Local incentive compatibility in non-convex type-spaces

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Abstract

We explore the equivalence of local incentive compatibility (LIC) (Carroll (2012)) and incentive compatibility (IC) in non-convex type-spaces. We provide a sufficient condition on a type-space called minimal richness for the said equivalence. Using this result, we show that LIC and IC are equivalent on large class of non-convex type-spaces such as type-spaces perturbed by modularity and concave-modularity. The gross substitutes type-space and the generalized gross substitutes and complements type-space are important examples of type-spaces perturbed by modularity and concave-modularity, respectively. Finally, we provide a geometric property consisting of three conditions for the equivalence of LIC and IC, and show that all the conditions are indispensable.

JEL Classification: D82, D44, D47

Keywords: local incentive compatibility, (global) incentive compatibility, non-convex type-spaces, minimally rich type-spaces, gross substitutes type-space, generalized gross substitutes and complements type-space

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1. INTRODUCTION

We consider standard mechanism design problem where a set of agents have valuations for each alternative in a finite set of alternatives. Based on these valuations, the planner has to select an alternative to be shared by all the agents and some payment for each agent. Such a decision scheme is called a mechanism.

Agents evaluate their net utilities by means of quasilinear utility functions. A mechanism is incentive compatible (IC) if no agent can increase his/her net utility by misreporting his/her type. A mechanism is locally IC (LIC) if no agent can increase his/her net utility by misreporting to a type that is "close" to his/her sincere type.

An important problem in mechanism design is to characterize all IC mechanisms for a given type-space. Except for the case when the type-space is $\mathbb{R}^{|A|}$, where A is the set of alternatives, this turns out to be a hard problem. As an intermediate step, researchers have got interested in exploring if the requirement of IC can be reduced considerably.[1] Local IC (LIC) (Carroll (2012)) turns out to be a way.

LIC ensures that a mechanism is IC on the types that are sufficiently close (with respect to Euclidean distance) to each other. More formally, LIC requires that for every type t there is a neighborhood of t such that the mechanism is IC on both (t,s) and (s,t) for every type s in that neighborhood.[2] Carroll (2012) showed that LIC is equivalent to IC on any convex type-space. To the best of our knowledge, nothing is known about the said equivalence on other type-spaces, despite the fact that there are several important non-convex type-spaces such as the gross substitute one in combinatorial auction.[3]

The crucial fact about convex type-space is that the line joining any two types lie in the type-space. A natural step to get out of the convex type-space would be to consider a type-space that is polygonally connected: between every two types there is a (finite) sequence of lines in the type-space that join them. However, polygonal connectedness alone cannot guarantee the equivalence of LIC and IC (See Examples 5.1, 5.2 and 5.3). We strengthen it by introducing a condition called minimal richness and show that it is sufficient for the equivalence of LIC and IC. As applications of our result, we show that LIC and IC are equivalent on large class of non-convex type-spaces such as type-spaces perturbed by modularity and concave-modularity. Further, we show that the gross substitutes type-space and the generalized gross substitutes and complements type-space are important examples of type-spaces perturbed by modularity and concave-modularity, respectively.

The Gross substitutes type-space has been extensively studied in the literature in various contexts such as matching, mechanism design, equilibrium and algorithms (see Ausubel and Milgrom (2002), Gul

[1] For the importance of identifying a minimal set of incentive constraints that will imply full incentive compatibility - see discussions in Chapter 7 of Fudenberg and Tirole (1991), Armstrong (2000) and Chapter 6 in Vohra (2011).

[2] A mechanism is IC on a pair of types (t,s) if an agent with sincere type t cannot manipulate (that is, cannot increase his/her net utility) by reporting the type as s.

[3] It is worth mentioning that characterizing all type-spaces where LIC and IC are equivalent is a long standing open problem and is considered as a hard problem as well.

and Stacchetti (1999), Paes Leme (2017)). The gross-substitutability condition was first introduced by Kelso and Crawford (1982) in the context of two sided matching markets of workers and firms. They showed that gross-substitutability is a sufficient condition for the existence of Walrasian equilibria. Later, Shioura and Yang (2015) generalized the gross-substitutability condition to generalized gross substitutes and complements condition where they allow multiple objects of the same kind and also allow for some complementarities across objects.

Recently Kushnir and Lokutsievskiy (2021) proved that every monotone allocation function defined on the gross-substitutes type space and the generalized gross substitutes and complements type-space is also cyclically monotone.[4] Our paper compliments their paper by establishing the equivalence of LIC and IC and thereby making the problem of designing mechanisms quite tractable on these domains.

Next, we provide a geometric condition on a type-space for the equivalence of LIC and IC. We identify three conditions and show that together these conditions ensure the equivalence of LIC and IC. Further, we show that these three conditions are indispensable, that is, if we drop any of the conditions, then the equivalence of LIC and IC is no longer guaranteed.

2. Preliminaries

We consider a one-agent model in this paper. This is without loss of generality for our analysis.[5]

Let A be a finite set of alternatives with $|A| = n$. For any given subset X of $\mathbb{R}^n$, by $\partial(X)$ we denote the boundary of X. A type t is a mapping from A to $\mathbb{R}$ that represents the valuation of each alternative in A. We view a type as an element of $\mathbb{R}^n$ (with an arbitrary but fixed indexation of the alternatives). By relative valuation of an alternative a with respect to another alternative b at a type t, we mean the number $t(a) - t(b)$. For two types t and t', we denote the line joining them by $[t,t']$.[6] A subset T of $\mathbb{R}^n$ is called a type-space. A polygonal path from t to t' in T is a finite collection of types $(t = t^1, \ldots, t' = t^k)$ such that $[t^l, t^{l+1}]$ lies in T for all $l \in \{1, \ldots, k-1\}$. A type-space T is polygonally connected if for every $t, t' \in T$, there exists a polygonal path from t to t' in T. An allocation rule is a map $f : T \to A$ and a payment rule is a map $p : T \to \mathbb{R}$. A (direct) mechanism μ is a pair consisting of an allocation rule f and a payment rule p.

[4] An allocation function f on a type-space T is monotone (or, 2-cycle monotone) if for all $t, t' \in T, t(f(t)) - t(f(t')) + t'(f(t')) - t'(f(t)) \geq 0$, and it is cyclically monotone if for any integer M and any points $t^0, t^1, \ldots, t^M = t^0$ in T, $\sum_{k=0}^{M-1} t^k(f(t^k)) - t^k(f(t^{k+1})) \geq 0$.

