An Implementation Approach to Rotation Programs

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Abstract

We study rotation programs within the standard implementation framework under complete information. A rotation program is a myopic stable set whose states are arranged circularly, and agents can effectively move only between two consecutive states. We provide characterizing conditions for the implementation of efficient rules in rotation programs. Moreover, we show that the conditions fully characterize the class of implementable multi-valued and efficient rules.

Keywords: Rotation Programs; Job Rotation; Assignment Problems; Implementation; rights structures; Stability.

JEL Codes: C71; D71; D82.

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1 Introduction

An economic department must distribute the administrative load among its members. However, due to the workload most want to avoid these task. This impasse is often resolved by implementing a rotating program : each professor will take a task for some time.

Rotation programs are widely used. A prominent example is given by the business practice of job rotations, which consists of periodically rotating the jobs assigned to the employees throughout their employment. This practice has been used in many industries for a wide array of employees, from factory line workers to executives (Osterman (1994, 2000), Gittleman, Horrigan and Joyce (1998)) and for different reasons.[1] Furthermore, rotation programs have been practiced in managing common-pool resources as an alternative to quota and lotteries. In many areas of the world, rotating groups are formed for farming, grazing, gaining access to water, and allocating fishing spots (Ostrom (1990), Berkes (1992), Sneath (1998)). Recently, Ely, Galeotti and Jakub (2021) show that rotation schemes can be used to prevent the spread of infections. In this view, a rotation scheme is a mechanism to shape social interactions to minimize the risk of contagion. Further, as illustrated by the problem of task allocation in the department, rotation programs can help achieve fairness in assignment problems. Indeed, we human beings tend to solve these kinds of conflicts either by using lotteries or implementing rotation schemes. However, the literature on assignment problems focuses mainly on randomization (Hofstee, 1990; Bogomolnaia and Moulin , 2001; Budish, Che, Kojima and Milgrom, 2013), though experimental evidence (Eliaz and Rubinstein, 2014; Andreoni, Aydin, Barton, Bernheim, and Naecker, 2020) shows that lotteries do not avoid ex-post envy.

In this paper, we propose an implementation approach to the study of rotation programs in which agents can rotate continuously among Pareto efficient

[1] From one side, employees who rotate accumulate more human capital because they are exposed to a broader range of experiences. On another side, the employer itself learns more about its employees if it can observe how they perform at different jobs (Arya and Mittendorf, 2004).

allocations. Therefore, our challenge lies in designing a mechanism (i.e., game form) in which the behavior of agents always coincides with the recommendation given by a social choice rule (SCR). If such a mechanism exists, the SCR is implementable.

The first difficulty in adopting this approach concerns the choice of the solution concept. Most of the game-theoretical solutions used in literature, such as the core, the (strong) Nash equilibrium, and the stable set (von Neumann and Morgenstern, 1944), satisfy a property called *internal stability*. Roughly speaking, a set of outcomes is internally stable if it is free of inner contradictions, i.e., for every outcome in the set, no agent or group can directly move to another outcome of the set and be better off. However, this property is incompatible with our objective to study how to rotate positions among agents. Thus, a theory of implementation in rotation programs cannot rely on solutions that satisfy internal stability. Internal stability is relaxed in solution concepts that are modifications, extensions, or generalizations of the stable set. One of the most prominent is the "absorbing set." As Inarra, Kuipers and Oilazola (2005) point out, the notion of absorbing sets appears in the literature under different names and settings. Kalai, Pazner, and Schmeidler (1976) study the "admissible set" in various bargaining situations, and Shenoy (1979) defines the "elementary dynamic solution" for coalitional games. More recently, Jackson and Watts (2002) study the "closed cycle" for network formation and Inarra, Larrea and Molis (2013) study the absorbing set for roommate problems. Finally, the myopic stable set (MSS), defined by Demuynck, Herings, Saulle and Seel (2019a) for a general class of games, includes all previous notions of absorbing sets. The MSS is the smallest set of states such that the following properties are satisfied: 1) There are no profitable deviations from a state **inside** the set to a state **outside** the set; and 2) for each state outside the set, red there is a sequence of agents' deviations converging to the set. Thus, the MSS is a valid prediction of agents' play, though it violates internal stability because it allows deviations within the set. Furthermore, the prediction offered by the MSS is robust in the following terms:

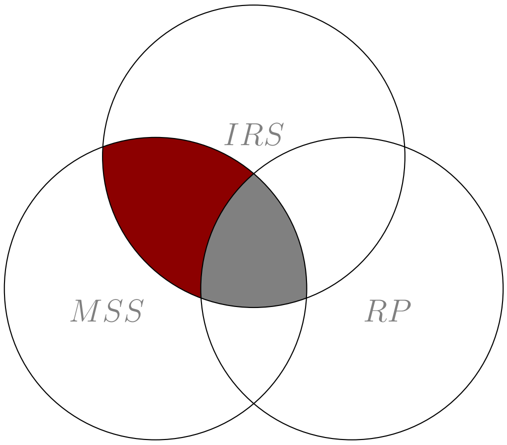
Though agents may reach an agreement on a state outside the set, a sequence of myopic improvements will bring them back to the MSS. For these reasons, we adopt the MSS as our solution concept.

From a methodological point of view, we exploit a novel implementation technique, named implementation via *rights structures* (Section 2), recently introduced by Koray and Yildiz (2018). A rights structure formalizes power distribution within society. Thus, differently from classical mechanism design, our design exercise consists of allocating rights to agents such that their behavior always coincides with the recommendation given by an SCR. We follow this approach for three reasons. Firstly, a persistent critique in economic design is that canonical mechanisms for implementing socially desirable outcomes have unnatural features (Jackson, 1992). Typically, canonical mechanisms are complex and difficult to explain in natural terms since they rely on tail-chasing constructions. By contrast, a rights structure can be easily explained to agents. Secondly, though rights structures do not model time, they effectively describe all the paths generated by agents' interactions. Finally, rights structures suit very well the environment of the MSS. Indeed, a rights structure together with a preference profile returns a social environment (Chwe, 1994), which is the natural setting of the MSS (Demuynck, Herings, Saulle and Seel, 2019a).

However, implementation in MSS cannot always guarantee the order of rotation. Indeed, it cannot exclude the possibility that a rotation gets stuck in a cycle which rules out some agents from the process. To solve this drawback, Section 4 introduces the notion of *implementation in rotation programs*. Implementation in rotation programs is a particular kind of implementation in MSS, in which every cycle generated within the MSS needs to be a rotation scheme.

Synopsis

The paper builds upon three blocks: implementation via rights structures (IRS), myopic stable sets (MSS) and rotation programs (RP). The paper's contribution lies in investigating the implications which stem from either $[IRS \cap$



MSS] or [$IRS \cap MSS \cap RP$]. This is depicted in the Venn diagram above.

Section 2 provides the model. Section 3 studies implementation in MSS via rights structures [$IRS \cap MSS$]. We show that *indirect monotonicity* is sufficient for implementation of efficient SCRs in MSS via a finite rights structure.[2] *Indirect monotonicity* is weaker than Maskin monotonicity and, for the finite case, our result encompasses implementation in core and generalized stable sets (van Deemen, 1991; Page and Wooders, 2009). Moreover, for marriage problems (Knuth, 1976) and a class of exchange economies with property rights (Balbuzanov and Kotowski, 2019), we show that the set of stable outcomes is implementable in MSS. It is worth stressing here that this implementation is obtained by devising a rights structure endowed with well-defined convergence properties (Appendix A). In Section 4, we study implementation in rotation programs via rights structures as a particular case of implementation in MSS [$IRS \cap MSS \cap RP$]. We identify a necessary condition, named *rotation monotonicity*, for implementation in rotation programs of efficient SCRs. When a multi-valued SCR describes the planner's goal, *rotation monotonicity* fully characterizes the class of implementable SCRs.[3] Finally, Section 5 studies two classes of assignment problems where efficient SCRs are implementable in rotation programs. Assignment problems in which agents share the same best/worst outcome, and assignment problems in which the planner knows that two agents have the same top-outcome. All proofs are relegated to the Appendix B.

[2] A finite rights structure is a rights structure in which the set of states is finite.
[3] See, for instance, Mukherjee, Muto, Ramaekers, and Sen (2019).

Related Literature

To the best of our knowledge, we are the first to study the economic design of rotation programs in an implementation framework that allows agents to rotate among Pareto efficient allocations continuously. The previous contributions that come closes to what we are doing are Yu and Zhang (2020a) and Yu and Zhang (2020b). In contrast to us, they study properties of one particular mechanism for task rotation, while we ask what kind of rotation schemes can be implemented in general.

Our contribution is also in line with Arya and Mittendorf (2004), who study job rotations within a principal-agent framework. In particular, they identify conditions under which job rotation and specialization are each optimal. In contrast to us, their job rotation scheme does not guarantee the circulation of employees through jobs.

Finally, our paper contributes to the literature on implementation via rights structure (Koray and Yildiz, 2018, 2019; Korpela, Lombardi and Vartiainen, 2019, 2020) and it is broadly related to the literature on assignments problems (Shapley and Shubik, 1971; Roth and Sotomayor, 1990; Abdulkadiroğlu and Sönmez, 1998).

2 The Setup

We consider a finite (nonempty) set of *agents*, denoted by N, and a finite (nonempty) set of *alternatives*, denoted by Z. We endow Z with a metric $\hat{d}$. For every set A, the power set of A is denoted by $\mathcal{A}$ and $\mathcal{A}_0 \equiv \mathcal{A} - \{\varnothing\}$ is the set of all nonempty subsets of A. Each element K of $\mathcal{N}_0$ is called a *coalition*. A *preference ordering* R_i is a complete and transitive binary relation over Z. Each agent $i (\in N)$ has a preference ordering R_i over Z. The *asymmetric* part P_i of R_i is defined by xP_iy if and only if xR_iy and not yR_ix, while the *symmetric* part I_i of R_i is defined by xI_iy if and only if xR_iy and yR_ix. A *preference profile* is thus an n-tuple of preference orderings $R \equiv (R_i)_{i \in N}$. For any profile R and $K \in \mathcal{N}_0$, we write xR_Ky

to denote that xR_iy holds for all $i \in K$ and xP_Ky to denote that xP_iy holds for all $i \in K$. As usual, $L_i(x, R)$ denotes the lower contour set of x at R for agent i. The *preference domain*, denoted by $\mathscr{R}$, consists of the set of admissible preference profiles satisfying the following property:

$$R \in \mathscr{R} \iff \text{for all } x, y \in Z: \text{ if } xI_Ny, \text{ then } x = y. \tag{1}$$

The domain of preferences underlying classical assignment problems, which are our main focus, satisfies the above property.

The goal of the planner is to implement a *social choice rule* (SCR) F, defined by $F : \mathscr{R} \longrightarrow \mathcal{Z}_0$. We refer to $x \in F(R)$ as an F-optimal outcome at R. The *range* of F is the set

$$F(\mathscr{R}) \equiv \{x \in Z | x \in F(R) \text{ for some } R \in \mathscr{R}\}.$$

The *graph* of F is the set

$$Gr(F) \equiv \{(x, R) | x \in F(R), R \in \mathscr{R}\}$$

We impose the following assumption on F:

Definition 1 (*Efficiency*)**.** We say that SCR F is *efficient*, if for all $R \in \mathscr{R}$, and all $z \in F(R)$, there does not exist any $x \in Z$ such that xR_Nz and xP_iz for at least one agent $i \in N$.

