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LOCAL INCENTIVE COMPATIBILITY IN ORDINAL TYPE-SPACES*

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Abstract

We explore the relation between different notions of local incentive compatibility (LIC) and incentive compatibility (IC) on ordinal type-spaces. In this context, we introduce the notion of ordinal local global equivalent (OLGE) and cardinal local global equivalent (CLGE) environments. First, we establish the equivalence between the two environments on strict ordinal type-spaces. Next, we consider ordinal type-spaces admitting indifference. We introduce the notion of almost everywhere IC and strong LIC, and provide a necessary and sufficient condition on ordinal type spaces for their equivalence. Finally, we provide results on how to (minimally) check the IC property of a given mechanism on any ordinal type-space and show that local types along with the boundary types form a minimal set of incentive constraints that imply full incentive compatibility.

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KEYWORDS: point-wise local incentive compatibility, adjusted local incentive compatibility, uniform local incentive compatibility, (global) incentive compatibility, ordinal type-spaces

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1. INTRODUCTION

We consider standard mechanism design problems when agents have quasi linear utility function. There is a finite set of alternatives and a finite set of agents. Agents' types are their valuations for the alternatives. A mechanism is incentive compatible (IC) if it is not possible for any agent to increase his/her (net) utility by misreporting his/her sincere type in any way. It is locally IC (LIC) if it is not possible for an agent to increase his/her (net) utility by misreporting to a type that lies in a small "neighborhood" of his/her sincere type. In other words, LIC is a weakening of IC where IC is required to be satisfied for deviations within a small neighborhood.

Characterizing all IC mechanisms on a given type-space is an important problem in mechanism design. However, despite its importance, the structure of IC mechanisms is known only for the case when the type-space is unrestricted (that is, $\mathbb{R}^{|A|}$, where A is the set of alternatives) (see Lavi et al. (2009) for details) and finding this structure on other type-spaces seem to be a hard problem. As an intermediate step, researchers have got interested in exploring if the requirement of IC can be reduced considerably.[1] Local IC (LIC) turns out to be a way.

The notion of LIC depends on the notion of neighborhood one intends to consider. Carroll (2012) considered neighborhoods with respect to Euclidean distance. We refer to this notion as pointwise LIC (PLIC). He showed that if the type-space is convex, then PLIC is equivalent to IC. To the best of our knowledge, nothing is known about the equivalence of LIC and IC on other type-spaces.

An ordinal domain is a collection of ordinal preferences. In contrast to cardinal environments (that is, for type-spaces), the relation between LIC and IC is well-explored for ordinal domains (see Kumar et al. (2021) for details). A type represents an ordinal preference if for any two alternatives a and b, a is preferred to b implies the valuation of a will be higher than b. A type-space is called ordinal if it is induced by an ordinal

[1]For the importance of identifying a minimal set of incentive constraints that imply full incentive compatibility - see discussions in Chapter 7 of Fudenberg and Tirole (1991), Armstrong (2000) and Chapter 6 in Vohra (2011).

domain, that is, it contains all types representing some preference in an ordinal domain.[2] An ordinal domain/type-space is strict if it does not admit indifference.

The mechanism design literature generally considers geometric restrictions such as connectedness and convexity on type-spaces. While these are simplifying technical assumptions, ordinal restrictions such as single-peaked or single-crossing or single-dipped preferences arise in several economic problems. For instance, in a problem where the location of a public good on a street needs to be decided, subsidies can be given to the people who reside far away from the chosen location. Similarly, in determining the budget for infrastructure, industrial development, etc., subsidies can be given to poor people (or whoever derives relatively lesser externalities from a decision). Barzel (1973), Stiglitz (1974), and Bearse et al. (2001) consider the problem of setting the level of tax rates to provide public funding in the education sector, and Ireland (1990) and Epple and Romano (1996) consider the same problem in the health insurance market.[3] Our analysis enables one to analyze these problems as a mechanism design problem with transfers. Mishra et al. (2016) explains how single-peakedness arises in a private good scheduling problem. Some other papers that deal with mechanism design in ordinal type-spaces are Mishra et al. (2014), Carbajal and Müller (2015), Mishra et al. (2016), etc.

An ordinal domain satisfies ordinal local global equivalence (OLGE) if every locally incentive compatible social choice function on that domain is incentive compatible. The notion of OLGE is defined in Kumar et al. (2021), where it is shown that a strict ordinal domain is OLGE if and only if it satisfies "Property L". Almost all well-known domains such as single-peaked, single-crossing, single-dipped, etc., satisfies OLGE. An ordinal type-space satisfies cardinal local global equivalence (CLGE) if every locally incentive compatible mechanism on that type-space is incentive compatible. We characterize strict CLGE type-spaces by showing that a strict ordinal domain is OLGE if and only if the corresponding strict type-space is CLGE. It is worth mentioning that our result applies to type-spaces that are not necessarily convex, not even connected. The relaxation of connectedness or convexity is not a trivial extension. For instance, the equivalence of

[2] All the results of this paper also hold if we consider types that are bounded below or bounded above (for instance, non-negative types).

[3] Individuals' preferences are considered to be single-peaked in such scenarios.

PLIC and IC does not hold on non-connected type-spaces, consequently we introduce the notion of uniform LIC (ULIC) and establish the equivalence of ULIC and IC on such type-spaces.

Indifference occurs naturally in preferences, therefore we explore the equivalence of LIC and IC on ordinal type-spaces admitting indifferences. We introduce the notion of almost everywhere IC. A mechanism is almost everywhere IC if it is IC outside a set of (Lebesgue) measure zero (thus, such a mechanism is IC except for some rare (measure zero) situations). We suitably define the notion of LIC to take care of indifference. We call it strong LIC and provide a necessary and sufficient condition on an ordinal type-space for the equivalence of strong LIC and almost everywhere IC. The closure of single-peaked or single-crossing type-spaces, single-plateaued type-spaces, etc., are non-convex type-spaces that satisfy the necessary and sufficient condition. As a corollary, we establish the equivalence of PLIC and almost everywhere IC on these type-spaces. To see the novelty of our analysis, note that the equivalence of PLIC and IC does not hold on such type-spaces (see Example 1 in Mishra et al. (2016)), and that is why it is important to see the extent to which IC can be ensured by PLIC on such non-convex type-spaces. What our result says is that the said equivalence actually holds but only in an almost everywhere sense. Mishra et al. (2016) consider the same problem for a particular type of mechanisms, called payments-only mechanisms and show that PLIC and IC are equivalent for such mechanisms. Our result complements their result by showing that one can drop payment-onlyness by requiring almost everywhere IC instead of IC.

Finally, we consider the problem of checking whether a given mechanism is IC or not on an arbitrary ordinal type-space. We show that to ensure IC of a mechanism, apart from checking the local types, one needs to check only the "boundary types". Thus, local types and boundary types form a minimal set of incentive constraints that imply full incentive compatibility. Since the boundary types have Lebesgue measure zero, this result reduces the complexity of checking whether a mechanism is IC or not in a considerable manner.

A salient feature of our result is that we deduce them for arbitrary notion of localness.

To see the importance of this framework, note that the notion of localness is very subjective and may vary from person to person. Not only that, the standard notion of adjacent localness does not apply to multi-dimensional ordinal domains (See Kumar et al. (2021) for details). Our general framework enables one to apply our results to any such scenario.

2. MODEL

We consider a one-agent model in this paper. This is without loss of generality for our analysis.[4]

Let A be a finite set of alternatives with $|A| = n$. For any given subset X of $\mathbb{R}^n$, by $cl(X)$ we denote the closure of X. A type t is a mapping from A to $\mathbb{R}$ that represents the valuation of each alternative in A. We view a type as an element of $\mathbb{R}^n$ (with an arbitrary but fixed indexation of the alternatives). A type t is strict if $t(a) \neq t(b)$ for all $a, b \in A$, otherwise it is a weak type.[5] By relative valuation of an alternative a with respect to another alternative b at a type t, we mean the number $t(a) - t(b)$. For two types t and t', we denote the line joining them by $[t, t']$.[6] A subset T of $\mathbb{R}^n$ is called a type-space. An allocation rule is a map $f : T \to A$ and a payment rule is a map $p : T \to \mathbb{R}$. A (direct) mechanism μ is a pair consisting of an allocation rule f and a payment rule p.

Definition 2.1. A mechanism (f, p) is *incentive compatible* (IC) on a pair of types (t, s) if

$$t(f(t)) - p(t) \geq t(f(s)) - p(s).$$

It is IC on a type-space T if it is IC on every pair of types $(t, s) \in T \times T$.

Below, we note some facts regarding the IC property. Facts 2.1 and 2.2 follow from the definition of IC property.

FACT 2.1. *Suppose that a mechanism (f, p) is IC on both pairs of types (t, s) and (s, t). Suppose further that $f(t) = f(s)$. Then, $p(t) = p(s)$.*

[4] All the results of this paper can be generalized to the case of more than one agent in a systematic manner (see Carroll (2012), Mishra et al. (2016), etc.).

[5] Note that strict types are not special cases of weak types.

[6] More formally, $[t, t'] = \{(1 - \alpha)t + \alpha t' \mid \alpha \in [0, 1]\}$.

Fact 2.2. *Suppose that a mechanism (f,p) is IC on both pairs of types (t,s) and (s,t). Suppose further that the relative valuation of $f(t)$ with respect to some alternative a is increased from t to s, that is, $s(f(t)) - s(a) > t(f(t)) - t(a)$. Then, $f(s) \neq a$.*

Our next fact provides a sufficient condition for a mechanism to be IC on a pair of types based on its IC property over a sequence of types. This fact was first used in Kumar and Roy (2021); see Appendix A.1 of the paper for a formal proof.

Fact 2.3. *A mechanism $\mu = (f,p)$ on a type-space T is IC on a pair of types (t,t') if there is a finite sequence of types $(t = t^1, \ldots, t^k = t')$ in T such that for all $l < k$,*
(i) μ is IC on (t^l, t^{l+1}), and
(ii) $t^1(f(t^{l+1})) - t^1(f(t^l)) \leq t^l(f(t^{l+1})) - t^l(f(t^l))$.

The relation between LIC and IC is well-studied for social choice functions on ordinal domains (see Kumar et al. (2021) for details); in this paper we extend this study for cardinal environments. In line with Kumar et al. (2021), we consider a general notion of localness represented by means of a graph on an ordinal domain.

A preference on A is a weak linear order, that is, a complete and transitive binary relation on A. If it is additionally antisymmetric, it is called a strict preference, otherwise it is called a weak preference.[7] For a weak preference R, we denote its strict part by P and the indifference part by I. We denote the set of all preferences on A by $\overline{\mathscr{P}}(A)$ and the set of all strict preferences on A by $\widehat{\mathscr{P}}(A)$. An ordinal domain $\overline{\mathscr{D}}$ is a subset of $\overline{\mathscr{P}}(A)$ and a strict ordinal domain $\widehat{\mathscr{D}}$ is a subset of $\widehat{\mathscr{P}}(A)$.