[5] All the results of this paper can be generalized to the case of more than one agent in a systematic manner (see Carroll (2012), Mishra et al. (2016), etc.).

[6] More formally, $[t,t'] = \{\alpha t + (1-\alpha)t' \mid \alpha \in [0,1]\}$.

Definition 2.1. A mechanism (f,p) is *incentive compatible* (IC) on a pair of types (t,s) if

$$t(f(t)) - p(t) \geq t(f(s)) - p(s).$$

It is IC on a type-space T if it is IC on every pair of types $(t,s) \in T \times T$.

The notion of local incentive compatibility (LIC) is introduced in Carroll (2012). A mechanism is LIC on a type-space T if for every $t \in T$, there exists an $\varepsilon > 0$ such that it is IC on (t,s) and (s,t) for every $s \in T$ with $||t-s|| < \varepsilon$.[7]

3. RESULT ON MINIMALLY RICH TYPE-SPACES

We introduce the notion of minimally rich type-spaces in this section and show that LIC and IC are equivalent on such type-spaces. As an application in Section 4, we consider type-spaces that arise in the context of combinatorial auctions and show that any type-space that is closed under scaling and closed under modular/concave-modular perturbations is also minimally rich. Gross substitutes (GS) and the generalized gross substitutes and complements (GGSC) type-spaces are important examples of such type-spaces.

A type-space is minimally rich if for any two types t and t' in it and for each alternative a, there is a type s satisfying the following two properties: (i) the lines joining s to both t and t' lie in the type-space, and (ii) for every alternative z, if the relative valuation of a with respect to z (weakly) increases from s to t', then it will also (weakly) increase from from t to s. Notice that minimally rich type-spaces are polygonally connected.

Definition 3.1. A type-space T is **minimally rich** if for all distinct $t,t' \in T$ and all $a \in A$, there exists $s \in T$ such that

(i) $[s,t]$ and $[s,t']$ lie in T, and

(ii) $s(a) - s(z) \geq t(a) - t(z)$ for all $z \in A$ such that $t'(a) - t'(z) \geq s(a) - s(z)$.

We explain the implication of minimal richness with some figures for the case where there are two-dimensions (that is, two objects). Let $A = \{a,b\}$. Suppose the valuation of a is represented on the horizonal axis and the valuation of b is represented on the vertical axis. Consider two types t and t'. Without loss of generality, assume $t(a) - t(b) \leq t'(a) - t'(b)$. See Figure 1 for such two types t and t'. Suppose that the line $[t,t']$ does not lie in the type-space (this is not shown in the figure). To satisfy the minimal richness condition for t and t', we need to find two types s and $\bar{s}$ (not necessarily distinct) for a

[7] We denote the Euclidean norm of a vector $t \in \mathbb{R}^n$ by $||t||$.

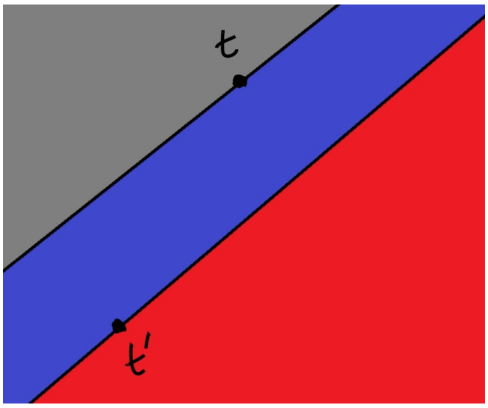

Figure 1

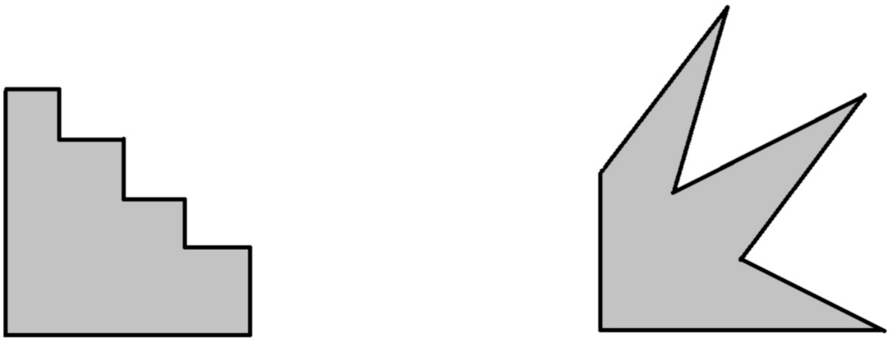

Figure 2

and b, respectively, such that the lines $[s,t]$, $[s,t']$, $[\bar{s},t]$ and $[\bar{s},t']$ lie in the type-space, s lies strictly below the slope 1 line passing through t, and $\bar{s}$ lies strictly above the slope 1 line passing through t'. In Figure 1, the shaded portions in red and blue are the feasible regions for s, and the shaded portions in grey and blue are the feasible regions for $\bar{s}$. It is worth noting that there are so many choices for s and $\bar{s}$, which in turn corroborates that the minimal richness condition is not much demanding. In Figure 2, we have provided two minimally rich type spaces (marked by the shaded region), one can verify in above discussed way that they are indeed minimally rich.

Theorem 3.1. *A mechanism on a minimally rich type-space is IC if and only if it is LIC.*

The proof of this Theorem is relegated to Appendix A.2.

4. Application to combinatorial auction model

Combinatorial auctions are mechanisms where agents are asked to report valuations for combinations of objects, often referred to as “bundles” or “packages”, instead of individual objects. Thus, agents are allowed to express their preferences more fully which often leads to greater auction revenues and improved economic efficiency. In what follows, we present two important classes of type-spaces that arise in the context of combinatorial auction model.

4.1 Type-spaces perturbed by modularity

Let $E = \{1,\ldots,k\}$ be the set of objects. The set of alternatives is $A = 2^E$, that is, the set of all possible subsets of E. For ease of presentation, let us denote the cardinality of A by n, that is, $n = 2^k$. Thus, a type is an element of $\mathbb{R}^n$.

We say that a type-space $T \subseteq \mathbb{R}^n$ is closed under scaling if for any $t \in T$ and any scalar $\lambda \geq 0$, we have $\lambda \cdot t \in T$.[8] Given $t \in T$ and a vector $m \in \mathbb{R}^k$, we define a type $t_m \in \mathbb{R}^n$ where $t_m(S) = t(S) + \sum_{i \in S} m(i)$ for every $S \subseteq E$. We say that T is closed under modular perturbations if for any $t \in T$ and any vector $m \in \mathbb{R}^k$, we have $t_m \in T$. A type t is **modular** if $t(S) = \sum_{i \in S} t(i)$ for all $S \subseteq E$.

