references therein). For instance, according to Mariotti and Diwakar (2016), in rural Ethiopia between 1997 and 2000, 15% of all households experienced a transitory poverty escape. A two-period cycle in Proposition 4 can be interpreted as the situation where an agent successes in escaping poverty but then falls back into it. By the way, Proposition 4 provides a theoretical explanation for the presence of transitory poverty escapes. To sum up, Propositions 1, 3 and 4 show that the rule of loans (credit market), fixed costs, productivity (including human capital, quality of machines, ...) and governance quality are important factors for escaping poverty.*

4 Conclusion

We have studied the nexus between the poverty trap, economic dynamics, and international aid by introducing fixed cost and development loans in a growth model à la Solow. Our tractable framework generates several insights. First, due to the fixed cost, the country may have a poverty trap. In such a case, development loans or grants may be helpful for the recipient's development process. However, whether or not the country can overcome the poverty trap depends not only on the rule of loan but also, and mainly, on the country's capacity (such as the TFP, the depreciation rate, the saving rate, and the governance quality). Moreover, under the presence of development loans, an endogenous cycle may arise.

In our framework, the poverty trap appears due to the presence of a fixed cost, which may be related to the lack of infrastructure (electricity, internet, road system, etc.). Therefore, it would be interesting to endogenize fixed costs in future research and introduce elements enabling fighting them. By doing this, we can investigate the optimal choice of development loans in reducing fixed costs and boosting capital accumulation. Such an analysis is particularly relevant when the effectiveness of aid strongly depends on the manners in which aid is used in recipient countries and on the absorptive capacity of these countries.

We have taken the rule of loan $(b_1, \bar{a}, R)$ as exogenous. Another avenue for the future research is to design the optimal rule of aid $(b_1, \bar{a}, R)$ in the bilateral relationship between the lenders (donors) and the borrowers (the recipient).

A Appendix: Formal proofs

Proof of Proposition 1. Point 1 is obvious. Let us prove point 2 (the proof of point 3 is similar). Note that the function $sAf((k-b_0)^+)-\delta k$ is concave on $[0,\infty)$ and strictly concave on $[b_0,\infty]$. Moreover, $\lim_{k\to b_0} sAf((k-b_0)^+)-\delta k = -\delta b_0 < 0$ and $\lim_{k\to\infty} sAf((k-b_0)^+) - \delta k < 0$.

[20]See Walde (2005) and references therein for endogenous growth cycles.

(2a) If $k_0 < k^0_{low}$, using the induction argument, we can prove that $k_t < k_{t+1}$ $\forall t$. So, k_t converges to zero.

(2b) If $k_0 > k^0_{low}$, we can see that $k_0 \leq (\geq) k^0_{high}$, then k_t increasingly (decreasingly) converges to k^0_{high}.

□

Proof of Proposition 2. Observe that when $(1-\delta)k_t + S_t + \lambda x > b_0$, the objective function $Af((1-\delta)k_t + S_t + \lambda x_t - b_0) - Rx_t$ has the derivative $\lambda Af'((1-\delta)k_t + S_t + \lambda x_t - b_0) - R$ which is decreasing in x_t. We now consider all possible cases:

1. If $(1-\delta)k_t + S_t + \lambda \bar{a} \leq b_0$ then $x_t = 0$. We obtain case 1.a in Proposition 2
2. If $(1-\delta)k_t + S_t \geq b_0$, there are three sub-cases:
   (a) If $\lambda Af'\big[(1-\delta)k_t + S_t - b_0\big] \leq R$, then we have $x_t = 0$. We obtain case 1.b in Proposition 2
   (b) If $\lambda Af'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] \geq R$, then we have $x_t = \bar{a}$. We obtain case 2 in Proposition 2
   (c) If $\lambda Af'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] < R < \lambda Af'\big[(1-\delta)k_t + S_t - b_0\big]$, then we have $x_t = a^*_t$ where a^*_t is uniquely determined by $\lambda Af'\big([(1-\delta)k_t + S_t + \lambda a^*_t - b_0]^+\big) = R$. We obtain case 3.a in Proposition 2
3. If $(1-\delta)k_t + S_t < b_0 < (1-\delta)k_t + S_t + \lambda\bar{a}$, then there are two sub-cases:
   (a) If $\lambda Af'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] \geq R$, then we have $x_t = \bar{a}$. We obtain case 2 in Proposition 2
   (b) If $\lambda Af'\big[(1-\delta)k_t + S_t + \lambda\bar{a} - b_0\big] < R$, then we have $x_t = a^*_t$ where a^*_t is uniquely determined by $(1-\delta)k_t + S_t + \lambda a^*_t - b_0 > 0$ and $\lambda Af'\big((1-\delta)k_t + S_t + \lambda a^*_t - b_0\big) = R$. We obtain case 3.b in Proposition 2