We deal with type-spaces that have some additional structure. For a type t and a preference R, we say that t represents R (or R represents t) if for all $a,b \in A$, aRb if and only if $t(a) \geq t(b)$. We denote the preference that a type t represents by $prfn(t)$, and the set of types that a preference R represents by $type(R)$.[8] Similarly, for an ordinal domain $\mathscr{D}$, we denote the set of all types that represent some preference in the domain by $type(\mathscr{D})$, that is, $type(\mathscr{D}) = \{t \in \mathbb{R}^{|A|} \mid prfn(t) \in \mathscr{D}\}$, and for a type-space T, the set

[7] Note that strict preferences are not special cases of weak preferences

[8] All the results of this paper also hold if we weaken the assumption on $type(R)$ to be the set of types that are bounded below (for instance, non-negative types) or bounded above. More formally, there exists a real number L such that $type(R) = \{t \mid t \text{ represents } R \text{ and } t(x) \geq L \text{ for every } x \in A\}$ or $type(R) = \{t \mid t \text{ represents } R \text{ and } t(x) \leq L \text{ for every } x \in A\}$.

of all preferences that are represented by some type in T by $prfn(T)$. A type-space T is called strict if $t(a) \neq t(b)$ for all $t \in T$ and all $a, b \in A$. We say that T is an ordinal type-space if $T = type(\mathscr{D})$ for some $\mathscr{D} \subseteq \overline{\mathscr{P}}(A)$.

Let $\mathscr{D}$ be a domain. An ordinal environment is a pair $(\mathscr{D}, G)$, where $\mathscr{D}$ is an ordinal domain and $G = \langle \mathscr{D}, E \rangle$ is an (undirected) graph on $\mathscr{D}$. Two preferences in $\mathscr{D}$ are called G-local if they form an edge in G. A path $(P^1, \ldots, P^k)$ from P^1 to P^k is G-local if every two consecutive preferences in it are G-local. An ordinal environment $(\mathscr{D}, G)$ is called strict if $\mathscr{D}$ is a collection of strict preferences.

We introduce the notion of cardinal environment in a natural way. A cardinal environment is a pair (T, G), where T is a type-space and G is an undirected graph on $prfn(T)$. Two types t and t' in T are said to be G-local if $prfn(t) = prfn(t')$ or $prfn(t)$ and $prfn(t')$ are G-local. A cardinal environment (T, G) is called strict if T is a strict type-space.

A mechanism μ on a cardinal environment (T, G) is LIC if it is IC on every pair of G-local types, that is, μ is IC on $type(\{R, R'\}) \cap T$ for all G-local preferences R and R' in $prfn(T)$.

A social choice function (SCF) on an ordinal domain $\mathscr{D}$ is a mapping $g : \mathscr{D} \to A$. It is IC on a pair of preferences (R, R') if $g(R) R g(R')$. An SCF $g : \mathscr{D} \to A$ is LIC on an ordinal environment $(\mathscr{D}, G)$ if it is IC on every pair of G-local preferences, and it is IC on $\mathscr{D}$ if it is IC on every pair of preferences in $\mathscr{D}$.

An ordinal environment $(\mathscr{D}, G)$ is called ordinal local global equivalent (OLGE) if every LIC SCF on $(\mathscr{D}, G)$ is IC on $\mathscr{D}$. Similarly, a cardinal environment (T, G) is called cardinal local global equivalent (CLGE) if every LIC mechanism on (T, G) is IC on T.

3. Two particular kinds of localness

The main objective of this paper is to characterize ordinal type-spaces so that a restricted version of IC, called local IC (LIC), becomes equivalent to IC. Although we present results for a general notion of localness by means of graphs, we specifically deal with two particular kinds of localness that are practically important. A mechanism is point-wise LIC (PLIC) on a type-space T if for every $t \in T$, there exists an $\varepsilon > 0$ such that it is IC on (t,s) and (s,t) for every $s \in T$ with $||t-s|| < \varepsilon$.[9] For a given $\varepsilon > 0$, a mechanism on a type-space T is called ε-LIC if it is IC on every pair of types $(t,s) \in T \times T$ having (Euclidean) distance less than ε, that is, for all $t,s \in T$, $||t-s|| < \varepsilon$ implies the mechanism is IC on (t,s). A mechanism is called uniformly LIC (ULIC) if it is ε-locally IC for some $\varepsilon > 0$.

FACT 3.1. *Carroll (2012) If a type-space T is convex, then every PLIC mechanism on T is IC on T.*

The notion of PLIC is introduced in Carroll (2012). According to this notion, one has the freedom to choose different ε for different types, whereas in ULIC one has to chose the same ε for all types. If the infimum value of the ε's chosen for different types in case of PLIC is positive, then that value can be taken as the choice of the ε in ULIC, and consequently, the two notions will become equivalent. On other hand, if the said infimum is zero, then ULIC becomes slightly stronger than PLIC. This slight strengthening of PLIC widens its applicability. To see this, consider the situation where there are just two alternatives, say a and b, and the type-space is $T = \{t \in \mathbb{R}^2 \mid t(a) \neq t(b)\}$. Thus, T is disconnected and can be written as a union of two disjoint open spaces $T^1 = \{t \in \mathbb{R}^2 \mid t(a) < t(b)\}$ and $T^2 = \{t \in \mathbb{R}^2 \mid t(a) > t(b)\}$. In such situations, one can define neighborhoods of the points in T^1 such that none of them intersects T^2, and those of the points in T^2 such that none of them intersects T^1. Thus, PLIC with such neighborhoods does not put any constraint on a pair of types (s,t) where $s \in T^1$ and $t \in T^2$, and consequently, cannot imply IC on T. However, ULIC imposes IC on certain pairs of types (sufficiently close ones) that are on the other sides of the boundary of T^1 and T^2, and thereby retains the "possibility" of implying IC on T. In fact, as we show in this paper, ULIC indeed implies IC on T.

It is worth mentioning that ULIC is as useful as PLIC for practical purposes. In reality, if one wants to check (by means of a program/device) whether some mechanism is LIC or not, he/she can only check it for some given neighborhood of each type, not for a sequence of neighborhoods whose size converges to zero.

[9] We denote the Euclidean norm of a vector $t \in \mathbb{R}^n$ by $||t||$.

4. A CHARACTERIZATION OF STRICT CLGE TYPE-SPACES

Kumar et al. (2021) provide a necessary and sufficient condition for a strict ordinal environment to be OLGE. We generalize their result for strict cardinal environments. We begin with defining some notions that are provided in Kumar et al. (2021).[10]

Definition 4.1. A pair of alternatives $\{a,b\}$ has a restoration in a G-local path $(P^1,\ldots,P^k)$ if there exist $1 \leq r < s < t \leq k$ such that either $[aP^rb, bP^sa$, and $aP^tb]$ or $[bP^ra, aP^sb$, and $bP^ta]$.

Definition 4.2. Given a strict ordinal environment $(\widehat{\mathscr{D}}, G)$, for $P, P' \in \widehat{\mathscr{D}}$ and $a \in A$, we say that a G-local path π from P to P' satisfies the Lower Contour Set no-restoration property (Property L) with respect to a if the path π has no $\{a,b\}$-restoration for all $b \in L(a,P)$ where $L(a,P) = \{z \in A \mid aPz\}$.

The strict ordinal environment $(\widehat{\mathscr{D}}, G)$ satisfies Property L if for all distinct $P, P' \in \widehat{\mathscr{D}}$ and all $a \in A$, there exists a G-local path from P to P' satisfying Property L with respect to a.

The following theorem in Kumar et al. (2021) provides a necessary and sufficient condition for a strict ordinal environment to be OLGE.

Theorem 4.1. *Kumar et al. (2021) A strict ordinal environment $(\widehat{\mathscr{D}}, G)$ is OLGE if and only if it satisfies Property L.*

Our next theorem generalizes Theorem 4.1 for strict cardinal environments.

Theorem 4.2. *A strict ordinal environment $(\widehat{\mathscr{D}}, G)$ is OLGE if and only if the strict cardinal environment $(type(\widehat{\mathscr{D}}), G)$ is CLGE.*

The proof of this theorem is relegated to Appendix A.1. It follows from Theorem 4.1 and Theorem 4.2 that a strict cardinal environment $(type(\widehat{\mathscr{D}}), G)$ is CLGE if and only if $(\widehat{\mathscr{D}}, G)$ satisfies Property L.

It is shown in Kumar et al. (2021) that well-known multi-dimensional ordinal domains such as the separable domain and the multi-dimensional single-peaked domain are OLGE. It follows from Theorem 4.2 that the cardinal environments of these domains are CLGE.

[10] See Kumar et al. (2021) for verbal explanations of these notions.

4.1 THE CASE OF ADJACENT LOCALNESS

In this subsection, we deal with a specific case where the notion of localness for the ordinal environment is "adjacency" and that for the cardinal environment is a weaker version of ULIC which we call adjusted LIC.

For some $1 \leq k \leq n$, we denote the k-th ranked alternative of a strict preference P by $P(k)$. Two strict preferences P and P' are said to be adjacent local if they differ by the ranking of two consecutively ranked alternatives, that is, there is $1 \leq k < n$ such that $P(k) = P'(k+1)$, $P(k+1) = P'(k)$, and $P(l) = P'(l)$ for all $l \notin \{k, k+1\}$. We write G^{ad} when localness is defined by adjacency, that is, there is an edge between P and P' in G^{ad} if and only if P and P' are adjacent.

A mechanism μ is said to be adjusted LIC (ALIC) on a strict type-space $\hat{T}$ if (i) for every type t in $\hat{T}$, there is a neighborhood around t such that μ is IC on both (t,s) and (s,t) for all types s in that neighborhood, and (ii) for every type $\bar{t}$ that lies on the boundary of $\hat{T}$ (that is, in $cl(\hat{T}) \setminus \hat{T}$), there is a neighborhood of $\bar{t}$ such that μ is IC on every pair of strict types in that neighborhood.

Part (i) of the definition of ALIC is the same as PLIC. As we have explained in Section 3, PLIC (with suitably chosen arbitrarily small neighborhoods) is unable to "spread" IC between two components of a disconnected type-space. Part (ii) of the definition of ALIC ensures the said spread in a natural way: it requires IC for types that are arbitrarily close but on the opposite sides of the boundary of the strict type-space. It does this by considering an arbitrary neighborhood of a type that lies on the boundary of $\hat{T}$ (and hence not in $\hat{T}$) and requiring IC for all pairs of types of $\hat{T}$ in this neighborhood.