To present our theory, we find it convenient to move away from classical mechanisms or game forms. From the implementation viewpoint, the rights structure is the design variable of the planner, playing the role of mechanism. Thus, we rely on an implementation framework which models rights distribution within the society. Roughly speaking, we assume that a planner first describes the available alternatives via a set of possible states. Then, he specifies which agent or group has the right to move from a state to another. The rights distribution is such that, for any state of the world, the prediction of the solution concept returns the socially desirable alternatives. Formally, to implement F, the planner constructs a *rights structure* $\Gamma = ((S, d), h, \gamma)$, where S is the *state space* equipped with a metric d, $h : S \to Z$ the *outcome function*, and γ a *code of rights*, which is a (possibly empty) correspondence $\gamma : S \times S \rightrightarrows \mathcal{N}$. Subsequently, a code of rights specifies, for each pair of distinct states (s, t), the family of coalitions $\gamma(s, t) \subseteq \mathcal{N}$ that is entitled to move from state s to t. If $\gamma(s, t) = \varnothing$ then no coalition is entitled to move from s to t. We denote by $\mathscr{G}$ the set of all possible rights structures.

The rights structure Γ presented here is an augmented version of the rights structure introduced by Koray and Yildiz (2018) which does not includes the metric d. Our formulation would allow us to properly define the solution concept over a possibly infinite state space. From an economic design perspective, the rights structure is the planner's design variable and corresponds to a "mechanism" in the economic theory jargon. A rights structure Γ is said to be an individual-based rights structure if, for each pair of distinct states $(s, t), \gamma(s, t)$ contains only unit coalitions if it is nonempty. A rights structure Γ is termed finite if the state space S is a finite set.

A rights structure together with a preference profile returns a *social environment* (Chwe, 1994), a general framework to model strategic interaction among agents or groups.

Definition 2 (Social Environment)**.** A *social environment* is a pair (Γ, R) consisting of a rights structure Γ together with a preference profile R.

Next, a model of behavior is needed to predict at what state the agents are going to end up with. This is often done by selecting a solution concept. Formally, a solution concept is a map $\phi : \mathscr{G} \times \mathscr{R} \longrightarrow \mathbb{X}$ such that for each social environment $(\Gamma, R) \in \mathscr{G} \times \mathscr{R}$ it returns a set of states $\phi(\Gamma, R) \subseteq \mathbb{X}$ which are the prediction of the game. Given (Γ, R), we denote by $\Phi(\Gamma, R)$ the union of all predictions ϕ of (Γ, R). Finally, we can define implementation via rights structures. An SCR is implementable in a solution ϕ by a finite rights structure if, at any preference profile, the set of outcomes induced by the game coincides with the set of socially optimal outcomes.

Definition 3 (Implementation)**.** A rights structure Γ implements F in ϕ if $F(R) = h \circ \Phi(\Gamma, R)$ holds for all $R \in \mathscr{R}$. If such a rights structure exists, F is implementable in ϕ by a rights structure.

3 Towards Implementation In Rotation Programs

As outlined above, the fundamental idea of our notion of implementation in rotation programs relies on the Myopic Stable Set (MSS) Demuynck, Herings, Saulle and Seel (2019a). As a first step, this section presents the MSS and studies its implementation via rights structures.

3.1 Implementation In Myopic Stable Set

To define the MSS, we need the notion of a *myopic improvement path*.[4] There is a myopic improvement path from a state s to a set T if a sequence of coalitional deviations from s to a state arbitrarily close to T exists such that every coalition involved in the sequence has the power as well as the incentive to move.

Definition 4 (*Myopic Improvement Path*)**.** Given a social environment (Γ, R), a sequence of states $s_1, \ldots, s_m$ is called a *myopic improvement path* from state s_1 to set $T \subseteq S$ at R, if for all $\epsilon > 0$ there exists a state $s \in T$ such that $d(s, s_m) < \epsilon$ and a collection of coalitions $K_1, \ldots, K_{m-1}$ such that, for $j = 1, \ldots, m-1$,

(i) $K_j \in \gamma(s_j, s_{j+1})$

(ii) $h(s_{j+1}) P_{K_j} h(s_j)$

An MSS can be defined as follows:[5]

[4]If the state space is finite then Definition 4 reduces to the following: A sequence of states $s_1, \ldots, s_m$ is called a *myopic improvement path* from state s_1 to set $T \subseteq S$ at R, if $s_m \in T$, and there exists a collection of coalitions $K_1, \ldots, K_{m-1}$ such that, for $j = 1, \ldots, m-1$,, (i) $K_j \in \gamma(s_j, s_{j+1})$ and (ii) $h(s_{j+1}) P_{K_j} h(s_j)$.

[5]When the set of states is finite, Condition 2 reduces to the following one: Iterated External stability: For all $t \in S \backslash M$, there exists a direct myopic improvement path from t to M.

Definition 5 (*Myopic Stable Set*)**.** A set $mss(\Gamma, R) \subseteq S$ is an MSS at (Γ, R) if it is closed and satisfies the following three conditions:

1. *Deterrence of external deviations*: For all $s \in mss(\Gamma, R)$, and all $t \in S \backslash mss(\Gamma, R)$, there is no coalition $K \in \gamma(s, t)$, such that $h(t) P_K h(s)$.

2. *Asymptotic external stability*: For all $t \in S \backslash mss(\Gamma, R)$, there exists a myopic improvement path from t to $mss(\Gamma, R)$.

3. *Minimality*: There is no set $M' \subset mss(\Gamma, R)$ that satisfies the two conditions above.

Deterrence of external deviations requires that from any state in the set, there are no coalitional deviations to states outside the set. *Asymptotic external stability* states a myopic improvement path to the set exists from any state outside the set. Finally, *Minimality* requires that the MSS is the smaller closed set satisfying the first two conditions.

Let MSS(Γ, R)=$\{s \in S \mid s \in mss(\Gamma, R)\}$ be the union of all MSSs at (Γ, R). Thus, according to Definition 3, an SCR is implementable in MSS by a finite rights structure if, for each preference profile, the outcomes selected by F coincide with those of the MSS.

Our characterization result is based on the following definition.

Definition 6 (*Chain*)**.** Given a triple $(z, R, R') \in Z \times \mathscr{R} \times \mathscr{R}$, a sequence of outcomes $z_1, \ldots, z_h$, with $z = z_1$ and $z \neq z_h$, is a *chain* if there are agents $i_1, \ldots, i_{h-1}$ such that:

(A.0) $z_{k+1} P'_{i_k} z_k$ for all $k \in \{1, \ldots, h-1\}$;

(A.1) $L_i(z_h, R) \nsubseteq L_i(z_h, R')$ for some $i \in N$.

Condition (A.0) states that for each outcome there is an agent preferring its successor in the sequence. Next, (A.1) requires that at the last element of the chain there is an agent who experiences a preference reversal when preferences moves from R to R'.

We will be using the following condition in our characterization result.

Definition 7 (*Indirect Monotonicity*)**.** An SCR F satisfies *indirect monotonicity* if for all $(z, R, R') \in Z \times \mathscr{R} \times \mathscr{R}$, the following is true: if $z \in F(R)$ and $z \notin F(R')$ with $L_i(z, R) \subseteq L_i(z, R')$ for all $i \in N$, then the sequence $z_1, \ldots, z_h$ with $z = z_1$, $z \neq z_h$ and $z_i \in F(R)$ for all $i = 1, ..., h$, is a *chain*.

Suppose that z is F-optimal at R. Further, suppose that preferences change from R to R' in such a way that the standing of z improves for every agent. Finally, suppose that z is not F-optimal at R'. Thus, we are in the case where Maskin monotonicity is violated. Then, *indirect monotonicity* says that from z there is a chain of F-optimal outcomes at R.

Note that *indirect monotonicity* is implied by Maskin monotonicity[6], and they are equivalent when F is single-valued. Also, our notion of *indirect monotonicity* resemble Condition α of Abreu and Sen (1990). However, in contrast to Abreu and Sen (1990), we requires sequence of F-optimal outcomes at R.

The following example is illustrative.

Example 1. Suppose that $N = \{1, 2, 3\}$, $Z = \{x, y, z\}$, and $\mathscr{R} = \{R, R'\}$. Preferences are defined in the table below.

R		R'	
1	2	1	2
y	x	x	y
x	z	z	z
z	y	y	x

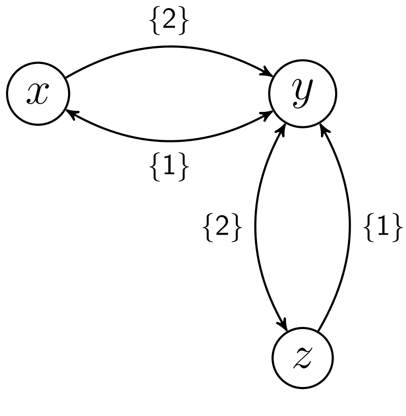


Figure 1: Example of preferences and implementing rights structure.

Let F be such that $F(R) = \{y, z\}$ and $F(R') = \{x, y\}$. Note that this SCR violates Maskin monotonicity. Indeed, the outcome z is F-optimal at R and it does not fall down in agents' preferences at R to R', however it is not F-optimal at R'.

[6]Maskin monotonicity says that if an outcome z is F-optimal at the profile R and this z does not strictly fall in preference of anyone when the profile changes to R', then z must remain a F-optimal outcome at R'

This SCR satisfies *indirect monotonicity* because there is a sequence z, y consisting of F-optimal elements at R such that z, y is a chain at R' (Condition (A.0)), i.e. at R', agent 2 prefers y to z . Moreover, for agent 1 it holds that $L_1(y, R) \nsubseteq L_1(y, R')$, (Condition (A.1)). Next, right part of Figure 1 is an example of implementing in MSS via a rights structure. We assume that states are outcomes. The rights structure is represented by an oriented graph in which vertices are the states and the edges illustrates the code of rights: agent 2 can move form x to y and from y to z and vice versa; agent 1 can move from z to y and from y to x and vice versa. According to this rights structure, the unique MSS at R and R' are respectively $mss(\Gamma, R) = \{y, z\}$ and $mss(\Gamma, R') = \{x, y\}$. To see this, take as an example the preference profile R. Then, the set $\{y, z\}$ satisfies deterrence of external deviations (only agent 1 can deviate to x but such a deviation is not a profitable for him), iterated external stability (from x there is a myopic improvement path to y by agent 1) and minimality (any subset of $\{y, z\}$ would violate deterrence of external stability).

The following result establishes our characterization result for the implementation in MSS via rights structures.[7]

Theorem 1. *Any efficient F satisfying indirect monotonicity is implementable in MSS by a finite rights structure.*

Indirect monotonicity is a sufficient condition for implementation in MSS via rights structures, though it is not necessary. Example 1 in Korpela, Lombardi and Saulle (2021) makes the point. Note that the implementing rights structure in Example 1 consists of a rotation scheme in which society rotate between x and y at R and between y and z at R'. However, there are circumstances in which the implementation in MSS is not always able to guarantee a rotation scheme. Before discussing this point and solving the drawback by elaborating a notion of rotation programs, we discuss in the following subsections the relevance of

[7] When Z is not a finite set, by using the rights structure designed in the proof of Theorem 1, it is possible to show that it implements F in MSS when F is closed valued and upper hemi-continuous, the set of alternatives Z is compact and the domain $\mathscr{R}$ is also compact.

Theorem 1. However, the impatient reader can move to Section 4 without loss of understanding.