□

Proof of Proposition 3. **Point 1**. We will prove that $k_t < b_0$, $a_t = 0$ $\forall t$. Indeed, we have

$$k_1 = (1-\delta)k_0 + sW_0 + \lambda a_0 \leq (1-\delta)k_0 + sW_0 + \lambda\bar{a} \leq b_0. \tag{19}$$

Since $k_1 \leq b_0$ and $R > 0$, we have $a_0 = 0$. So, our claim holds at the initial date. Suppose now that $k_t < b_0$, $a_{t-1} = 0$. Consider date $t+1$. We have

$$k_{t+1} = (1-\delta)k_t + s\Big(Af((k_t - b_0)^+) - Ra_{t-1}\Big) + \lambda a_t \tag{20}$$

$$= (1-\delta)k_t + \lambda a_t \leq (1-\delta)b_0 + \lambda\bar{a} \leq b_0 \tag{21}$$

where the last inequality follows from $\lambda\bar{a} \leq \delta b_0$. Since $k_{t+1} \leq b_0$ and $R > 0$, we have $a_t = 0$. Therefore, our claim is true. By consequence, we have $k_{t+1} = (1-\delta)k_t, \forall t \geq 1$, and, hence, k_t converges to zero.

Point 2. To prove point 2, we need an intermediate step.

Lemma 2. *Assume that $b_0 < k^a_{low} < k^a_{high} < b_1$, $k_t \in [k^a_{low}, k^a_{high}]$ and*

$$k_{t+1} = (1-\delta)k_t + s\big(Af(k_t - b_0) - R\bar{a}\big) + \lambda\bar{a} \tag{22a}$$

then $k_{t+1} \geq k_t$ and $k_{t+1} \in [k^a_{low}, k^a_{high}]$.

Proof. According to Lemma 1, for all $k_t \in [k^a_{low}, k^a_{high}]$, we have $k_{t+1} = H(k_t) \geq k_t$. Moreover, we have $k_{t+1} \in [k^a_{low}, k^a_{high}]$ because

$$k^a_{low} = H(k^a_{low}) \leq H(k_t) = k_{t+1} \leq H(k^a_{high}) = k^a_{high}. \tag{23}$$

□

According to point 2 of Proposition 2, conditions $b_0 < (1-\delta)k_0 + sW_0 + \lambda\bar{a} < k^{bind}$ imply that $a_0 = \bar{a}$. Hence, $k_1 = (1-\delta)k_0 + sW_0 + \lambda\bar{a}$. Therefore, $k_1 \in [k^a_{low}, k^a_{high}]$ because $k^a_{low} \leq (1-\delta)k_0 + sW_0 + \lambda\bar{a} \leq k^a_{high}$ (condition (14)).

Since $a_0 = \bar{a}$, we have $W_1 = Af(k_1 - b_0) - R\bar{a}$ and hence

$$k_2 = (1-\delta)k_1 + sW_1 + \lambda a_1 = (1-\delta)k_1 + s\big(Af(k_1 - b_0) - R\bar{a}\big) + \lambda a_1 \tag{24}$$

We have

$$(1-\delta)k_1 + s\big(Af(k_1 - b_0) - R\bar{a}\big) + \lambda\bar{a} = H(k_1) \geq k^a_{low} > b_0 \tag{25}$$

$$(1-\delta)k_1 + s\big(Af(k_1 - b_0) - R\bar{a}\big) + \lambda\bar{a} = H(k_1) \leq k^a_{high} < k^{bind} \tag{26}$$

According to point 2 of Proposition 2, these conditions imply that $a_1 = \bar{a}$, and hence $k_2 = H(k_1)$.

By induction and using Lemma 2, we have $a_t = \bar{a}\ \forall t \geq 0$, and $k_t = H(k_{t-1}) \in [k^a_{low}, k^a_{high}]$, $\forall t \geq 1$. According to Lemma 2, it is easy to see that k_t increasingly converges to k^a_{high}.