Definition 4.3. A mechanism μ on a strict type-space $\hat{T}$ is said to adjusted locally IC (ALIC) if it is PLIC and for every $\bar{t} \in cl(\hat{T}) \setminus \hat{T}$, there exists an open neighborhood $N(\bar{t}) \subseteq cl(\hat{T})$ of $\bar{t}$ such that for all $t', t'' \in N(\bar{t}) \cap \hat{T}$, μ is IC on (t', t'').

The following corollary is obtained from Theorem 4.2.

Corollary 4.1. *If a strict ordinal environment $(\widehat{\mathscr{D}}, G^{ad})$ is OLGE, then a mechanism on $type(\widehat{\mathscr{D}})$ is IC if and only if it is ALIC.*

The proof of this corollary is relegated to Appendix A.2.

Since ULIC implies ALIC by definition, it follows from Corollary 4.1 that if a strict ordinal environment satisfies OLGE with respect to adjacency localness, then ULIC and IC are equivalent on its cardinal version.

Corollary 4.2. *If a strict ordinal environment* $(\widehat{\mathscr{D}}, G^{ad})$ *is OLGE, then a mechanism on* $type(\widehat{\mathscr{D}})$ *is IC if and only if it is ULIC.*

A large class of strict ordinal environments of practical importance, such as single-peaked, single-dipped, single-crossing, etc., are OLGE with respect to the adjacency localness.[11] Corollary 4.1 implies that ALIC (and hence, ULIC) and IC are equivalent on their corresponding type-spaces. It should be noted that PLIC and IC are not equivalent on these type-spaces as the type-spaces are not connected.

4.2 A STRONGER VERSION OF THEOREM 4.2

Note that the statement of Theorem 4.2 requires *all* types to be present for each ordinal preference. Since requiring all types representing an ordinal preference might be restrictive for practical applications, we extract out the types, the presence of which is sufficient for the proof of Theorem 4.2. The objective is to emphasize that Theorem 4.2 holds for much weaker environments rather than just strict ordinal type-spaces.

We introduce a property called $\hat{L}$ for a cardinal environment and show that any cardinal environment satisfying this property is CLGE. Property $\hat{L}$ is a suitable adaptation of Property L for cardinal environments. Thus, instead of providing a sufficient condition on an ordinal domain to ensure CLGE on its corresponding type-space, we provide a sufficient condition on a strict type-space directly.

We say an alternative a overtakes another alternative b from a strict preference P to another strict preference P' if bPa and $aP'b$. Recall that Property L says that between any two strict preferences P and P' and any alternative a, there exists a local path from P to P' with no $\{a,b\}$-restoration for all $b \in L(a,P)$. Below, we introduce the cardinal version of Property L, which we call Property $\hat{L}$.

Definition 4.4. A strict cardinal environment $(\hat{T}, G)$ satisfies Property $\hat{L}$ if for all t, $t' \in \hat{T}$ and all $a \in A$, there exists a G-local path $(P^1, \ldots, P^k)$ in $prfn(\hat{T})$ with $P^1 \in prfn(t)$

[11] For the definition of these domains see Mishra et al. (2016) and Carroll (2012).

and $P^k \in prfn(t')$ satisfying Property L with respect to a such that for all $l < k$ and all $t^{l+1} \in type(P^{l+1}) \cap \hat{T}$, there exists $t^l \in type(P^l) \cap \hat{T}$ such that

(i) $t^l(a) - t^l(x) > t^{l+1}(a) - t^{l+1}(x)$ for all x that a does not overtake from P^l to P^{l+1}, and

(ii) $t^l(a) - t^l(y) \geq t(a) - t(y)$ for all y that a overtakes from P^l to P^{l+1}.

Theorem 4.3. *A strict cardinal environment $(\hat{T}, G)$ is CLGE if $\hat{T}$ satisfies Property $\hat{L}$.*

The proof of this theorem follows by using similar arguments as in the proof of Theorem 4.2.

5. ORDINAL DOMAINS WHEN LOCALNESS IS DEFINED ON STRICT PREFERENCES

In this section, we consider ordinal environments admitting indifference where local structure is given by means of a graph over the strict preferences. Since in such an environment, LIC does *not* impose any restriction on weak preferences, it cannot ensure IC on the whole domain as well. So, we impose additional requirements on weak preferences in order to ensure IC.

For an ordinal domain $\overline{\mathscr{D}}$, we denote its maximal strict ordinal subset by $strict(\overline{\mathscr{D}})$, that is, $strict(\overline{\mathscr{D}}) = \{P \in \overline{\mathscr{D}} \mid P \text{ is a strict preference}\}$.

In order to generalize Theorem 4.2 for ordinal domains allowing indifferences, we introduce the notion of weak-compatibility. For a weak preference R, we say a strict preference $\hat{P}$ is compatible with R if aPb implies $a\hat{P}b$ for all $a, b \in A$. For instance, if $R = [ab]c[de]f$, then the following preferences are compatible with R: $abcdef$, $abcedf$, $bacdef$, and $bacedf$.[12] Weak compatibility says that for every weak preference R in $\overline{\mathscr{D}}$, there exists a strict preference in $\overline{\mathscr{D}}$ that is compatible with R.

Let $\overline{\mathscr{D}}$ be an ordinal domain. Let G be an (undirected) graph on $strict(\overline{\mathscr{D}})$. A mechanism μ is **strong LIC** on the cardinal environment $(type(\overline{\mathscr{D}}), G)$ if it is LIC, and

[12] By $[ab]c$, we denote a weak preference where a and b are indifferent, and are preferred to c. Similarly by abc, we denote a strict preference where a is preferred b, and b is preferred to c.

additionally IC on every pair of types $(\bar{t}, \hat{t})$ such that there is $P \in strict(\overline{\mathscr{D}})$ so that $\hat{t}$ is a strict type in $type(P)$ and $\bar{t}$ is a weak type in $cl(type(P)) \cap type(\overline{\mathscr{D}})$.

Now, we introduce the notion of almost everywhere IC. We use the following notation to ease the presentation. For a type-space $\bar{T}$, we denote its maximal strict subset by $strict(\bar{T})$, that is, $strict(\bar{T}) = \{t \in \bar{T} \mid t(a) \neq t(b)$ for all distinct $a, b \in A\}$. A mechanism on a type-space $\bar{T}$ is **almost everywhere IC**, if it is IC on every pair of types in $\bar{T} \times strict(\bar{T})$. Thus, an almost everywhere IC mechanism might fail to become IC on a pair of types $(t, \bar{t})$ only if $\bar{t}$ is a weak type that lies in $\bar{T}$. Since the (Lebesgue) measure of the set $\bar{T} \setminus strict(\bar{T})$ is zero, the measure (in the product space) of the set of pairs on which an almost everywhere IC mechanism may fail to be IC is also zero, which justifies the name. Note that almost everywhere IC implies strong LIC by definition.

As we have mentioned in Subsection 5.1, the equivalence between strong LIC and IC does not hold on ordinal type-spaces admitting indifference. Our next theorem establishes the extent to which IC is implied by strong LIC. It turns out that strong LIC implies IC "almost everywhere".

Theorem 5.1. *Let $\overline{\mathscr{D}}$ be an ordinal domain. Then, the following two statements are equivalent.*

(i) Every strong LIC mechanism on the environment $(type(\overline{\mathscr{D}}), G)$ is almost everywhere IC.

(ii) The domain $\overline{\mathscr{D}}$ satisfies weak-compatibility and the environment $(strict(\overline{\mathscr{D}}), G)$ is OLGE.

The proof of this theorem is relegated to Appendix A.3.

5.1 CLOSURE OF TYPE-SPACES OF STRICT ORDINAL DOMAINS

Let $\widehat{\mathscr{D}}$ be a strict ordinal domain and let $cl(type(\widehat{\mathscr{D}}))$ be the closure of $type(\widehat{\mathscr{D}})$. Since $cl(type(\widehat{\mathscr{D}}))$ is closed, Part (ii) of Definition 4.3 is vacuously true, and consequently, the notion of ALIC boils down to that of PLIC. However, Corollary 4.1 does not hold anymore, that is, PLIC does not imply IC on $cl(type(\widehat{\mathscr{D}}))$ (see Example 1 in Mishra et al. (2016) for details). It is worth mentioning that PLIC implies strong LIC in adjacency environments.[13] Therefore the equivalence of strong LIC and IC cannot hold in such environments. The following corollary, which is obtained from Theorem 5.1, says that a version of Corollary 4.1 holds if we weaken IC by almost everywhere IC.

Corollary 5.1. *If a strict ordinal environment $(\widehat{\mathscr{D}}, G^{ad})$ is OLGE, then every PLIC mechanism on $cl(type(\widehat{\mathscr{D}}))$ is almost everywhere IC.*

The proof of this corollary is relegated to Appendix A.4.

Corollary 5.1 applies to closure of single-peaked or single-crossing type-spaces or single-plateaued type-spaces.[14] Corollary 5.1 also applies to closure of single-peaked domain on a tree (Demange (1982)).

5.2 LIC vs. IC for a given mechanism

Having a characterization of ordinal type-spaces such that the equivalence of strong LIC and almost everywhere IC holds, the next natural step is to look at the extent to which we can push the almost everywhere IC property on such type-spaces. As discussed earlier, an almost everywhere IC mechanism might fail to be IC only on the pairs $(t, \bar{t})$ where $\bar{t}$ is a weak type in the type-space.[15] We show that if $\bar{t}$ is a weak type lying in the *interior* of the type-space, then such mechanisms are bound to be IC on pairs $(t, \bar{t})$. Hence, we establish the fact that an almost everywhere IC mechanism might fail to be IC only on the pairs $(t, \bar{t})$ where $\bar{t}$ is a weak type lying on the boundary of the type-space, thereby modifying our previous result. We further identify the possible outcomes of a given mechanism at such a weak type $\bar{t}$ (that is, we identify possible values of $f(\bar{t})$) such that the mechanism is IC on pairs $(t, \bar{t})$.

For a set $\bar{T} \subseteq \mathbb{R}^n$, by $\bar{T}^o$ we denote the interior of the set $\bar{T}$, that is, $\bar{T}^o = \{t \in \bar{T} \mid$ there exists $\varepsilon > 0$ such that $s \in \bar{T}$ for every s with $d(t,s) < \varepsilon\}$. By $\widehat{\partial}\bar{T}$ we denote the points in $\bar{T}$ that lie on the boundary of $\bar{T}$, that is, $\widehat{\partial}\bar{T} = \bar{T} \setminus \bar{T}^o$.

Theorem 5.2. *Let a strict ordinal environment $(\widehat{\mathscr{D}}, G^{ad})$ be OLGE and let $\bar{T} = cl(type(\widehat{\mathscr{D}}))$. Suppose μ is an arbitrary PLIC mechanism on $\bar{T}$. Then,*

[13] For a formal proof, see the proof of Corollary 5.1.

