Formation of committees under constraints through random voting rules

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FORMATION OF COMMITTEES UNDER CONSTRAINTS THROUGH RANDOM VOTING RULES*

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Abstract

We consider the problem of choosing a committee from a set of available candidates through a randomized social choice function when there are bounds on the size (the number of members) of the committee to be formed. We show that for *any* (non-vacuous) restriction on the size of the committee, a random social choice function (RSCF) is onto and strategy-proof if and only if it is a range-restricted random dictatorial rule. Next, we consider the situation where an "undesirable committee" can be chosen with positive probability *only if* everyone in the society wants it as his best committee. We call this property strong unanimity. We characterize all strongly unanimous and strategy-proof RSCFs when there is exactly one undesirable committee. A common situation where a single committee is undesirable is one where the null committee is not allowed to be formed. We further show that there is no RSCF satisfying strong unanimity and strategy-proofness when there are more than one undesirable committees. Finally, we extend all our results when strategy-proofness is strengthened with group strategy-proofness.

JEL Classification: D71, D82.
Keywords: Committee Formation; Random Social Choice Function; Strategy-proofness; Ontoness; Strong unanimity; Group strategy-proofness

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1. INTRODUCTION

A group of individuals (or a society) needs to form a committee consisting of members from a fixed set of candidates. There are restrictions on the size (number of members) of the committee. Such constraints arise naturally as committees with more than certain members (say 15) tend to be unwieldy and difficult to operate. On the other hand, committees with less than certain people (say 6) tend to be unrepresentative. At the extreme, a "null committee" (that is, a committee with no members) will not serve the basic purpose of forming a committee.

Different individuals have different preferences over the feasible committees. A *random social choice function* (RSCF) decides a probability distribution over the feasible committees for every collection of preferences of the individuals. It is *onto* if each feasible committee can be a potential outcome (with probability one) at some collection of preferences. It is *strategy-proof* if no individual can improve the outcome (with respect to first order stochastic dominance comparison) by misreporting his preference. Thus, strategy-proofness incentivises individuals to reveal their preferences truthfully making truth telling a weakly dominant strategy equilibrium.

We consider situations where the size of the committee to be formed lies between k_1 and k_2 where $k_1 \leq k_2$ and there are at least 3 members. To avoid vacuous restrictions we assume that either $k_1 > 0$ or $k_2 < m$ or both. We show in Theorem 1 that an RSCF is onto and strategy-proof if and only if it is *range-restricted random dictatorial*. *Range-restricted dictatorial* rules are suitable modification of *dictatorial* rules when there is a restriction on the range. While a dictatorial *deterministic social choice funtion* (DSCF) always selects the best alternative of a particular agent (the dictator), a range-restricted dictatorial DSCF selects the best alternative of the dictator from the feasible set. A range-restricted random dictatorial RSCF is a convex combination of range-restricted dictatorial DSCFs. In other words, an RSCF is range-restricted random dictatorial if each individual has a fixed probability weight such that his most preferred feasible committee receives that probability at any profile. It is worth emphasizing that Theorem 1 holds as long as there is some non-vacuous feasibility restriction on the size of the committees to be selected, for instance, it applies to situations with the mildest requirement that the null committee cannot be formed. It is important to note from our result that although the domain under committee formation is a "nice" possibility domain (allowing a large class of onto and strategy-proof rules), it becomes an impossibility domain when restriction on the size of the committee is imposed.

As we have mentioned, some committees are not allowed to be selected as they are perceived as undesirable by the designer. But, what if everyone in a society wants such a committee? The designer can make an exception for such special cases. This way he can achieve unanimity which is sacrificed in Theorem 1, as well as respect his concern for undesirability. In view of this, we consider situations where an undesirable committee can be given positive probability *only when* everyone wants it as his best. We call this property *strong unanimity* since it is stronger than unanimity. We prove two results in this frame work depending on the number of undesirable committees. In Theorem 2, we characterize all RSCFs satisfying strategy-proofness and strong unanimity when there is exactly only one undesirable committee. In Theorem 3, we show that when there are more than one undesirable committees, there is no RSCF that satisfies strategy-proofness and strong unanimity.

1.1 Related literature

The committee formation model is introduced in Barbera et al. (1991). The main result of this paper says that when preferences of agents are separable, unanimous and strategy-proof DSCF must be *decomposable*. In other words, whether a candidate will be included or not will be solely decided by the (marginal) preferences of agents over that candidate, in particular, such a decision will not depend on the preferences, as well as decisions, for any other candidate. Serizawa (1995) considers a generalization of separable preferences which he calls cross-shaped preferences and shows that if a voting by committees rule without a dummy voter is strategy-proof on some rich domain, then any preference of a voter is cross-shaped on some set. Later, Breton and Sen (1999) show that the decomposability property of strategy-proof social choice functions is very general–it holds for all multi-dimensional models with separable preferences. In a seminal paper, Barberà et al. (2005) consider the deterministic version of the same model as ours, that is, a (deterministic) committee formation problem where not all committees are feasible. They consider arbitrary sets of infeasible sets and provide the structure of onto and strategy-proof DSCFs in this setting. However, Barberà et al. (2005) present the structure of onto and strategy-proof DSCFs in terms of a “minimal Cartesian decomposition of the range set of the DSCF", and we do not see any obvious way to derive (the deterministic version) of our result from this result. Recently, Roy et al. (2019) consider the random version of the committee formation problem (without any constraints). They chatacterize the set of unanimous and strategy-proof RSCFs as the RSCFs satisfying monotonicity and marginal decomposability. They also consider a special case of the our main result (Theorem 1) where committees of a fixed size are feasible.

2. The Model

Let $M = \{1, \ldots, m\}$ be a finite set of m components. For each component k, $A^k = \{0, 1\}$ is the set of alternatives available in component k. For any $K \subseteq M$, $A^K = \prod_{k \in K} A^k$, denotes the set of alternatives available in components in K. The set of all (multi-dimensional) alternatives is given by A^M. For ease of presentation, we write A instead of A^M. By definition, the number of alternatives in A is 2^m. Throughout this paper, we do not use braces for singleton sets.

The set M denotes the set of possible candidates from which a committee has to be formed. Thus each component refers to a possible candidate for a committee, where the numbers 0 and 1 for a component refer to the social states where the corresponding member is excluded and included in the committee, respectively. Similarly, every alternative $a = (a^1, \ldots, a^m) \in A$ refers to a committee in which the member k is present if and only if $a^k = 1$. The size (the number of members) of a committee a is denoted by $|a|$, that is, $|a| = \sum_{i=1}^{m} a^i$. For $a, b \in A$, we define $[a, b]$ as the set $\{c \in A \mid \text{ either } a^k \leq c^k \leq b^k \text{ or } b^k \leq c^k \leq a^k \text{ for all } k \in M\}$. We denote by $\underline{a}$ we denote the null committee, that is, the committee with no candidate. More formally, $\underline{a}^k = 0$ for all $k \in M$. For two alternatives $a, b \in A$ and a component $k \in M$, we say that a is the *k-th deviation* of b if $a^k \neq b^k$, and for all $l \neq k$, $a^l = b^l$.

A complete, reflexive, antisymmetric, and transitive binary relation over A (also called a linear order) is called a preference. For a preference P and for distinct $a, b \in A$, aPb is interpreted as "a is strictly preferred to b according to P". We denote by $\mathcal{P}(A)$ the set of all preferences over A. A subset $\mathcal{D}$ of $\mathcal{P}(A)$ is called a domain. For a preference P and a subset B of A, we denote by $P|_B$ the preference in $\mathcal{P}(B)$ such that for all $b, b' \in B$, $bP|_B b'$ if and only if bPb'. In other words, $P|_B$ is the restriction of P to B. Similarly, for a domain $\mathcal{D}$, we denote by $\mathcal{D}|_B$ the restrictions of the preferences in $\mathcal{D}$ to B, that is,

$\mathcal{D}|_B = \{P|_B \mid P \in \mathcal{D}\}$.

A preference P is **separable** if for all $a^{-k}, b^{-k} \in A^{M-k}$ and all $x^k, y^k \in A^k$, we have $(x^k, a^{-k})P(y^k, a^{-k})$ if and only if $(x^k, b^{-k})P(y^k, b^{-k})$. We denote the set of all separable preferences by $\mathcal{S}$. We let $\tau(P)$ denote the top-ranked alternative in P. For all $t \in \{1, 2, \ldots, 2^m\}$, we denote the t-th ranked alternative in P by $r_t(P)$. The upper contour set $U(a, P)$ of an alternative a at a preference P consists of the alternatives that are "at least as good as" a according to P, that is, $U(a, P) := \{b \mid bPa\}$.[1]

Let $N = \{1, \ldots, n\}$ be a set of n agents. An element of $\mathcal{S}^n$ is called a preference profile and is denoted by P_N. A random social choice function (RSCF) φ is a mapping $\varphi : \mathcal{S}^n \to \Delta A$ where ΔA denotes the set of probability distributions over A. We define the notion of strategy-proofness of an RSCF. The notion is familier in the literature.

Definition 2.1. An RSCF $\varphi : \mathcal{S}^n \to \Delta A$ is **strategy-proof** if for all $i \in N$, all $P_i, P'_i \in \mathcal{S}$, and all $P_{-i} \in \mathcal{S}^{n-1}$, $\varphi(P_i, P_{-i})$ first order stochastically dominates $\varphi(P'_i, P_{-i})$ according to P_i, that is,

$$\sum_{t=1}^{j} \varphi_{r_t(P_i)}(P_i, P_{-i}) \geq \sum_{t=1}^{j} \varphi_{r_t(P_i)}(P'_i, P_{-i}) \text{ for all } j = 1, \ldots, 2^m.$$

Our notion of strategy-proofness for RSCFs is the standard one introduced in Gibbard (1977). No agent can strictly increase the aggregate probability over any upper contour set according to her true preferences. If it were possible to do, there would exist a utility representation of her true preferences with the property that the expected utility from misrepresentation strictly exceeds that from truth-telling.

3. A CHARACTERIZATION OF ONTO AND STRATEGY-PROOF RSCFS

We denote by $A(k_1, k_2)$, where $0 \leq k_1 \leq k_2 \leq m$, the set of all committees having at least k_1 and at most k_2 members. More formally, $A(k_1, k_2) = \{a \in A \mid k_1 \leq |a| \leq k_2\}$. Throughout the paper, we assume that $A(k_1, k_2)$ is arbitrary but fixed. In order to have the size restriction non-vacuous, we assume that $A(k_1, k_2) \neq A$, that is, either $0 < k_1$ or $k_2 < m$ or both. For two alternatives $a, b \in A$, we write $b \gg a$ if $b^r \geq a^r$ for all $r \in M$. Throughout this section we assume that there are at least three components, that is, $|M| \geq 3$.

[1] Observe that $a \in U(a, P)$ by reflexivity.

In our model, agents have preferences over the committees in A, however, only some committee in $A(k_1, k_2)$ is allowed to receive positive probability. Clearly unanimity is incompatible with this range restriction. We therefore need to replace it by the onto property.

Definition 3.1. An RSCF $\varphi : \mathcal{S}^n \to \Delta A(k_1, k_2)$ is **onto** if for all $b \in A(k_1, k_2)$, there is $P_N \in \mathcal{S}^n$ such that $\varphi_b(P_N) = 1$.

Definition 3.2. A DSCF $f : \mathcal{S}^n \to A(k_1, k_2)$ is called **range-restricted dictatorial** if there exists $i \in N$ such that $f(P_N)$ chooses the most preferred alternative of agent i from the set $A(k_1, k_2)$. An RSCF is called **range-restricted random dictatorial** if it is a convex combination of range-restricted dictatorial DSCFs.

Theorem 1. *Let $\varphi : \mathcal{S}^n \to \Delta A(k_1, k_2)$ be an onto and strategy-proof RSCF. Then, φ is range-restricted random dictatorial.*

Proof. The proof of the theorem is relegated to Appendix 6.1; we provide an outline of the same here. Without loss of generality we assume that $k_1 > 0$. Let φ be an onto and strategy-proof RSCF on $\mathcal{S}^n$. We obtain the domain $\mathcal{S}|_{A(k_1,k_2)}$ by restricting the preferences of the agents to the set of alternatives $A(k_1, k_2)$ and define the the restriction of φ on this restricted domain as the RSCF $\hat{\varphi}$. We show that $\hat{\varphi}$ is well-defined and satisfies unanimity and strategy-proofness. Note that to show φ is range-restricted random dictatorial, it is enough to show that $\hat{\varphi}$ is random dictatorial. Theorem 5 in Chatterji et al. (2014) shows that if a domain $\mathcal{D}$ satisfies "Condition α" then the domain is random dictatorial for arbitrary number of agents if it is random dictatorial for two agents.[2] We show that the domain $\mathcal{S}|_{A(k_1,k_2)}$ satisfies Condition α. To show $\hat{\varphi}$ is random dictatorial for two agents we first show that the same holds over the profiles for which the size of the top-ranked alternatives of both agents are equal to k_1 (that is, the minimum size possible). Next we show it over the profiles where the size of the top-ranked alternative of one agent is k_1 and that of the other agent is arbitrary. Finally, we complete the proof by showing this for arbitrary profiles. ■

[2] A domain $\mathcal{D}$ is random dictatorial for n agents if every unanimous and strategy-proof RSCF $\bar{\varphi} : \mathcal{D}^n \to \Delta A$ is random dictatorial.