Proposition 4.1. *Let $T \subseteq \mathbb{R}^n$ be a type-space that is closed under scaling and closed under modular perturbations. Then, T is minimally rich.*

The proof of this proposition is relegated to Appendix A.3.

An important example of a type-space that is closed under scaling and closed under modular perturbations is the gross substitutes type-space.[9]

The gross substitutes type-space is well-studied in the literature in the context of matching, auction, etc., (see, e.g., Murota (2016) and Paes Leme (2017) for extensive surveys). This notion was introduced by Kelso and Crawford (1982) as a sufficient condition for the existence of Walrasian equilibrium.

A price p (vector) for individual objects in E is an element of $\mathbb{R}^k$. The price of a bundle S is $p(S) = \sum_{i \in S} p(i)$. The demand correspondence for a price $p \in \mathbb{R}^k$ and a type t is defined as

$$D(t,p) = \arg\max_{S \subseteq E} \{t(S) - p(S)\}.$$

In other words, the demand correspondence for p and t contains those bundles whose net valuation (valuation minus price) according to p and t is the maximum.

A type t satisfies the gross-substitutability condition if, roughly speaking, its demand correspondence satisfies a (partial) independence property with respect to (increasing) price. This is in the sense that if we increase the price of some objects (while keeping that unchanged for the others), then, in some sense, the "demand" of the objects whose prices are not changed will not be affected. More formally, if we go from one price vector to a higher price vector (that is, if we weakly increase the price of each object),

[8] For any type $t = (t_1,\ldots,t_n) \in T$ and any scalar $\lambda \geq 0$, $\lambda \cdot t = (\lambda t_1,\ldots,\lambda t_n)$.

[9] The fact that gross substitutes type-space is closed under scaling and closed under modular perturbations is well known in the literature.

then for each demanded bundle S at the former price there will be a demanded bundle S' at the higher price that contains all objects in S whose prices are not changed.

Definition 4.1 (Kelso and Crawford (1982))**.** A type t satisfies the gross-substitutability condition if for all $p, p' \in \mathbb{R}^k$ with $p' \geq p$, we have $S \in D(t,p)$ implies there exists $S' \in D(t,p')$ with $\{i \in S \mid p(i) = p'(i)\} \subseteq S'$.

We now introduce the notion of submodular types. A type t is **submodular** if for all $S \subseteq E$ and all distinct $i, j \notin S$, we have $t(\{i,j\} \cup S) + t(S) \leq t(\{i\} \cup S) + t(\{j\} \cup S)$.

In what follows, we present a characterization of gross-substitutability condition purely in terms of inequalities involving the agent's valuations given by Reijnierse et al. (2002) and Fujishige and Yang (2003). It says that a type t satisfies gross-substitutability condition if and only if it is submodular and satisfies a technical condition. For an intuitive understanding of the technical condition consider a type t, a set of objects S, and three objects i, j, k that are not in S. Consider the three numbers $t(\{i, j\} \cup S) + t(\{k\} \cup S)$, $\{t(\{i,k\} \cup S) + t(\{j\} \cup S)$, and $t(\{j,k\} \cup S) + t(\{i\} \cup S)\}$. The technical condition (part (ii) of Proposition 4.2) says that out of these three numbers two will be equal and the other will be weakly lower.

Proposition 4.2 (Reijnierse et al. (2002))**.** *A type t satisfies the gross-substitutability condition if and only if*

(i) t is submodular, and

(ii) for all $S \subseteq E$ and all distinct $i, j, k \notin S$,

$$t(\{i,j\} \cup S) + t(\{k\} \cup S) \leq \; max\{t(\{i,k\} \cup S) + t(\{j\} \cup S), t(\{j,k\} \cup S) + t(\{i\} \cup S)\}.$$

REMARK 4.1. Modular types satisfy the gross-substitutability condition. Furthermore, if $|E| = 2$ (say $E = \{i, j\}$), then any type t satisfies the gross-substitutability condition if and only if $t(\{i, j\}) + t(\emptyset) \leq t(\{i\}) + t(\{j\})$.

A type-space is gross substitutes if it contains all types satisfying the gross-substitutability condition. It is well-known that the gross substitutes type-space is not convex.

The following corollary is obtained from Theorem 3.1, Proposition 4.1 and the fact that the gross substitutes type-space is closed under scaling and closed under modular perturbations.

Corollary 4.1. *A mechanism on the gross substitutes type-space is IC if and only if it is LIC.*

4.2 TYPE-SPACES PERTURBED BY CONCAVE-MODULARITY

As before, let the set of objects be $E = \{1,\dots,k\}$. The number of units available for object j is a_j. The set of alternatives A is the set of all feasible object bundles which is defined as $A = \{(z_1,\dots,z_k) \mid z_i \in \mathbb{Z} \text{ and } 0 \leq z_i \leq a_i \text{ for all } i \in E\}$.[10] Let $n = |A|$. Thus, a type is an element of $\mathbb{R}^n$.

Given concave functions $g_i : \{0,1,\dots,a_i\} \to \mathbb{R}$ for each $1 \leq i \leq k$, we define $\tilde{g} = (g_1,\dots,g_k)$. For any $t \in T$, we define a type $t_{\tilde{g}} \in \mathbb{R}^n$ where $t_{\tilde{g}}(z) = t(z) + \sum_{i=1}^{k} g_i(z_i)$ for all $z \in A$. We say that T is closed under concave-modular perturbations if for any $t \in T$ and concave functions $g_i : \{0,1,\dots,a_i\} \to \mathbb{R}$ for each $1 \leq i \leq k$, we have $t_{\tilde{g}} \in T$. A type $m : A \to \mathbb{R}$ *modular-concave* if there exists a concave function $g_i : \{0,1,\dots,a_i\} \to \mathbb{R}$ for each $1 \leq i \leq k$ such that $m(z) = \sum_{i=1}^{k} g_i(z_i)$ for all $z \in A$.

Proposition 4.3. *Let $T \subseteq \mathbb{R}^n$ be a type-space that is closed under scaling and closed under concave-modular perturbations. Then, T is minimally rich.*

The proof of this proposition is relegated to Appendix A.4.

An important example of a type-space that is closed under scaling and closed under concave-modular perturbations is the generalized gross substitutes and complements type-space (for details see the proof of Theorem 3 in Kushnir and Lokutsievskiy (2021)).