[14] For the definition of single-plateau domain see Berga (1998).

[15] We use t here to denote a generic element of the type-space.

(i) μ *is IC on* $\bar{T} \times \bar{T}^o$*, and*

(ii) μ *is IC on* $\bar{T} \times \{\bar{t}\}$ *for all* $\bar{t} \in \widehat{\partial}\bar{T}$ *such that there exists* $P \in \widehat{\mathscr{D}}$ *with* $\bar{t} \in cl(type(P))$ *and* $f(\bar{t})Pz$ *for every* z *with* $\bar{t}(f(\bar{t})) = \bar{t}(z)$.

The proof of this theorem is relegated to Appendix A.5.

REMARK 5.1. For simplicity we present Theorem 5.2 for adjacent localness and PLIC mechanisms but it can be suitably formulated for arbitrary notion of localness and strong LIC mechanisms.

REMARK 5.2. Example 1 in Mishra et al. (2016) presents a single-peaked type-space where they construct a PLIC mechanism that fails to be IC. It follows from part (i) of Theorem 5.2 in our paper that a PLIC mechanism on such a type-space $\bar{T}$ can violate IC only on types lying in $\bar{T} \times \widehat{\partial}\bar{T}$. For such a violation on any pair of types (t,t'), it must be the case that t' lie in either $type(c[ba])$ or $type(a[bc])$.[16] It further follows from part(ii) of Theorem 5.2 that the outcome at type t' must be either a if $t' \in type(c[ba])$ or c if $t' \in type(a[bc])$. Thus, the counter example (Example 1 in Mishra et al. (2016)) was the only way (upto symmetry) to construct a PLIC mechanism that violates IC.

A. APPENDIX

A.1 PROOF OF THEOREM 4.2

Proof: If part: The proof of the if part is rather straightforward; we provide it for the sake of completeness. Let $\big(type(\widehat{\mathscr{D}}),G\big)$ be a CLGE strict cardinal environment. We show that $(\widehat{\mathscr{D}},G)$ is OLGE. Suppose not. Then there exists an SCF $\varphi : \widehat{\mathscr{D}} \to A$ that is LIC on $(\widehat{\mathscr{D}},G)$ but fails to be IC on $\widehat{\mathscr{D}}$. Therefore, there exists $P,P' \in \widehat{\mathscr{D}}$ and $x,y \in A$ such that $\varphi(P') = x$, $y = \varphi(P)$, and xPy. We complete the proof of the if part by constructing a mechanism (f,p) that is LIC on $\big(type(\widehat{\mathscr{D}}),G\big)$ but fails to be IC on $type(\widehat{\mathscr{D}})$, which will lead to a contradiction to the fact that $\big(type(\widehat{\mathscr{D}}),G\big)$ is CLGE.

Define $f(s) = \varphi(prfn(s))$ and $p(s) = 0$ for all $s \in type(\widehat{\mathscr{D}})$. The fact that (f,p) is LIC on $\big(type(\widehat{\mathscr{D}}),G\big)$ follows straightforwardly from the definition of G-local types and

[16] Recall that by $c[ba]$, we denote a weak preference where c is preferred to both a and b, and a and b are indifferent.

the fact that φ is LIC on $(\widehat{\mathscr{D}}, G)$. Fix any $t \in type(P)$ and $t' \in type(P')$. By the definition of f, $f(t) = y$ and $f(t') = x$. This, together with the facts that xPy and $p(t) = 0 = p(t')$, implies that $t(f(t)) - p(t) < t(f(t')) - p(t')$, and hence, it follows that (f, p) is not IC on (t, t'), a contradiction. This completes the proof of the if part of the theorem.

Only if part: Let $(\widehat{\mathscr{D}}, G)$ be an OLGE strict ordinal environment. We show that the environment $\big(type(\widehat{\mathscr{D}}), G\big)$ is CLGE. Let (f, p) be an LIC mechanism on $\big(type(\widehat{\mathscr{D}}), G\big)$. We need to show that (f, p) is IC on $type(\widehat{\mathscr{D}})$.

Consider arbitrary $t, t' \in type(\widehat{\mathscr{D}})$. By the definition of $type(\widehat{\mathscr{D}})$, there are $P, P' \in \widehat{\mathscr{D}}$ such that $t \in type(P)$ and $t' \in type(P')$. Without loss of generality, let us assume that the alternatives in A are indexed as $a_1, \ldots, a_n$ such that $a_1 P a_2 P \ldots P a_n$. Suppose $f(t') = a_j$ for some $j \in \{1, \ldots, n\}$. We proceed to show that (f, p) is IC on (t, t').

If $P = P'$, then t and t' are G-local types, and hence the proof follows by the assumption of the theorem that (f, p) is IC on every pair of G-local types. So, assume $P \neq P'$. Since $(\widehat{\mathscr{D}}, G)$ is OLGE, by Theorem 4.1, it satisfies Property L. This, together with the fact that $f(t') = a_j$, implies that there exists a G-local path $\pi = (P^1, \ldots, P^k)$ from P to P' satisfying Property L with respect to a_j. Since π has no (a_j, x)-restoration for all $x \in \{a_{j+1}, \ldots, a_n\}$, it follows that $L(a_j, P^{l+1}) \setminus \{a_1, \ldots, a_{j-1}\} \subseteq L(a_j, P^l) \setminus \{a_1, \ldots, a_{j-1}\}$ for all $l \in \{1, \ldots, k-1\}$.

Let $l_1 \geq 1$ be the minimum number with the property that for each $l \in \{l_1 + 1, \ldots, k\}$ there exist $t^l \in type(P^l)$ such that $f(t^l) = a_j$. Note that such an l_1 will always exist as $f(t') = a_j$ and $t' \in type(P^k)$.

Claim 1. There exists $\tilde{t}^1 \in type(P^{l_1+1})$ such that $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$.

Proof of Claim 1. By the definition of l_1, for each $l \in \{l_1 + 1, \ldots, k\}$ there exist $t^l \in type(P^l)$ such that $f(t^l) = a_j$. Now we show that $p(t^l) = p(t')$ for each $l \in \{l_1 + 1, \ldots, k\}$. First we show that $p(t^{k-1}) = p(t')$. Since P^k and P^{k-1} are G-local preferences and (f, p) is LIC on $\big(type(\widehat{\mathscr{D}}), G\big)$, (f, p) is IC on $type(\{P^k, P^{k-1}\})$, and hence, is IC on (t^{k-1}, t') and (t', t^{k-1}). Since $f(t^{k-1}) = f(t')$, the fact that $p(t^{k-1}) = p(t')$ now follows from Fact 2.1. By using this argument repeatedly, it follows that for all $l \in \{l_1 + 1, \ldots, k\}$, $p(t^l) = p(t')$. Set $t^{l_1+1} = \tilde{t}^1$. This completes the proof of the claim. □

Also note that since $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$, by the definition of incentive compatibility it follows that (f,p) is IC on $(\tilde{t}^1,t')$.

If $l_1 = 1$, then by Claim 1, there exists $\tilde{t}^1$ in $type(P^2)$ such that $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$. Since $t \in type(P)$, $\tilde{t}^1 \in type(P^2)$, and (f,p) is IC on $type(\{P = P^1, P^2\})$, it follows that $t(f(t)) - p(t) \geq t(f(\tilde{t}^1)) - p(\tilde{t}^1)$. Since $f(\tilde{t}^1) = f(t')$ and $p(\tilde{t}^1) = p(t')$, this implies $t(f(t)) - p(t) \geq t(f(t')) - p(t')$, which shows that (f,p) is IC on (t,t') thereby completing the proof of the Theorem.

Suppose $l_1 > 1$. Then by Claim 1 there exists $\tilde{t}^1 \in type(P^{l_1+1})$ such that $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$. We proceed to Step 1.

Step 1. In this step, we show that if $l_1 > 1$, then there exists $\tilde{t}^2 \in type(P^{l_1})$ such that $f(\tilde{t}^2) \in \{a_1,\dots,a_{j-1}\}$ and $\tilde{t}^2(f(\tilde{t}^1)) - \tilde{t}^2(f(\tilde{t}^2)) \geq t(f(\tilde{t}^1)) - t(f(\tilde{t}^2))$.

Since $l_1 > 1$, by the definition of l_1, we must have $f(s) \neq a_j$ for all $s \in type(P^{l_1})$.[17]

Claim 2. $L(a_j,P^{l_1+1}) \nsubseteq L(a_j,P^{l_1})$.

Proof of Claim 2. Assume for contradiction that $L(a_j,P^{l_1+1}) \subseteq L(a_j,P^{l_1})$. Consider a type $t^{l_1} \in type(P^{l_1})$ such that $t^{l_1}(a_j) - t^{l_1}(x) > \tilde{t}^1(a_j) - \tilde{t}^1(x)$ for all $x \in A \setminus \{a_j\}$. Such a type can be found since $L(a_j,P^{l_1+1}) \subseteq L(a_j,P^{l_1})$. Since P^{l_1} and P^{l_1+1} are G-local preferences, (f,p) is IC on $type(P^{l_1},P^{l_1+1})$, and hence, is IC on $(t^{l_1},\tilde{t}^1)$ and $(\tilde{t}^1,t^{l_1})$. This, together with the facts that $t^{l_1}(a_j) - t^{l_1}(x) > \tilde{t}^1(a_j) - \tilde{t}^1(x)$ for all $x \in A \setminus \{a_j\}$ and $f(\tilde{t}^1) = a_j$, implies that $f(t^{l_1}) = a_j$. This leads to a contradiction to the fact that $f(s) \neq a_j$ for all $s \in type(P^{l_1})$. This completes the proof of the claim. □

Since $L(a_j,P^{l_1+1}) \nsubseteq L(a_j,P^{l_1})$, it must be that $a_l P^{l_1} a_j$ and $a_j P^{l_1+1} a_l$ for some $l \in \{1, \dots, n\}$. Let $B_1 = \{a_l \mid a_l P^{l_1} a_j \text{ and } a_j P^{l_1+1} a_l\}$. Note that since $L(a_j,P^{l_1+1}) \setminus \{a_1,\dots, a_{j-1}\} \subseteq L(a_j,P^{l_1}) \setminus \{a_1,\dots,a_{j-1}\}$, we must have $B_1 \subseteq \{a_1,\dots,a_{j-1}\}$. Choose $\tilde{t}^2 \in type(P^{l_1})$ such that

(i) $\tilde{t}^2(a_j) - \tilde{t}^2(x) > \tilde{t}^1(a_j) - \tilde{t}^1(x)$ for all $x \in A \setminus B_1$, and

(ii) $\tilde{t}^2(a_j) - \tilde{t}^2(y) \geq t(a_j) - t(y)$ for all $y \in B_1$.