4. WEAKENING RANGE RESTRICTION BY UNANIMITY

As we have mentioned in Section 1, we consider restrictions on the size of a committee in this paper as committees with certain size might not be perceived as desirable. However, this cost us unanimity. In this section, we explore what happens if we impose unanimity in a minimal way.

Suppose that the designer has just one restriction on the size of the chosen committee: it cannot be zero. It is a basic requirement for any committee formation problem. Theorem 1 says that every onto and strategy-proof RSCFs in this case are range-restricted random dictatorial. Note that here a null committee is not allowed to be formed even if every individual wants it the most. Suppose that the designer relaxes the restriction in the following way: an undesirable committee (for the current instance, the null committee) can only be formed if everyone unanimously agree on it as their most preferred committee. Thus, the designer imposes unanimity in a strong sense. The question arises as to what type of RSCFs will satisfy strategy-proofness and (minimal) unanimity. We answer this question in this section.

An RSCF $\varphi : \mathcal{S}^n \to \Delta A$ is **unanimous** if for all P_N and all $a \in A$,

$$[\tau(P_i) = a \text{ for all } i \in N] \implies [\varphi_a(P_N) = 1].$$

If all agents have a common top-ranked committee at a profile, a unanimous RCSF picks that committee at that profile. It is clearly a weak form of efficiency. Let $B \subset A$ be the set of desirable committees. An RSCF $\varphi : \mathcal{S}^n \to \Delta A$ satisfies **strong unanimity with respect to B** if

(i) φ is unanimous, and

(ii) for all $a \in A \setminus B$, $\varphi_a(P_N) > 0$ if and only if $\tau(P_i) = a$ for all $i \in N$.

In what follows, we distinguish two cases depending on whether exactly one committee is undesirable, or more than one committees are undesirable. Our results vary significantly over these two cases.

4.1 Exactly one committee is undesirable

Suppose $A \setminus B$ is singleton. For arbitrary $A \setminus B$, say $\underset{\sim}{a}$, we introduce the definition of the *aversive to $\underset{\sim}{a}$ rule*. As the name suggests, this rule avoids selecting the undesirable committee in every possible manner. More precisely, for each candidtae of the undesirable committee, it selects that candidate only if *everyone* wants him, and for each candidate outside the undesirable committee, it selects that candidate if *at least one* agent wants him.

Definition 4.1. An RSCF $f_{\underset{\sim}{a}} : \mathcal{S}^n \to \Delta A$ is called the **aversive to $\underset{\sim}{a}$ rule** if for each $P_N \in \mathcal{S}^n$, $f_{\underset{\sim}{a}}(P_N) = a$ implies that for each $s \in M$, we have

$$a^s = \begin{cases} \underset{\sim}{a}^s \text{ if } \tau(P_i)^s = \underset{\sim}{a}^s \text{ for all } i \in N, \\ 1 - \underset{\sim}{a}^s \text{ if } \tau(P_i)^s = 1 - \underset{\sim}{a}^s \text{ for some } i \in N. \end{cases}$$

Theorem 2. *Let $A \setminus B = \underset{\sim}{a}$ and $\varphi : \mathcal{S}^n \to \Delta A$ be an RSCF. Then φ satisfies strong unanimity with respect to B and strategy-proofness if and only if $\varphi = f_{\underset{\sim}{a}}$.*

Proof. The proof is relegated to Appendix 6.2. ■

Remark 4.1. It is worth noting that $f_{\underset{\sim}{a}}$ is anonymous. This in particular implies that strong unanimity and strategy-proofness together imply anonymity.

Remark 4.2. Note that $f_{\underset{\sim}{a}}$ is a DSCF. Thus, strong unanimity and strategy-proofness force an RSCF to be deterministic.

4.2 More than one committees are undesirable

Suppose $|A \setminus B| > 1$. Our next theorem says that there is no RSCF that is strong unanimous with respect to B, as well as strategy-proof.

Theorem 3. *There is no RSCF $\varphi : \mathcal{S}^n \to \Delta A$ satisfying strong unanimity with respect to B and strategy-proofness.*

Proof. Let $\varphi : \mathcal{S}^n \to \Delta A$ be an RSCF satisfying strategy-proofness and strong unanimity with respect to B. Without loss of generality assume that $\underline{a} \in A \setminus B$. Using similar arguments as in the proof of Theorem 2, we can show that $\varphi(P_N) = a$ where for all $k \in M$, $a^k = 0$ if $\tau(P_i)^k = 0$ for all $i \in N$ and $a^k = 1$ if $\tau(P_i)^k = 1$ for some $i \in N$. Let

$x \in A \setminus B$ be such that $x \neq \underline{a}$. Consider the profile $\bar{P}_N \in \mathcal{S}^n$ such that $\tau(\bar{P}_1) = x$ and $\tau(\bar{P}_i) = \underline{a}$ for all $i \neq 1$. Thus $\varphi(\bar{P}_N) = x$. However, since x is undesirable and it is not that everyone wants it as their top-ranked alternative (in fact, nobody other than agent 1 wants it as his best), this a contradiction to strong unanimity. This completes the proof of the theorem. ■

5. GROUP STRATEGY-PROOFNESS

In this section we analyze the structure of group strategy-proof RSCFs on the domain under committee formation. A group of agents manipulate an RSCF at a profile if they can (simultaneously) misreport their preferences so that for each agent in the group the probability of some upper-contour set strictly increases. An RSCF is group strategy-proof if no group of agents can manipulate it. In other words no matter how a group of agents misreport their preferences, there will be one agent in the group for whom the new outcome will be stochastically dominated.

Definition 5.1. An RSCF $\varphi : \mathcal{S}^n \to \Delta A$ is **group strategy-proof** if for all $S \subseteq N$, all $P_S, P'_S \in \mathcal{S}$, and all $P_{-S} \in \mathcal{S}^{n-|S|}$, there exists $i \in S$ such that $\varphi(P_S, P_{-S})$ first order stochastically dominates $\varphi(P'_S, P_{-S})$ according to P_i, that is,

$$\sum_{t=1}^{j} \varphi_{r_t(P_i)}(P_S, P_{-S}) \geq \sum_{t=1}^{j} \varphi_{r_t(P_i)}(P'_S, P_{-S}) \text{ for all } j = 1, \ldots, 2^m.$$

We obtain the following corollary from Theorem 1. It says that every onto and group strategy-proof RSCF on the domain under committee formation will be range-restricted dictatorial. In particular, such rules will be DSCFs. As in the case of Theorem 1, we assume that $|M| \geq 3$ and either $0 < k_1$ or $k_2 < m$ or both.

Corollary 1. *Let $\varphi : \mathcal{S}^n \to \Delta A(k_1, k_2)$ be an onto and group strategy-proof RSCF. Then, φ is range-restricted dictatorial.*

Proof. In view of Theorem 1, to prove the corollary it is enough to show that all range restricted random dictatorial rules other than the dictatorial ones are group strategy-proof. Let φ be such a rule and let $\{\epsilon_i\}_{i=1}^{n}$ be the coefficients of the agents. Since φ is not a range-restricted dictatorial rule, there exist $j, j' \in N$ with $\epsilon_j > 0$ and $\epsilon_{j'} > 0$. We proceed to show that φ is not group strategy-proof. Let $a, b, c \in A$ be such that

$|a| = |b| = |c| = k_1$. Consider $P, \bar{P} \in \mathcal{S}$ such that (i) $\tau(P) = a$ and $\tau(\bar{P}) = c$, and (ii) bPc and $b\bar{P}a$. Note that such preferences exist as $|a| = |b| = |c|$. Let $P_N \in \mathcal{S}^n$ be such that $P_j = P$ and $P_i = \bar{P}$ for all $i \neq j$. By the definition of φ, $\varphi_a(P_N) = \epsilon_j$ and $\varphi_c(P_N) = \sum_{i \neq j} \epsilon_i$. This implies $\varphi_{U(b,P_j)}(P_N) = \epsilon_j$ and $\varphi_{U(b,P_k)}(P_N) = \sum_{i \neq j} \epsilon_j$. Let $\hat{P} \in \mathcal{S}$ be such that $\tau(\hat{P}) = b$. Consider $P'_j, P'_{j'} \in \mathcal{S}$ such that $P'_j = P'_{j'} = \hat{P}$. By the definition of φ, this means $\varphi_b(P'_j, P'_{j'}, P_{-\{j,j'\}}) = \epsilon_j + \epsilon_{j'}$ and $\varphi_c(P'_j, P'_{j'}, P_{-\{j,j'\}}) = \sum_{i \neq j,j'} \epsilon_i$. Therefore, $\varphi_{U(b,P_j)}(P'_j, P'_{j'}, P_{-\{j,j'\}}) = \epsilon_j + \epsilon_{j'}$ and $\varphi_{U(b,P_{j'})}(P'_j, P'_{j'}, P_{-\{j,j'\}}) = 1$. Since $\epsilon_j > 0$ and $\epsilon_{j'} > 0$, this means agents j and j' together can manipulate at P_N via $(P'_j, P'_{j'})$. This completes the proof of the corollary. ■

In Theorem 3, we have shown that if there are more than one undesirable committees, then there is no strategy-proof RSCF satisfying strong unanimity with respect to the desirable committees. Our next corollary says that if we replace strategy-proofness by group strategy-proofness, then the same result holds even when there exists exactly one undesirable committee. Thus, there is no group strategy-proof and strong unanimous RSCF on the committee formation domains when there are undesirable committees.

Corollary 2. *Let $B \subset A$. Then there is no RSCF $\varphi : \mathcal{S}^n \to \Delta A$ satisfying strong unanimity with respect to B and group strategy-proofness.*

Proof. In view of Theorem 2 and Theorem 3, it is enough to show that when there is one undesirable committee, then the aversive to undesirable committee rule (f) is not group strategy-proof. Without loss of generality we assume that the undesirable committee is $\underline{a}$. Consider $a, b, c \in A$ such that (i) $|a| = 1$ with $a_1 = 1$ and $a_k = 0$ for all $k \neq 1$, (ii) $|b| = 1$ with $b_2 = 1$ and $b_k = 0$ for all $k \neq 2$, and (iii) $|c| = 2$ with $c_1 = c_2 = 1$ and $c_k = 0$ for all $k \neq 1, 2$. Let $P, \bar{P} \in \mathcal{S}$ be such that (i) $\tau(P) = a$, $\tau(\bar{P}) = b$, and (ii) $\underline{a}Pc$ and $\underline{a}\bar{P}c$. Note that such preferences exist as $c \gg a \gg \underline{a}$ and $c \gg b \gg \underline{a}$. Let $P_N \in \mathcal{S}^n$ be such that $P_1 = P$, $P_2 = \bar{P}$, and $P_i = \tilde{P}$ for all $i \neq 1, 2$ where $\tau(\tilde{P}) = \underline{a}$. By the definition of f, this means $f(P_N) = c$. Consider $P'_1, P'_2 \in \mathcal{S}$ such that $P'_1 = P'_2 = \tilde{P}$. By the definition of f, this means $f(P'_1, P'_2, P_{-\{1,2\}}) = \underline{a}$. Since $\underline{a}P_1c$ and $\underline{a}P_2c$, this means agents 1 and 2 together can manipulate at P_N via (P'_1, P'_2). This completes the proof of the corollary. ■

6. Appendix

We use the following notation to facilitate the presentation of the proofs. We write

$P \equiv ab\cdots$ to mean that $\tau(P) = a$ and $r_2(P) = b$, and write $P \equiv ab\cdots cd \cdots e$ to mean that $\tau(P) = a$, $r_2(P) = b$, c and d are consecutively ranked with cPd, and e is the bottom ranked alternative in P. We use similar notations without further explanation. For $a \gg b$, we write $P \equiv [a,b]\cdots$ to mean that $\tau(P) = a$, and any alternative in the set $[a,b]$ is preferred to any alternative outside the set according to P, that is, the alternatives in the set $[a,b]$ form an upper contour set in P.

6.1 PROOF OF THEOREM 1

We only show the only-if part. We first introduce a few terminologies to ease the presentation of the proof. For $P \in \mathcal{S}|_{A(k_1,k_2)}$, we use the notation Q to denote *an* "extension" of P to a preference in $\mathcal{S}$, that is, Q is such that if we restrict Q to the alternatives in $A(k_1,k_2)$, we obtain P, more formally, $Q|_{A(k_1,k_2)} = P$. Similarly, for $Q \in \mathcal{S}$, by P we denote the restriction of Q to the alternatives in $A(k_1,k_2)$.

Proof. By our assumption, we have either $k_1 > 0$ or $k_2 < m$ (or both). We assume $k_1 > 0$; the arguments for the case $k_2 < m$ are symmetric. We start with a claim.