Shioura and Yang (2015) introduces the notion of generalized gross substitutes and complements (GGSC) type-space. A set $C \subseteq \mathbb{Z}^k$ is a integer convex set if it contains all integer vectors in its convex hull.[11]

The objects are partitioned into two classes E_1 and E_2, that is, $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$. The objects are substitutes within classes and complements across the classes. For instance, in the problem of allocation of spectrum licenses, radio spectrum licenses are substitutes within each region, but complements across regions.[12] We denote the total number of units in a class $E_r \in \{E_1, E_2\}$ in a bundle $z \in A$ by $z(E_r)$, that is, $z(E_r) = \sum_{l \in E_r} z_l$.

We now extend the notion of demand correspondence defined in Subsection 4.1. Note that the price of a bundle $z \in A$ is $z \cdot p$, where $p \in \mathbb{R}^k$ is the price vector of individual objects. Therefore, for a price $p \in \mathbb{R}^k$ and a type t, we define demand correspondence as

$$D(p,t) = \arg\max_{z \in A} \{t(z) - p \cdot z\}.$$

For $r \in \{1,2\}$ and $i \in E_r$, a bundle z' is an improvement of a bundle z with respect to E_r except for i if $z'_l \geq z_l$ for all $l \in E_r \setminus \{i\}$, and $z'_l \leq z_l$ for all $l \in E_r^c$. Let us denote by $\chi_i \in \mathbb{R}^k$ the vector whose i-th

[10] We denote by $\mathbb{Z}$ the set of all integers.

[11] Shioura and Yang (2015) use the term discrete convex set instead of integer convex set.

[12] This example is taken from Kushnir and Lokutsievskiy (2021).

component is 1 and other components are 0. Let $\chi_0 = (0,\ldots,0) \in \mathbb{R}^k$ be the null vector.

Definition 4.2. A type t satisfies the generalized gross substitutes and complements condition if for each price $p \in \mathbb{R}^k$,

(i) $D(p,t)$ is an integer convex set, and

(ii) for each $z \in D(p,t)$, each $r = 1,2$, each $i \in E_r$, and each $\delta > 0$, there exists an improvement z' of z with respect to E_r except for i such that $z' \in D(p+\delta\chi_i,t)$ and

$$z(E_r) - z(E_r^c) \geq z'(E_r) - z'(E_r^c).$$

Similar to the characterization of gross-substitutability condition purely in terms of inequalities involving agent's valuations provided in Proposition 4.2, Shioura and Yang (2015) (Theorem 3.3) proves that any type t satisfies the generalized gross substitutes and complements condition if and only if it is GM-concave.

Let $U = \text{diag}(1,\ldots,1,-1,\ldots,-1)$ be a diagonal $k \times k$ matrix that contains 1 in the first $|E_1|$ diagonal entries and -1 in the remaining $|E_2|$ diagonal entries. For $z = (z_1,\ldots,z_k) \in \mathbb{Z}^k$, define supp(z) $= \{i \in E \mid z_i > 0\}$.

A type $t : A \to \mathbb{R}$ is called GM-concave if for all $z, z' \in A$ and all $i \in \text{supp}\big(U(z-z')\big)$, there exists $j \in \text{supp}\big(U(z'-z)\big) \cup \{0\}$ such that

$$t(z) + t(z') \leq t\big(z - U(\chi_i - \chi_j)\big) + t\big(z' + U(\chi_i - \chi_j)\big). \tag{1}$$

A type-space is generalized gross substitutes and complements if it contains all GM-concave types. It can be verified that the gross substitutes type-space is a special case of the generalized gross substitutes and complements type-space.

We obtain the following corollary from Theorem 3.1, Proposition 4.3 and the fact that the generalized gross substitutes and complements type-space is closed under scaling and closed under concave-modular perturbations.

Corollary 4.2. *A mechanism on the generalized gross substitutes and complements type-space is IC if and only if it is LIC.*

5. A SIMPLE GEOMETRIC STRUCTURE OF NON-CONVEX TYPE-SPACES FOR THE EQUIVALENCE OF LIC AND IC

For ease of presentation, we assume in this section that $A = \{1,\ldots,n\}$, that is, the alternatives are indexed by the numbers $1,\ldots,n$. For a type $t \in \mathbb{R}^n$, we denote by $D(t) = \{s \in \mathbb{R}^n \mid$ there exists $c \in$

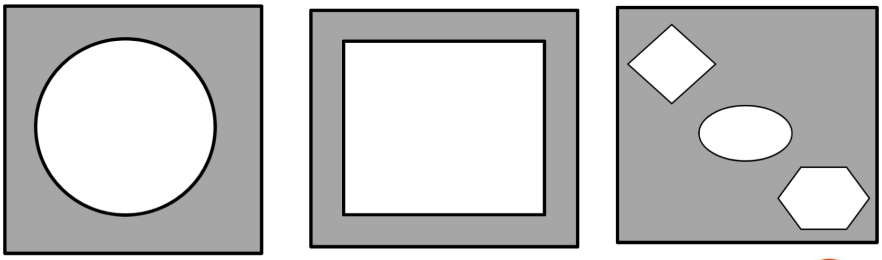

Figure 3

$\mathbb{R}$ such that $s(i) = t(i) + c$ for all $i \in \{1, \ldots, n\}\}$ the set of points that have the same relative difference between the alternatives as in t. Let $C = \prod_{i=1}^{n} [a_i, b_i]$, where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$, be a set in $\mathbb{R}^n$, and let $\partial(C)$ denote the boundary of C. Suppose that $T \subseteq C$ is such that T is open in C, $\partial(C) \subseteq T$, and for each $t \in T$, there is $s \in \partial(C) \cap D(t)$ such that the line $[t, s]$ lies in T. In Figure 3, we provide some examples of such a set T (marked by the shaded region) in two dimensions to illustrate its structure, and in Figures 4, 5 and 6, we provide examples of a set (marked by the shaded region) that does not satisfy the above mentioned property.[13] The type-spaces marked by shaded portion in Figures 3, 4, 5 and 6 does not include the boundary of the inner shape(s).

Theorem 5.1. *Let $T \subseteq C$ is such that*

(i) T is open in C,

(ii) $\partial(C) \subseteq T$, and

(iii) for each $t \in T$, there is $s \in \partial(C) \cap D(t)$ such that the line $[t, s]$ lies in T.

A mechanism on T is IC if and only if it is LIC.

The proof of Theorem 5.1 is relegated to Appendix A.5.