Point 3. Since $k_0 < b_1$, we have

$$a_0 = \arg\max_{x \in [0,\bar{a}]} Af\Big(\big((1-\delta)k_0 + sW_0 + \lambda x - b_0\big)^+\Big) - Rx. \tag{27}$$

By assumptions (15b) and (15c), we have $(1-\delta)k_0 + sW_0 + \lambda\bar{a} - b_0 > 0$ and $\lambda Af'\big((1-\delta)k_0 + sW_0 + \lambda\bar{a} - b_0\big) \geq R$. By consequence, Proposition 2 implies that $a_0 = \bar{a}$. Therefore, the capital stock at date 1 is $k_1 = (1-\delta)k_0 + sW_0 + \lambda\bar{a}$.

Condition (15c) implies that $k_1 > b_1$, and hence $a_1 = 0$. Therefore, we have

$$k_2 = (1-\delta)k_1 + s\Big(Af\big((1-\delta)k_0 + sW_0 + \lambda a_0 - b_0\big) - Ra_0\Big) + \lambda a_1 \tag{28}$$

$$= (1-\delta)k_1 + s\Big(Af\big((1-\delta)k_0 + sW_0 + \lambda\bar{a} - b_0\big) - R\bar{a}\Big) \tag{29}$$

$$> (1-\delta)k_1 + \delta\big((1-\delta)k_0 + sW_0 + \lambda\bar{a}\big) = k_1 \tag{30}$$

The last inequality follows from (15d). So, $k_2 > k_1$. By induction, we have $k_{t+1} > k_t\ \forall t \geq 1$. Hence, $k_t > b_1$, $\forall t \geq 1$, which implies that $a_t = 0\ \forall t \geq 1$.

Since k_t is increasing and bounded, it converges to a steady state. Since $k_t > k^0_{low}$, we have $\lim_{t\to\infty} k_t = k^0_{high}$.

□

Proof of Proposition 4. First of all, since $F(k) = A(k - b_0)^+$ and $\lambda A > R$, according to Proposition 2, we can easily compute x_t as follows

$$x_t = \arg\max_{0 \le x \le \bar{a}} \left\{ A[(1-\delta)k_t + sW_t + \lambda x - b_0]^+ - Rx \right\} = \begin{cases} \bar{a} & \text{if } (1-\delta)k_t + sW_t + \lambda\bar{a} > b_0 \\ 0 & \text{if } (1-\delta)k_t + sW_t + \lambda\bar{a} \le b_0 \end{cases}$$

Since we are assuming that $\lambda A > R$, we have $x_t = \bar{a}\ \forall t \ge 0$.

1. Let $k_0 \in (0, b_1)$. Since $k_0 < b_1$ and $b_1 < b_0$, we have $k_0 < b_0$ which implies that $W_0 = 0$. Again, by $k_0 < b_1$, we have $a_0 = x_0 = \bar{a}$. Thus,

$$k_1 = (1-\delta)k_0 + sW_0 + a_0 = (1-\delta)k_0 + \lambda\bar{a}$$

Observe that $k_1 \ge \lambda\bar{a} > b_0 > b_1$. This implies that $a_1 = 0$, and hence

$$\begin{aligned} k_2 &= (1-\delta)k_1 + sW_1 + a_1 = (1-\delta)k_1 + s\Big(A(k_1 - b_0) - R\bar{a}\Big) \\ \text{and } k_2 - \gamma_0 &= (1-\delta)k_1 + s\Big(A(k_1 - b_0) - R\bar{a}\Big) - \gamma_0 \\ &= (1-\delta+sA)((1-\delta)k_0 + \lambda\bar{a}) - sAb_0 - sR\bar{a} - \gamma_0 \\ &= (1-\delta+sA)(1-\delta)(k_0 - \gamma_0) \end{aligned}$$