3.2 Convergence Property

As Jackson (1992) and Moore (1992) point out, canonical mechanisms for implementing socially desirable outcomes have unnatural futures: they are highly complex and challenging to explain in natural terms. In particular, when agents are boundedly rational, such mechanisms may lead to the convergence of undesirable outcomes. Our result shows that even unsophisticated agents, using elementary adjustment rules, can reach F-optimal outcomes; our mechanism is robust to some bounded rationality. Indeed, Theorem 1 demonstrates that the implementing rights structure guarantees the convergence to a myopic stable state in a finite number of transitions among states. The reason is that our implementation problems are solved by devising a finite rights structure. This property assures that the MSS can be reached in a finite sequence of myopic improvements from any state outside of it.

Corollary 1. *Every efficient and monotonic $F : \mathscr{R} \longrightarrow \mathcal{Z}_0$ is implementable in MSS via a finite rights structure.*

This result can be thought of as the counterpart of recurrent implementation in better-response dynamics studied by Cabrales and Serrano (2011), in which agents myopically adjust their actions in the direction of better-responses. When combined with a "no-worst-alternative condition," these authors show that a variant of monotonicity is a key condition for implementation in recurrent strategies. Corollary 1 shows that for assignment problems of indivisible goods, monotonicity, together with Pareto efficiency, is sufficient for a similar type of implementability.

In Appendix A, we study two models where convergence is desirable. In particular, we consider exchange economies with complex endowment systems recently introduced by Balbuzanov and Kotowski (2019) as well as the class of "pure marriage problems" studied by Knuth (1976). Both models do not satisfy any converge property. We show that the direct exclusion core of Balbuzanov and Kotowski (2019) and the solution that selects all stable matchings in the sense of (Knuth, 1976), can be implemented in MSS.

3.3 Connections To Other Notions Of Implementation

We conclude this section by showing that implementation in MSS by a finite rights structure is equivalent to implementation in absorbing set and implementation in generalized stable set (van Deemen, 1991; Page and Wooders, 2009). Before showing it, let us formally introduce these alternative notions of stability.

Definition 8 (Absorbing Set)**.** Let us assume that S is finite. The set $A(\Gamma, R) \subseteq S$ is an absorbing set at (Γ, R) if it satisfies the following two conditions:
(a) For all $s \in A(\Gamma, R)$, if $|A(\Gamma, R)| > 1$ then there exists a finite myopic improvement path from s to $A(\Gamma, R)$.
(b) For all $s \in A(\Gamma, R)$ there does not exist any finite myopic improvement path from s to $S \backslash A(\Gamma, R)$.

Condition (a) affirms that for any state in the absorbing set it is possible to reach any other state via a myopic improvement path; Finally, by Condition (b), from any state in the absorbing set it is not possible to leave the set via a myopic improvement path.

van Deemen (1991) and Page and Wooders (2009) propose an extension of the stable set (von Neumann and Morgenstern, 1944) which replace the standard dominance relation with its transitive closure. The following is an equivalent definition of the generalized stable set based on our notion of improvement path.

Definition 9 (*Generalized Stable Set*)**.** Let us assume that S is finite. The set $V(\Gamma, R) \subseteq S$ is a generalized stable set at (Γ, R) if it satisfies the following two conditions:

1. *Iterated Internal Stability:* For all $s \in V(\Gamma, R)$ there is not a $t \in V(\Gamma, R)$ with $s \neq t$ such that $s = s_1, ..., s_m = t$ is a myopic improvement path from s to $V(\Gamma, R)$.
2. *Iterated External Stability:* For all $s \in S \backslash V(\Gamma, R)$ there exists a finite myopic improvement path from s to $V(\Gamma, R)$.

Inarra, Kuipers and Oilazola (2005) and Nicolas (2009) study the relation between absorbing sets and generalized stable sets. Korpela, Lombardi and Saulle (2021) (Theorem 2) provide further insights into the relationship between these solution concepts. In particular, they show that when the state space is finite, the union of generalized stable sets is equivalent to the union of absorbing sets, which, in turn, is equivalent to the unique myopic stable set.

Theorem 1, when combined with Theorem 2 in Korpela, Lombardi and Saulle (2021), gives us the following significant result.[8]

Corollary 2. *Any efficient F satisfying indirect monotonicity is implementable in absorbing sets by a finite rights structure, and in generalized stable sets by a finite rights structure.*

4 Rotation Programs

As noted earlier, implementation in MSS is only a preliminary step towards implementation in rotation programs. Indeed, on the one hand, implementation in MSS gives the planner the ability to design cycles among socially optimal outcomes. On the other hand, the planner does not have complete control of the cycles, in the sense that he cannot always guarantee that all agents circulate through all socially optimal outcomes. We illustrate this point through the following example.

Example 2. Suppose that $N = \{1, 2, 3\}$, $Z = \{x, y, z\}$, and $\mathscr{R} = \{R, R'\}$. Preferences are defined in the table below.

Let F be such that $F(R) = \{x, y, z\}$ and $F(R') = \{x, y\}$. This SCR satisfies *indirect monotonicity* because $F(R') \subseteq F(R)$, $F(R) \backslash F(R') = \{z\}$ and $L_3(z, R) \nsubseteq$

[8]The proof of Corollary 2 is omitted.

R			R'		
1	2	3	1	2	3
x	z	y	x	x	y
y	x	z	y	y	x
z	y	x	z	z	z

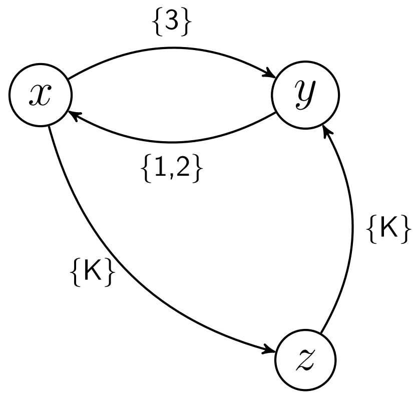


Figure 2: Example of preferences and implementing rights structure. $\#K \geqslant 2$

$L_3(z, R')$. Note that at R' only agent 3 wants to move from x to y, and agents 1 and 2 want to move from y to x. Therefore, to produce a rotation among $\{x, y\}$ at R', it is necessary to give to agent 3 the power to move from x to y and to agent 1 or 2 the power to move from y to x. A rights structure that implements F in MSS is depicted in Figure 2, in which the set of states is $S = Z$, the outcome function is the identity map, and in which γ is represented by the arrows. Note that at R, such a rights structure generates a sub-cycle in which the outcome z is ruled out. Consequently, the rotation among states $\{x, y, z\}$ cannot be guaranteed.

We solve this drawback by focusing on a refinement of the MSS.

4.1 Implementation In Rotation Programs

We start by defining a rotation program as follows.

Definition 10 (*Rotation Program*)**.** A *rotation program* for (Γ, R) is an ordered subset of states $\bar{S} = \{s_1, ..., s_m\} \subseteq S$ such that for all $s_i, s_{i+1} \in \bar{S}$:

(i) For all $s \in \bar{S} \setminus \{s_i\}$, $h(s_i) \neq h(s)$.

(ii) For all $s \in S \setminus \{s_i, s_{i+1}\}$ and all $K \in \mathcal{N}_0$, if $K \in \gamma(s_i, s)$, then not $h(s)\, P_K h(s_i)$.

(iii) There exists $K \in \mathcal{N}_0$ such that $K \in \gamma(s_i, s_{i+1})$ and $h(s_i)\, P_K h(s_{i+1})$.

Condition (i) says that in a rotation program there are no two states providing the same outcome; Conditions (ii) and (iii) together require that the only possible transitions occur among adjacent states, in a uni-directional way, following the cycle. Our notion of implementation in rotation programs can be stated as follows.

Definition 11 (*Implementation in Rotation Programs*)**.** A rights structure Γ implements F in rotation programs if the following requirements are satisfied:

- Γ implements F in MSS.
- For all $R \in \mathscr{R}$, $MSS(\Gamma, R)$ is partitioned in rotation programs $\{S_1, ..., S_m\}$ such that $h \circ S_i = F(R)$ for all $i = 1, ..., m$.

If such a rights structure exists, we say that F is *implementable in rotation programs*.

Roughly speaking, the above notion of implementation refines our notion of implementation in MSS, in the sense all myopic stable states must be arranged circularly. Thus, and irrespective of agents' preferences, the core is empty when an implementable $F(R)$ has more than one outcome.

4.2 Characterization Results

In what follows, we introduce the notion of *Rotation Monotonicity* and *Property M*, which are at the heart of the theory we develop here. To this purpose, we define the notion of *ordered chain*.

Definition 12 (*Ordered Chain*)**.** Given a pair $(R, R') \in \mathscr{R} \times \mathscr{R}$ and ordered outcomes $z_1, ..., z_m$, a sequence $z_k, \ldots, z_{k+h}$ (modulo m) with $1 \leqslant k \leqslant m$ and $1 \leqslant h \leqslant m-1$ is an *ordered chain* if there are agents $i_k, \ldots, i_{k+h}$ and an outcome $z \in Z$ such that:

(B.0) $z_{k+1+\ell} P'_{i_{k+\ell}} z_{k+\ell}$ for $\ell \in \{0, ..., h-1\}$;

(B.1) $z_{k+h} R_{i_{k+h}} z$ and $z P'_{i_{h+h}} z_{k+h}$

The notion of ordered chain recall the notion of a chain with the main difference that the former must respect an initial order. Condition (B.0) is similar to (A.0) in Definition 6: for each outcome in the sequence there is an agent preferring its successor. In contrast to (A.0), the new condition (B.0) does include the last element of the sequence. Finally, Condition (B.1) requires that at the last element of the sequence there is an agent who experiences a preference reversal.

We will be using the following condition in our characterization result.

Definition 13 (*Rotation Monotonicity*)**.** An SCR F satisfies *rotation monotonicity* if for all $R \in \mathscr{R}$, $F(R)$ can be ordered as $z_{1,R}, \ldots, z_{m,R}$ for some integer $m \geqslant 1$, and for all $(R, R') \in \mathscr{R} \times \mathscr{R}$, the following requirement is satisfied: if $F(R) \neq F(R')$ and either $\#F(R') > 1$ or $[\#F(R') = 1$ and $F(R') \notin F(R)]$, then for each $z_{k,R} \in F(R)$ for $1 \leqslant k \leqslant m$, the sequence $z_{k,R}, \ldots, z_{k+h,R}$ (modulo m) with $1 \leqslant k \leqslant m$ is an *ordered chain*.

Fix any two preference profiles: R and R'. First, note that *rotation monotonicity* applies only when $F(R) \neq F(R')$ and either (i) $F(R')$ is not a single-tone or (ii) $F(R')$ is a single-tone but it is not F-optimal at R. When *rotation monotonicity* has a bite, it requires that the F-optimal outcomes $F(R)$ can be ordered so that for every $z \in F(R)$ there is a an ordered chain of F-optimal components.

Rotation monotonicity implies *indirect monotonicity* when $\#F(R) \neq 1$ for all $R \in \mathscr{R}$. With respect to *indirect monotonicity*, *rotation monotonicity* requires that all F-optimal outcomes at R must be arranged circularly. The next result shows that only SCRs satisfying *rotation monotonicity* are implementable in rotation programs.

Theorem 2 (*Necessity*)**.** *If F is implementable in rotation programs, then it satisfies rotation monotonicity.*

Recall that the SCR in Example 2 is not implementable in rotation programs while Example 1 is. It is illustrative to study these examples in the light of Theorem 2.