We explain how such a choice of $\tilde{t}^2$ is possible. Note that (i) implies that the relative valuation of a_j with respect to each alternative in $A \setminus B_1$ is increased from $\tilde{t}^1$ to $\tilde{t}^2$. This can be assured by the fact that there is no $z \in A \setminus B_1$ such that $z P^{l_1} a_j$ and $a_j P^{l_1+1} z$.

[17] Otherwise $l_1 - 1$ would satisfy the requirement of the definition of l_1 contradicting the fact that l_1 is the minimum number satisfying this requirement.

Similarly, (ii) can be assured by means of the fact that the relative ordering of a_j with any alternative in B_1 is the same in both P and P^{l_1}.

Since (f,p) is IC on $type(\{P^{l_1},P^{l_1+1}\})$ and $f(s) \neq a_j$ for all $s \in type(P^{l_1})$, (i) implies that $f(\tilde{t}^2) \in B_1$. This, together with (ii) and the fact that $a_j = f(\tilde{t}^1)$, implies that $\tilde{t}^2(f(\tilde{t}^1)) - \tilde{t}^2(f(\tilde{t}^2)) \geq t(f(\tilde{t}^1)) - t(f(\tilde{t}^2))$. This completes Step 1. □

Note that since $\tilde{t}^2 \in type(P^{l_1})$, $\tilde{t}^1 \in type(P^{l_1+1})$ and P^{l_1} and P^{l_1+1} are G-local preferences, (f,p) is IC on $(\tilde{t}^2,\tilde{t}^1)$.

We now complete the proof of the Theorem by using Step 1 recursively. Let $f(\tilde{t}^2) = b_2$, where $b_2 \in \{a_1,\ldots,a_{j-1}\}$. Since $(\widehat{\mathscr{D}},G)$ is OLGE, by Theorem 4.1, it satisfies Property L. This, together with the fact that $f(\tilde{t}^2) = b_2$, implies there exists a G-local path $(\hat{P}^1,\ldots,\hat{P}^r)$ from P to P^{l_1} satisfying Property L with respect to b_2.

Let $l_2 \geq 1$ be the minimum number with the property that for each $l \in \{l_2+1,\ldots,r\}$ there exist $t^l \in type(\hat{P}^l)$ such that $f(t^l) = b_2$. Note that such an l_2 will always exist as $f(\tilde{t}^2) = b_2$ and $\tilde{t}^2 \in type(\hat{P}^r)$.

Using similar logic as in Claim 1, it follows that there exists $\tilde{t}^3 \in type(\hat{P}^{l_2+1})$ such that $f(\tilde{t}^3) = f(\tilde{t}^2) = b_2$ and $p(\tilde{t}^3) = p(\tilde{t}^2)$. This, together with the definition of incentive compatibility implies that (f,p) is IC on $(\tilde{t}^3,\tilde{t}^2)$.

If $l_2 = 1$, we have $\tilde{t}^3 \in type(\hat{P}^2)$ such that $f(\tilde{t}^3) = f(\tilde{t}^2) = b_2$ and $p(\tilde{t}^3) = p(\tilde{t}^2)$. Since $t \in type(P)$, $\tilde{t}^3 \in type(\hat{P}^2)$, and (f,p) is IC on $type(\{P = \hat{P}^1,\hat{P}^2\})$, it follows that $t(f(t)) - p(t) \geq t(f(\tilde{t}^3)) - p(\tilde{t}^3)$. Since $f(\tilde{t}^3) = f(\tilde{t}^2)$ and $p(\tilde{t}^3) = p(\tilde{t}^2)$, this implies $t(f(t)) - p(t) \geq t(f(\tilde{t}^2)) - p(\tilde{t}^2)$, which shows that (f,p) is IC on $(t,\tilde{t}^2)$. Hence we have a finite sequence of types $(t,\tilde{t}^2,\tilde{t}^1,t')$ such that (f,p) is IC on $(t,\tilde{t}^2)$, $(\tilde{t}^2,\tilde{t}^1)$ and $(\tilde{t}^1, t')$. Therefore the sequence of types $(t,\tilde{t}^2,\tilde{t}^1,t')$ satisfies condition (i) of Fact 2.3. Further note that since $f(\tilde{t}^1) = f(t')$, the fact that $\tilde{t}^1(f(t')) - \tilde{t}^1(f(\tilde{t}^1)) \geq t(f(t')) - t(f(\tilde{t}^1))$ is trivially satisfied (both sides being 0). This, together with step 1 implies that we have
(i) $\tilde{t}^2(f(\tilde{t}^1)) - \tilde{t}^2(f(\tilde{t}^2)) \geq t(f(\tilde{t}^1)) - t(f(\tilde{t}^2))$, and
(ii) $\tilde{t}^1(f(t')) - \tilde{t}^1(f(\tilde{t}^1)) \geq t(f(t')) - t)f(\tilde{t}^1)$.
Hence the sequence of types $(t,\tilde{t}^2,\tilde{t}^1,t')$ satisfies condition (ii) of Fact 2.3. Therefore, by Fact 2.3, it follows that (f,p) is IC on (t,t') thereby completing the proof of the Theorem.

Suppose $l_2 > 1$. Then, by using similar logic as in Step 1, there exists $\tilde{t}^4 \in type(\hat{P}^{l_2})$ such that $f(\tilde{t}^4) \in U(b_2,P)$ where $U(b_2,P) = \{z \in A \mid zPb_2\}$ and $\tilde{t}^4(f(\tilde{t}^3) - \tilde{t}^4(f(\tilde{t}^4)) \geq t(f(\tilde{t}^3)) - t(f(\tilde{t}^4))$.

Continuing in this manner, either we end up showing (f,p) is IC on (t,t') or we can construct a finite sequence $(\tilde{t}^{2u},\tilde{t}^{2u-1},\ldots,\tilde{t}^4,\tilde{t}^3,\tilde{t}^2,\tilde{t}^1)$ such that
(i) t^{2j} and t^{2j-1} are G-local types for all $j \in \{1,\ldots,u\}$,
(ii) $f(\tilde{t}^{2j}) = b_{2j}$ for all $j \in \{1,\ldots,u\}$,
(iii) $b_{2(j+1)}Pb_{2j}$ for all $j \in \{1,\ldots,u-1\}$,
(iv) $f(\tilde{t}^{2j+1}) = f(\tilde{t}^{2j})$ and $p(\tilde{t}^{2j+1}) = p(\tilde{t}^{2j})$ for all $j \in \{1,\ldots,u-1\}$, and
(v) $\tilde{t}^{2j}(f(\tilde{t}^{2j-1})) - \tilde{t}^{2j}(f(\tilde{t}^{2j})) \geq t(f(\tilde{t}^{2j-1})) - t(f(\tilde{t}^{2j}))$ for all $j \in \{1,\ldots,u\}$.

Since $b_{2(j+1)}Pb_{2j}$ for all $j \in \{1,\ldots,u-1\}$ and the process has not terminated, it must be that $f(\tilde{t}^{2u}) = a_1 = r_1(P)$. Let $\tilde{t}^{2u} \in type(P^{l_u})$ for some $P^{l_u} \in \widehat{\mathscr{D}}$. Since $(\widehat{\mathscr{D}},G)$ is OLGE and $f(\tilde{t}^{2u}) = a_1$, by Theorem 4.1, there must exist a G-local path $\pi = (\bar{P}^1,\ldots,\bar{P}^w)$ from P to P^{l_u} satisfying Property L with respect to a_1. This means $L(a_1,\bar{P}^{l+1}) \subseteq L(a_1,\bar{P}^l)$ for all $l \in \{1,\ldots,w-1\}$.

Claim 3. (f,p) is IC on $(t,\tilde{t}^{2u})$.

Proof of Claim 3. Using similar logic as in the proof of Claim 2 (by using w in place of l_1+1, $w-1$ in place of l_1 and a_1 in place of a_j), it follow that there exists $t^{w-1} \in t(\bar{P}^{w-1})$ such that $f(t^{w-1}) = a_1 = f(\tilde{t}^{2u})$. The fact that $p(t^{w-1}) = p(\tilde{t}^{2u})$ now follows from Fact 2.1.

By using this fact repeatedly, it follows that for all $l \in \{2,\ldots,w-2\}$, there exists $t^l \in type(\bar{P}^l)$ such that $f(t^l) = a_1 = f(\tilde{t}^{2u})$ and $p(t^l) = p(\tilde{t}^{2u})$, which in particular means that there exists $t^2 \in type(\bar{P}^2)$ such that $f(t^2) = f(\tilde{t}^{2u}) = a_1$ and $p(t^2) = p(\tilde{t}^{2u})$. Since $t \in type(P)$, $t^2 \in type(\bar{P}^2)$, and (f,p) is IC on $type(\{P = \bar{P}^1,\bar{P}^2\})$, it follows that $t(f(t)) - p(t) \geq t(f(t^2)) - p(t^2)$. Since $f(t^2) = f(\tilde{t}^{2u}) = a_1$ and $p(t^2) = p(\tilde{t}^{2u})$, this implies $t(f(t)) - p(t) \geq t(f(\tilde{t}^{2u})) - p(\tilde{t}^{2u})$, which shows that (f,p) is IC on $(t,\tilde{t}^{2u})$ thus proving the claim. □

Now we show that (f,p) is IC on (t,t'). Consider the sequence of types $(t,\tilde{t}^{2u},\tilde{t}^{2u-1},\ldots,\tilde{t}^4,\tilde{t}^3,\tilde{t}^2,\tilde{t}^1,t')$. By construction, (f,p) is IC on each pair of consecutive types, and hence, the sequence of types $(t,\tilde{t}^{2u},\tilde{t}^{2u-1},\ldots,\tilde{t}^4,\tilde{t}^3,\tilde{t}^2,\tilde{t}^1,t')$ satisfies condition (i) of Fact 2.3. Moreover, since $f(\tilde{t}^{2j+1}) = f(\tilde{t}^{2j})$ for all $j \in \{1,\ldots,u-1\}$, it follows that $\tilde{t}^{2j+1}(f(\tilde{t}^{2j})) - \tilde{t}^{2j+1}(f(\tilde{t}^{2j+1})) = 0 \geq 0 = t(f(\tilde{t}^{2j})) - t(f(\tilde{t}^{2j+1}))$ for all $j \in \{1, \ldots, u-1\}$. Similarly, since $f(\tilde{t}^1) = f(t')$, it follows that $\tilde{t}^1(f(t')) - \tilde{t}^1(f(\tilde{t}^1)) = 0 \geq 0 = t(f(t')) - t(f(\tilde{t}^1))$. These, together with the fact that $\tilde{t}^{2j}(f(\tilde{t}^{2j-1})) - \tilde{t}^{2j}(f(\tilde{t}^{2j})) \geq t(f(\tilde{t}^{2j-1})) - t(f(\tilde{t}^{2j}))$ for all $j \in \{1,\ldots,u\}$, imply that the sequence of types $(t, \tilde{t}^{2u}, \tilde{t}^{2u-1}, \ldots, \tilde{t}^4, \tilde{t}^3, \tilde{t}^2, \tilde{t}^1, t')$ satisfies condition (ii) of Fact 2.3. Hence, by Fact 2.3, it follows that (f,p) is IC on (t,t'), which completes the proof of the only if part of Theorem.