From this, we have

$$k_2 - b_1 = (1-\delta+sA)(1-\delta)(k_0 - \gamma_0) - (b_1 - \gamma_0) < 0$$

because $(1-\delta+sA)(1-\delta) < 1$, $k_0 - \gamma_0 < b_1 - \gamma_0$, and $b_1 - \gamma_0 > 0$. It means that $k_2 < b_1$. This condition implies that $a_2 = x_2 = \bar{a}$ and $k_2 < b_0$. Thus, we have $W_2 = 0$ and

$$\begin{aligned} k_3 &= (1-\delta)k_2 + sW_2 + a_2 = (1-\delta)k_2 + \lambda\bar{a} \\ &= (1-\delta)\Big((1-\delta)k_1 + s\Big(A(k_1 - b_0) - R\bar{a}\Big)\Big) + \lambda\bar{a} \\ &= (1-\delta)(1-\delta+sA)k_1 - (1-\delta)sAb_0 - (1-\delta)sR\bar{a} + \lambda\bar{a} \\ \text{and } k_3 - \gamma_1 &= (1-\delta)(1-\delta+sA)(k_1 - \gamma_1). \end{aligned}$$

By induction, we obtain conditions (17a-17d), and hence

$$\begin{aligned} k_{2t+2} - \gamma_0 &= (1-\delta)(1-\delta+sA)(k_{2t} - \gamma_0) \\ k_{2t+1} - \gamma_1 &= (1-\delta)(1-\delta+sA)(k_{2t-1} - \gamma_1) \end{aligned}$$

By consequence, we obtain the convergence.

2. Let $k_0 > b_0$ and $(1-\delta)k_0 + sW_0 < b_1$.

Observe that $k_0 > b_1$ because $b_0 > b_1$. So, the country is not eligible for the development loan. This implies that $a_0 = 0$, and hence

$$k_1 = (1-\delta)k_0 + sW_0 + a_0 = (1-\delta)k_0 + sW_0 = (1-\delta)k_0 + s\Big(A(k_0 - b_0) - R\bar{a}\Big).$$

Since $(1-\delta)k_0 + sW_0 < b_1$, we have $k_1 < b_1$. So, the country is now eligible for the development loan. This condition implies that $a_1 = x_1 = \bar{a}$ and $k_1 < b_0$. Thus, we have $W_1 = 0$ and

$$\begin{aligned}
k_2 &= (1-\delta)k_1 + sW_1 + a_1 = (1-\delta)k_1 + \lambda\bar{a} \\
&= (1-\delta)\Big((1-\delta)k_0 + s\Big(A(k_0 - b_0) - R\bar{a}\Big)\Big) + \lambda\bar{a} \\
&= (1-\delta)(1-\delta+sA)k_0 - (1-\delta)sAb_0 - (1-\delta)sR\bar{a} + \lambda\bar{a} \\
\text{and } k_2 - \gamma_1 &= (1-\delta)(1-\delta+sA)(k_0 - \gamma_1).
\end{aligned}$$

Since $k_2 = (1-\delta)k_1 + \lambda\bar{a} > \lambda\bar{a} > b_0$, we have $a_2 = 0$, and hence,

$$\begin{aligned}
k_3 &= (1-\delta)k_2 + sW_2 + a_2 = (1-\delta)k_2 + sW_2 = (1-\delta)k_2 + s\Big(A(k_2 - b_0) - R\bar{a}\Big) \\
&= (1-\delta+sA)k_2 - sAb_0 - sR\bar{a} \\
&= (1-\delta+sA)\Big((1-\delta)k_1 + \lambda\bar{a}\Big) - sAb_0 - sR\bar{a} \\
&= (1-\delta+sA)(1-\delta)k_1 + (1-\delta+sA)\lambda\bar{a} - sAb_0 - sR\bar{a} \\
k_3 - \gamma_0 &= (1-\delta+sA)(1-\delta)(k_1 - \gamma_0)
\end{aligned}$$

Observe that

$$\begin{aligned}
&k_3 - b_1 = (1-\delta+sA)(1-\delta)(k_1 - \gamma_0) + \gamma_0 - b_1 \\
&\leq (1-\delta+sA)(1-\delta)(b_1 - \gamma_0) - (b_1 - \gamma_0) = \Big((1-\delta+sA)(1-\delta) - 1\Big)(b_1 - \gamma_0) < 0
\end{aligned}$$

because $(1-\delta+sA)(1-\delta) < 1$ and $b_1 > \gamma_0$.

By induction, we get (18a-18d) and hence

$$k_{2t+2} - \gamma_0 = (1-\delta)(1-\delta+sA)(k_{2t} - \gamma_1),\ k_{2t+1} - \gamma_1 = (1-\delta)(1-\delta+sA)(k_{2t-1} - \gamma_0)$$

By consequence, we obtain the convergence.

□

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