Claim 1. *Let $\varphi : \mathcal{S}^n \to \Delta A$ be an RSCF and Q_N, $\bar{Q}_N$ be such that $P_i = \bar{P}_i$ for all $i \in N$. Then $\varphi(Q_N) = \varphi(\bar{Q}_N)$.*

Proof. We show that $\varphi(Q_N) = \varphi(\bar{Q}_i, Q_{-i})$ where $P_i = \bar{P}_i$. Suppose not. Let $b \in A(k_1,k_2)$ be such that $\varphi_b(Q_N) \neq \varphi_b(\bar{Q}_i, Q_{-i})$ and $\varphi_a(Q_N) = \varphi_a(\bar{Q}_i, Q_{-i})$ for all $a \in A(k_1,k_2)$ with aQ_ib. In other words, b is the maximal element of $A(k_1,k_2)$ according to Q_i that violates the assertion of the claim. Without loss of generality, assume that $\varphi_b(Q_N) < \varphi_b(\bar{Q}_i, P_{-i})$. However, since $\varphi_a(Q_N) = \varphi_a(\bar{Q}_i, Q_{-i})$ for all $a \notin A(k_1,k_2)$ with aQ_ib, we have $\varphi_{U(b,Q_i)}(Q_N) < \varphi_{U(b,Q_i)}(\bar{Q}_i, Q_{-i})$. This means agent i manipulates at Q_N via $\bar{Q}_i$, which is a contradiction. This completes the proof of the claim. ■

Consider an onto and strategy-proof RSCF $\varphi : \mathcal{S}^n \to \Delta A(k_1,k_2)$. Construct the RSCF $\hat{\varphi} : (\mathcal{S}|_{A(k_1,k_2)})^n \to \Delta A(k_1,k_2)$ as follows: for all $P_N \in \mathcal{S}|_{A(k_1,k_2)}$,

$$\hat{\varphi}(P_N) = \varphi(Q_N). \tag{1}$$

This is well-defined by Claim 1. Because φ is strategy-proof, $\hat{\varphi}$ is also strategy-proof. Moreover, since φ is onto with range $A(k_1,k_2)$, strategy-proofness of φ implies $\hat{\varphi}$ is unanimous. In order to show that φ is range-restricted random dictatorial, it is enough to show that $\hat{\varphi}$ is random dictatorial. In what follows we prove this when $n = 2$ (Proposition 1), and finally, conclude the proof for arbitrary n by using a result from Chatterji et al. (2014) (Proposition 2).

We start with a lemma which we repeatedly use in the proof.

Lemma 1. *Let $k_1 < k_2$ with $k_1 < m - 1$. Then, $\varphi(P_1, P_2) = \varphi(P_1', P_2')$ for all P_1, P_1', P_2, $P_2' \in \mathcal{S}^{A(k_1,k_2)}$ with $\tau(P_1) = \tau(P_1')$ and $\tau(P_2) = \tau(P_2')$.*

Proof. Since every onto and strategy-proof RSCF $\varphi : \mathcal{S}^2 \to \Delta A(k_1, k_2)$ satisfies unanimity on the restricted domain $\mathcal{S}^{A(k_1,k_2)}$, it is sufficient to show that every unanimous and strategy-proof RSCF $\bar{\varphi} : (\mathcal{S}^{A(k_1,k_2)})^2 \to \Delta A(k_1, k_2)$ is tops-only.

Proposition 2 of Chatterji and Zeng (2019) states that every unanimous and strategy-proof RSCF on a connected$^+$ domain satisfies tops-only property. To complete the proof of this lemma we show that $\mathcal{S}^{A(k_1,k_2)}$ is a connected$^+$ domain. We first provide some definitions from Chatterji and Zeng (2019).

Definition 6.1. A preference P_i, say $r_1(P_i) \equiv (x_s)_{s \in M}$ is **top-separable** if for all $a, b \in A$ such that a is s-th deviation of b for some $s \in M$, we have $[a^s = x^s] \implies [aP_ib]$.

A domain is **multidimensional** if all preferences in it are top-separable.

Let $\Gamma(P, P') = \{\{a, b\} \in A^2 \mid aPb \text{ and } bP'a\}$ denote the set collecting all pairs of alternatives that are oppositely ranked across P and P'. Two Preferences P and P' are *adjacent*, denoted $P \sim P'$, if $\Gamma(P, P') = \{a, b\}$ for some $a, b \in A$. Preferences P and P' are *adjacent*$^+$, denoted $P \sim^+ P'$, if they are separable preferences, and $\Gamma(P, P') = \{\{(a^s, z^{-s}), (b^s, z^{-s})\}\}_{z^{-s} \in A^{-s}}$ for some $s \in M$ and $a^s, b^s \in A^s$.

Given two distinct preferences P and P', a sequence of preferences $\{P^k\}_{k=1}^t$, which is required to contain no repetition, is referred to as a *path* connecting P and P' if for all $1 \le k \le t - 1$, either $P^k \sim P^{k+1}$ or $P^k \sim^+ P^{k+1}$.

Definition 6.2. A domain $\mathcal{D}$ satisfies the **Interior$^+$ property** if given distinct $P, P' \in \mathcal{D}$ with $r_1(P) = r_1(P') \equiv a$, there exists a path $\{P^w\}_{w=1}^z \in \mathcal{D}^a$ connecting P and P'.

Definition 6.3. A domain $\mathcal{D}$ satisfies the **Exterior$^+$ property** if for all $P, P' \in \mathcal{D}$ with $r_1(P) \neq r_1(P')$ and for all $a, b \in A$ with aPb and $aP'b$, there exists a path $\{P^w\}_{w=1}^z$ connecting P and P' such that aP^kb for all $1 \le w \le z$. In addition, when $r_1(P)$ and $r_1(P')$

are such that $r_1(P) = (x^s, z^{-s})$ and $r_1(P') = (y^s, z^{-s})$, the path $\{P^w\}_{w=1}^z$ satisfies the **no-detour property**, i.e., $r_1(P^w) \in (A^s, z^{-s})$ for all $1 \le w \le z$.

A multidimensional domain satisfying the Interior$^+$ property and the Exterior$^+$ property is referred to as a connected$^+$ domain.

We are now ready to show that $\mathcal{S}^{A(k_1,k_2)}$ is a connected$^+$ domain. Since $\mathcal{S}^{A(k_1,k_2)}$ is a separable domain, it is $\mathcal{S}^{A(k_1,k_2)}$ is multidimensional. We proceed to show that it satisfies Interior$^+$ and Exterior$^+$ property.

Interior$^+$ Property: Note that for $a \in A(k_1, k_2)$, the set of preferences in $\mathcal{S}$ with a as the top-ranked alternative is the same as those in $\mathcal{S}^{A(k_1,k_2)}$ with the same corresponding top-ranked alternative. It is shown in Chatterji and Zeng (2019) (see Appendix E.2) that $\mathcal{S}$ satisfies the Interior$^+$ property. Therefore, $\mathcal{S}^{A(k_1,k_2)}$ also satisfies the Interior$^+$ property.

Exterior$^+$ Property: Since $\mathcal{S}$ satisfies the Exterior$^+$ property (Appendix E.2. in Chatterji and Zeng (2019)), we have the following fact.

Fact 1. *For every $P \equiv x \cdots a \cdots b \cdots$ and $P' \equiv y \cdots a \cdots b \cdots$ where x is s-th deviation of y for some $s \in M$, there exists a path $\{P^w\}_{w=1}^z$ connecting P and P' such that aP^wb and $r_1(P^w) \in \{x, y\}$ for all $1 \le w \le z$.*

Take $P \equiv x \cdots a \cdots b \cdots$ and $P' \equiv y \cdots a \cdots b \cdots$ for some $a, b \in A$ such that x, $y \in A(k_1, k_2)$. We show that there exists a path $\{P^w\}_{w=1}^z$ in $\mathcal{S}^{A(k_1,k_2)}$ such that aP^wb for all $1 \le w \le z$. Since $k_1 < k_2$ by the assumption of the lemma, there exists a sequence of alternatives $\{x_p\}_{p=1}^q$ in $A(k_1, k_2)$ connecting x and y such that for all $p < q$, x_p is the s-th deviation of x_{p+1} for some $s \in M$. In view of the Fact 1, to show the existence of the path $\{P^w\}_{w=1}^z$ in $\mathcal{S}^{A(k_1,k_2)}$, it is enough to show that for every x_p there exists a preference $\bar{P} \equiv x_p \cdots a \cdots b \cdots$. Moreover, by separability, showing the existence of the preference $\bar{P}$ is equivalent to showing that for every x_p there exists $s \in M$ such that $x_p^s = a^s \neq b^s$. Since $P \equiv x \cdots a \cdots b \cdots$ and $P' \equiv y \cdots a \cdots b \cdots$, there exists $u, v \in M$ such that $x^u = a^u \neq b^u$ and $y^v = a^v \neq b^v$. If $y^u = a^u$ or $x^v = a^v$, then for each $p \le q$ we have either $x_p^u = a^u \neq b^u$ or $y_p^v = a^v \neq b^v$, and we are done. Therefore, assume $x^v \neq a^v$ and $y^u \neq a^u$. Note that if we can show that along the sequence $\{x_p\}_{p=1}^q$ the v^{th} component of an alternative is changed from x^v to y^v before the u^{th} component is changed from x^u to y^u, then it will follow that for all $p \le q$, either $x_p^u = a^u \neq b^u$ or

$x_p^v = a^v \neq b^v$. In what follows, we show this by considering different cases.

Case 1: Suppose $x^u = x^v = 1$.
If $|x| > k_1$, then from x_1 to x_2, we can change the v^{th} component of x_1 from 1 to 0. If $|x| = k_1$, then from x_1 to x_2, we can change some component other than u of x_1 from 0 to 1 (which is possible as $|x| = k_1 < m$), and then from x_2 to x_3, we can change the v^{th} component of x_2 from 1 to 0.

Case 2: Suppose $x^u = 1$ and $x^v = 0$.
If $|x| < k_2$, then from x_1 to x_2, we can change the v^{th} component of x_1 from 0 to 1. If $|x| = k_2$, then from x_1 to x_2, we can change some component other than u of x_1 from 1 to 0 (this is possible as $k_1 > 0$ (by our initial assumption), and hence, $k_2 > 1$), and then from x_2 to x_3, we can change the v^{th} component of x_2 from 0 to 1.

Case 3: Suppose $x^u = 0$ and $x^v = 1$.
If $|x| > k_1$ then from x_1 to x_2, we can change the v^{th} component of x_1 from 1 to 0. If $|x| = k_1$ then from x_1 to x_2, we can change some component other than u of x_1 from 0 to 1 (this is possible as by the assumption of lemma $k_1 < m - 1$) and then from x_2 to x_3, we change the v^{th} component of x_2 from 1 to 0.

Case 4: Suppose $x^u = x^v = 0$.
If $|x| < k_2$ then from x_1 to x_2, we can change the v^{th} component of x_1 from 1 to 1. If $|x| = k_2$ then from x_1 to x_2, we change some component of x_1 from 1 to 0 (which is different from u as $x^u = 0$) and then from x_2 to x_3, we change the v^{th} component of x_2 from 0 to 1.

Since Cases 1 to 4 are exhaustive, this completes the proof of the fact that for all $x, y \in A(k_1, k_2)$, all $a, b \in A$, and all $P \equiv x \cdots a \cdots b \cdots$, $P' \equiv y \cdots a \cdots b \cdots$, there exists a path $\{P^w\}_{w=1}^z$ in $\mathcal{S}^{A(k_1,k_2)}$ such that aP^wb for all $1 \leq w \leq z$. We are left to show that the same holds with the added restriction that $r_1(P^w) \in \{x, y\}$ for all $1 \leq w \leq z$ for the case when x is s-th deviation of y for some $s \in M$. This holds because if there is some $s \in M$ such that x is s-th deviation of y, then by the above construction we get a sequence of alternatives $\{x_1, x_2\}$ where $x_1 = x$ and $x_2 = y$. This completes the proof of Lemma 1. ■

In the rest of the proof, we use the following notation: $A(k) = \{a \in A \mid |a| = k\}$.

Proposition 1. *$\hat{\varphi} : (\mathcal{S}|_{A(k_1,k_2)})^2 \to \Delta A(k_1, k_2)$ is random dictatorial.*

Proof. If $k_1 = m$, then $|A(k_1, k_2)| = 1$, and hence, we are done by unanimity. Therefore, we assume $k_1 < m$. To ease the presentation of the rest of the proof, by $\mathcal{P}(u, v)$, where $u, v \in [k_1, k_2]$, we denote all profiles where the size of agent 1's top-ranked alternative is u and that of agent 2's top-ranked alternative is v. We complete the proof in two steps. In Step 1, we show that $\hat{\varphi}$ is random dictatorial over the profiles in $\mathcal{P}(u, v)$ where at least one of u and v is k_1. In Step 2, we use this to establish that $\hat{\varphi}$ is random dictatorial over $\mathcal{P}(u, v)$ for arbitrary $u, v \in [k_1, k_2]$.

Step 1. Consider the profiles $\mathcal{P}(u, v)$ where at least one of u and v is k_1. Assume without loss of generality that $u = k_1$. We distinguish cases based on the structure of the top-ranked alternative of agent 2.

Case 1. Suppose $v = k_1$.

We complete the proof for this case by means of two lemmas. In Lemma 2, we show that for all profiles where agents 1 and 2 have fixed top-ranked alternatives with size k_1, say a and b, the rule $\hat{\varphi}$ behaves like a random dictatorial rule, that is, for all such profiles agent 1's top-ranked alternative gets a fixed probability, say ϵ_{ab}, and agent 2's top-ranked alternatives gets the remaining probability $1 - \epsilon_{ab}$. In Lemma 3, we show that the number ϵ_{ab} does not depend on a and b, and thereby complete the proof for Case 1.