■

A.2 Proof of Corollary 4.1

Proof: IC implies ALIC by definition, we show the converse. Consider an ALIC mechanism $\mu = (f,p)$ on $type(\widehat{\mathscr{D}})$. We show that μ is IC. It follows from Theorem 4.2 that the cardinal environment $(type(\widehat{\mathscr{D}}), G^{ad})$ is CLGE. Therefore, to show that μ is IC, it is sufficient to show that it is IC on any pair of G^{ad}-local types. Consider two G^{ad}-local types t and t', and the line $[t,t']$. Let P and P' (not necessarily distinct), respectively, be the adjacent preferences that t and t' represent. Since P and P' are adjacent, there will be at most one point in the line $[t,t']$ that does not lie in $type(\widehat{\mathscr{D}})$. Such a point (or type) will lie on the boundary of $type(P)$ and $type(P')$ and will represent some weak preference and hence outside the domain $\widehat{\mathscr{D}}$. Let t^* be that point (if it exists). By means of ALIC, we can choose $\bar{t}$ and $\hat{t}$ in $type(\widehat{\mathscr{D}}) \cap [t,t']$ such that μ is IC on $(\bar{t},\hat{t})$ and $(\hat{t},\bar{t})$, and the lines $[t,\bar{t}]$ and $[\hat{t},t']$ lie in $type(\widehat{\mathscr{D}})$. These, together with Fact 3.1 and the fact that implications of ALIC and PLIC are the same in the interior of a type-space, implies that μ is IC on any two types of the sequence $(t,\bar{t},\hat{t},t')$. Let us rename the sequence $(t,\bar{t},\hat{t},t')$ as $(t = t^1, \bar{t} = t^2, \hat{t} = t^3, t' = t^4)$. Note that the sequence (t^1,t^2,t^3,t^4) satisfies Condition (i) of Fact 2.3 because of the fact that μ is IC on any two consecutive types of the sequence. Let $f(t^i) = a^i$ for each $i \in \{1,2,3,4\}$.

Claim. (t^1,t^2,t^3,t^4) satisfies Condition (ii) of Fact 2.3.

Proof of the claim. Since μ is IC on any two consecutive types of the sequence (t^1,t^2, t^3,t^4),

$$t^i(a^i) - t^i(a^{i+1}) \geq t^{i+1}(a^i) - t^{i+1}(a^{i+1}) \text{ for every } i \in \{1,2,3\}. \tag{1}$$

We need to show that $t^i(a^{i+1}) - t^i(a^i) \geq t^1(a^{i+1}) - t^1(a^i)$ for all $i \in \{1,2,3\}$ which would then establish that (t^1,t^2,t^3,t^4) satisfies Condition (ii) of Fact 2.3. Since $(t = t^1, \ldots, t^4 = t')$ is a finite sequence of types in $[t,t']$, there exists $0 = \beta_1 < \beta_2 < \beta_3 < \beta_4 = 1$ such that $t^i = (1-\beta_i)t^1 + \beta_i t^4$ for all $i \in \{1,2,3\}$. Fix any $i \in \{1,2,3\}$. By (1), we have

$$t^i(a^i) - t^i(a^{i+1}) \geq t^{i+1}(a^i) - t^{i+1}(a^{i+1}). \tag{2}$$

Substituting $t^i = (1-\beta_i)t^1 + \beta_i t^4$ and $t^{i+1} = (1-\beta_{i+1})t^1 + \beta_{i+1}t^4$ in (2), we get

$$\begin{aligned} &t^1(a^i) - t^1(a^{i+1}) + \beta_i[(t^4(a^i) - t^4(a^{i+1})) - (t^1(a^i) - t^1(a^{i+1}))] \\ &\geq\ t^1(a^i) - t^1(a^{i+1}) + \beta_{i+1}[(t^4(a^i) - t^4(a^{i+1})) - (t^1(a^i) - t^1(a^{i+1}))]. \end{aligned} \tag{3}$$

Since $\beta_i < \beta_{i+1}$, from (3) we conclude that

$$(t^4(a^i) - t^4(a^{i+1})) - (t^1(a^i) - t^1(a^{i+1})) \leq 0. \tag{4}$$

Substituting $t^i = (1-\beta_i)t^1 + \beta_i t^4$ in $t^i(a^{i+1}) - t^i(a^i)$, we get

$$t^i(a^{i+1}) - t^i(a^i) = (1-\beta_i)(t^1(a^{i+1}) - t^1(a^i)) + \beta_i(t^4(a^{i+1}) - t^4(a^i)). \tag{5}$$

Since $\beta_i \geq 0$, (4) and (5) together imply $t^1(a^{i+1}) - t^1(a^i) \leq t^i(a^{i+1}) - t^i(a^i)$. This implies that (t^1,t^2,t^3,t^4) satisfies Condition (ii) of Fact 2.3 which completes the proof of the claim. □

Therefore, by applying Fact 2.3 to the sequence (t^1,t^2,t^3,t^4), we obtain that μ is IC on (t,t'). This completes the proof of the corollary. ■

A.3 Proof of Theorem 5.1

Proof of (i) implies (ii): Suppose (i) holds but (ii) does not hold. Since (ii) does not hold, either the domain $\overline{\mathscr{D}}$ does not satisfy weak-compatibility or the environment

$(strict(\overline{\mathscr{D}}),G)$ is not OLGE. We distinguish these two cases.

Case A. Suppose that the domain $\overline{\mathscr{D}}$ does not satisfy weak-compatibility.

Since $\overline{\mathscr{D}}$ does not satisfy weak-compatibility, there exists a weak preference $R \in \overline{\mathscr{D}}$ for which there is no strict preference in $strict(\overline{\mathscr{D}})$ that is compatible with R. First, note that R cannot be indifferent over all the alternatives in A, that is, it is not possible that aIb for all $a,b \in A$. This is because, if R is so, then any strict preference in $strict(\overline{\mathscr{D}})$ (recall that $strict(\overline{\mathscr{D}}) \neq \emptyset$ by our assumption) is compatible with R. So, let us assume that aPb for some $a,b \in A$.

Consider the mechanism $\mu = (f,p)$ such that $f(t) = b$ for all $t \in type(R)$ and $f(t) = a$ for all other types, and $p(t) = 0$ for all $t \in type(\overline{\mathscr{D}})$. We claim that this mechanism is strong LIC but not almost everywhere IC.

Since both f and p are constant (equal to a and 0, respectively) over all strict types in $type(\overline{\mathscr{D}})$, the mechanism μ is LIC. To see that μ is strong LIC, consider any pair of types $(\bar{t},\hat{t})$ where $\hat{t}$ is a strict type in $type(P^*)$ and $\bar{t}$ is a weak type in $cl(type(P^*)) \cap type(\overline{\mathscr{D}})$ for some $P^* \in strict(\overline{\mathscr{D}})$. Since there is no strict preference in $strict(\overline{\mathscr{D}})$ that is compatible with R, we have $\bar{t} \notin type(R)$.

By the construction of f, this means $f(\bar{t}) = f(\hat{t}) = a$. This, together with the fact that the payment function is constant everywhere (equal to 0), implies that μ is IC on $(\bar{t},\hat{t})$. Therefore, μ is strong LIC. Finally, we show that μ is not almost everywhere IC. Consider any type $\bar{\bar{t}}$ in $type(R)$ and any strict type $\hat{\hat{t}}$ in $type(\overline{\mathscr{D}})$. By the definition of f, we have $f(\bar{\bar{t}}) = b$ and $f(\hat{\hat{t}}) = a$. Since aPb, we have $\bar{\bar{t}}(a) > \bar{\bar{t}}(b)$. This, together with the fact that $p(\bar{\bar{t}}) = p(\hat{\hat{t}}) = 0$, implies $\bar{\bar{t}}(f(\bar{\bar{t}})) - p(\bar{\bar{t}}) < \bar{\bar{t}}(f(\hat{\hat{t}})) - p(\hat{\hat{t}})$, and hence, μ is not IC on the pair $(\bar{\bar{t}},\hat{\hat{t}})$. Since $\hat{\hat{t}}$ is a strict type in $type(\overline{\mathscr{D}})$, this means μ is not almost everywhere IC, which is a contradiction to (i). This completes the proof for Case A.

Case B. Suppose that the environment $(strict(\overline{\mathscr{D}}),G)$ is not OLGE.

Since the environment $(strict(\overline{\mathscr{D}}),G)$ is not OLGE, there is an SCF g on $strict(\overline{\mathscr{D}})$ that is LIC on $(strict(\overline{\mathscr{D}}),G)$ but not IC. Let P and P' be two preferences in $strict(\overline{\mathscr{D}})$ on which g fails to be IC, that is, $g(P')Pg(P)$.

In what follows, we construct a mechanism $\mu = (f,p)$ that is strong LIC on $(type(\overline{\mathscr{D}}), G)$ but not IC, and thereby arrive at a contradiction to (i). Consider a strict type

$\hat{t} \in type(\overline{\mathscr{D}})$. Define $f(\hat{t}) = g(prfn(\hat{t}))$. This is well-defined as there is a unique $prfn(\hat{t})$ in $\overline{\mathscr{D}}$ for such strict types $\hat{t}$. Next, consider a weak type $\bar{t}$ and consider the (strict) preferences in $\overline{\mathscr{D}}$ that is compatible with the weak preference that represent $\bar{t}$, that is, the preferences $\mathscr{P}(\bar{t}) = \{\hat{P} \in strict(\overline{\mathscr{D}}) \mid \hat{P} \text{ is compatible with } prfn(\bar{t})\}$. Let $P^* \in \mathscr{P}(\bar{t})$ be such that $\bar{t}(g(P^*)) \geq \bar{t}(g(\hat{P}))$ for all $\hat{P} \in \mathscr{P}(\bar{t})$. Define $f(\bar{t}) = g(P^*)$. Take the payment function p to be identically zero, that is, $p(t) = 0$ for all $t \in type(\overline{\mathscr{D}})$.