Example 1 (Continued)**.** The social choice rule F in Example 1 satisfies *rotation monotonicity*. To see this, first note that $F(R) \neq F(R')$ and both are multi-valued. At R, the F-optimal element can be ordered as y, z. Then, from y there is a chain

$\{y,z\} \subseteq F(R)$ ($zP_1'y$) and a preference reversal for agent 1 (yR_1z). Moreover, from z there is a chain $\{y,z\} \subseteq F(R)$ ($yP_2'z$) and a preference reversal for agent 2 (zR_2y). At R', the F-optimal element can be ordered as x, y. Then, from x there is a chain $\{x,y\} \subseteq F(R)$ (yP_1x) and a preference reversal for agent 1 ($xR_1'y$). Finally, from y there is a chain $\{y,x\} \subseteq F(R)$ (xP_2y) and a preference reversal for agent 2 ($yR_2'x$).

Example 2 (Continued)**.** The social choice rule F in Example 2 does not satisfy *rotation monotonicity*. To see this, notice that there are two cyclic orderings of $F(R) - x, y, z$ and x, z, y. Both violate *rotation monotonicity*. Ordering x, y, z violates *rotation monotonicity* because $L_i(y,R) \subseteq L_i(y,R')$ and $z \in L_i(y,R')$ for all $i \in N$, and x, z, y violates *rotation monotonicity* because $L_i(x,R) \subseteq L_i(x,R')$ and $z \in L_i(x,R')$ for all $i \in N$.

We already pointed out that *rotation monotonicity* has a bite only when either $\#F(R') > 1$ or $[\#F(R') = 1$ but $F(R') \notin F(R)]$. It follows that *rotation monotonicity* alone is not a sufficient condition for implementation in rotation programs. However, we show that it is sufficient together with another auxiliary condition termed *Property M*, which can be defined as follows.

Definition 14 (*Property M*)**.** An SCR F satisfies *property M* if for all $R \in \mathscr{R}$, the set $F(R)$ can be ordered as $z_{1,R}, \ldots, z_{m,R}$ for $m = F(R)$, and for all $(R,R') \in \mathscr{R} \times \mathscr{R}$, the following requirement is satisfied: if $F(R) \neq F(R')$, $\#F(R') = 1$ and $F(R') = z_{j,R}$ for $1 \leqslant j \leqslant m$ then for each $z_{k,R} \in F(R) \backslash F(R')$ for $1 \leqslant k \leqslant m$ and $k \neq j$

- either the sequence $z_{k,R}, \ldots, z_{k+h,R}$ (modulo m) is an ordered chain;
- or there is a sequence of agents $i_1, \ldots, i_\ell$ such that:

  1. $F(R')P_{i_\ell}'z_{j-1,R}P_{i_{\ell-1}}'...P_{i_2}'z_{k+1,R}P_{i_1}'z_{k,R}$

     and

  2. $L_i(z_{j,R},R) \cup \{z_{j+1,R}\} \subseteq L_i(z_{j,R},R') \quad \forall i \in N.$

Fix any two preference profiles R and R'. *Property M* applies only when $F(R')$ consists of an unique outcome, $z_{j,R}$, that is also F-optimal at R. Then *Property M* requires that for each outcome that is in $F(R)$ but not in $F(R')$ either the conclusion of *rotation monotonicity* holds or there exists a sequence of agents who myopically prefer under R' to move from $z_{k,R}$ to $F(R')$ via a sequence of F-optimal outcomes at R and, moreover, for every agent, there is a monotonic transformation at $z_{j,R}$ when preferences change from R to R' and $z_{j+1,R}$ is not strictly preferred to $z_{j,R}$ at R'.

Theorem 3 (*Sufficiency*)**.** *If F is efficient and it satisfies rotation monotonicity and Property M with respect to the same ordered set of outcomes in $F(R)$, for all $R \in \mathscr{R}$, then it is implementable in rotation programs by a finite rights structure.*

We conclude this section by considering the case that a multi-valued SCR describes the planner's goal at any preference profile. As discussed by Mukherjee, Muto, Ramaekers, and Sen (2019), this is a relevant case. Under these circumstances, since *Property M* is always satisfied, *rotation monotonicity* fully characterizes the class of implementable rules in rotation programs. The following result establishes the point.[9]

Corollary 3. *Suppose $\#F(R) > 1$ for all $R \in \mathcal{R}$. Then F is implementable in rotation programs if and only if F satisfies rotation monotonicity.*

5 Assignment Problems

A basic yet widely applicable problem in economics is to allocate indivisible objects to agents. This problem is referred to as the assignment problem. In this setting, there is a set of objects, which we term as "jobs", and the goal is to allocate them among the agents in an optimal manner without allowing transfers of money. The assignment problem is a fundamental setting that is not an economic environment. Since the model applies to many resource allocation

[9] The proof of Corollary 3 is omitted.

settings in which the objects can be public houses, school seats, course enrollments, car park spaces, chores, joint assets of a divorcing couple, or time slots in schedules, we now apply Corollary 3 to this fundamental setting.

A job rotation problem (N, J, P) is a triplet where $N = \{1, ..., n\}$ is a finite set of agents with $n \geqslant 2$, $J = \{j_1, ..., j_n\}$ is a finite set of jobs, $P = (P_i)_{i \in N}$ is a profile of linear orderings such that every $P_i \subseteq J \times J$. Let (N, J, P) be a job rotation problem. Every agent i's preferences over J at P_i can be extended to an ordering over the set of allocations $\bar{J} = \{j \in J^n | j_k \neq j_l \text{ for all } k, l \in N\}$ in the following natural way:

$$jR_ij' \Leftrightarrow \text{either } j_iP_ij'_i \text{ or } j_i = j'_i, \quad \text{for all } j, j' \in \bar{J}.$$

Let $\mathscr{R}$ denote the set of all (extended) preference profiles.

Example 3 in Appendix A shows that not every efficient F on $\mathscr{R}$ is implementable in rotation programs. Given this, we focus on two classes of job rotation problem that satisfy *rotation monotonicity* and thus can be implemented in rotation programs.

5.1 A Job Rotation Problem With Restricted Domain

There are situations in which there is a common best/worst job among the available ones. For instance, suppose that the head of an economics department needs to allocate one microeconomics course to each of its microeconomics teachers. Courses can be ranked according to their sizes. The best possible assignment for everyone is to be assigned to the PhD course with the lowest number of students, whereas the common worst possible outcome for every teacher is to be assigned to the largest possible class at the undergraduate level.

In what follows, we consider situations in which there is a common best job, which is denoted by j_1^*. Since situations in which there is a common worst job can be treated symmetrically, we omit their analysis here. The set of jobs J is given by $\{j_1^*, j_2, ..., j_n\}$. Let $\bar{\mathscr{R}}$ be preference domain such that

$\bar{\mathscr{R}} = \{R \in \mathscr{R} | \text{for all } i \in N, \ \arg\max_J R_i = \{j_1^*\}\}$. With abuse of notation, we also use $\bar{\mathscr{R}}$ to denote the set of all (extended) preference profiles. The next result show that the efficient solution F defined over $\bar{\mathscr{R}}$ is implementable in rotation programs.

Theorem 4. *$F : \bar{\mathscr{R}} \to \bar{\mathcal{J}}_0$ is implementable in rotation programs.*

The intuition behind this theorem is that for each R, elements of $F(R)$ can be arranged circularly as $x(1, R), ..., x(m, R), x(1, R)$ such that no two consecutive allocations of the arrangement allocate j_1^* to the same agent. Thus, the ordered set required by *rotation monotonicity* can be set as $x(1, R), ..., x(m, R)$. Take any R' such that $F(R) \neq F(R')$. Since F is monotonic, it follows that there exists an $x(i, R) \in F(R)$ for which it holds that $x(i, R) R_\ell z$ and $z P'_\ell x(i, R)$ for some agent $\ell \in N$ and an allocation $z \in \bar{J}$. Since, by the way we arranged the elements of $F(R)$, it holds that for all $k \neq i$, $x(k+1, R) P'_j x(k, R)$ for some agent j, it is clear that F satisfies *rotation monotonicity*.

In the context of auction design, Milgrom (2004) has stated that, in contrast to much of the theoretical literature, the set of outcomes is almost never fixed in practice but it is itself subject to design. This observation extends also to tasks. To see it, let us go back to our problem of task allocation in the department. In this context, tasks can be designed by the head of the department in a way that there is a common best task, in the sense that it is the less time consuming one. Since in many cases tasks can be designed in a way of meeting the requirements of Theorem 4, the set of its applications is wide.

5.2 A Job Rotation Problem With Partially Informed Planner

As another application we consider a scenario in which the planner knows that two agents have the same top choice. Specifically, for agent i's linear ordering $R_i \subseteq J \times J$, let $\tau(R_i)$ denote the top-ranked job of agent i at R_i. We assume that planner knows that both agent 1 and agent 2 have a common top-ranked job, although he does not necessarily know which job this is, and that the domain of admissible profiles of linear orderings is given by $\hat{\mathscr{R}} = \{R \in \mathscr{R} | \tau(R_1) = \tau(R_2)\}$. With abuse of notation, we also use $\hat{\mathscr{R}}$ to denote the set of all (extended) preference profiles over $\bar{J}$.

We are interested in implementing a subsolution $\phi : \hat{\mathscr{R}} \longrightarrow \mathcal{J}_0$ of the efficient solution. We construct ϕ at R by following three sequential steps: **Step 1:** Assign $\tau(R_1)$ either to agent 1 or to agent 2. **Step 2:** Assign the remaining jobs $J \backslash \{\tau(R_1)\}$ to $N \backslash \{1, 2\}$ in a Pareto efficient way. **Step 3:** Assign the remaining job to agent 2 if agent 1 has received his top-ranked job, otherwise, assign it to agent 1. The set $\phi(R)$ can be thought of as the set of outcomes generated by an underlying random *serial dictatorship mechanism* (Abdulkadiroğlu and Sönmez, 1998), in which the only permutations that are admissible are those in which the first agent and the last agent of the ordering are respectively either agent 1 and agent 2 or agent 2 and agent 1.

Theorem 5. *$\phi : \hat{\mathscr{R}} \rightarrow \bar{\mathcal{J}}_0$ is implementable in rotation programs.*

6 Concluding Remarks

This paper studies rotation programs in an implementation framework. A rotation program is an MSS (Demuynck, Herings, Saulle and Seel, 2019a) in which states are arranged circularly. We identify conditions for implementation in MSS of Pareto efficient SCRs by a finite rights structure (Koray and Yildiz, 2019). Implementation in MSS is robust in the following sense: at any preference profile, every non-stable allocation converges to a stable allocation via a sequence of myopic deviations. Moreover, implementation in MSS encompasses implementation in absorbing sets and in generalized stable sets.

We identify a sufficient condition for implementation in MSS, named *indirect monotonicity*. This condition is weaker than (Maskin) monotonicity. Furthermore, we show that *rotation monotonicity*, when combined with an auxiliary condition, is sufficient for implementation in rotation programs. Rotation monotonicity is necessary and sufficient for implementation when the SCR never selects a single outcome. Finally, we study some welfare implications of this characterization result. We learn that implementation in rotation programs is somewhat restrictive when the set of outcome is fixed. However, as in the context of auction design (Milgrom, 2004), the outcome space is important when SCRs are implemented in rotation programs. By a clever design of new outcomes a host of nice rules become implementable.

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Appendix A

Convergence in Exchange Economy

Let us consider the class of exchange economies studied by Balbuzanov and Kotowski (2019) and consider the notion of *direct exclusion core*. We show, by means of an example, that free exchange of goods do not necessary converge to the direct exclusion core. However, the direct exclusion core is implementable in MSS via a finite rights structure. This implies that irrespective of the initial allocation of objects, it is possible to converge to a direct exclusion core allocation in a finite sequence of coalitional moves.