Lemma 2. *For all distinct $a, b \in A(k_1, k_2)$ with $|a| = |b| = k_1$, there exists $\epsilon_{ab} \geq 0$ such that for all $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$, we have*

$$\hat{\varphi}_a(P_1, P_2) = \epsilon_{ab} \text{ and } \hat{\varphi}_b(P_1, P_2) = 1 - \epsilon_{ab}.$$

Proof of Lemma 2. Let $\bar{P}_1, \bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that
(a) $|\tau(\bar{Q}_1)| = |\tau(\bar{Q}_2)| = \underline{a}$,
(b) $a\bar{Q}_1 x$ for all $x \in A(k_1, k_2) \setminus a$ and $b\bar{Q}_1 x$ for all $x \in A(k_1, k_2) \setminus a$, and
(c) $b\bar{Q}_2 x$ for all $x \in A(k_1, k_2) \setminus b$ and $a\bar{Q}_2 x$ for all $x \in A(k_1, k_2) \setminus a$.
Note that such a preference exists as $|a| = |b| = k_1$. This implies $\bar{P}_1 = ab \cdots$ and $\bar{P}_2 = ba \cdots$. By strategy-proofness, $\hat{\varphi}_{\{a,b\}}(\bar{P}_1, \bar{P}_2) = 1$, otherwise agent 1 will manipulate at $(\bar{P}_1, \bar{P}_2)$ via $\bar{P}_2$, thereby obtaining probability 1 on b by unanimity. Let $\epsilon_{ab} \geq 0$ be such that $\hat{\varphi}_a(\bar{P}_1, \bar{P}_2) = \epsilon_{ab}$. Since $\hat{\varphi}_{\{a,b\}}(\bar{P}_1, \bar{P}_2) = 1$, we have $\hat{\varphi}_b(\bar{P}_1, \bar{P}_2) = 1 - \epsilon_{ab}$. Consider $P_1 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $\tau(P_1) = a$. By strategy-proofness, $\hat{\varphi}_a(P_1, \bar{P}_2) = \hat{\varphi}_a(\bar{P}_1, \bar{P}_2) = \epsilon_{ab}$.

Again by strategy-proofness $\hat{\varphi}_{\{a,b\}}(P_1, \bar{P}_2) = 1$ as otherwise agent 2 will manipulate at $(P_1, \bar{P}_2)$ via P_1 which would assign probability one to a. This implies $\hat{\varphi}_b(P_1, \bar{P}_2) = 1 - \epsilon_{ab}$.

Consider $P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $\tau(P_2) = b$. By a symmetric argument as before, $\hat{\varphi}_a(\bar{P}_1, P_2) = \epsilon_{ab}$ and $\hat{\varphi}_b(\bar{P}_1, P_2) = 1 - \epsilon_{ab}$. Since $\tau(P_1) = \tau(\bar{P}_1) = a$ and $\tau(P_2) = \tau(\bar{P}_2) = b$, by strategy-proofness $\hat{\varphi}_b(P_1, P_2) = \hat{\varphi}_b(P_1, \bar{P}_2) = 1 - \epsilon_{ab}$ and $\hat{\varphi}_a(P_1, P_2) = \hat{\varphi}_a(\bar{P}_1, P_2) = \epsilon_{ab}$. This completes the proof of Lemma 2. ■

Lemma 3. *For all $a \neq b \in A(k_1, k_2)$ and $c \neq d \in A(k_1, k_2)$ with $|a| = |b| = |c| = |d| = k_1$, we have $\epsilon_{cd} = \epsilon_{ab}$.*

Proof of Lemma 3. Suppose $\{c, d\} \neq \{a, b\}$. Consider $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $\tau(P_1) = a$ and $\tau(P_2) = d$. Using similar arguments as in Lemma 2, we have there exists $\epsilon_{ad} \geq 0$ such that $\hat{\varphi}_a(P_1, P_2) = \epsilon_{ad}$ and $\hat{\varphi}_d(P_1, P_2) = 1 - \epsilon_{ad}$. Since this is true for all preferences in $\mathcal{S}|_{A(k_1,k_2)}$ with top-ranked alternative d, without loss of generality we can take $P_2 \equiv db \cdots$. Let $\tilde{P}_2 \equiv bd \cdots$. Therefore, by strategy-proofness $\hat{\varphi}_d(P_1, P_2) \geq \hat{\varphi}_b(P_1, \tilde{P}_2)$. By Lemma 2, this implies $1 - \epsilon_{ad} \geq 1 - \epsilon_{ab}$ and hence, $\epsilon_{ab} \geq \epsilon_{ad}$. Similarly, by strategy-proofness $\hat{\varphi}_b(P_1, \tilde{P}_2) \geq \hat{\varphi}_d(P_1, P_2)$ which gives $\epsilon_{ad} \geq \epsilon_{ab}$. Combining these two we have $\epsilon_{ab} = \epsilon_{ad}$.

Now starting with $\hat{\varphi}_a(\hat{P}_1, \hat{P}_2) = \epsilon_{ad}$ and $\hat{\varphi}_d(\hat{P}_1, \hat{P}_2) = 1 - \epsilon_{ad}$ for all $\hat{P}_1, \hat{P}_2$ with $\tau(\hat{P}_1) = a$ and $\tau(\hat{P}_2) = d$ and using similar arguments as above with interchanging the roles of agent 1 and 2, we have $\hat{\varphi}_c(P_1, P_2) = \epsilon_{cd}$ and $\hat{\varphi}_d(P_1, P_2) = 1 - \epsilon_{cd}$ where $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = c$ and $\tau(P_2) = d$. This shows that Lemma 3 holds for all $c, d \in A$ with $\{c, d\} \neq \{a, b\}$.

We complete the proof by showing that Lemma 3 holds for $c = b$ and $d = a$. As $|A(k_1)| \geq 3$ (which is because $|M| \geq 3$ and $k_1 < m$), take $x \in A(k_1) \setminus \{a, b\}$. Since $\{a, x\} \notin \{a, b\}$, by similar arguments above we have Lemma 3 holds for $c = a$ and $d = x$. Again as $\{b, x\} \notin \{a, x\}$ we have Lemma 3 holds for $c = b$ and $d = x$. Lastly, as $\{b, a\} \neq \{b, x\}$ we have Lemma 3 holds for $c = b$ and $d = a$. This completes the proof of Lemma 3. ■

Lemmas 2 and 3 complete the proof for Case 1. □

Since $\epsilon_{ab} = \epsilon_{cd}$ for all $a \neq b$ and all $c \neq d$ under Case 1, we denote the common value (under Case 1) by ϵ. This completes the proof for the case when $k_1 = k_2$. In the rest of the proof we assume that $k_1 < k_2$.

Case 2. Suppose $v > k_1$ and $\tau(P_2) \gg \tau(P_1)$.

Let $a, b \in A$ be such that $|a| = k_1$, $|b| > k_1$, and $b \gg a$. We show that for all $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$, we have

$$\hat{\varphi}_a(P_1, P_2) = \epsilon \text{ and } \hat{\varphi}_b(P_1, P_2) = 1 - \epsilon,$$

where ϵ is as obtained in Case 1.

We first show this for $k_1 = m - 1$. Suppose $k_1 = m - 1$. Then $v = m$. Since $v = m$, we have $b \gg a$. Let $\bar{Q}_1, \bar{Q}_2 \in \mathcal{S}$ be such that $\bar{Q}_1 \equiv ab \cdots$ and $\bar{Q}_2 \equiv ba \cdots$. Therefore, since $a, b \in A(k_1, k_2)$, $\bar{P}_1 \equiv ab \cdots$ and $\bar{P}_2 \equiv ba \cdots$, and hence, using similar arguments as in Lemma 2, we can show that there exists $\epsilon_1 \geq 0$ such that for all $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $r_1(P_1) = a$ and $r_1(P_2) = b$, we have

$$\hat{\varphi}_a(P_1, P_2) = \epsilon_1 \text{ and } \hat{\varphi}_b(P_1, P_2) = 1 - \epsilon_1. \tag{2}$$

It is left to show that $\epsilon = \epsilon_1$. Let $c \in A(k_1)$ with $c \neq a$. Since $|b| - |c| = 1$, there exist $\hat{P}_2, \tilde{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $\hat{P}_2 \equiv bc \cdots$ and $\tilde{P}_2 \equiv cb \cdots$. Let $P_1 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that $r_1(P_1) = a$. As $r_1(P_1) = a$ and $r_1(\hat{P}_2) = b$, by (2) we have

$$\hat{\varphi}_a(P_1, \hat{P}_2) = \epsilon_1 \text{ and } \hat{\varphi}_b(P_1, \hat{P}_2) = 1 - \epsilon_1. \tag{3}$$

Moreover, as $r_1(P_1) = a$ and $r_1(\tilde{P}_2) = c$, by Case 1 we have

$$\hat{\varphi}_a(P_1, \tilde{P}_2) = \epsilon \text{ and } \hat{\varphi}_c(P_1, \tilde{P}_2) = 1 - \epsilon. \tag{4}$$

Now, since $\hat{P}_2 = bc \cdots$ and $\tilde{P}_2 = cb \cdots$, by strategy-proofness we have $\hat{\varphi}_{\{b,c\}}(P_1, \hat{P}_2) = \hat{\varphi}_{\{b,c\}}(P_1, \tilde{P}_2)$. This, together with (3) and (4), implies $\epsilon = \epsilon_1$, which completes the proof of the proposition for the when $k_1 = m - 1$.

Now, we proceed to prove the proposition when $k_1 < m - 1$.

We first complete the proof of Case 2 assuming $k_1 < m - 1$. We consider different cases based on the structure of the extensions of the preferences P_1 and P_2.

Case 2.1. Suppose that both P_1 and P_2 have extensions where the top-ranked alternatives remain the same, that is, there are extensions Q_1 and Q_2 of P_1 and P_2, respectively, with $\tau(Q_1) = a$ and $\tau(Q_2) = b$.

It follows from Lemma 1 that under the assumption of Case 2.1, the outcome $\varphi(Q_1, Q_2)$ depends only on the top-ranked alternatives of Q_1 and Q_2. Therefore, assuming $Q_1 = [a, b] \cdots$, by unanimity and strategy-proofness we have, $\varphi_w(Q_1, Q_2) > 0$ implies $w \in [a, b]$. Since $\varphi = \hat{\varphi}$, we have $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in [a, b]$. In what follows, we show that

$$\hat{\varphi}_w(P_1, P_2) = 0 \text{ for all } w \in (a, b). \tag{5}$$

Consider $c \in A(k_1) \setminus a$ with $b \gg c$ and consider the alternatives in (c, b). Our next claim shows that the alternatives in $(a, b) \setminus (c, b)$ will not receive any positive probability at $\hat{\varphi}(P_1, P_2)$. Since the outcome $\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, and hence let us assume that the alternatives in $[b, c]$ form an upper contour set in P_2, i.e., $P_2 \equiv [b, c] \cdots$.

Claim 2. *$\hat{\varphi}_w(P_1, P_2) = 0$ for all $w \in (a, b) \setminus (c, b)$.*

Proof of Claim 2. Consider $\hat{P}_1, \hat{\hat{P}}_1 \in \mathcal{S}|_{A(k_1,k_2)}$ such that (i) $\hat{P}_1 \equiv ac \cdots$, $\hat{\hat{P}}_1 \equiv ca \cdots$, (ii) $r_t(\hat{P}_1) = r_t(\hat{\hat{P}}_1)$ for all $t \geq 3$, and (iii) for all $w \in (a, b) \setminus (c, b)$ and all $z \in (a, b) \cap (c, b)$, we have $x\hat{P}_1 y$ (and hence $x\hat{\hat{P}}_1 y$). To see why such a preference in $\mathcal{S}|_{A(k_1,k_2)}$ will exist note that since $|a| = |c|$, there are preferences $\hat{Q}_1$ and $\hat{\hat{Q}}$ in $\mathcal{S}$ with top-ranked alternative as $\underline{a}$ whose restrictions will satisfy the mentioned properties. As $\tau(\hat{\hat{P}}_1) = c$ and $P_2 \equiv [b, c] \cdots$, by unanimity and strategy-proofness, we have

$$\hat{\varphi}_{[b,c]}(\hat{\hat{P}}_1, P_2) = 1. \tag{6}$$

Since $r_t(\hat{P}_1) = r_t(\hat{\hat{P}}_1)$ for all $t \geq 3$, by strategy-proofness, we have

$$\hat{\varphi}_{\{a,c\}}(\hat{P}_1, P_2) = \hat{\varphi}_{\{a,c\}}(\hat{\hat{P}}_1, P_2) \text{ and} \tag{7}$$

$$\hat{\varphi}_x(\hat{P}_1, P_2) = \hat{\varphi}_x(\hat{\hat{P}}_1, P_2) \text{ for all } x \notin \{a, c\}. \tag{8}$$

Note that as $|a| = |c| = k_1$ and $b \gg c$, we have $a \notin [b, c]$. Therefore, by (6), $\hat{\varphi}_a(\hat{\hat{P}}_1, \hat{P}_2) = 0$, and hence, by (7),

$$\hat{\varphi}_c(\hat{\hat{P}}_1, P_2) = \hat{\varphi}_a(\hat{P}_1, P_2) + \hat{\varphi}_c(\hat{P}_1, P_2). \tag{9}$$