Since g is not IC on (P,P'), by the definition, μ is not IC on any pair of types (t,t') such that $t \in type(P)$ and $t' \in type(P')$. Therefore, μ is not almost everywhere IC. We claim that the mechanism μ is strong LIC. The fact that μ is LIC follows from the fact that g is LIC. To see that μ is IC on every pair of types $(\bar{t},\hat{t})$ where $\hat{t}$ is a strict type in $type(\tilde{P})$ and $\bar{t}$ is a weak type in $cl(type(\tilde{P})) \cap type(\overline{\mathscr{D}})$ for some (strict) preference $\tilde{P} \in \overline{\mathscr{D}}$, consider such a pair of types $(\bar{t},\hat{t})$. We need to show $\bar{t}(f(\bar{t})) \geq \bar{t}(f(\hat{t}))$. Since $\hat{t}$ is a strict type, by the definition of f, $f(\hat{t}) = g(\tilde{P})$. Moreover, since $\bar{t}$ is a weak type and $\tilde{P}$ is compatible with the weak preference that represent $\bar{t}$, by the definition of f, we have $\bar{t}(f(\bar{t})) \geq \bar{t}(g(\tilde{P}))$. Combining these observations, we obtain $\bar{t}(f(\bar{t})) \geq \bar{t}(f(\hat{t}))$. This shows that the mechanism μ is strong LIC, completing the proof by contradicting (i).

Proof of (ii) implies (i): Consider an OLGE environment $(strict(\overline{\mathscr{D}}),G)$ such that $\overline{\mathscr{D}}$ satisfies weak-compatibility. We show that every strong LIC mechanism on $(type(\overline{\mathscr{D}}),G)$ is almost everywhere IC. Consider a strong LIC mechanism $\mu = (f,p)$. To show that it is almost everywhere IC, we need to show that it is IC on every pair of types $(t,\hat{t})$ where $\hat{t}$ is a strict type in $type(\overline{\mathscr{D}})$. Fix any pair of types $(t,\hat{t})$ such that $\hat{t}$ is a strict type in $type(\overline{\mathscr{D}})$. We distinguish two cases based on the structure of t.

Case 1. Suppose t is a strict type.

Since $(strict(\overline{\mathscr{D}}),G)$ OLGE, by Theorem 4.2, the environment $(type(strict(\overline{\mathscr{D}})),G)$ is CLGE. This means every LIC mechanism on $(type(strict(\overline{\mathscr{D}})),G)$ is IC. Since strong LIC implies LIC, it follows that μ is IC on $type(strict(\overline{\mathscr{D}}))$, and in particular, IC on $(t,\hat{t})$. This completes the proof for Case 1.

Case 2. Suppose t is a weak type.

For notational convenience, let us denote t by $\bar{t}$. If $\bar{t}$ is such that $\bar{t}(x) = \bar{t}(y)$ for every $x,y \in A$, then $\bar{t} \in cl(type(prfn(\hat{t}))) \cap type(\overline{\mathscr{D}})$. Since μ is strong LIC, μ is IC on

$(\bar{t},\hat{t})$, which completes the proof of the theorem. Now assume that $\bar{t}(x) \neq \bar{t}(y)$ for some $x,y \in A$.

Let P^* be a strict preference in $\overline{\mathscr{D}}$ that is compatible with the weak preference representing $\bar{t}$. Such a preference exists since $\overline{\mathscr{D}}$ satisfies the weak-compatibility property. Let B be the set of alternatives that have the same valuation as $f(\hat{t})$ in $\bar{t}$, and are preferred to $f(\hat{t})$ in P^*, that is, $B = \{b \in A \mid \bar{t}(b) = \bar{t}(f(\hat{t}))$ and $bP^*f(\hat{t})\}$. Notice that since $\bar{t}(x) \neq \bar{t}(y)$ for some $x,y \in A$, $A \setminus (B \cup f(\hat{t})) \neq \emptyset$. Let $\widetilde{T}$ be the set of types $\tilde{t}$ representing the preference P^* such that the relative valuation of $f(\hat{t})$ with respect to any alternative in $A \setminus (B \cup f(\hat{t}))$ strictly increases from $\bar{t}$ to $\tilde{t}$, that is, $\widetilde{T} = \{\tilde{t} \in type(P^*) \mid \tilde{t}(f(\hat{t})) - \tilde{t}(z) > \bar{t}(f(\hat{t})) - \bar{t}(z)$ for all $z \in A \setminus (B \cup f(\hat{t}))\}$. Since P^* is a strict preference that is compatible with the weak preference representing $\bar{t}$ and $A \setminus (B \cup f(\hat{t})) \neq \emptyset$, we have $\widetilde{T} \neq \emptyset$. We distinguish two further subcases.

Case 2.1. Suppose that there exists $\tilde{t} \in \widetilde{T}$ such that $f(\tilde{t}) \notin B$.

Since $f(\tilde{t}) \notin B$, by the definition $\widetilde{T}$, we have

$$\tilde{t}(f(\hat{t})) - \tilde{t}(f(\tilde{t})) \geq \bar{t}(f(\hat{t})) - \bar{t}(f(\tilde{t})), \tag{6}$$

where the equality holds only when $f(\hat{t}) = f(\tilde{t})$. Consider the sequence $(\bar{t},\tilde{t},\hat{t})$. We apply Fact 2.3 to this sequence. Since $\tilde{t}$ is a strict type in $type(P^*)$ and $\bar{t}$ is a weak type in $cl(type(P^*)) \cap type(\overline{\mathscr{D}})$ and μ is strong LIC, μ is IC on $(\bar{t},\tilde{t})$. Moreover, since both $\tilde{t}$ and $\hat{t}$ are strict types, by Case 1, it follows that μ is IC on $(\tilde{t},\hat{t})$. Thus, μ is IC on both the pairs $(\bar{t},\tilde{t})$ and $(\tilde{t},\hat{t})$, and thereby satisfies the Condition (i) of Fact 2.3. Furthermore, Condition (ii) of Fact 2.3 follows from (6). Therefore, the sequence $(\bar{t},\tilde{t},\hat{t})$ satisfies the conditions of Fact 2.3 and hence μ is IC on $(\bar{t},\hat{t})$. This completes the proof for Case 2.1.

Case 2.2. Suppose that Case 2.1 does not hold, that is, for all $\tilde{t} \in \widetilde{T}$, we have $f(\tilde{t}) \in B$.

Let $\widetilde{B}$ be the set of outcomes of f on $\widetilde{T}$, that is, $\widetilde{B} = \{f(\tilde{t}) \mid \tilde{t} \in \widetilde{T}\}$. Let $\tilde{b}$ be the worst alternative in $\widetilde{B}$ according to P^*, that is, $bP^*\tilde{b}$ for all $b \in \widetilde{B} \setminus \{\tilde{b}\}$. Let $t_{\tilde{b}} \in \widetilde{T}$ be a type such that $f(t_{\tilde{b}}) = \tilde{b}$. Let $T_{\tilde{b}}$ be the set of strict types in $type(\overline{\mathscr{D}})$ such that the relative valuation of $\tilde{b}$ with repect to any other alternative in $\widetilde{B}$ is greater than that in $t_{\tilde{b}}$, that is, $T_{\tilde{b}} = \{\tilde{\tilde{t}} \in type(\overline{\mathscr{D}}) \mid \tilde{\tilde{t}}$ is a strict type and $\tilde{\tilde{t}}(\tilde{b}) - \tilde{\tilde{t}}(z) > t_{\tilde{b}}(\tilde{b}) - t_{\tilde{b}}(z)$ for all $z \in \widetilde{B} \setminus \{\tilde{b}\}\}$. Note that since μ is strong LIC and the types in $T_{\tilde{b}}$ are strict, by Case 1, μ is IC on any pair of types in $T_{\tilde{b}}$. By the construction of the type-space $T_{\tilde{b}}$ and Fact 2.2, this implies that the outcome of f at any type in $T_{\tilde{b}}$ cannot be in the set $\widetilde{B} \setminus \{\tilde{b}\}$.

Consider the types in $\widetilde{T}_{\tilde{b}} = T_{\tilde{b}} \cap \widetilde{T}$. Since there is no restriction on the types in $\widetilde{T}$ about the relative valuation of $\tilde{b}$ with respect to any other alternative in $\widetilde{B} \setminus \{\tilde{b}\}$, we have $\widetilde{T}_{\tilde{b}} \neq \emptyset$. Moreover, since both $\widetilde{T}$ and $T_{\tilde{b}}$ put no restriction on the relative valuation of $\tilde{b}$ with respect to $f(\hat{t})$ (except that the said relative valuation is positive), the difference of the valuation of $\tilde{b}$ and $f(\hat{t})$ can be arbitrarily small in the types in $\widetilde{T}_{\tilde{b}}$, that is, $\inf_{\tilde{\tilde{t}} \in \widetilde{T}_{\tilde{b}}} \tilde{\tilde{t}}(\tilde{b}) - \tilde{\tilde{t}}(f(\hat{t})) = 0$. By the definition of $T_{\tilde{b}}$, the outcome of f at any type in $T_{\tilde{b}}$ cannot be in the set $\widetilde{B} \setminus \{\tilde{b}\}$. Moreover, by the assumption of Case 2.2, the outcome of f at any type in $\widetilde{T}$ has to be in the set $\widetilde{B}$, it follows that the outcome of any type in $\widetilde{T}_{\tilde{b}}$ is $\tilde{b}$.

Consider any type $\tilde{t}_{\tilde{b}}$ in $\widetilde{T}_{\tilde{b}}$. Since μ is strong LIC and both $\tilde{t}_{\tilde{b}}$ and $\hat{t}$ are strict types, by Case 1, μ must be IC on $(\tilde{t}_{\tilde{b}}, \hat{t})$. Therefore,

$$p(\tilde{t}_{\tilde{b}}) - p(\hat{t}) \leq \tilde{t}_{\tilde{b}}(\tilde{b}) - \tilde{t}_{\tilde{b}}(f(\hat{t})). \tag{7}$$

Since $f(\tilde{t}_{\tilde{b}}) = \tilde{b}$ for all types $\tilde{t}_{\tilde{b}} \in \widetilde{T}_{\tilde{b}}$ and μ is IC on $\widetilde{T}_{\tilde{b}}$, by Fact 2.1 we have $p(\tilde{t}_{\tilde{b}})$ must be the same for all types in $\widetilde{T}_{\tilde{b}}$. Let $p(\tilde{t}_{\tilde{b}}) = c$ for all $\tilde{t}_{\tilde{b}} \in \widetilde{T}_{\tilde{b}}$ and for some $c \in \mathbb{R}$. Taking infimum on both sides of (7) and doing some rearrangement, we obtain

$$-p(\hat{t}) \leq -c \tag{8}$$

Since $\bar{t}(f(\hat{t})) = \bar{t}(\tilde{b})$, adding $\bar{t}(f(\hat{t}))$ to the left side of (8) and $\bar{t}(\tilde{b})$ to the right side of (8), we get

$$\bar{t}(f(\hat{t})) - p(\hat{t}) \leq \bar{t}(\tilde{b}) - c.$$

Fix any $\tilde{t}_{\tilde{b}} \in \widetilde{T}_{\tilde{b}}$. Since $f(\tilde{t}_{\tilde{b}}) = \tilde{b}$, this implies

$$\bar{t}(f(\hat{t})) - p(\hat{t}) \leq \bar{t}(f(\tilde{t}_{\tilde{b}})) - c \tag{9}$$