An *economy* is a quadruplet (N, H, P, ω) where $N = \{1, ..., n\}$ is a finite nonempty set of agents, $H = \{h_1, ..., h_m\}$ is a finite set of indivisible objects, called houses, that can be allocated among the agents, $P = (P_i)_{i \in N}$ is a profile of linear orderings, where each linear ordering is defined over $H \cup \{h_0\}$, and the endowment system $\omega : 2^N \longrightarrow 2^H$ is a function that specifies the houses owned by each coalition. For each coalition $K \in \mathcal{N}_0$, we write $\omega(K) = \bigcup_{T \in \mathcal{K}_0} \omega(T)$. Let us assume that the endowment system ω satisfies the following four properties: (A1) *Agency*: $\omega(\varnothing) = \varnothing$, (A2) *Monotonicity*: $K \subseteq K' \implies \omega(K) \subseteq \omega(K')$, (A3) *Exhaustivity*: $\omega(N) = H$, and (A4) *Non-contestability*: For each $h \in H$, there exists $K^h \in \mathcal{N}_0$ such that $h \in \omega(K) \iff K^h \subseteq K$.

Property A1 restricts ownership to agents or groups. Property A2 requires that a coalition has in its endowment anything that belongs to any sub-coalition. Property A3 states that the grand coalition N jointly owns everything. In property A4, coalition K^h is called the minimal controlling coalition of house h. It guarantees that each house has a set of one or more "co-owners" without opposing and mutually exclusive claims. As Balbuzanov and Kotowski (2019, Lemma 1) show, these properties are needed to assure that the direct exclusion core is nonempty.

We assume that each agent may live in at most one house and each house $h \in H$ may accommodate at most one agent. A house may be vacant and an agent can be homeless. We can model this latter outcome by the agent's assignment to an outside option $h_0 \notin H$, which has unlimited capacity.

An allocation $\mu : N \longrightarrow H \cup \{h_0\}$ is an assignment of agents to houses such that $\#\mu^{-1}(h) \leqslant 1$ for all $h \in H$. We write $\mu(K)$ to denote $\bigcup_{i \in K} \mu(i)$ for any $K \in \mathcal{N}_0$. Let (N, H, R, ω) be an economy. Every linear ordering R_i can be extended to an ordering over the collection $\mathcal{M}$ of allocations in the following way: $\mu R_i \mu' \iff$ either $\mu(i) P_i \mu'(i)$ or $\mu(i) = \mu'(i)$, for all $\mu, \mu' \in \mathcal{M}$. With little abuse of notation, we denote both by R_i. Let $\mathscr{R}$ denote the class of admissible preference profiles of extended preferences.

Definition 15. Given an economy (N, H, R, ω), a coalition $K \in \mathcal{N}_0$ can *directly exclusion block* the allocation μ at R with allocation σ if
(a) $\sigma(i) P_i \mu(i)$ for all $i \in K$ and
(b) $\mu(j) P_j \sigma(j) \implies \mu(j) \in \omega(K)$ for all $j \in N \backslash K$.

To speak, a coalition can directly exclusion block an assignment whenever each member strictly gains from an alternative and anyone harmed by the reallocation is excluded from a house belonging to the coalition. The *direct exclusion core* is the set of allocations that cannot be directly exclusion blocked by any nonempty coalition.

Definition 16 (*Direct Exclusion core*)**.** Given an economy (N, H, R, ω), its *direct exclusion core*, denoted by $CO(R, \omega)$, is defined by $CO(R, \omega) = \{\mu \in \mathcal{M} |$no coalition can directly exclusion block μ at $R\}$.

Thus, no coalition can gainfully destabilize a direct exclusion core allocation by invoking their collective exclusion rights. Balbuzanov and Kotowski (2019, Lemma 1) show that the direct exclusion core is never empty and all its allocations are Pareto efficient.

Let us show that the direct exclusion core does not satisfy any external stability requirement. To this end, let us represent an allocation μ by a permutation matrix with columns indexed by elements of N and rows indexed by elements of $H \cup \{h_0\}$, where h_0 is the last row. If for some $h \in H \cup \{h_0\}$ and some $i \in N$, entry $\mu_{hi} = 1$, then good h has been assigned to agent i.

Let us consider an economy with three agents and three houses.[10] Each house $i \in H$ is owned by agent i and agents' preferences are given in the table below. It can be checked that the direct exclusion core at R consist of the

R

1	2	3
2	3	1
3	1	2
1	2	3
h_0	h_0	h_0

$$\mu = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

allocation μ. Let us consider the following allocations:

$$\sigma^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \sigma^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Although the direct exclusion core is not empty, the process of 'free' exchange of houses may not lead to μ because such a process may cycle. Indeed, agents may myopically cycle around σ^1, σ^2 and σ^3.

To see it, note that for each agent i, his endowment $\omega(i) = i$ corresponds to his third choice–his last choice is to become homeless. Therefore, given this initial situation, coalition $\{1, 2\}$ can trade so that they can achieve the allocation σ^1. At σ^1, agent 1 obtains his first best choice. Thus, coalition $\{2, 3\}$ is the only coalition that can achieve a strict improvement. The only allocation that $\{2, 3\}$ can move to is allocation σ^2, where agent 2 obtains is first best choice. At σ^2, only coalition $\{1, 3\}$ can achieve a strict improvement by moving to the only attainable allocation σ^3, where agent 3 obtains is first best choice. At σ^3, only coalition $\{1, 2\}$ can achieve a strict improvement by moving to the only attainable allocation σ^1. Therefore, free exchange may lock agents in a cycle of exchanges.

A natural question that arises from the preceding example is whether it is possible to achieve the direct exclusion core by means of a different exchange process. The answer is provided by Corollary 4, which shows that the direct

[10]We borrow this example from Demuynck, Herings, saulle and Seel (2019b, pp.12-13).

exclusion core is implementable in MSS via a finite rights structure. To formalize our answer, fix any endowment system ω satisfying the above four properties. Let us define F_{ω}^{CO} by $F_{\omega}^{CO}(R) = CO(R, \omega)$ for all $R \in \mathscr{R}$.

Corollary 4. *Fix any endowment system ω satisfying properties A1-A4. F_{ω}^{CO} is implementable in MSS via a finite rights structure.*

Convergence In Matching

As a second application, we consider a two-sided, one-to-one matching model, namely the "marriage problem". A marriage problem is a market without transfers where the sides of the market are, for example, workers and firms (job matching), medical students and hospitals (matching of students to internships), students and advisors (matching of students to thesis advisors). The two sided of the markets are simply referred as "men" and "women", hence the name "marriage problem". An output of the model is termed a matching, which pairs each woman with at most one man, and each man with at most one woman. Roughly speaking, a matching is stable when there is no blocking pair, that is, no pair of agents are better off with each other than with their assigned partners. A formal description of this matching model is presented in **??**. There are two prominent models describing the marriage problem: the Gale-Shapley model (Gale and Shapley, 1962) and the Knut model (Knuth, 1976). The former studies stability for marriage problems allowing the possibly for agents to be single. The latter is a pure matching model in which no agents is allowed to be single (and thus the number of men and women is assumed to be the same). Roth and Vande Vate (1990) show that, the set of stable matching in the Gale-Shapley model exhibits a convergence property, that is, for any non stable matching there exist a myopic improvement path to a stable matching. On the contrary, for the Knut model, no general convergence result is provided. Moreover Tamura (1993) shows that, under usual matching rules, when there are at least four women, there exists preferences such that agents cycle among non stable matchings. Our next result fills the gap. Indeed, since a stable matching in the marriage problem is monotonic and efficient, we establish, as a corollary to Theorem 1, that the set of stable matching in the Knut model is implementable in MSS and thus there exists a mechanism such that a converge property in the Knut model is restored.[11]

Corollary 5. *The set of stable matching in the Knut model is implementable in MSS via a finite rights structure.*

Note that, under usual matching rules, Demuynck, Herings, Saulle and Seel (2019a) show that the MSS is a superset of the set of stable matchings. From this point of view, Corollary 5 further enlighten the relation between the MSS and the set of stable matchings. Moreover, it suggests that implementation by rights structures could represent a tool to refine the MSS whenever its prediction under canonical rules is too loose. Since this conjecture overcomes the purposes of the present work, we leave it as an avenue for future research.

A Not Implementable Efficient SCR

Example 3. Let F be the efficient SCR defined over $\mathscr{R}$. Suppose that there are three agents. Let the profiles P, P', P'' be defined as follows:

P				P'				P''			
1	2	3		1	2	3		1	2	3	
j_1	j_1	j_2	,	j_1	j_1	j_3	and	j_1	j_1	j_2	.
j_3	j_2	j_3		j_3	j_2	j_2		j_3	j_3	j_3	
j_2	j_3	j_1		j_2	j_3	j_1		j_2	j_2	j_1	

It can easily be checked that $F(R) = \{(j_3, j_1, j_2), (j_1, j_2, j_3), (j_1, j_3, j_2)\}$, $F(R') = \{(j_3, j_1, j_2), (j_1, j_2, j_3)\}$ and $F(R'') = \{(j_3, j_1, j_2), (j_1, j_3, j_2)\}$. F is not implementable in rotation programs because it violates *rotation monotonicity*. To see it, assume, to the contrary, that F satisfies *rotation monotonicity*. Then, the elements of $F(R)$ can be ordered as $x(1, R), x(2, R), x(3, R)$.

[11]The proof of Corollary 5 is omitted.

Let us consider R''. Select $i \in N$ such that $x(i, R) = (j_3, j_1, j_2)$. We show that $x(i+1, R) = (j_1, j_3, j_2)$. Since $x(i, R)$ has not fallen strictly in anyone's preference ordering because R'' is a monotonic transformation of R at $(j_3, j_1, j_2) = x(i, R)$–$L_i((j_3, j_1, j_2), R) \subseteq L_i((j_3, j_1, j_2), R')$ for each agent i, it follows that we can only move to the next element of the ordered set, that is, to $x(i+1, R)$. Since the top-ranked job for agent 2 at P'' is j_1 and since, moreover, the top-ranked job for agent 3 at P'' is j_2, it follows that only agent 1 can move to $x(i+1, R)$ at R'', which implies that $x(i+1, R)$ must coincide with (j_1, j_2, j_3), that is, we have that $x(i+1, R)\, P_1'' x(i, R)$ and $x(i+1, R) = (j_1, j_3, j_2)$.[12]

Let us now consider R'. Let us consider the allocation $x(i+1, R) = (j_1, j_2, j_3)$. Since R' is a monotonic transformation of R at $x(i+1, R)$, it follows that we can only move to the next element of the ordered set, that is, to $x(i+2, R)$. Note that the top-ranked job for agent 1 at R' is j_1. Also, note that the top-ranked job for agent 3 at R' is j_3. This implies that only agent 2 can move to $x(i+2, R)$, and so $x(i+2, R)$ must coincide with $(j_3, j_1, j_2) = x(i, R)$, which contradicts the assumption that the elements of $F(R)$ can be ordered as $x(1, R), x(2, R), x(3, R)$. Thus, F does not satisfy *rotation monotonicity*.