Let $\hat{\varphi}_a(P_1, P_2) = \epsilon_1$. By strategy-proofness, we have

$$\hat{\varphi}_a(\tilde{P}_1, P_2) = \epsilon_1 \text{ for all } \tilde{P}_1 \in \mathcal{S}|_{A(k_1,k_2)} \text{ with } \tau(\tilde{P}_1) = a. \tag{10}$$

By using similar arguments as we have used in obtaining (10), we can show that there exists $\epsilon_1' \geq 0$ such that

$$\hat{\varphi}_c(\bar{P}_1, P_2) = \epsilon_1' \text{ for all } \bar{P}_1 \in \mathcal{S}|_{A(k_1,k_2)} \text{ with } \tau(\bar{P}_1) = c. \tag{11}$$

By (10), we have $\hat{\varphi}_a(\hat{P}_1, P_2) = \epsilon_1$, and by (11), we have $\hat{\varphi}_c(\hat{\hat{P}}_1, P_2) = \epsilon_1$. Therefore, by (9), we have

$$\epsilon_1' \geq \epsilon_1. \tag{12}$$

Let $P_2' \equiv [b,a] \cdots$. Starting with P_2' in place of P_2, and using similar arguments as we have used in obtaining (12), we can show that

$$\epsilon_1 \geq \epsilon_1'. \tag{13}$$

Combining (12) and (13), we get $\epsilon_1 = \epsilon_1'$, and hence,

$$\hat{\varphi}_a(\hat{P}_1, P_2) = \hat{\varphi}_c(\hat{\hat{P}}_1, \hat{P}_2). \tag{14}$$

By (6) and (8), we have

$$\hat{\varphi}_x(\hat{P}_1, P_2) = 0 \text{ for all } x \notin [b,c] \cup a. \tag{15}$$

We are now ready to complete the proof of the claim that $\hat{\varphi}_w(P_1, P_2) = 0$ for all $w \in (a,b) \setminus (c,b)$. Assume for contradiction that $\hat{\varphi}_w(P_1, P_2) > 0$ for some $w \in (a,b) \setminus (c, b)$. Consider $U(w, \hat{P}_1)$. Recall that by the construction of $\hat{P}_1$, $x\hat{P}_1 y$ for all $x \in (a,b) \setminus (b,c)$ and all $y \in (a,b) \cap (c,d)$. Therefore, $w\hat{P}_1 y$ for all $y \in (a,b) \cap (c,b)$. By (14) and (15), this yields $\hat{\varphi}_{U(w,\hat{P}_1)}(\hat{P}_1, P_2) = \epsilon_1$. Moreover, as $\hat{\varphi}_a(P_1, P_2) = \epsilon_1$ and $\hat{\varphi}_w(P_1, P_2) > 0$, we have $\hat{\varphi}_{U(w,\hat{P}_1)}(P_1, P_2) > \epsilon_1$. However, then agent 1 manipulates at $(\hat{P}_1, P_2)$ via P_1, a contradiction. This completes the proof of the claim. $\square$

In view of Claim 2, to complete the proof of (5), that is, to show that $\hat{\varphi}_w(P_1, P_2) = 0$ for all $w \in (a,b)$, it is sufficient to show that for all $w \in (a,b)$, there exists $d \in A(k_1) \setminus a$ with $b \gg d$ such that $w \notin (d,b)$. We show this in the following claim.

Claim 3. *For each $w \in (a,b)$, there exists $d \in A(k_1)$ with $b \gg d$ such that $w \notin (d,b)$.*

Proof of Claim 3. Consider $w \in (a,b)$. Since $w \in (a,b)$, there must exist $s \in M$ such that $a_s = w_s = 0$ and $b_s = 1$. Let $d \in A(k_1)$ be such that $b \gg d$ and $d_s = 1$. Since $w_s = 0$ and $d_s = b_s = 1$, it does not hold that $b \gg w \gg d$, and hence, $w \notin (d, b)$. This completes the proof of the claim. $\square$

We now complete the proof of the lemma for Case 2.1. Suppose $\hat{\varphi}_a(P_1, P_2) = \epsilon'$. We show that $\epsilon' = \epsilon$. By (5), we have $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b\}$, and hence, $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon'$. Let $\tilde{P}_2 \equiv [c, b] \cdots$. By Case 1, $\hat{\varphi}_c(P_1, \tilde{P}_2) = 1 - \epsilon$. Since $[b, c]$ is an upper contour set in P_2, by strategy-proofness this implies $\hat{\varphi}_{[b,c]}(P_1, P_2) \geq 1 - \epsilon$. By (5), only a and b can receive positive probability at (P_1, P_2). This, together with the fact that $a \notin [b, c]$, yields $\hat{\varphi}_b(P_1, P_2) \geq 1 - \epsilon$, and hence, $\epsilon \geq \epsilon'$. Using similar logic as used in obtaining $\hat{\varphi}_b(P_1, P_2) \geq 1 - \epsilon$, we have $\hat{\varphi}_c(P_1, \tilde{P}_2) \geq 1 - \epsilon'$, which implies $\epsilon' \geq \epsilon$. Combining these observations, we obtain $\epsilon = \epsilon'$, and hence, $\hat{\varphi}_a(\bar{P}_1, \bar{P}_2) = \epsilon$ and $\hat{\varphi}_b(\bar{P}_1, \bar{P}_2) = 1 - \epsilon$. This completes the proof for Case 2.1.

Case 2.2. Suppose that exactly one of P_1 and P_2 has an extension where the top-ranked alternative remains the same.

Assume without loss of generality, Q_2 is an extension of P_2 with $\tau(Q_2) = b$ and there is no extension Q_1 of P_1 with $\tau(Q_1) = a$. Let $\tilde{P}_1 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that there is extension $\tilde{Q}_1$ of $\tilde{P}_1$ with $\tau(\tilde{Q}_1) = a$. By Case 2.1, $\hat{\varphi}_a(\tilde{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\tilde{P}_1, P_2) = 1 - \epsilon$. Since $\tau(P_1) = \tau(\tilde{P}_1) = a$, by strategy-proofness,

$$\hat{\varphi}_a(P_1, P_2) = \epsilon. \tag{16}$$

It remains to show $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. Suppose not. Since $\tau(P_2) = b$, by means of strategy-proofness, we can assume that $P_2 \equiv [b, a] \cdots$. Since $[b, a]$ is an upper contour set in P_2 and $\tau(P_1) = a$, by unanimity and strategy-proofness, we have $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in [a, b]$. Suppose $\hat{\varphi}_w(P_1, P_2) > 0$ for some $w \in (a, b)$. Consider the upper contour set $U(w, \tilde{P}_1)$. Since $w \in (a, b)$ and $\tau(\tilde{P}_1) = a$, we have $b \notin U(w, \tilde{P}_1)$. This, together with the facts that $\hat{\varphi}_a(\tilde{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\tilde{P}_1, P_2) = 1 - \epsilon$, implies $\hat{\varphi}_{U(w,\tilde{P}_1)}(\tilde{P}_1, P_2) = \epsilon$. By (16) and our assumption that $\hat{\varphi}_w(P_1, P_2) > 0$, we obtain $\hat{\varphi}_{U(w,\tilde{P}_1)}(P_1, P_2) > \epsilon$. However, then agent 1 manipulates at $(\tilde{P}_1, P_2)$ via P_1, a contradiction. Therefore, $\hat{\varphi}_w(P_1, P_2) = 0$ for all $w \in (a, b)$. Since $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in [a, b]$, this implies

$$\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon. \tag{17}$$

This completes the proof for Case 2.2.

Case 2.3. Suppose that none of P_1 and P_2 has an extension where the top-ranked alternative remains the same.

Let $\tilde{P}_1, \tilde{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that there are extensions $\tilde{Q}_1$ and $\tilde{Q}_2$ of $\tilde{P}_1$ and $\tilde{P}_2$, respectively, with $\tau(\tilde{Q}_1) = a$ and $\tau(\tilde{Q}_2) = b$. By Case 2.2, $\hat{\varphi}_a(\tilde{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(P_1, \tilde{P}_2) = 1 - \epsilon$. Since $\tau(\tilde{P}_i) = \tau(P_i)$ for all $i \in \{1, 2\}$, by strategy-proofness $\hat{\varphi}_a(P_1, P_2) = \epsilon$ and $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. This completes the proof of the lemma for Case 2.3.

Since Cases 2.1, 2.2, 2.3 exhaust Case 2, the proof for Case 2 is complete. □

Case 3. Suppose $v > k_1$ and $\tau(P_2) \not\gg \tau(P_1)$.

We use induction for the proof of this case. We consider the base case as the maximum feasible size of $\tau(P_2)$, and use induction on the size of $\tau(P_2)$ in a decreasing manner. Note that if $|\tau(P_2)| = m$, then it is not possible that $\tau(P_2) \not\gg \tau(P_2)$. Therefore, maximum possible value of v under Case 3 is $\min\{k_2, m - 1\}$. We consider the case $v = \min\{k_2, m - 1\}$ as the base case of the induction.

Base case for Case 3. Suppose $v = \min\{k_2, m - 1\}$.

Let $a, b \in A$ be such that $|a| = k_1$, $|b| = \min\{k_2, m - 1\}$, and $b \not\gg a$. We show that for all $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$, we have

$$\hat{\varphi}_a(P_1, P_2) = \epsilon \text{ and } \hat{\varphi}_b(P_1, P_2) = 1 - \epsilon,$$

where ϵ is as obtained in Case 1. We distinguish two subcases of the base case depending on the value of k_2. We enumerate these cases as 3B.1, 3B.2, etc., to emphasize the fact that they are subcases of the base case.

Case 3B.1. Suppose $k_2 = m$.

This implies $|b| = m - 1$. Note that since $k_1 < |b| < k_2$, for each $\tilde{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ any extension $\tilde{Q}_2$ of $\tilde{P}_2$ must have b as the top alternative, i.e., $\tau(\tilde{Q}_2) = b$. Consider $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$. We show that $\hat{\varphi}_a(P_1, P_2) = \epsilon$ and $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. Let $c \in A(m)$, i.e., $|c| = m$. We consider two cases based on the structure of the extensions of the preference P_1.

Case 3B.1.1. Suppose that P_1 has extension where the top-ranked alternative remain the same, that is, there is extension Q_1 of P_1 with $\tau(Q_1) = a$.

It follows from Lemma 1 that under the assumption of Case 3B.1.1, the outcome

$\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, and hence let us assume that the alternatives b and c form an upper contour set in P_2, i.e., $P_2 \equiv bc\cdots$.

Our first claim shows that for all $w \in A(k_1, k_2) \setminus \{a, b, c\}$ there exists a preference $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $\bar{P}_2 \equiv cb\cdots w\cdots a\cdots$.

Claim 4. *For each $w \in A(k_1, k_2) \setminus \{a, b, c\}$, there exists $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $\bar{P}_2 \equiv cb\cdots w\cdots a\cdots$.*

Proof of Claim 4. Since $w \in A(k_1, k_2)$, $|w| \geq k_1$. Moreover, since $w \neq a$, there must be $r \in M$ such that $a_r = 0$ and $w_r = 1$. As $|c| = m$, by separability it follows that there is a preference $\bar{Q}_2 \in \mathcal{S}$ with $\tau(\bar{Q}_2) = c$ such that $w\bar{Q}_2 a$. Additionally, as $|b| = m - 1$, we can have b as the second ranked alternative in $\bar{Q}_2$. The preference $\bar{P}_2$ can be obtained by taking the restriction of $\bar{Q}_2$ to $A(k_1, k_2)$. This completes the proof of Claim 4. □

In the next claim we show that $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$ and $\hat{\varphi}_a(P_1, P_2) = \epsilon$.

Claim 5. *$\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$ and $\hat{\varphi}_a(P_1, P_2) = \epsilon$.*

Proof of Claim 5. Assume for contradiction there exists $w \notin \{a, b, c\}$ such that $\hat{\varphi}_w(P_1, P_2) > 0$. By the definition of $\hat{\varphi}$ this implies $w \in A(k_1, k_2)$. Let $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that $\bar{P}_2 \equiv cb\cdots w\cdots a\cdots$. Such a preference exists by Claim 4. As by our assumption $P_2 \equiv bc\cdots$, strategy-proofness implies

$$\hat{\varphi}_{\{b,c\}}(P_1, P_2) = \hat{\varphi}_{\{b,c\}}(P_1, \bar{P}_2). \tag{18}$$

Since $c \gg a$ and $\tau(\bar{P}_2) = c$, by Case 2, we have $\hat{\varphi}_a(P_1, \bar{P}_2) = \epsilon$ and $\hat{\varphi}_c(P_1, \bar{P}_2) = 1 - \epsilon$. This and (18) together with strategy-proofness imply

$$\hat{\varphi}_{\{b,c\}}(P_1, P_2) = 1 - \epsilon. \tag{19}$$

Consider $U(w, \bar{P}_2)$. Since $\bar{P}_2 \equiv cb\cdots w\cdots a\cdots$, we have $a \notin U(w, \bar{P}_2)$. This together with $\hat{\varphi}_a(P, \bar{P}_2) = \epsilon$ and $\hat{\varphi}_c(P, \bar{P}_2) = 1 - \epsilon$ imply $\hat{\varphi}_{U(w,\bar{P}_2)}(P_1, \bar{P}_2) = 1 - \epsilon$.