The equivalence between LIC and IC in Theorem 5.1 is guaranteed by the existence of a certain kind of polygonal path between every two types lying in the type-space. The specified polygonal path satisfies some monotonic condition over the relative valuation between alternatives.[14]

As we have demonstrated by Figure 3, the main importance of Theorem 5.1 is that its conditions are geometrically easy to check. Additionally, Theorem 5.1 provides a geometric insight on the kind of subsets of $\mathbb{R}^n$ (say X) for which LIC and IC are equivalent on the complement, that is, $\mathbb{R}^n \setminus X$.

It is worth mentioning that conditions of Theorem 5.1 are indispensable, that is, if we drop any condition of Theorem 5.1, then the equivalence of LIC and IC is no longer guaranteed. We provide examples below to support this statement. For simplicity, let us assume $A = \{a, b\}$. The valuation of the alternative a is represented on the horizontal axis and the valuation of the alternative b is represented on the vertical axis.

[13] For more details, see Examples 5.1, 5.2 and 5.3.

[14] For details, see the proof of Theorem 5.1.

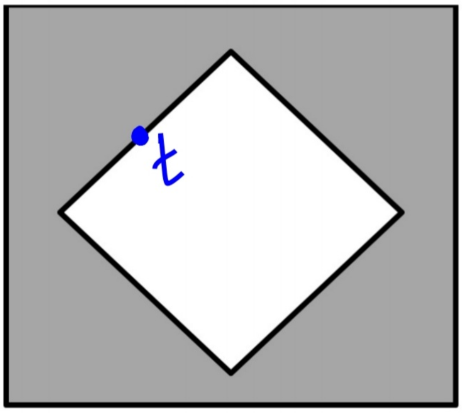

Figure 4

Example 5.1. Suppose we drop Condition (i) of Theorem 5.1. Consider the type-space T (marked by shaded portion) in Figure 4 where T includes the boundary of the inner square. Notice that this figure satisfies Conditions (ii) and (iii) of Theorem 5.1. The inner square has sides of slopes 1 and -1. Note that T includes the boundary of the inner square, and hence it is not open in C. Suppose that the side of the inner square containing the point t marked in Figure 4 lie on the line having slope 1 and passing through origin. Define a mechanism $\mu = (f, p)$ such that $f(t) = a$, $f(t') = b$ for every $t' \in T \setminus \{t\}$, and $p(\bar{t}) = 0$ for every $\bar{t} \in T$. Consider a neighborhood of t such that it does not intersect with any type $\bar{t}$ with $\bar{t}(a) - \bar{t}(b) > 0$, and similarly for any $\bar{t}$ with $\bar{t}(a) - \bar{t}(b) > 0$, consider its neighborhood such that it does not intersect t. It can be verified that μ is LIC with such neighbourhoods. However, μ violates IC on every pair (t', t) with $t'(a) - t'(b) > 0$.

Example 5.2. Suppose we drop Condition (ii) of Theorem 5.1. Consider the type-space T (marked by shaded portion) in Figure 5 where T does not include the boundary of the cut out portion from the square. Clearly, T does not contain the boundary of the square. Notice that this figure satisfies Conditions (i) and (iii) of Theorem 5.1. The red line has slope 1 and the vertical intercept is 0, i.e., slope 1 line passing through the origin. The blue line also has slope 1 but the vertical intercept is 1. T_1, T_2 and T_3 forms a partition of T as depicted in Figure 5. Define a mechanism $\mu = (f, p)$ such that $f(t) = a$ for every $t \in T_1 \cup T_2$, $f(t') = b$ for every $t' \in T_3$, $p(t) = 2$ for every $t \in T_1 \cup T_3$, and $p(t') = 1$ for every $t' \in T_2$. For any given type $t \in T_1$, consider a neighborhood of t such that it does not intersect with any type belonging to T_2, and similarly for any type $\bar{t} \in T_2$, consider its neighborhood such that it does not intersect with any type belonging to T_1. It can be verified that μ is LIC with such neighbourhoods. However, μ violates IC on every pair (t, t') with $t \in T_1$ and $t' \in T_2$.

Example 5.3. Suppose we drop Condition (iii) of Theorem 5.1. Consider the type-space T (marked

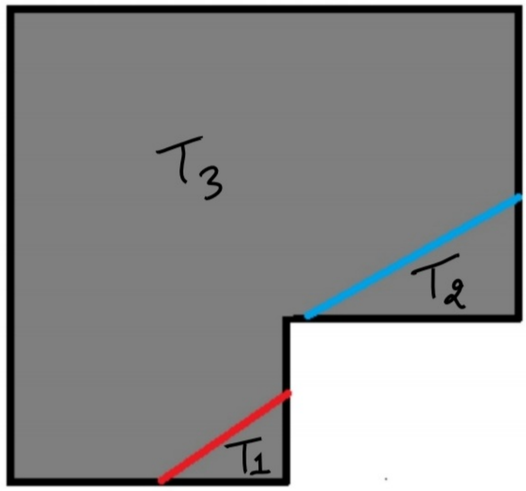

Figure 5

by shaded portion) in Figure 6. It does not include the boundary of inner shape. Notice that this figure satisfies Conditions (i) and (ii) of Theorem 5.1. The red line has slope 1 and the vertical intercept is 0, i.e., slope 1 line passing through the origin. The blue line also has slope 1 but the vertical intercept is 1. The subsets T_1, T_2 and T_3 form a partition of T as depicted in Figure 6. Notice that every type in T_1 violates Condition (iii) of Theorem 5.1. Define the mechanism $\mu = (f, p)$ such that $f(t) = b$ for every $t \in T_1 \cup T_3$, $f(t') = a$ for every $t' \in T_2$, $p(t) = 2$ for every $t \in T_1 \cup T_2$, and $p(t') = 3$ for every $t' \in T_3$. For any given type $t \in T_1$, consider a neighborhood of t such that it does not intersect with any type belonging to T_3, and similarly for any type $\bar{t} \in T_3$, consider its neighborhood such that it does not intersect with any type belonging to T_1. It can be verified that μ is LIC with such neighbourhoods. However, μ violates IC on every pair (t, t') with $t \in T_3$ and $t' \in T_1$.