Appendix B

Proofs

Proof of Theorem 1. The state space S consists of $S = Gr(F) \cup Z$. Since Z finite, it follows that S is finite as well. The outcome function h is defined such that $h(z, R) = z$ for all $(z, R) \in S$ and $h(z) = z$ for all $z \in Z$. The code of rights γ is given by the following five rules:

RULE 1: $\{i\} \in \gamma((z, R), (x, R))$ for all $R \in \mathscr{R}$, all $z, x \in F(R)$, and all $i \in N$,

[12] It cannot be that $x(i+1, R) = (j_1, j_3, j_2)$ because this would lead to the contradiction that $x(i+2, R) = (j_3, j_1, j_2)$. The reason is that there cannot be any preference reversal around (j_1, j_2, j_3) because R'' is a monotonic transformation of R at (j_1, j_3, j_2). Thus, we can only move to next element of the ordered set. Since the top-ranked job for agent 1 at P'' is j_1 and since, moreover, the top-ranked job for agent 3 at P'' is j_2, the allocation $x(i+2, R)$ must coincide with (j_3, j_1, j_2) because $(j_3, j_1, j_2)\, P_2''\, (j_1, j_3, j_2)$.

RULE 2: $\{i\} \in \gamma((z, R), x)$ if $x \in L_i(z, R)$,

RULE 3: $\{i\} \in \gamma(x, (z, R))$ for all $x, (z, R) \in S$, and all $i \in N$,

RULE 4: $\{i\} \in \gamma(x, y)$ for all $x, y \in S$, and all $i \in N$, and

RULE 5: $\gamma(s, s') = \varnothing$ for any other $s, s' \in S$.

Let us show that the rights structure $\Gamma = (S, h, \gamma)$ defined above implements F in MSS if F is efficient and indirect monotonic. To this end, suppose that F is efficient and indirect monotonic. The following lemmata will be useful in proving our result. To proceed with our lemmata, we need the following additional definitions. For each $R, R' \in \mathscr{R}$:

$$M(R) \equiv \{(z, R) \mid z \in F(R)\} \subseteq S \quad U(R) \equiv \{z \in Z \mid Z \subseteq L_i(z, R) \text{ for all } i \in N\};$$

$$Q(R, R') \equiv \left\{ (z', R') \in M(R') \;\middle|\; \begin{array}{l} \text{there does not exist any myopic improvement} \\ \text{path from } (z', R') \text{ to } M(R) \cup U(R) \text{ at } R \end{array} \right\};$$

$$Q(R) \equiv \bigcup_{R' \in \mathscr{R}} Q(R, R').$$

Since S is finite, the property of asymptotic external stability of Definition 5 is equivalent to the property of iterated external stability, which is defined in a footnote of Section 3. Fix any profile R. The objective of the following lemmata is to show that

$$MSS(\Gamma, R) = M(R) \cup U(R) \cup Q(R) \text{ and } F(R) = h \circ (M(R) \cup U(R) \cup Q(R)).$$

Lemma 1. *There is a finite myopic improvement path to $M(R) \cup U(R)$ at R from every state $s \in Z \backslash U(R)$.*

Proof of Lemma 1. Take any $s \in Z \backslash U(R)$. If $U(R) \neq \varnothing$, then there exists a one step myopic improvement path from s to $U(R)$, by Rule 4. Otherwise, suppose that $U(R) = \varnothing$. We divide the rest of the proof in two parts according to whether $s \notin F(R)$ or not.

Case 1: $s \notin F(R)$. Suppose that $sR_ih(s')$ for all $i \in N$ and all $s' \in M(R)$. Since $s' \in M(R)$ and F satisfies efficiency, it holds that $sI_ih(s')$ for all $i \in N$. Since $R \in \mathscr{R}$, it follows that $s = h(s')$, and so $s \in F(R)$, which is a contradiction. Therefore, it must be the case that there exists an $s' \in M(R)$ such that $h(s')P_is$ for some $i \in N$. Hence, by Rule 3, there exists a one-step improvement path from s to $M(R)$ at R.

Case 2: $s \in F(R)$. Suppose that there exists an agent $i \in N$ such that $h(s')P_is$ for some $s' \in M(R)$. By Rule 3, there exists a one step myopic improvement path from s to $M(R)$ at R. Otherwise, suppose that $sR_ih(s')$ for all $s' \in M(R)$ and for all $i \in N$. Efficiency of F implies that $h(s')I_Ns$ for all $s' \in M(R)$, and so $h(s') = s$ because $R \in \mathscr{R}$. However, since $U(R) = \varnothing$, there exists $s'' \in Z$ and an agent $i \in N$ such that $s''P_is$. Note that agent i has the power to move from s to s' by Rule 4 and the incentive to do so since $s''P_is$. Since F satisfies efficiency and $s \in F(R)$, there must exist another agent $j \in N\backslash\{i\}$ such that sP_js''. Since $s \in F(R)$, by assumption, it follows that $(s, R) \in M(R)$. By Rule 3, agent j can move from s'' to (s, R). Hence, we have established a two-step myopic improvement path at R from s to $(s, R) \in M(R)$—that is, $i \in \gamma(s, s'')$ and $s''P_is$ and $j \in \gamma(s'', (s, R))$ and $h(s, R)P_js''$. ■

Lemma 2. *For any $R' \in \mathscr{R}$, the set $Q(R, R')$ satisfies deterrence of external deviations and $h(Q(R, R')) = \{h(s) \in Z | s \in Q(R, R')\} \subseteq F(R)$.*

Proof of Lemma 2. Suppose that $Q(R, R') \neq \varnothing$ for some $R' \in \mathscr{R}$. Otherwise, there is nothing to be proved. Let us first prove that $h(Q(R, R')) \subseteq F(R)$. By definition, $Q(R, R') \subseteq M(R')$. Take any $(z', R') \in Q(R, R')$. Assume, to the contrary, that $h(z', R') = z' \notin F(R)$. Suppose that there exists an agent $i \in N$ such that yP_iz' for some $y \in L_i(z', R')$. Then, by Rule 2, agent $i \in \gamma((z', R'), y)$ since $y \in L_i(z', R')$. An immediate contradiction is obtained if $y \in U(R)$ because there is a one step myopic improvement from $Q(R, R')$ to $U(R)$. Suppose $y \in Z\backslash U(R)$. By Lemma 1, there is a finite myopic improvement path from y to

$M(R) \cup U(R)$. Therefore, there exists a finite myopic improvement path from (z', R') to $M(R) \cup U(R)$, which contradicts the definition of $Q(R, R')$. Thus, it has to be that $L_i(z', R') \subseteq L_i(z', R)$ for all $i \in N$.

Let us proceed according to whether $\{z\} = F(R')$ or not. Suppose that $\{z\} = F(R')$. Since F satisfies *indirect monotonicity* and $L_i(z', R') \subseteq L_i(z', R)$ for all $i \in N$, it must be the case that $z \in F(R)$, which is a contradiction. Suppose that $\{z\} \neq F(R')$. Since $z' \in F(R') \backslash F(R)$ and since $L_i(z', R') \subseteq L_i(z', R)$ for all $i \in N$, *indirect monotonicity* implies that there exist a sequence of outcomes $\{z_1 \ldots, z_h\} \subseteq F(R')$ with $z' = z_1$ and $z \neq z_h$ a sequence of agents $i_1, \ldots, i_{h-1}$ such that (i) $z_{k+1} P_{i_k} z_k$ for all $k \in \{1, \ldots, h-1\}$ and (ii) $L_i(z_h, R') \nsubseteq L_i(z_h, R)$ for some $i \in N$.

By Rule 1, part (i) of *indirect monotonicity* implies that there exists a finite myopic improvement path from (z', R') to $(z_h, R') \in M(R')$ at R. Part (ii) of *indirect monotonicity* implies that there exists a state $y \in L_i(z_h, R')$ such that $y P_i z_h$. By Rule 2, $\{i\} \in \gamma((z_h, R'), y)$. An immediate contradiction is obtained whenever $y \in U(R)$ because there is a finite myopic improvement path from (z', R') to $U(R)$ at R. Suppose that $y \in Z \backslash U(R)$. Then, by Lemma 1, there exists a finite myopic improvement path from y to $M(R) \cup U(R)$ at R. Therefore, there exists a finite myopic improvement path from (z', R') to $M(R) \cup U(R)$ at R, which contradicts our initial supposition that $(z', R') \in Q(R, R')$. We conclude that $h(Q(R, R')) \subseteq F(R)$.

To complete the proof of Lemma 2, let us show that $Q(R, R') \subseteq M(R')$ satisfies deterrence of external deviations at R. The only way to get out of this set is to use either Rule 1 or Rule 2. Therefore, from any state of $Q(R, R')$, agents can only deviate to $M(R') \backslash Q(R, R')$ or Z. Note that if $M(R') \backslash Q(R, R') \neq \varnothing$, then there exists a myopic improvement path to $M(R) \cup U(R)$ at R, by the definition of $Q(R, R')$. Also, note that from any state in $Z \backslash U(R)$, there exists a finite myopic improvement path to $M(R) \cup U(R)$ at R, by Lemma 1. Hence, if an agent could benefit by deviating from a state $s \in Q(R, R')$ to a state outside of $Q(R, R')$ at R, there would exist a myopic improvement path from s to $M(R) \cup U(R)$ at R, which would contradict the definition of $Q(R, R')$. ■

Lemma 3. *If V is a nonempty subset of S satisfying both deterrence of external deviations and iterated external stability at (Γ, R), then $M(R) \subseteq V$.*

Proof of Lemma 3. Let V be a nonempty subset of S satisfying both deterrence of external deviations and iterated external stability at (Γ, R). We show that $M(R) \subseteq V$. We proceed in two steps.

Step 1: $M(R) \cap V \neq \varnothing$. For the sake of contradiction, let $M(R) \cap V = \varnothing$. Then, by iterated external stability of V, there exists a sequence of states $s_1, \ldots, s_m$ with $s_1 \in M(R)$ and a collection of coalitions $K_1, \ldots, K_{m-1}$ such that, for $j = 1, \ldots, m-1$, $K_j \in \gamma(s_j, s_{j+1})$ and $h(s_{j+1}) P_{K_j} h(s_j)$. Moreover, $s_m \in V$. By definition of γ, by the fact that $s_1 \in M(R)$ and that $h(s_{j+1}) P_{K_j} h(s_j)$, we have that only Rule 1 applies, and so it has to be that $\{s_1, ..., s_m\} \subseteq M(R)$. Therefore, $s_m \in M(R) \cap V$, which is a contradiction.