Recall that by our assumption $\hat{\varphi}_w(P_1, P_2) > 0$. This together with (19) imply $\hat{\varphi}_{U(w,\bar{P}_2)}(P_1, P_2) > 1 - \epsilon$. However, this is a contradiction as otherwise agent 2 manipulates at $(P_1, \bar{P}_2)$ via P_2. This shows that

$$\hat{\varphi}_w(P_1, P_2) > 0 \text{ implies } w \in \{a, b, c\}. \tag{20}$$

Moreover, as by (19), $\hat{\varphi}_{\{b,c\}}(P_1, P_2) = 1 - \epsilon$, (20) implies $\hat{\varphi}_a(P_1, P_2) = \epsilon$. This completes the proof of the claim. □

Our next claim shows that $\hat{\varphi}_c(P_1, P_2) = 0$.

Claim 6. $\hat{\varphi}_c(P_1, P_2) = 0$.

Proof of Claim 6. Let $d \in A(k_1)$ be such that $b \gg d$. Further, let $\tilde{P}_1 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that $\tilde{P}_2 \equiv b \cdots d \cdots c \cdots a \cdots$. Such a preference is feasible as there exists $r \in M$ where $d_r = b_r = 0$ and $c_r = 1$. As $\tau(\tilde{P}_2) = \tau(P_2)$, by Lemma 1, $\hat{\varphi}(P_1, P_2) = \hat{\varphi}(P_1, \tilde{P}_2)$. This together with Claim 5 imply,

$$\begin{aligned} &\hat{\varphi}_a(P_1, \tilde{P}_2) = \epsilon \text{ and} \\ &\hat{\varphi}_{\{b,c\}}(P_1, \tilde{P}_2) = 1 - \epsilon. \end{aligned} \tag{21}$$

Assume for contradiction $\hat{\varphi}_c(P_1, P_2) > 0$ and hence, by Lemma 1 $\hat{\varphi}_c(P_1, \tilde{P}_2) > 0$. This together with (21) imply $\hat{\varphi}_b(P_1, \tilde{P}_2) < 1 - \epsilon$. Consider $U(d, \tilde{P}_2)$. Since $a, c \notin U(d, \tilde{P}_2)$, we have $\hat{\varphi}_{U(d,\tilde{P}_2)}(P_1, \tilde{P}_2) = \hat{\varphi}_b(P_1, \tilde{P}_2) < 1 - \epsilon$.

Let $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that $\tau(\bar{P}_2) = d$. By Case 1, this means $\hat{\varphi}_d(P_1, \bar{P}_2) = 1 - \epsilon$. However, this is a contradiction as agent 2 manipulates at $(P_1, \tilde{P}_2)$ via $\bar{P}_2$. This shows $\hat{\varphi}_c(P_1, P_2) = 0$ and completes the proof of the claim. □

We are now ready to complete the proof for Case 3.1.1. By Claim 5, $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$ and $\hat{\varphi}_a(P_1, P_2) = \epsilon$, and by Claim 6, $\hat{\varphi}_c(P_1, P_2) = 0$. Combining these two observations, we get $\hat{\varphi}_a(P_1, P_2) = \epsilon$ and $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. This completes the proof for Case 3B.1.1.

Case 3B.1.2. Suppose that P_1 does not have an extension where the top-ranked alternative remains the same.

Let $\bar{P}_1 \in \mathcal{S}|_{A(k_1,k_2)}$ be such that there is an extension of $\bar{Q}_1 \in \mathcal{S}$ with $\tau(\bar{Q}_1) = a$. By Case 3B.1.1, $\hat{\varphi}_a(\bar{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\bar{P}_1, P_2) = 1 - \epsilon$. Therefore, by strategy-proofness

$$\hat{\varphi}_a(P_1, P_2) = \hat{\varphi}_a(\bar{P}_1, P_2) = \epsilon. \tag{22}$$

It remains to show $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. Suppose not. Since $\tau(P_2) = b$, by means of strategy-proofness, we can assume that $P_2 \equiv bc\cdots$. Using similar arguments as in the proof of Claim 5, we have $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$. Suppose $\hat{\varphi}_c(P_1, P_2) > 0$. Note that as $\bar{P}_1$ has an extension $\bar{Q}_1$ such that $\tau(\bar{Q}_1) = a$, it must be that $c\bar{Q}_1 b$. To see this recall that there exists $s \in M$ such that $b_s = 0$, $c_s = 1$ and for all $r \neq s$, $b_r = c_r = 1$. Further, by our assumption on b, there exists $t \in M$ such that $a_t = 1$ and $b_t = 0$. Therefore, we have $t = s$. This means $a_r = b_r$ implies $a_r = c_r$ for all $r \in M$ and hence, $c\bar{Q}_1 b$. Consider $U(c, \bar{P}_1)$. Since $b \notin U(c, \bar{P}_1)$, by Case 3B.1.1, $\hat{\varphi}_{U(c,\bar{P}_1)}(\bar{P}_1, P_2) = \epsilon$. As by our assumption $\hat{\varphi}_c(P_1, P_2) > 0$, (22) implies $\hat{\varphi}_{U(c,\bar{P}_1)} > \epsilon$. However, this is a contradiction as agent 1 manipulates at $(\bar{P}_1, P_2)$ via P_1. Thus, $\hat{\varphi}_c(P_1, P_2) = 0$ and hence, $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. This completes the proof for the Case 3B.1.2.

Since Cases 3B.1.1 and 3B.1.2 exhaust Case 3B.1, the proof for Case 3B.1 is complete.

Case 3B.2. Suppose $k_2 \leq m - 1$.

This means $|b| = k_2$. Consider $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$. We show that $\hat{\varphi}_a(P_1, P_2) = \epsilon$ and $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. We complete the proof by considering different cases based on the structure of the extensions of the preferences P_1 and P_2.

Case 3B.2.1. Suppose that both P_1 and P_2 have extensions where the top-ranked alternatives remain the same, that is, there are extensions Q_1 and Q_2 of P_1 and P_2, respectively, with $\tau(Q_1) = a$ and $\tau(Q_2) = b$.

Let $c \in A(k_1)$ be such that $b \gg c$. In view of Lemma 1, under the assumption of Case 3B.2.1, the outcome $\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, and hence, let us assume $P_1 \equiv ac\cdots$. Our next claim shows that $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$.

Claim 7. $\hat{\varphi}_w(P_1, P_2) > 0$ *implies* $w \in \{a, b, c\}$ *and* $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$.

Proof of Claim 7. Consider $\hat{P}_1 \equiv ca\cdots b$, i.e., $\tau(\hat{P}_1) = c$, $r_2(\hat{P}_1) = a$, and for all $x \in A(k_1, k_2) \setminus b$, we have $x\hat{P}_1 b$. Note that such a $\hat{P}_1 \in \mathcal{S}|_{A(k_1,k_2)}$ exists where an extension is $\hat{Q}_1$ with $|\tau(\hat{Q}_1)| = 0$. The constraints on a, c, and b can satisfied as $|a| = |c| = k_1$ and $|b| = k_2$. Since $P_1 \equiv ac\cdots$, by strategy-proofness

$$\hat{\varphi}_{\{a,c\}}(\hat{P}_1, P_2) = \hat{\varphi}_{\{a,c\}}(P_1, P_2). \tag{23}$$

Since $|c| = k_1$ and $b \gg c$, by Case 2, we have $\hat{\varphi}_c(\hat{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\hat{P}_1, P_2) = 1 - \epsilon$.

Therefore, by (23)

$$\hat{\varphi}_{\{a,c\}}(P_1, P_2) = \epsilon. \tag{24}$$

Assume for contradiction there exists $w \notin \{a, b, c\}$ such that $\hat{\varphi}_w(P_1, P_2) > 0$. Consider $U(w, \hat{P}_1)$. Since $\hat{P}_1 \equiv ca \cdots b$, we have $c \in U(w, \hat{P}_1)$ and $b \notin U(w, \hat{P}_1)$. Therefore, by $\hat{\varphi}_c(\hat{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\hat{P}_1, P_2) = 1 - \epsilon$, we have $\hat{\varphi}_{U(w,\hat{P}_1)}(\hat{P}_1, P_2) = \epsilon$. However, as by (24), $\hat{\varphi}_{\{a,c\}}(P_1, P_2) = \epsilon$ and by our assumption $\hat{\varphi}_w(P_1, P_2) > 0$, we have $\hat{\varphi}_{U(w,\hat{P}_1)}(\hat{P}_1, P_2) > \epsilon$. But this is a contradiction as agent 1 manipulates at $(\hat{P}_1, P_2)$ via P_1. This shows $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$.

Since by (24), $\hat{\varphi}_{\{a,c\}}(P_1, P_2)\epsilon$, we have by $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$, $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. This completes the proof of the claim. □

We are now ready to complete the proof for Case 3B.2.1. Note that there exists $d \neq c$ such that $|d| = k_1$ and $b \gg d$. This is because $|c| = k_1 < k_2 = |b|$. Therefore, using similar arguments as in Claim 7, we can show that $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b, c\}$. Combining this with Claim 7, we have $\hat{\varphi}_w(P_1, P_2) > 0$ implies $w \in \{a, b\}$. Moreover, since by Claim 7 $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$, we have $\hat{\varphi}_a(P_1, P_2) = \epsilon$. This completes the proof for Case 3B.2.1.

The remaining two cases are as follows.

Case 3B.2.2 Suppose that exactly one of P_1 and P_2 has an extension where the top-ranked alternative remains the same.

Case 3B.2.3. Suppose that none of P_1 and P_2 has an extension where the top-ranked alternative remains the same.

The proofs for Cases 3B.2.2 and 3B.2.3 are similar to the proofs for Cases 2.2 and 2.3, respectively, and consequently are omitted.

Since Cases 3B.2.1, 3B.2.2, 3B.2.3 exhaust Case 3B.2, the proof for Case 3B.2 is complete. Moreover, since Cases 3B.1 and 3B.2 exhaust the base case, the proof for the base case is complete.

Induction step for Case 3. Suppose that $\hat{\varphi}$ is random dictatorial over all profiles (under Case 3) in $\mathcal{P}(k_1, v)$ when $v \geq k$ for some $k_1 < k \leq k_2$. We show that the same holds when $v = k - 1$.

Let $a, b \in A$ with $|a| = k_1$, $|b| = k - 1$. Consider $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$. We show that $\hat{\varphi}_a(P_1, P_2) = \epsilon$ and $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. Since $|b| < k_2$, for each $\tilde{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ any extension $\tilde{Q}_2$ of $\tilde{P}_2$ must have b as the top alternative, i.e., $\tau(\tilde{Q}_2) = b$.

We complete the proof by considering different cases based on the structure of the extensions of the preference P_1. Following our enumeration terminology, we enumerate these cases as 3I.1 and 3I.2 to emphasize the fact that they are subcases of the induction step.

Case 3I.1. Suppose that P_1 has extension where the top-ranked alternatives remain the same, that is, there is extension Q_1 of P_1 with $\tau(Q_1) = a$.
Let $c \in A(k_1, k_2)$ such that $c \gg b$ and $|c| = k$. It follows from Lemma 1 that under the assumption of Case 1, the outcome $\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, and hence let us assume that the alternatives b and c form an upper contour set in P_2, i.e., $P_2 \equiv bc \cdots$. Our first claim shows that for all $w \in A(k_1, k_2) \setminus \{a, b\}$ such that there exists $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\bar{P}_2 = cb \cdots w \cdots a \cdots$, we have $\hat{\varphi}_w(P_1, P_2) = 0$.

Claim 8. *$\hat{\varphi}_w(P_1, P_2) = 0$ for all $w \in A(k_1, k_2) \setminus \{a, b\}$ such that there exists $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\bar{P}_2 = cb \cdots w \cdots a \cdots$.*

Proof of Claim 8. Let $w \in A(k_1, k_2) \setminus \{a, b\}$ be such that there exists $\bar{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\bar{P}_2 \equiv cb \cdots w \cdots a \cdots$. As by our assumption $P_2 \equiv bc \cdots$, strategy-proofness implies

$$\hat{\varphi}_{\{b,c\}}(P_1, P_2) = \hat{\varphi}_{\{b,c\}}(P_1, \bar{P}_2). \tag{25}$$

Since $|a| = k_1$ and $|c| = k$, by base case, we have $\hat{\varphi}_a(P_1, \bar{P}_2) = \epsilon$ and $\hat{\varphi}_c(P_1, \bar{P}_2) = 1 - \epsilon$. This together with (25) imply

$$\hat{\varphi}_{\{b,c\}}(P_1, P_2) = 1 - \epsilon. \tag{26}$$

Assume for contradiction $\hat{\varphi}_w(P_1, P_2) > 0$. Consider $U(w, \bar{P}_2)$. Since $\bar{P}_2 \equiv cb \cdots w \cdots a \cdots$, we have $c, b \in U(w, \bar{P}_2)$ and $a \notin U(w, \bar{P}_2)$. Therefore, by $\hat{\varphi}_a(P_1, \bar{P}_2) = \epsilon$ and $\hat{\varphi}_c(P_1, \bar{P}_2) = 1 - \epsilon$, we have $\hat{\varphi}_{U(w,\bar{P}_2)}(P_1, \bar{P}_2) = 1 - \epsilon$. However, as by (26), $\hat{\varphi}_{\{b,c\}}(P_1, P_2) = 1 - \epsilon$ and by our assumption $\hat{\varphi}_w(P_1, P_2) > 0$, we have $\hat{\varphi}_{U(w,\bar{P}_2)}(P_1, \bar{P}_2) > 1 - \epsilon$. But this is a contradiction as agent 2 manipulates at $(P_1, \bar{P}_2)$ via P_2. This completes the proof of the claim. $\square$

We now show that

$$\hat{\varphi}_w(P_1, P_2) = 0 \text{ for all } w \notin \{a, b\}. \tag{27}$$

In view of Claim 8 and the fact that under the assumption of Case 1 the outcome of $\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, to show (27), it is sufficient to show that for all $w \in A(k_1, k_2) \setminus \{a, b\}$ there exists $d \in A(k)$ and $\hat{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $d \gg b$ and $\hat{P}_2 \equiv db \cdots w \cdots a \cdots$. Our next claim shows this.