A. APPENDIX

A.1 A USEFUL LEMMA

In this section, we present a lemma that we will use in deriving the rest of the results of the paper. The lemma provides a sufficient condition for a mechanism to be IC on a pair of types based on its IC property over a sequence of types. As we show in Sections 3 and 5, this simple lemma is quite powerful in

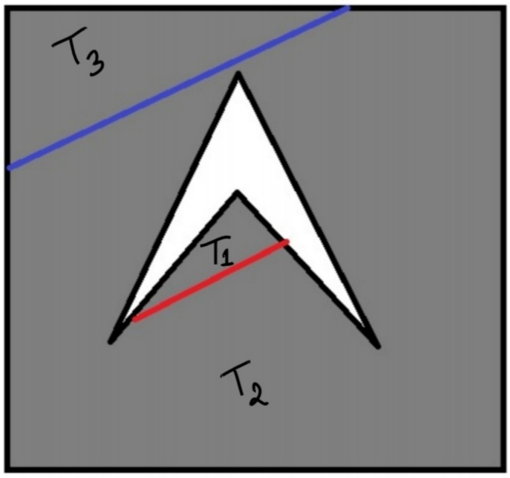

Figure 6

deducing a wide range of results.

Lemma A.1. *A mechanism $\mu = (f, p)$ on a type-space T is IC on a pair of types (t, t') if there is a finite sequence of types $(t = t^1, \ldots, t^k = t')$ in T such that for all $l < k$,*
(i) μ is IC on (t^l, t^{l+1}), and
(ii) $t^1(f(t^{l+1})) - t^1(f(t^l)) \leq t^l(f(t^{l+1})) - t^l(f(t^l))$.

The proof of this lemma is quite straightforward; we provide it here for the sake of completeness.

Proof: Consider a mechanism $\mu = (f, p)$ on a type-space T. Let (t, t') be a pair of types in T for which there is a finite sequence of types $(t = t^1, \ldots, t^k = t')$ in T such that for all $l < k$, (i) μ is IC on (t^l, t^{l+1}), and (ii) $t^1(f(t^{l+1})) - t^1(f(t^l)) \leq t^l(f(t^{l+1})) - t^l(f(t^l))$. We show that μ is IC on the pair (t, t').

We prove this by induction. By the assumption, μ is IC on (t^1, t^2). Suppose μ is IC on (t^1, t^l) for some $l < k$. This yields

$$t^1(f(t^1)) - p(t^1) \geq t^1(f(t^l)) - p(t^l). \tag{2}$$

Since μ is IC on (t^l, t^{l+1}), we have

$$t^l(f(t^l)) - p(t^l) \geq t^l(f(t^{l+1})) - p(t^{l+1}). \tag{3}$$

Adding $t^1(f(t^l))$ to both sides of (3) and doing some rearrangement, we obtain

$$t^1(f(t^l)) - p(t^l) \geq t^1(f(t^l)) + t^l(f(t^{l+1})) - t^l(f(t^l)) - p(t^{l+1}). \tag{4}$$

Combining (4) and Part (ii) of the condition in the lemma, we have $t^1(f(t^l)) - p(t^l) \geq t^1(f(t^{l+1})) - p(t^{l+1})$. This, together with (2) gives $t^1(f(t^1)) - p(t^1) \geq t^1(f(t^{l+1})) - p(t^{l+1})$, which implies μ is IC on (t^1, t^{l+1}). This completes the proof. ■

A.2 Proof of Theorem 3.1

Proof: The "only if" part of the theorem follows from the definition. We proceed to prove the "if" part of the theorem. Let (f, p) be an LIC mechanism on a minimally rich type-space $T \subseteq \mathbb{R}^n$. We will show that for any $t, t' \in T$, (f, p) is IC on (t, t').

Fix any $t, t' \in T$. If $[t, t'] \subseteq T$, then by Carroll (2012) it follows that (f, p) is IC on (t, t').[15] Suppose $[t, t'] \nsubseteq T$. Let $f(t') = a$. Since T satisfies minimal richness, there exists $s \in T$ satisfying conditions (i) and (ii) of Definition 3.1. Assume $f(s) = b$. By condition (i), the line $[s, t']$ lies in T. Therefore, by Carroll (2012), (f, p) is IC on both (t', s) and (s, t'). This implies $t'(a) - t'(b) \geq s(a) - s(b)$.

Therefore, by condition (ii) we have that,

$$s(a) - s(b) \geq t(a) - t(b). \tag{5}$$

By condition (i), both lines $[t, s]$ and $[s, t']$ lie in T. Therefore, by Carroll (2012), (f, p) is IC on both (t, s) and (s, t'). This, together with the facts that $f(t') = a, f(s) = b$ and $s(a) - s(b) \geq t(a) - t(b)$, Lemma A.1 implies that (f, p) is IC on (t, t'). This completes the proof of the theorem. ■

A.3 Proof of Proposition 4.1

Proof: Let T denote a type-space that is closed under scaling and closed under modular perturbations. Let us denote the zero vector by $\mathbf{0}$. Suppose M is the set of all modular types. First we show that $M \subseteq T$. Since T is closed under scaling, $\mathbf{0} \in T$. Also, since T is closed under modular perturbations, $\mathbf{0}_m \in T$ for every $m \in \mathbb{R}^k$. By definition, every modular type can be written as $\mathbf{0}_m$ for some $m \in \mathbb{R}^k$. Hence, $M \subseteq T$.

Next we show that $[t, m]$ lies in T for all $t \in T$ and $m \in M$. Take any $t \in T$ and $m \in M$. Pick any $0 < \lambda < 1$. Since $m \in M$, we have $(1 - \lambda) \cdot m \in M$, and since T is closed under scaling, we have $\lambda \cdot t \in T$. These, together with the fact that T is closed under modular perturbations, imply that $\lambda \cdot t + (1 - \lambda) \cdot m \in T$. Since $0 < \lambda < 1$ is arbitrary, this implies that the line $[t, m] \subseteq T$.

Now we show that T satisfies minimal richness. Fix any $t, t' \in T$ and $H \subseteq E$. To prove minimal richness we need to show that there exists a modular type $m \in T$ satisfying $m(H) - m(F) \geq t(H) - t(F)$ for all $F \subseteq E$ such that $t'(H) - t'(F) \geq m(H) - m(F)$. We prove something even stronger: there exists a modular type $m \in T$ such that $m(H) - m(F) \geq t(H) - t(F)$ for all $F \subseteq E$.

[15] Carroll (2012) shows that any LIC mechanism on a type-space T is IC on (t, t') if $[t, t'] \subseteq T$.

Since A is finite, there exists $c > 0$ such that $c \geq t(H) - t(F)$ for all $F \subseteq E$. Define a modular type $m \in M$ such that $m(\emptyset) = 0$, $m(i) = c$ for $i \in H$, and $m(i) = -c$ for $i \in E \setminus H$. Clearly $m(H) - m(F) \geq c$ for all $F \subseteq E$ with $F \neq H$. Therefore, $m(H) - m(F) \geq t(H) - t(F)$ for all $F \subseteq E$. Since $m \in M$, both $[t,m]$ and $[m,t']$ lie in T, and hence T is minimally rich. ■

A.4 PROOF OF PROPOSITION 4.3

Proof: Let T denote a type-space that is closed under scaling and closed under concave-modular perturbations. Let M denote the set of all modular-concave types. Similar to the proof of Proposition 4.1, it follows that $M \subseteq T$ and $[t,m]$ lies in T for all $m \in M$ and $t \in T$.