Step 2: $M(R) \subseteq V$. Take any $s \in M(R)$. Assume, to the contrary, that $s \notin V$. Since, by Step 1, $M(R) \cap V \neq \varnothing$, take any $s' \in M(R) \cap V$. Since $s, s' \in M(R)$, it must be the case that $h(s) \neq h(s')$. Suppose that for some $i \in N$, $h(s) P_i h(s')$. By Rule 1, agent i can move from s' to s, which contradicts the property of deterrence of external deviations of V. Therefore, it has to be that $h(s') R_N h(s)$. Since $R \in \mathscr{R}$ and $h(s) \neq h(s')$, it follows that $h(s') P_i h(s)$ for some $i \in N$. Since F is efficient, it follows that $h(s) \notin F(R)$, and so $s \notin M(R)$, which is a contradiction. Since the choice of s' is arbitrary and since, moreover, $s \in M(R)$, it follows that $M(R) \cap V = \varnothing$, which is a contradiction. Thus, it has to be that $M(R) \subseteq V$. ■

Lemma 4. *The set $M(R) \cup U(R) \cup Q(R)$ satisfies both deterrence of external deviations and iterated external stability at (Γ, R). Moreover, $F(R) = h \circ (M(R) \cup U(R) \cup Q(R))$.*

Proof of Lemma 4. By definition of Γ, the set $M(R)$ satisfies deterrence of external deviations. By Lemma 2, the set $Q(R)$ satisfies deterrence of external deviations. By definition, the set $U(R)$ satisfies deterrence of external deviations. Deterrence of external deviations is therefore satisfied by $M(R) \cup U(R) \cup Q(R)$. By Lemma 1, there is a finite myopic improvement path from $Z \backslash U(R)$ to

$M(R) \cup U(R)$ at R. For any $R' \in \mathscr{R}$, by the definition of $Q(R, R')$, there is a myopic improvement path from $M(R') \backslash Q(R, R')$ to $M(R) \cup U(R)$ at R. This implies that for any state outside of $M(R) \cup U(R) \cup Q(R)$ there is a myopic improvement path to $M(R) \cup U(R)$ at R, and so iterated external stability is satisfied by $M(R) \cup U(R) \cup Q(R)$. ■

Lemma 5. *If V is a nonempty subset of S satisfying both deterrence of external deviations and iterated external stability at (Γ, R), then $M(R) \cup U(R) \cup Q(R) \subseteq V$.*

Proof of Lemma 5. By Lemma 3, we already know that $M(R) \subseteq V$. By iterated external stability of V, it has to be that $U(R) \subseteq V$—the reason is that no myopic improvement path can begin from a unanimously best outcome. We are left to show that $Q(R) \subseteq V$. To this end, take any $R' \in \mathscr{R}$. Since $Q(R, R')$ satisfies deterrence of external deviations at (Γ, R) by Lemma 2, it follows that $Q(R, R') \subseteq V$, otherwise, iterated external stability of V is violated by the fact that $Q(R, R')$ satisfies deterrence of external deviations. Since R' is arbitrary, we conclude that $Q(R) \subseteq V$. Thus, $M(R) \cup U(R) \cup Q(R) \subseteq V$. ■

Lemma 6. $M(R) \cup U(R) \cup Q(R) = MSS(\Gamma, R)$

Proof of Lemma 6. Lemma 4 implies that the set $M(R) \cup U(R) \cup Q(R)$ satisfies both deterrence of external deviations and iterated external stability at (Γ, R). Lemma 5 implies that the set $M(R) \cup U(R) \cup Q(R)$ is the smallest nonempty set satisfying these two properties. Therefore, the unique MSS of (Γ, R) consists of $M(R) \cup U(R) \cup Q(R)$. ■

Lemma 7. $F(R) = h \circ (M(R) \cup U(R) \cup Q(R))$.

Proof of Lemma 7. Let us show that $F(R) = h \circ M(R) \cup U(R) \cup Q(R)$. Clearly, $F(R) \subseteq h \circ M(R)$, and so $F(R) \subseteq h \circ M(R) \cup U(R) \cup Q(R)$. For the converse, Lemma 2 implies that $h \circ Q(R, R') \subseteq F(R)$ for all $R' \in \mathscr{R}$. Since F is efficient, it follows that $U(R) \subseteq F(R)$. Moreover, by definition of $M(R)$, it follows that $h \circ M(R) \subseteq F(R)$. Therefore, $F(R) = h \circ M(R) \cup U(R) \cup Q(R)$. ■

Proof of Corollary 4. Fix any endowment system ω satisfying properties A1-A4. F_{ω}^{CO} is Pareto efficient because the direct exclusion core is efficient. In light of

Corollary 1, we need only to show that F_{ω}^{CO} is monotonic. To this end, take any $\mu \in F_{\omega}^{CO}(R)$ for some $R \in \mathscr{R}$. Take any $R' \in \mathscr{R}$ such that $L_i(\mu, R) \subseteq L_i(\mu, R')$ for all i. Let us show that $\mu \in F_{\omega}^{CO}(R') = CO(R', \omega)$. Since $\mu \in CO(R, \omega)$, it follows that no coalition can directly exclusion block μ at R. That is, for all $K \in \mathcal{N}_0$ and for all $\sigma \in \mathcal{M}$, $\mu(i) R_i \sigma(i)$ for some $i \in K$ or $[\mu(j) P_j \sigma(j)$ for some $j \in N \backslash K$ and $\mu(j) \notin \omega(K)]$. If $\mu(i) R_i \sigma(i)$ for some $i \in K$, it follows from the fact that R' is a monotonic transformation of R at μ that $\mu(i) R'_i \sigma(i)$ for some $i \in K$. If $\mu(j) P_j \sigma(j)$ for some $j \in N \backslash K$ and $\mu(j) \notin \omega(K)$, it follows from the the fact that R' is a monotonic transformation of R at μ and the fact that R_j is a linear ordering that $\mu(j) P'_j \sigma(j)$ for some $j \in N \backslash K$ and $\mu(j) \notin \omega(K)$. We have that no coalition can directly exclusion block μ at R'. Thus, F_{ω}^{CO} is monotonic. ■

Proof of Theorem 2. Suppose that Γ implements F in rotation program. Fix any R. Then, the set $MSS(\Gamma, R)$ is partitioned in rotation programs $\{S_1, ..., S_m\}$ such that $h \circ S_i = F(R)$ for all $i = 1, ..., J$. Fix any rotation program $S_j = \{s_1, ..., s_m\}$ for some $m \in \mathbb{N}$. Let $x(i, R) = s_i = h(s_i)$ for all $s_i \in S_j$. Thus, $F(R)$ is an ordered set of $\#S_j = m \geqslant 1$ outcomes. Fix any R' such that $F(R') \neq F(R)$. Suppose that either $\#F(R') > 1$ or $[\#F(R') = 1$ and $F(R') \notin F(R)]$. Fix any $s_i \in S_j$. We proceed according to whether $s_i \in MSS(\Gamma, R')$ or not.

Case 1: $s_i \in MSS(\Gamma, R')$ By the implementability of F, $h(s_i) \in F(R) \cap F(R')$. Since by the assumption that $F(R') \notin F(R)$ whenever $\#F(R') = 1$, it must be that $\#F(R') > 1$. Since Γ implements F in rotation program, the set $MSS(\Gamma, R')$ is partitioned in rotation programs $\{\bar{S}_1, ..., \bar{S}_m\}$ such that $h \circ \bar{S}_i = F(R')$ for all $i = 1, ..., m$. Then, there exists a unique j such that $s_i \in \bar{S}_j$. Without loss of generality, let $s_i = s_1 \in \bar{S}_j$.

Step 1:. Since $\bar{S}_j$ is a rotation program and since $\#F(R') > 1$, it follows that there exist $s_2 \in \bar{S}_j \backslash \{s_1\}$ and a coalition K_1 such that $K_1 \in \gamma(s_1, s_2)$ and $h(s_2) P'_{K_1} h(s_1)$. Suppose that there exists $i_1 \in K_1$ such that $h(s_1) R_{i_1} h(s_2)$. Then, there exists $h(s_2) \in Z$ such that $h(s_2) P'_{i_1} h(s_1)$ and $h(s_1) R_{i_1} h(s_2)$, where $h(s_1) = h(s_i) = x(i, R)$. Otherwise, suppose that $h(s_2) P_{K_1} h(s_1)$. Since S_j is a rotation program, it follows that $s_2 = s_{i+1} \in S_j$ and $h(s_{i+1}) = x(i+1, R)$.

The above Step 1 can be applied to $s_2 = s_{i+1} \in \bar{S}_j$ to derive a state $s_3 \in \bar{S}_j \backslash \{s_2\}$ and a coalition K_2 such that $K_2 \in \gamma(s_2, s_3)$ and $h(s_3) P'_{K_2} h(s_2)$ where $h(s_2) = x(i+1, R)$. Suppose that $s_3 = s_1$. Since $\bar{S}_j$ is a rotation program, it follows that $\bar{S}_j = \{s_1, s_2\}$. Since $F(R') \neq F(R)$, it follows that $s_3 = s_1 \neq s_{i+2} \in S_j$. It follows that there exists $i_2 \in K_2$ such that $h(s_1) P'_{i_2} h(s_2)$ and $h(s_2) R_{i_2} h(s_1)$. Thus, $z P'_{i_2} x(i+1, R) P'_{i_1} x(i, R)$ and $x(i+1, R) R_{i_2} z$ where $z = h(s_1) = x(i, R) \in Z$. Suppose that $s_3 \neq s_1$. Then, $s_3 \in \bar{S}_j \backslash \{s_1, s_2\}$. Suppose that there exists $i_2 \in K_2$ such that $h(s_2) R_{i_2} h(s_3)$. Thus, there exists $h(s_3) = z \in Z$ such that $h(s_3) P'_{i_2} h(s_2) P'_{i_1} h(s_1)$ and $h(s_2) R_{i_2} h(s_3)$, where $h(s_1) = h(s_i) = x(i, R)$ and $h(s_2) = h(s_{i+1}) = x(i+1, R)$. Otherwise, suppose that $h(s_3) P_{K_2} h(s_2)$. Since S_j is a rotation program, it follows that $s_3 = s_{i+2} \in S_j$ and $h(s_{i+2}) = x(i+2, R)$. And, so on.

Since $\bar{S}_j \neq S_j$, after a finite number $1 \leqslant h \leqslant m$ of iterations, $s_1, s_2, ..., s_{h+1}$ states and $i_1, i_2, .., i_h$ agents can be derived such that $s_1, ..., s_h \in \bar{S}_j \cap S_j$, with $h(s_\ell) = h(s_{i+\ell-1}) = x(i+\ell-1, R)$ for all $\ell = 1, ..., h$, $s_{h+1} \in \bar{S}_j$, $h(s_{h+1}) = z \in Z$ and for all $\ell \in \{1, ..., h\}$,$h(s_{\ell+1}) P'_{i_\ell} h(s_\ell)$ and $h(s_h) R_{i_h} h(s_{h+1})$.

Case 2: $s_i \notin MSS(\Gamma, R')$. By iterated external stability of $MSS(\Gamma, R')$, there exists a finite myopic improvement path from s_i to $t \in MSS(\Gamma, R')$; that is, there are coalitions $\{K_1, ..., K_{q-1}\}$ and states $\{s_i = t_1, t_2, ..., t_q = t\}$ such that for all $p = 1, ..., q-1$, $K_p \in \gamma(t_p, t_{p+1})$ and $h(t_{p+1}) P'_{K_p} h(t_p)$. Since Γ implements F in rotation program, the set $MSS(\Gamma, R')$ is partitioned in rotation programs $\{\bar{S}_1, ..., \bar{S}_m\}$ such that $h \circ \bar{S}_i = F(R')$ for all $i = 1, ..., m$. Then, there exists a unique j such that $t_q \in \bar{S}_j$.

Step 1: Suppose that $t_2 \neq s_{i+1}$. Since S_j is a rotation program and $s_i = t_1 \in S_j$, it follows that there exists $i_1 \in K_1$ such that $h(t_1) R_{i_1} h(t_2)$ where $h(t_1) = h(s_i) = x(i, R)$. Therefore, $h(t_2) P'_{i_1} h(t_1)$ and $h(t_1) R_{i_1} h(t_2)$, as we sought. Otherwise, suppose that $t_2 = s_{i+1} \in S_j$. If there exists $i_1 \in K_1$ such that $h(t_1) R_{i_1} h(t_2)$, then again $h(t_2) P'_{i_1} h(t_1)$ and $h(t_1) R_{i_1} h(t_2)$. Otherwise, suppose that $t_2 = s_{i+1} \in S_j$, $h(t_2) = x(i+1, R)$ and $h(t_2) P_{K_1} h(t_1)$.