Now, since μ is strong LIC and $\bar{t}$ is a weak type in $cl(type(P^*)) \cap type(\overline{\mathscr{D}})$ and $\tilde{t}_{\tilde{b}}$ is a strict type in $type(P^*)$, μ is IC on $(\bar{t}, \tilde{t}_{\tilde{b}})$. This implies

$$\bar{t}(f(\bar{t})) - p(\bar{t}) \geq \bar{t}(f(\tilde{t}_{\tilde{b}})) - c \tag{10}$$

By (9), this yields

$$\bar{t}(f(\bar{t})) - p(\bar{t}) \geq \bar{t}(f(\hat{t})) - p(\hat{t}), \tag{11}$$

which concludes that μ is IC on $(\bar{t}, \hat{t})$. ∎

A.4 Proof of Corollary 5.1

Proof: Let $(\widehat{\mathscr{D}}, G^{ad})$ be a strict ordinal OLGE environment and let μ be a PLIC mechanism on $cl(type(\widehat{\mathscr{D}}))$. We show that μ is almost everywhere IC. Let $\overline{\mathscr{D}}$ be the set of all preferences representing the types in $cl(type(\widehat{\mathscr{D}}))$, that is, $\overline{\mathscr{D}} = prfn(cl(type(\widehat{\mathscr{D}})))$. By definition, $\overline{\mathscr{D}}$ satisfies weak compatibility. Since $(\widehat{\mathscr{D}}, G^{ad})$ is OLGE, by Theorem 5.1, this implies that every strong LIC mechanism on $(cl(type(\widehat{\mathscr{D}})), G^{ad})$ is almost everywhere IC. So, to show that μ is almost everywhere IC on $cl(type(\widehat{\mathscr{D}}))$, it is sufficient to show that μ is strong LIC on $(cl(type(\widehat{\mathscr{D}})), G^{ad})$.

Consider any pair of G^{ad}-local preferences (P, P') in $\widehat{\mathscr{D}}$. Since P and P' are adjacent local, $cl(type(\{P, P'\}))$ is convex (see Fact 1 in Mishra et al. (2016) for details). Because, μ is PLIC, it follows from Fact 3.1 that it is IC on $cl(type(\{P, P'\}))$. In particular, it is IC on (i) any pair of strict types (t, t') in $type(\{P, P'\})$, and (ii) every pair of types $(\bar{t}, \hat{t})$ in $cl(type(P))$ where $\bar{t}$ is a weak type and $\hat{t}$ is a strict type. Since P and P' are arbitrary G^{ad}-local types, it follows that μ is strong LIC. ∎

A.5 Proof of Theorem 5.2

Proof: Let $(\widehat{\mathscr{D}}, G^{ad})$ be a strict ordinal OLGE environment and let $\bar{T} = cl(type(\widehat{\mathscr{D}}))$. Consider a PLIC mechanism μ on $\bar{T}$. By Corollary 5.1, μ is almost everywhere IC on $\bar{T}$. We first prove a claim.

Claim A.1. *Let a weak type $\bar{t} \in \bar{T} \setminus strict(\bar{T})$ be such that there exists a strict type $\hat{t} \in strict(\bar{T})$ with the property that* *(i)* $\hat{t}(f(\bar{t})) - \hat{t}(x) > \bar{t}(f(\bar{t})) - \bar{t}(x)$ *for all* $x \in A \setminus \{f(\bar{t})\}$*, and*

(ii) μ *is IC on* $(\hat{t}, \bar{t})$*.*

Then, μ *is IC on* $\bar{T} \times \{\bar{t}\}$*.*

Proof of the claim: Since μ is almost everywhere IC on $\bar{T}$, it is IC on the pair $(\bar{t}, \hat{t})$. Moreover, by Condition (ii) of the claim, μ is IC on the pair $(\hat{t}, \bar{t})$. Thus, μ is IC on both $(\bar{t}, \hat{t})$ and $(\hat{t}, \bar{t})$. By Condition (i) of the claim, the relative valuation of $f(\bar{t})$ with respect to any other alternative is increased from $\bar{t}$ to $\hat{t}$. Therefore, by Fact 2.2, we have $f(\bar{t}) = f(\hat{t})$. This, together with the fact that μ is IC on both pairs $(\bar{t}, \hat{t})$ and $(\hat{t}, \bar{t})$, implies by Fact 2.1 that $p(\bar{t}) = p(\hat{t})$.

Now, since μ is almost everywhere IC, it is IC on $\bar{T} \times \{\hat{t}\}$. Because $f(\bar{t}) = f(\hat{t})$ and $p(\bar{t}) = p(\hat{t})$, it follows that μ is IC on $\bar{T} \times \{\bar{t}\}$. This completes the proof of the claim. □

We are now ready to prove the theorem.

Proof of (i): We show that μ is IC on $\bar{T} \times \bar{T}^o$. Since μ is almost everywhere IC on $\bar{T}$, by definition μ is IC on $\bar{T} \times strict(\bar{T})$. Note that $\bar{T}^o$ might contain weak types. So, we need to show that μ is IC on $\bar{T} \times (\bar{T}^o \setminus strict(\bar{T}))$. Consider any $\bar{t} \in \bar{T}^o \setminus strict(\bar{T})$. By the definition of $\bar{T}^o$, there exists $\varepsilon_1 > 0$ such that $\{s \in \mathbb{R}^n \mid d(\bar{t}, s) < \varepsilon_1\} \subset \bar{T}$. Also, by the definition of a PLIC mechanism, there exists $\varepsilon_2 > 0$ such that μ is IC on $(\bar{t}, s)$ and $(s, \bar{t})$ for every $s \in \bar{T}$ with $d(\bar{t}, s) < \varepsilon_2$. Consider a type $\hat{t}$ in $strict(\bar{T})$ with $d(\bar{t}, \hat{t}) < min\{\varepsilon_1, \varepsilon_2\}$ such that $\hat{t}(f(\bar{t})) - \hat{t}(x) > \bar{t}(f(\bar{t})) - \bar{t}(x)$ for all $x \in A \setminus \{f(\bar{t})\}$. Such a type can be constructed from $\bar{t}$ by lowering the valuation of each alternative other than $f(\bar{t})$ by an arbitrarily small amount. This, together with the facts that $\bar{t} \in \bar{T} \setminus strict(\bar{T})$, μ is almost everywhere IC on $\bar{T}$, and μ is IC on $(\hat{t}, \bar{t})$, implies by Claim A.1 that μ is IC on $\bar{T} \times \{\bar{t}\}$. Since $\bar{t} \in \bar{T}^o \setminus strict(\bar{T})$ is arbitrary, it follows that μ is IC on $\bar{T} \times (\bar{T}^o \setminus strict(\bar{T}))$. This completes the proof of Part (i) of the theorem.

Proof of (ii): Let $\bar{t} \in \widehat{\partial} \bar{T}$ such that there exists $P \in \widehat{\mathscr{D}}$ with $\bar{t} \in cl(type(P))$ and $f(\bar{t})Pz$ for every z with $\bar{t}(f(\bar{t})) = \bar{t}(z)$. Since μ is PLIC and $cl(type(P))$ is convex, by Fact 3.1, μ is IC on $cl(type(P))$. Since $f(\bar{t})Pz$ for every z with $\bar{t}(f(\bar{t})) = \bar{t}(z)$, starting from the type $\bar{t}$, we can construct a type $\hat{t} \in type(P)$ by suitably lowering the valuation of each alternative other than $f(\bar{t})$ such that $\hat{t}(f(\bar{t})) - \hat{t}(x) > \bar{t}(f(\bar{t})) - \bar{t}(x)$ for all $x \in A \setminus \{f(\bar{t})\}$.

This, together with the facts that μ is almost everywhere IC on $\bar{T}$ and μ is IC on $(\hat{t}, \bar{t})$, implies by Claim A.1 that μ is IC on $\bar{T} \times \{\bar{t}\}$. This completes the proof of Part (ii) of the theorem. ■

REFERENCES

ARMSTRONG, M. (2000): "Optimal multi-object auctions," *The Review of Economic Studies*, 67, 455–481.

BARZEL, Y. (1973): "Private schools and public school finance," *Journal of Political Economy*, 81, 174–186.

BEARSE, P., G. GLOMM, AND E. JANEBA (2001): "Composition of Government Budget, Non-Single Peakedness, and Majority Voting," *Journal of Public Economic Theory*, 3, 471–481.

BERGA, D. (1998): "Strategy-proofness and single-plateaued preferences," *Mathematical Social Sciences*, 35, 105–120.

CARBAJAL, J. C. AND R. MÜLLER (2015): "Implementability under monotonic transformations in differences," *Journal of Economic Theory*, 160, 114–131.

CARROLL, G. (2012): "When are local incentive constraints sufficient?" *Econometrica*, 80, 661–686.

DEMANGE, G. (1982): "Single-peaked orders on a tree," *Mathematical Social Sciences*, 3, 389–396.

EPPLE, D. AND R. E. ROMANO (1996): "Public provision of private goods," *Journal of political Economy*, 104, 57–84.

FUDENBERG, D. AND J. TIROLE (1991): "Game Theory," *MIT Press, Cambridge*.

IRELAND, N. J. (1990): "The mix of social and private provision of goods and services," *Journal of Public Economics*, 43, 201–219.

KUMAR, U. AND S. ROY (2021): "Local incentive compatability in non-convex type-spaces," https://mpra.ub.uni-muenchen.de/110872/1/MPRA_paper_110872.pdf, unpublished manuscript.

KUMAR, U., S. ROY, A. SEN, S. YADAV, AND H. ZENG (2021): "Local global equivalence in voting models: A characterization and applications," *Theoretical Economics*, 16, 1195–1220.

LAVI, R., A. MUALEM, AND N. NISAN (2009): "Two simplified proofs for Roberts' theorem," *Social Choice and Welfare*, 32, 407–423.

MISHRA, D., A. PRAMANIK, AND S. ROY (2014): "Multidimensional mechanism design in single peaked type spaces," *Journal of Economic Theory*, 153, 103–116.

——— (2016): "Local incentive compatibility with transfers," *Games and Economic Behavior*, 100, 149–165.

STIGLITZ, J. (1974): "The demand for education in public and private school systems," *Journal of Public Economics*, 3, 349–385.

VOHRA, R. V. (2011): *Mechanism design: a linear programming approach*, vol. 47, Cambridge University Press.