We show that T satisfies minimal richness. Fix any $t,t' \in T$ and $z \in A$. To prove minimal richness we need to show that there exists a modular-concave type $m \in T$ satisfying $m(z) - m(z') \geq t(z) - t(z')$ for all $z' \in A$ such that $t'(z) - t'(z') \geq m(z) - m(z')$. As we did in the case of proving Proposition 4.1, we prove something stronger: there exists a modular-concave type $m \in T$ such that $m(z) - m(z') \geq t(z) - t(z')$ for all $z' \in A$.

Since A is finite, there exists $c > 0$ such that $c \geq t(z) - t(z')$ for all $z' \in A$. For each $i = 1, \ldots, k$, define the concave function g_i such that $g_i(0) = 0$ and $g_i(j) = -c|z_i - j|$ for all $j = 1, \ldots, a_i$. Consider the modular-concave type m defined by $m(\bar{z}) = \sum_{i=1}^{k} g_i(\bar{z}_i)$ for all $\bar{z} \in A$. We have $m(z) - m(z') = \sum_{i=1}^{k} (g_i(z_i) - g_i(z'_i)) = c \sum_{i=1}^{k} |z_i - z'_i| \geq c$ for all $z' \in A \setminus \{z\}$. Therefore, $m(z) - m(z') \geq t(z) - t(z')$ for all $z' \in A$. Since $m \in M$, both $[t,m]$ and $[m,t']$ lie in T, and hence T is minimally rich. ■

A.5 PROOF OF THEOREM 5.1

Proof: First we prove a claim that will be used in the proof of the Theorem 5.1. We use the following terminologies in the proof. A polygonal path from t to t' in T is a finite collection of types $(t = t^1, \ldots, t' = t^k)$ such that $[t^l, t^{l+1}]$ lies in T for all $l \in \{1, \ldots, k-1\}$. An alternative i weakly (or strictly) improves from a type t to another type t' if $t(i) - t(j) \leq t'(i) - t'(j)$ for all $j \in A \setminus \{i\}$ (or, $t(i) - t(j) < t'(i) - t'(j)$ for all $j \in A \setminus \{i\}$). An alternative i weakly (or strictly) improves along a polygonal path $(t^1, \ldots, t^k)$ if i weakly (or strictly) improves from the type t^l to t^{l+1} for all $l \in \{1, \ldots, k-1\}$. For any alternative $i \in A$, let $\alpha_i = \inf\{t(i) \mid t \in C \setminus T\}$ and $\beta_i = \sup\{t(i) \mid t \in C \setminus T\}$. Since $\partial(C) \subseteq T$, $a_i < \alpha_i \leq \beta_i < b_i$. Let $U(i) = \{t \in T \mid t(i) \in (\beta_i, b_i]\}$ and $L(i) = \{t \in T \mid t(i) \in [a_i, \alpha_i)\}$. For any $i \in A$, we will often refer to $U(i)$ and $L(i)$ as hollow faces of the cuboid.

Claim A.1. *For every $s, s' \in T$ and $i \in A$, there exist (not necessarily distinct) $t^1, t^2, t^3 \in T$ such that*

(i) $t^1 \in D(s)$,

(ii) *i weakly improves from* t^1 *to* t^2*,*

(iii) $[s,t^1],[t^1,t^2]$*, and* $[t^2,t^3]$ *lie in* T*, and*

(iv) *there exists a polygonal path from* s' *to* t^3 *along which* i *strictly improves.*

Proof: Fix $s,s' \in T$ and $i \in A$. We need to find $t^1,t^2,t^3 \in T$ satisfying conditions $(i),(ii),(iii)$, and (iv). Let $L = \{t \in C \mid t(i) = b_i,\ t(j) = a_j \text{ for some } j \in A \setminus \{i\}\}$. Notice that $L \subseteq T$. We distinguish the following two cases:

Case (i): Suppose $s' \in L$. Take any $s \in T$. Set $t^3 = s'$. Hence condition (iv) is vacuously satisfied. By the assumption on T, there is $t^1 \in \partial(C) \cap D(s)$ such that the line $[s,t^1]$ lies in T.[16] Since $t^1 \in D(s)$, condition (i) is satisfied. Now we have to find $t^2 \in T$ such that the lines $[t^1,t^2],[t^2,s']$ lie in T and i weakly improves from t^1 to t^2. Since $t^1 \in \partial(C)$, there exists $j \in A$ such that $t^1(j) \in \{a_j,b_j\}$. We further distinguish the following two subcases:

Case (i.a): Suppose $j \neq i$. Define $t^2 \in T$ such that $t^2(i) = b_i$ and $t^2(l) = t^1(l)$ for all $l \in A \setminus \{i\}$.[17] Note that for any type t lying on the line $[t^1,t^2]$, $t(j) = t^1(j) \in \{a_j,b_j\}$. Therefore, the line $[t^1,t^2]$ lies in $\partial(C)$. Since $\partial(C) \subseteq T$, it follows that the line $[t^1,t^2]$ lies in T. Since $s'(i) = b_i = t^2(i)$, it follows by using a similar logic that the line $[t^2,s']$ lies in T. These, together with the fact that the line $[s,t^1]$ lies in T, implies that condition (iii) is satisfied. By the construction of t^2, $t^2(i) - t^2(l) \geq t^1(i) - t^1(l)$ for all $l \in A \setminus \{i\}$. Hence condition (ii) is also satisfied, and thereby the proof for this subcase is complete.

Case (i.b): Suppose $j = i$. Then $t^1(i) \in \{a_i,b_i\}$. Since $s' \in L$, there exists $k \in A \setminus \{i\}$ such that $s'(k) = a_k$. Define $t^2 \in T$ such that $t^2(i) = t^1(i)$, $t^2(k) = s'(k) = a_k$ and $t^2(l) = t^1(l)$ for all $l \in A \setminus \{i, k\}$.[18] Since $t^2(i) = t^1(i)$ and $t^2(k) = s'(k) = a_k$, by using a similar logic as in Case (i.a), it follows that the lines $[t^1,t^2]$ and $[t^2,s']$ lie in T. This, together with the fact that the line $[s,t^1]$ lies in T, implies that condition (iii) is satisfied. By the construction of t^2, $t^2(i) - t^2(l) \geq t^1(i) - t^1(l)$ for all $l \in A \setminus \{i\}$. Hence condition (ii) is also satisfied, and thereby the proof for the subcase is completed.