Claim 9. *For all $w \in A(k_1, k_2) \setminus \{a, b\}$ there exists $d \in A(k)$ and $\hat{P}_2 \in \mathcal{S}|_{A(k_1,k_2)}$ such that $d \gg b$ and $\hat{P}_2 \equiv db \cdots w \cdots a \cdots$.*

Proof of Claim 9. Since $w \in A(k_1, k_2) \setminus a$ and $|a| = k_1$, there exists $s \in M$ such that $a_s = 0$ and $w_s = 1$. If $b_s = 1$ then choose d to be any alternative such that $d \gg b$ and $|d| = l + 1$. Otherwise, choose d such that $d_s = 1$ and $b_r = d_r$ for all all $r \neq s$. Note that as $b_s = 0$ and $d_s = 1$, we have $|d| = l + 1$. Since $a_s = 0$ $w_s = d_s = 1$, there exists a preference $\hat{Q}_2 \in \mathcal{S}$ such that $\tau(\hat{Q}_2) = d$ and $w\bar{Q}_2 a$. Moreover since $|d| - |b| = 1$ and $d \gg b$, we can have b to be the second ranked alternative in $\hat{Q}_2$. The preference $\hat{P}_2$ can be obtained by taking the restriction of $\hat{Q}_2$ to $A(k_1, k_2)$. This completes the proof of the claim. $\square$

Now we are ready to complete the proof for Case 3I.1. Recall that by (26), we have $\hat{\varphi}_{\{b,c\}}(P_1, P_2) = 1 - \epsilon$. Combining this with (26), we have $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$ and hence, $\hat{\varphi}_a(P_1, P_2) = \epsilon$. This completes the proof for Case 3I.1.

Case 3I.2. Suppose that P_1 does not have an extension where the top-ranked alternative remains the same.

The proof for this case is similar to the proof for Case 2.2, and consequently is omitted. Since Cases 3I.1 and 3I.2 exhaust the induction step for Case 3, the proof of the induction step is complete. Moreover, Cases 3B and 3I complete the proof for Case 3, and Cases 1,2, and 3 together complete the proof of Step 1. $\square$

Step 2. Consider the profiles in $\mathcal{P}(u, v)$ where both u and v are greater than k_1. Note that if $k_2 < m$ and at least one of u and v is k_2, then using similar arguments as in Step 1, we can show that $\hat{\varphi}$ is random dictatorial over the profiles in $\mathcal{P}(u, v)$. Therefore, assume that if $k_2 < m$, then both u and v are less than k_2. This implies both $P_1, P_2 \in \mathcal{P}(u, v)$ have extensions with the same corresponding top-ranked alternatives, that is, there are extensions Q_1 and Q_2 of P_1 and P_2, respectively, with $\tau(Q_1) = a$ and $\tau(Q_2) = b$.

We distinguish cases based on the structures of $\tau(P_1)$ and $\tau(P_2)$.

Case 1. Suppose either $\tau(P_2) \gg \tau(P_1)$ or $\tau(P_1) \gg \tau(P_2)$.
Without loss of generality we assume that $\tau(P_2) \gg \tau(P_1)$. Let $a, b \in A$ be such that $|a| > k_1, |b| > k_1$, and $b \gg a$. We show that for all $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$, we have

$$\hat{\varphi}_a(P_1, P_2) = \epsilon \text{ and } \hat{\varphi}_b(P_1, P_2) = 1 - \epsilon.$$

Let $c \in A(k_1)$ be such that $a \gg c$. Since both P_1, P_2 have extensions with the same top-ranked alternatives, it follows from Lemma 1 that the outcome $\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, and hence, we can assume that $P_1 = [a, c] \cdots$ and $P_2 = [b, a] \cdots$. As $\tau(P_1) = a$ and $P_2 = [b, a] \cdots$, by unanimity and strategy-proofness, we have

$$\hat{\varphi}_{[a,b]}(P_1, P_2) = 1. \tag{28}$$

Let $\bar{P}_1 = [c, a] \cdots \in \mathcal{S}|_{A(k_1,k_2)}$. By Step 1, we have $\hat{\varphi}_c(\bar{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\bar{P}_1, P_2) = 1 - \epsilon$. Since $P_1 = [a, c] \cdots$ and $\bar{P}_1 = [c, a] \cdots$, strategy-proofness implies $\hat{\varphi}_{[a,c]}(P_1, P_2) = \hat{\varphi}_{[a,c]}(\bar{P}_1, P_2)$. Therefore, as $b \in [a, c]$, combining all these observations, we obtain

$$\hat{\varphi}_{[a,c]}(P_1, P_2) = \epsilon. \tag{29}$$

Since $b \gg a \gg c$, we have $[a, c] \cap [a, b] = a$. Thus by (28) and (29), we get

$$\hat{\varphi}_a(P_1, P_2) = \epsilon. \tag{30}$$

We now prove a claim.

Claim 10. *For all $w \in [a, b] \setminus b$, we have $w\bar{P}_1 b$.*

Take $w \in [a, b] \setminus b$. Since $b \gg a$, it must be that $b \gg w \gg a$. Moreover, as $a \gg c$, we have $w \gg c$. Therefore, $b \gg w \gg c$. As $\tau(\bar{P}_1) = c$, this implies $w\bar{P}_1 b$. This completes the proof of the claim. □

To complete the proof for Case 1, by (28) and (30), it is enough to show that $\hat{\varphi}_{(a,b)}(P_1, P_2) = 0$. Assume for contradiction $\hat{\varphi}_w(P_1, P_2) > 0$ for some $w \in (a, b)$. Consider $U(w, \bar{P}_1)$. As $\tau(\bar{P}_1) = c$ and $w\bar{P}_1 b$ (by Claim 10), we have $\hat{\varphi}_{U(w,\bar{P}_1)}(\bar{P}_1, P_2) = \epsilon$. Moreover, by (30) and the fact that $\hat{\varphi}_w(P_1, P_2) > 0$, we have $\hat{\varphi}_{U(w,\bar{P}_1)}(P_1, P_2) > \epsilon$. However, this is a contradiction as agent 1 manipulates at $(\bar{P}_1, P_2)$ via P_1. This completes the proof for Case 1.

Case 2. Suppose $\tau(P_2) \not\gg \tau(P_1)$ and $\tau(P_1) \not\gg \tau(P_2)$.
Let $a, b \in A$ be such that $|a| > k_1, |b| > k_1$, $b \not\gg a$, and $a \not\gg b$. We show that for all $P_1, P_2 \in \mathcal{S}|_{A(k_1,k_2)}$ with $\tau(P_1) = a$ and $\tau(P_2) = b$, we have

$$\hat{\varphi}_a(P_1, P_2) = \epsilon \text{ and } \hat{\varphi}_b(P_1, P_2) = 1 - \epsilon. \tag{31}$$

Suppose $|a| = k_2$. Then k_2 must be strictly less than m as otherwise $a \gg b$. Therefore, using similar arguments as in Step 1 we can show that (31) holds. In view of this, we assume $|a|, |b| \neq k_2$ for the case at hand.

Let $c, d \in A(k_1)$ be such that (i) $a \gg c$ but $b \not\gg c$, and (ii) $b \gg d$ but $a \not\gg d$. The existence of such c and d is guaranteed by the fact that $|a| > k_1, |b| > k_1$, $a \not\gg b$, and $b \not\gg a$. Our assumptions on c, d imply that $[a, c] \cap [b, d] = \emptyset$. Since both P_1, P_2 have extensions with the same corresponding top-ranked alternatives, it follows from Lemma 1 that the outcome $\hat{\varphi}(P_1, P_2)$ depends only on the top-ranked alternatives of P_1 and P_2, and hence, we can assume that $P_1 \equiv [a, c] \cdots$ and $P_2 \equiv [b, d] \cdots$. Let $\bar{P}_1 \equiv [c, a] \cdots$. Since $|c| = k_1$, by Step 1 we have $\hat{\varphi}_c(\bar{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\bar{P}_1, P_2) = 1 - \epsilon$. Moreover, as $P_1 \equiv [a, c] \cdots$ and $\bar{P}_1 = [c, a] \cdots$, by strategy-proofness we have $\hat{\varphi}_{[a,c]}(P_1, P_2) = \hat{\varphi}_{[a,c]}(\bar{P}_1, P_2)$. Combining all these observations, we obtain

$$\hat{\varphi}_{[a,c]}(P_1, P_2) = \epsilon. \tag{32}$$

Let $\bar{P}_2 \equiv [d, b] \cdots$. Since $|d| = k_1$, by Step 1, $\hat{\varphi}_a(P_1, \bar{P}_2) = \epsilon$ and $\hat{\varphi}_d(P_1, \bar{P}_2) = 1 - \epsilon$. Thus, using similar arguments as in the derivation of (32), we have

$$\hat{\varphi}_{[b,d]}(P_1, P_2) = 1 - \epsilon. \tag{33}$$

Since $[a, c] \cap [b, d] = \emptyset$, by (32) and (33), we have

$$\hat{\varphi}_{[a,c] \cup [b,d]}(P_1, P_2) = 1. \tag{34}$$

Let $\bar{c} \in A(k_2)$ be such that $\bar{c} \gg a$ and $\hat{P}_1 \equiv [\bar{c}, a] \cdots$. If $k_2 = m$ then $\bar{c} \gg b$, and hence by

Case 1, $\hat{\varphi}_{\bar{c}}(\hat{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\hat{P}_1, P_2) = 1 - \epsilon$. If $k_2 < m$ then using similar arguments as in Step 1, we have $\hat{\varphi}_{\bar{c}}(\hat{P}_1, P_2) = \epsilon$ and $\hat{\varphi}_b(\hat{P}_1, P_2) = 1 - \epsilon$.

Let $\tilde{P}_1 = [a, \bar{c}] \cdots$. Since $\hat{P}_1 \equiv [\bar{c}, a] \cdots$, by strategy-proofness, $\hat{\varphi}_{[a,\bar{c}]}(\hat{P}_1, P_2) = \hat{\varphi}_{[a,\bar{c}]}(\tilde{P}_1, P_2)$. Moreover, as both $\tilde{P}_1$ and P_2 have extensions with the same corresponding top-ranked alternatives, we have $\hat{\varphi}(P_1, P_2) = \hat{\varphi}(\tilde{P}_1, P_2)$. Therefore, as $b \notin [a, \bar{c}]$, combining all these observations, we have

$$\hat{\varphi}_{[a,\bar{c}]}(P_1, P_2) = \epsilon. \tag{35}$$

We now prove a claim. The claim shows that the sets $[a, \bar{c}]$ and $[a, c] \cup [b, c]$ have only a in their intersection.

Claim 11. $[a, \bar{c}] \cap \big[[a, c] \cup [b, d]\big] = a.$

It is straightforward that $a \in [a, \bar{c}] \cap \big[[a, c] \cup [b, d]\big]$. We show that no other alternative belongs to this set. Take $w \in [a, \bar{c}] \setminus a$. Since $\bar{c} \gg a$ and $|a| < |\bar{c}|$, by the definition of $[a, \bar{c}] \setminus a$, $w \gg a$ and $|w| > |a|$. Using a similar logic, as $a \gg c$ and $|c| < |a|$, we have $|x| \leq |a|$ for all $x \in [a, c]$. Since $w \gg a$, we have $|w| > |a|$, and hence it follows that $w \notin [a, c]$. To complete the proof of the claim it is left to show that $w \notin [b, d]$. Recall that by our assumption $a \not\gg b$. This means there exists $s \in M$ such that $a_s = 1$ and $b_s = 0$. Since $w \gg a$ and $a_s = 1$, it must be that $w_s = 1$. Moreover, as $b \gg d$, this implies $w \notin [b, d]$. This completes the proof of the claim. □

By Claim 11 and (34), we have $\hat{\varphi}_{[a,\bar{c}]}(P_1, P_2) = \hat{\varphi}_a(P_1, P_2)$. Combining this with (35), we obtain

$$\hat{\varphi}_a(P_1, P_2) = \epsilon. \tag{36}$$

Considering $\bar{d} \in A(k_2)$ such that $\bar{d} \gg b$ and using similar arguments as in the derivation of (36), we can show that $\hat{\varphi}_b(P_1, P_2) = 1 - \epsilon$. This completes the proof for Case 2.