Case (ii): Suppose $s' \in T \setminus L$. Take any $s \in T$. Define $t^3 \in T$ such that $t^3(i) = b_i$ and $t^3(l) = a_l$ for all $l \in A \setminus \{i\}$. Since by construction $t^3 \in L$, by Case (i) there exist $t^1,t^2 \in T$ such that conditions (i), (ii) and (iii) are satisfied. Therefore, we only need to show that condition (iv) is also satisfied, i.e., there exists a polygonal path from s' to t^3 along which i strictly improves. By the assumption on T, there exists $\bar{t} \in \partial(C) \cap D(s')$ such that the line $[s',\bar{t}]$ lies in T. Since $\bar{t} \in \partial(C)$, there exists $j \in A$ such that $\bar{t}$ belongs to $L(j) \cup U(j)$. Since $[s',\bar{t}]$ lies in T and $\bar{t} \in \partial(C)$, there exists a type $\tilde{t} \notin \partial(C)$ on the line $[s',\bar{t}]$ such that $\tilde{t}$ belongs to the same hollow face as $\bar{t}$. Since $\tilde{t}$ lies on the line $[s',\bar{t}]$, $\tilde{t} \in D(s')$.

[16] If s itself is a point in $\partial(C)$, then we can set $s = t^1$.

[17] In $\mathbb{R}^3$, we can view t^2 as the foot of the perpendicular from t^1 to the face of the cuboid having b_i as the valuation of the alternative i for every type.

[18] In $\mathbb{R}^3$, we can view t^2 as the foot of the perpendicular from t^1 to the face of the cuboid having a_k as the valuation of the alternative k for every type.

Define $\widehat{T} = \{t \in T \mid \tilde{t}(i) < t(i),\, t(l) = \tilde{t}(l)$ for all $l \in A \setminus \{i\}$ and t belongs to the same hollow face as $\tilde{t}\}$. Since T is open in C, there exists $\hat{t} \in \widehat{T} \setminus \partial(C)$ such that the line $[s', \hat{t}]$ lies in T. Note that $\hat{t}$ belongs to $L(j) \cup U(j)$ and $s'(i) - s'(l) < \hat{t}(i) - \hat{t}(l)$ for all $l \in A \setminus \{i\}$. We further distinguish the following two subcases:

Case (ii.a): Suppose $j \neq i$. Define $t^4 \in T$ such that $t^4(i) = b_i$ and $t^4(l) = \hat{t}(l)$ for all $l \in A \setminus \{i\}$. Since $\hat{t} \notin \partial(C)$, we have $a_l < \hat{t}(l) < b_l$ for every $l \in A$. Hence $\hat{t}(i) - \hat{t}(l) < t^4(i) - t^4(l)$ for all $l \in A \setminus \{i\}$ and $t^4 \notin L$. Note that for any type t lying on the line $[\hat{t}, t^4]$, $t \in L(j) \cup U(j)$. Therefore, the line $[\hat{t}, t^4]$ lies in $L(j) \cup U(j)$. Since $L(j) \cup U(j) \subseteq T$, it follows that the line $[\hat{t}, t^4]$ lies in T. Since $t^3(i) = b_i = t^4(i)$, it follows by using a similar logic that the line $[t^4, t^3]$ lies in T. Since $t^4 \notin L$, we have $t^4(i) - t^4(l) < t^3(i) - t^3(l)$ for all $l \in A \setminus \{i\}$. Therefore, $(s', \hat{t}, t^4, t^3)$ is a polygonal path from s' to t^3 along which i strictly improves. Hence, condition (iv) is also satisfied, and thereby the proof for this subcase is complete.

Case (ii.b): Suppose $j = i$. Define $t^4 \in T \cap (L(i) \cup U(i))$ such that $t^4(i) = \hat{t}(i) + \varepsilon < b_i$, $t^4(k) = a_k$ and $t^4(l) = \hat{t}(l)$ for some $\varepsilon > 0$, $k \in A \setminus \{i\}$ and all $l \in A \setminus \{i, k\}$. Note that such a type t^4 can always be found since T is open in C and $t^4 \notin L$. Since $t^3(k) = a_k = t^4(k)$ and $t^4 \notin L$, by using a similar logic as in Case (ii.a), it follows that $(s', \hat{t}, t^4, t^3)$ is a polygonal path from s' to t^3 along which i strictly improves. Hence, condition (iv) is satisfied, and thereby the proof for this subcase is complete.

Since Cases (i) and (ii) are exhaustive, this completes the proof of the claim. ■

Having proved Claim A.1, now we proceed towards the proof of Theorem 5.1. Consider an LIC mechanism $\mu = (f, p)$ on T and consider two arbitrary types s and s' in T. We show that μ is IC on (s, s'). Let $f(s') = i$. By Claim A.1, there exist $t^1, t^2, t^3 \in T$ such that (i), (ii), (iii), and (iv) are satisfied. Suppose $(s' = s^1, \ldots, s^k = t^3)$ be a polygonally connected path from s' to t^3 satisfying (iv). By Carroll (2012), it follows that every LIC mechanism is IC on a line. Hence, μ is IC on both (s^l, s^{l+1}) and (s^{l+1}, s^l) for all $l \in \{1, \ldots, k-1\}$. This, together with the facts that $f(s') = i$ and i strictly improves along $(s' = s^1, \ldots, s^k = t^3)$, implies that $f(s^l) = i$ for every $l \in \{1, \ldots, k\}$. Now consider the polygonally connected path $(s, t^1, t^2, t^3 = s^k, \ldots, s^1 = s')$ from s to s'. Since this path is polygonally connected, μ is IC on every pair of consecutive types in $(s, t^1, t^2, t^3 = s^k, \ldots, s^1 = s')$, thereby satisfying condition (i) of Lemma A.1. We show that $(s, t^1, t^2, t^3 = s^k, \ldots, s^1 = s')$ satisfies condition (ii) of Lemma A.1. By condition (i), (ii), and (iii) of Claim A.1 and the fact that $f(s^l) = i$ for every $l \in \{1, \ldots, k\}$, it follows that condition (ii) of Lemma A.1 is also satisfied. Therefore, by Lemma A.1, we obtain that μ is IC on (s, s'). This completes the proof of the Theorem. ■

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