Since Cases 1 and 2 are exhaustive, the proof of Step 2 is complete. □

Step 1 and Step 2 together complete the proof of Proposition 1. ■

Proposition 2. *$\hat{\varphi} : (\mathcal{S}|_{A(k_1,k_2)})^n \to \Delta A(k_1, k_2)$ is random dictatorial.*

Proof. Note that by Proposition 1, for $n = 2$ every unanimous and strategy-proof RSCF on $\mathcal{S}|_{A(k_1,k_2)}$ is random dictatorial.

Theorem 5 in Chatterji et al. (2014) states that if, for $n = 2$, every unanimous and strategy-proof RSCF on a domain satisfying 'Condition α' is random dictatorial, then the same is true for $n > 2$. This Condition α requires that there are distinct alternatives $a, b, c \in A$ and preferences P_1, P_2, and P_3, such that (i) $P_1 \equiv a \cdots b \cdots c \cdots$, $P_2 \equiv b \cdots c \cdots a \cdots$, and $P_3 \equiv c \cdots a \cdots b \cdots$, and (ii) for every $x \in A \setminus \{a, b, c\}$, either bP_1x or cP_2x or aP_3x. It is not hard to verify that $\mathcal{S}|_{A(k_1,k_2)}$ satisfies Condition α. Hence, by Theorem 5 in Chatterji et al. (2014), we have $\hat{\varphi}$ is random-dictatorial for any arbitrary n. ■

This completes the proof of the theorem. ■

6.2 Proof of Theorem 2

Proof. For the ease of the presentation of the proof we assume that $A \setminus B = \underline{a}$.

(If part) It is easy to see that $f_{\underline{a}}$ satisfies strong unanimity with respect to B and strategy-proofness.

(Only-if part) Let $\varphi : \mathcal{S}^n \to \Delta A$ be an RSCF satisfying strong unanimity with respect to B and strategy-proofness. We show that $\varphi = f_{\underline{a}}$, i.e., for each $P_N \in \mathcal{S}^n$, $\varphi(P_N) = a$ implies that for each $s \in M$, we have

$$a^s = \begin{cases} 0 \text{ if } \tau(P_i)^s = 0 \text{ for all } i \in N, \\ 1 \text{ if } \tau(P_i)^s = 1 \text{ for some } i \in N. \end{cases}$$

It follows from Appendix E.2. in Chatterji and Zeng (2019) that φ is tops-only. For $P_N \in \mathcal{S}^n$, let $T(P_N) = \{i \in N \mid \tau(P_i) \neq \underline{a}\}$. The theorem follows trivially by unanimity when $T(P_N) = \emptyset$. To prove the theorem for the cases $|T(P_N)| \geq 1$ we use induction on the number agents in $T(P_N)$. Here we consider $|T(P_N)| = 1$ as the base case.

Base case for the theorem: Suppose $|T(P_N)| = 1$.

Without loss of generality assume that $T(P_N) = 1$. Suppose $\tau(P_1) = a$. By the definition of $f_{\underline{a}}$, this means $f_{\underline{a}}(P_N) = a$. We use induction on $|a|$ to show that $\varphi(P_N) = a$.

Base case for the proof of the base case of the theorem: Suppose $|a| = 1$.

Let $P_1' \in \mathcal{S}$ be such that $\tau(P_1') = \underline{a}$ and $r_2(P_1') = a$. Since $|a| - |\underline{a}| = 1$ and φ is tops-only without loss of generality we can assume that $r_2(P_1) = \underline{a}$. This means by strategy-proofness, $\varphi_{\{a,\underline{a}\}}(P_N) = \varphi_{\{a,\underline{a}\}}(P_1', P_{-1})$. Since (P_1', P_{-1}) is a unanimous profile we have $\varphi(P_1', P_{-1}) = \underline{a}$. Therefore, $\varphi_{\{a,\underline{a}\}}(P_N) = 1$. If $\varphi_{\underline{a}}(P_N) > 0$ then we have a contradiction to strong unanimity with respect to B as $\tau(P_1) \neq \underline{a}$ and $A \setminus B = \underline{a}$. Thus, $\varphi(P_N) = a$. $\square$

Induction step for the proof of the base case of the theorem: Suppose $\varphi(P_N) = f_{\underline{a}}(P_N)$ for all $P_N \in \mathcal{S}^n$ and all $a \in A$ with $|T(P_N)| = 1$ and $|a| \leq t - 1$ for some $t \leq m$. We proceed to show that $\varphi(P_N) = f_{\underline{a}}(P_N)$ when $|T(P_N)| = 1$ and $|a| \leq t$.

Let $b, c \in A$ be such that $|b| = |c| = t - 1$ with $a \gg b$ and $a \gg c$. Let $P \in \mathcal{S}$ be such that $\tau(P) = b$ and $r_2(P) = a$. By the induction hypothesis we have $\varphi(P, P_{-1}) = b$. Since φ is tops-only and $a \gg b$ with $|a| - |b| = 1$ without loss of generality we can assume that $r_2(P_1) = b$. Therefore, by strategy-proofness, we have $\varphi_{\{a,b\}}(P_N) = \varphi_{\{a,b\}}(P, P_{-1})$. This together with $\varphi(P, P_{-1}) = b$ imply $\varphi_{\{a,b\}}(P_N) = 1$. Using similar arguments we can show that $\varphi_{\{a,c\}}(P_N) = 1$. Combining these two facts with $b \neq c$, we have $\varphi(P_N) = a$. This completes the proof of the base case. $\square$

Induction step for the theorem: Suppose $\varphi(P_N) = f_{\underline{a}}(P_N)$ for all P_N with $|T(P_N)| \leq n_1 - 1$ for some $n_1 \leq n$. We show that the same holds for all P_N with $|T(P_N)| = n_1$.

Let $P_N \in \mathcal{S}^n$ be such that $T(P_N) = n_1 - 1$. Assume without loss of generality that $T(P_N) = \{2, \ldots, n_1\}$. It is sufficient to show that $\varphi(P_1', P_{-1}) = f_{\underline{a}}(P_1', P_{-1})$ for all $P_1' \in \mathcal{S}$. We show this by using induction on $|\tau(P_1')|$. If $\tau(P_1') = \underline{a}$, we are done by our induction hypothesis. Suppose that $\varphi(P_1', P_{-1}) = f_{\underline{a}}(P_1', P_{-1})$ for all $P_1' \in \mathcal{S}$ such that $|\tau(P_1')| \leq l - 1$. We prove the same for all $P_1' \in \mathcal{S}$ with $|\tau(P_1')| = l$. Let $P_1'' \in \mathcal{S}$ be such that $|\tau(P_1'')| = l$. Suppose $\tau(P_1'') = a''$. Let $P_1' \in \mathcal{S}$ be such that $\tau(P_1'') \gg \tau(P_1')$ and $\tau(P_1'')$ be the t-th deviation of $\tau(P_1')$. Suppose $\tau(P_1') = a'$.

Let $\varphi(P_1', P_{-1}) = b$. We distinguish two cases based on the value of b^t.

Case 1: Suppose $b^t = 1$.

Since a'' is the t-th deviation of a' with $a'' \gg a'$, to complete the proof for this case we need to show that $\varphi(P_1'', P_{-1}) = b$. Let $c \in A$ be the t-th deviation of b. Since φ is tops-only and $\tau(P_1')$ and c are t-th deviations of $\tau(P_1'')$ and b, respectively, without loss of generality we can assume that $P_1' = \cdots cb \cdots$ and $P_1'' = \cdots bc \cdots$ and $U(b, P_1') = U(c, P_1'')$. This together with strategy-proofness imply $\varphi_b(P_1', P_{-1}) \leq \varphi_b(P_1'', P_{-1})$. Combining this with the fact $\varphi(P_1', P_{-1}) = b$, we get $\varphi(P_1'', P_{-1}) = b$. This completes the proof of Case 1. $\square$

Case 2: Suppose $b^t = 0$.

Let $c \in A$ be t-th deviation of b. Since $b^t = 0$ and a'' is the t-th deviation of a' with $a'' \gg$

a', to complete the proof for this case we need to show that $\varphi(P_1'', P_{-1}) = c$. Moreover, as φ is tops-only and $\tau(P_1')$ and b are t-th deviations of $\tau(P_1'')$ and c, respectively, without loss of generality we can assume $P_1' = \cdots bc \cdots$ and $P_1'' = \cdots cb \cdots$ and $U(c, P_1') = U(b, P_1'')$. This together with strategy-proofness imply $\varphi_{\{b,c\}}(P_1', P_{-1}) = \varphi_{\{b,c\}}(P_1, P_{-1})$. Therefore, as $\varphi(P_1', P_{-1}) = b$, we have $\varphi_{\{b,c\}}(P_1'', P_{-1}) = 1$. Assume for contradiction $\varphi(P_1'', P_{-1}) \neq c$ which together with $\varphi_{\{b,c\}}(P_1'', P_{-1}) = 1$ imply $\varphi_b(P_1, P_{-1}) > 0$.

Let $\tau(P_2) = d$. Consider a sequence of alternatives $(x_1 = d, x_2, \ldots, x_p = \underline{a})$ such that $x_j \gg x_{j+1}$ and $|x_j| - |x_{j+1}| = 1$ for all $j < p$. Let $P_2' \in \mathcal{S}$ be such that $\tau(P_2') = x_2$. As $x_1 \gg x_2$ and $|x_1| - |x_2| = 1$ there exists $r \in M$ such that x_2 is the r-th deviation of x_1. Let $b' \in A$ be the r-th deviation of b. Since φ is tops-only and x_1 and b are r-th deviations of x_2 and b', respectively, without loss of generality we can assume that b and b' are consecutively ranked alternatives in both P_2 and P_2', and for all $x \notin \{b, b'\}$, xP_2b if and only if $xP_2'b$. Thus, by strategy-proofness $\varphi_{\{b,b'\}}(P_1'', P_{-1}) = \varphi_{\{b,b'\}}(P_1'', P_2', P_{-\{1,2\}})$. As $\varphi_b(P_1'', P_{-1}) > 0$, this means $\varphi_{\{b,b'\}}(P_1'', P_2', P_{-\{1,2\}}) > 0$. Note that as the t^{th} component of b is 0, the t^{th} component of b' is also 0.

As $x_2 \gg x_3$ and $|x_2| - |x_3| = 1$ there exists $s \in M$ such that x_3 is the s-th deviation of x_2. Let $\bar{b}$ and $\bar{b}'$ be the s-th deviations of b, and b', respectively. Further let $\bar{P}_2$ and $\bar{P}_2'$ be two preferences such that (i) $\tau(\bar{P}_2) = x_2$ and $\tau(\bar{P}_2') = x_3$, (ii) b and $\bar{b}$ are consecutively ranked alternatives in both $\bar{P}_2$ and $\bar{P}_2'$, and for all $x \notin \{b, \bar{b}\}$, $x\bar{P}_2b$ if and only if $x\bar{P}_2'\bar{b}$, and (iii) b' and $\bar{b}'$ are consecutively ranked alternatives in both $\bar{P}_2$ and $\bar{P}_2'$, and for all $x \notin \{b', \bar{b}'\}$, $x\bar{P}_2b'$ if and only if $x\bar{P}_2'\bar{b}'$. Such preferences exist as x_2, b, and b' are s-th deviations of x_3, $\bar{b}$, and $\bar{b}'$, respectively. Since φ is tops-only and $\tau(P_2') = \tau(\bar{P}_2)$, we have $\varphi_{\{b,b'\}}(P_1'', \bar{P}_2, P_{-\{1,2\}}) > 0$. Moreover, by strategy-proofness, $\varphi_{\{b,\bar{b}\}}(P_1'', \bar{P}_2, P_{-\{1,2\}}) = \varphi_{\{b,\bar{b}\}}(P_1'', \bar{P}_2', P_{-\{1,2\}})$ and $\varphi_{\{b',\bar{b}'\}}(P_1'', \bar{P}_2, P_{-\{1,2\}}) = \varphi_{\{b',\bar{b}'\}}(P_1'', \bar{P}_2', P_{-\{1,2\}})$. This together with $\varphi_{\{b,b'\}}(P_1'', \bar{P}_2, P_{-\{1,2\}}) > 0$ imply $\varphi_{\{b,b',\bar{b},\bar{b}'\}}(P_1'', \bar{P}_2', P_{-\{1,2\}}) > 0$. Note that as for both b and b', the t^{th} components are 0, the t^{th} components of both $\bar{b}$ and $\bar{b}'$ are also 0.

Continuing in this manner we can show that $\varphi_X(P_1'', \tilde{P}_2, P_{-\{1,2\}}) > 0$ where $\tau(\tilde{P}_2) = x_p = \underline{a}$ and for all $z \in X$ we have $z^t = 0$. However, since $T((P_1'', \tilde{P}_2, P_{-\{1,2\}})) = \{1, 3, 4 \ldots, n_1\}$, by the induction hypothesis $\varphi(P_1'', \tilde{P}_2, P_{-\{1,2\}}) = f_{\underline{a}}(P_1'', \tilde{P}_2, P_{-\{1,2\}})$. This together with $\tau(P_1'')^t = 1$ implies $\varphi_X(P_1'', \tilde{P}_2, P_{-\{1,2\}}) = 0$, a contradiction to $\varphi_X(P_1'', \tilde{P}_2, P_{-\{1,2\}}) > 0$. Therefore, $\varphi(P_1'', P_{-1}) = c$. This completes the proof of Case 2. $\square$

Since Cases 1 and 2 are exhaustive, the proof of the induction step and hence, the proof the theorem is complete. ■

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