The reasoning used in the above Step 1 can be applied to t_3 to conclude that either there exists $i_2 \in K_2$ such that $h(t_2)\, R_{i_2} h(t_3)$ for some $i_2 \in K_2$ or $h(t_3)\, P_{K_2} h(t_2)$ and $t_3 = s_{i+2} \in S_j$.

In the former case, we have that $h(t_3)\, P'_{i_2} h(t_2)\, P'_{i_1} h(t_1)$ and $h(t_2)\, R_{i_2} h(t_3)$, where $h(t_1) = x(i, R)$ and $h(t_2) = x(i+1, R)$. In the latter case, we have that $h(t_3) = x(i+2, R)$ and $h(t_3)\, P_{K_2} h(t_2)$.

Since the myopic improvement path from s_i to $t \in MSS(\Gamma, R')$ is finite, after a finite number $1 \leqslant r \leqslant q-1$ of iterations, we have that $h(t_{p+1})\, P'_{i_p} h(t_p)$ for all $p = 1, ..., r$, and either $[h(t_r)\, R_{i_r} h(t_{r+1})$ for some $i_r \in K_r]$ or $[r = q-1$, $h(t_{p+1})\, P_{K_p} h(t_p)$ and $t_p = s_{i+p-1} \in S_j$ for all $p = 1, ..., r$, and $t_q \in S_j \cap \bar{S}_j]$. In the former case, we have that for all $p = 1, ..., r$, $h(t_{p+1})\, P'_{i_p} h(t_p)$ and $h(t_r)\, R_{i_r} h(t_{r+1})$, where $h(t_p) = h(s_{i+p-1}) = x(i+p-1)$ for all $p = 1, ..., r$. In the latter case, since $t_q \in \bar{S}_j$, it follows that $t_q \in MSS(\Gamma, R')$. Case 1 above can be applied to the outcome $h(t_q) = h(s_{i+q-1}) = x(i+q-1) \in F(R)$ to complete the proof.

Proof of Theorem 3. The implementing rights structure is a variant of the rights structure constructed in the proof of Theorem 1. What changes is only the definition of Rule 1. The state space is $S = Gr(F) \cup Z$. The outcome function is $h(x, R) = x$ for all $(x, R) \in Gr(F)$ and $h(x) = x$ for all $x \in Z$. The code of rights γ is defined as follows. For all $i \in N$, all $R \in \mathscr{R}$ and all $s, t \in S$:

RULE 1: If $s = (x(k, R), R)$ and $t = (x(k+1, R), R)$ for some $1 \leqslant k \leqslant m$, then $\{i\} \in \gamma((x(k, R), R), (x(k+1, R), R))$, where the outcomes $x(k, R)$ are those specified by properties 1 and 2.

RULE 2: If $s = (z, R)$, $t = x$ and $x \in L_i(z, R)$, then $\{i\} \in \gamma((z, R), x)$.

RULE 3: If $s = x$ and $t = (z, R)$, then $\{i\} \in \gamma(x, (z, R))$.

RULE 4: If $s = z$ and $t = x$, then $\{i\} \in \gamma(s, t)$.

RULE 5: Otherwise, $\gamma(s, t) = \varnothing$.

Rule 1 allows agent i to be effective only between two consecutive socially optimal outcomes at R, that is, between $(x(k, R), R)$ and $(x(k+1, R), R)$ for all $1 \leqslant k \leqslant m$. Fix any R. Let us show that Γ implements F in rotation programs. We first show that $F(R) = h \circ MSS(\Gamma, R)$ and then we show that Γ

partitions $MSS(\Gamma, R)$ in rotation programs such that for each rotation program S, it holds that $F(R) = h \circ S$. To show that $F(R) = h \circ MSS(\Gamma, R)$ and that $MSS(\Gamma, R) = M(R) \cup U(R) \cup Q(R)$, we need to show that Lemmata 1-7 still hold under the new rights structure Γ. It can be checked that the only proofs that need to be amended are the proofs of Lemma 2 and Lemma 3. As far as the proof of Lemma 3 is concerned, the arguments provided to prove Step 2 of Lemma 3 no longer hold. However, the statement of this step is still true under the new Γ. To show this, take any $s = (x(i,R), R) \in M(R) \cap V$, which exists by Step 1 of the proof of Lemma 3. We show that $M(R) \subseteq V$. Assume, to the contrary, there exists $s' = (x(i',R), R) \in M(R)$ such that $s' \notin V$. To complete the proof of Lemma 3, let us first show that $M(R)$ is a rotation program. Since F is efficient and since $\mathscr{R}$ satisfies the restriction in (1), it follows that for all $1 \leqslant k \leqslant m$ and all $(x(k,R), R), (x(k+1,R), R) \in M(R)$, there exists $j \in N$ such that $x(k+1,R) P_j x(k,R)$. By definition of Rule 1, it follows that for each $1 \leqslant k \leqslant m$, there exists $j \in N$ such that $\{j\} \in \gamma((x(k,R), R), (x(k+1,R), R))$ and $x(k+1,R) P_j x(k,R)$. Moreover, by definition of γ, it follows that $M(R)$ is a rotation program because for each $(x(k,R), R)$, there do not exist any $K \in \mathcal{N}_0$ and any $s \in S$, with $s \neq (x(k,R), R)$ and $s \neq (x(k+1,R), R)$, such that $K \in \gamma((x(k,R), R), s)$ and $h(s) P_K x(k,R)$. Let us now complete the proof of Lemma 3. Since for each $1 \leqslant k \leqslant m$ there exists $j \in N$ such that $\{j\} \in \gamma((x(k,R), R), (x(k+1,R), R))$ and $x(k+1,R) P_j x(k,R)$, it follows that there exist $s_0, s_1, ..., s_{p-1}, s_p$, with $s_0 = s$ and $s_p = s'$, and $i_0, ..., i_{p-1}$ such that $i_h \in \gamma(s_h, s_{h+1})$ and $h(s_{h+1}) P_{i_h} h(s_h)$ for all $h = 0, ..., p-1$, where $s_h \in M(R)$ for all $h = 0, 1, ..., p$. Since $s_0 \in M(R) \cap V$ and $s_p \in M(R) \backslash V$, there exists the smallest index $h^* \in \{0, ..., p-1\}$ such that $s_{h^*} \in M(R) \cap V$ and $s_{h^*+1} \in M(R) \backslash V$. Since $i_{h^*} \in \gamma(s_{h^*}, s_{h^*+1})$ and $h(s_{h^*+1}) P_{i_{h^*}} h(s_{h^*})$, this contradicts our initial supposition that V satisfies the property of deterrence of external deviations. Thus, we have that $M(R) \subseteq V$, and so Lemma 3 holds as well.

As far as the proof of Lemma 2 is concerned, it needs to be amended as follows. Fix any $R' \in \mathscr{R}$. The proof of Lemma 2 holds if $\#F(R) \neq 1$ or if $\#F(R) = 1$

and $F(R) \notin F(R')$. The reason is that in these cases *rotation monotonicity* implies *indirect monotonicity*. To complete the proof of Lemma 2, let us suppose that $\#F(R) = 1$ and $F(R) \in F(R')$. Suppose that $F(R) = \{a\} \neq F(R') = \{z(1,R'), ..., z(m,R')\}$. Without loss of generality, let $a = z(1,R')$. Suppose that *Property M* implies that for each $z(i,R') \in F(R') \setminus \{z(1,R')\}$, there exist $x \in Z$ and $i_1, ..., i_h$, with $1 \leqslant h \leqslant m$, such that:

$$z(i+\ell+1,R')\,P_{\ell+1}z(i+\ell,R') \text{ for all } \ell \in \{0,...,h-1\} \quad \text{and}$$

$$z(i+h,R')\,P_h x \text{ and } xR'_h z(i+h,R').$$

By definition of γ, we have that for each $z(i,R') \in F(R') \setminus \{z(1,R')\}$, there exists a finite myopic improvement path from $(z(i,R'), R')$ to x. Suppose that $U(R) \neq \varnothing$. Since F is efficient and since, moreover, $\mathscr{R}$ satisfies the restriction in (1), it follows that $U(R) = \{z(1,R')\}$. Since by Rule 2 there exists a finite myopic improvement path from x to $z(1,R')$, it follows that there exists a finite myopic improvement path from $z(i,R') \in F(R') \setminus \{z(1,R')\}$ to $M(R) \cup U(R)$. Suppose that $U(R) = \varnothing$. Since Lemma 1 implies that there exists a finite myopic improvement path from x to $M(R) \cup U(R)$, we conclude that there exists a finite myopic improvement path from $z(i,R') \in F(R') \setminus \{z(1,R')\}$ to $M(R) \cup U(R)$. It follows from the definition of $Q(R,R') \subseteq M(R')$ that $Q(R,R') = \varnothing$ if there exists a finite myopic improvement path from $(z(1,R'), R')$ to $M(R) \cup U(R)$, otherwise, $Q(R,R') = \{(z(1,R'), R')\}$. In either case, we have that $h \circ Q(R,R') \subseteq F(R)$ and that $Q(R,R')$ satisfies the property of deterrence of external deviations. Note that $Q(R,R') = \{(z(1,R'), R')\}$ satisfies this property for the following two reasons: 1) Since every agent i is effective in move the state from $(z(1,R'), R')$ to $(z(2,R'), R')$, it cannot be that $z(2,R')\,P_i z(1,R')$ for some i, otherwise, since we have already shown that there exists a finite myopic improvement path from $(z(1,R'), R')$ to $M(R) \cup U(R)$, it follows that $Q(R,R') = \varnothing$, which is a contradiction; and 2) it cannot be that $xP_i z(1,R')$ for some i and some $x \in L_i(z(1,R'), R')$, otherwise, since Rule 2 implies that

$\{i\} \in \gamma((z(1, R'), R'), x)$ and $xP_i z(1, R')$ and since, moreover, Lemma 1 implies that there exists a finite myopic improvement path from x to $M(R) \cup U(R)$, since we have already shown that there exists a finite myopic improvement path from $(z(1, R'), R')$ to $M(R) \cup U(R)$, it follows that $Q(R, R') = \varnothing$, which is a contradiction. Suppose that the above arguments do not hold for some $z(i, R') \in F(R') \setminus \{z(1, R')\}$. Clearly, for each $z(i, R') \in F(R') \setminus \{z(1, R')\}$ such that the above arguments hold, we have that there exists a finite myopic improvement path from $z(i, R') \in F(R') \setminus \{z(1, R')\}$ to $M(R) \cup U(R)$. *Property M* implies that $L_i(z(1, R'), R') \cup \{z(2, R')\} \subseteq L_i(z(1, R'), R)$ for all $i \in N$. For each $z(i, R') \in F(R') \setminus \{z(1, R')\}$ for which the above arguments do not hold, *Property M* implies that there exists a sequence of agents $i_1, ..., i_\ell$ such that

$$z(1, R') P_{i_\ell} z(m, R') P_{i_{\ell-1}} \cdots P_{i_2} z(i+1, R') P_{i_1} z(i, R') \tag{2}$$

Since every agent i can be effective in moving the state from $(z(1, R'), R')$ to $(z(2, R'), R')$, it follows that no agent has an incentive to do so because $z(2, R') \in L_i(z(1, R'), R)$ for all $i \in N$. Since, by Rule 1, each agent $i \in \{i_1, ..., i_\ell\}$ is effective in moving between two consecutive states in $M(R')$, it follows from (2) that there exists a finite myopic improvement path from $(z(i, R'), R')$ to $(z(1, R'), R')$. We conclude that for each $z(i, R') \in F(R) \setminus \{z(1, R')\}$, there exists a finite myopic improvement path from $(z(i, R'), R')$ to either $M(R) \cup U(R)$ or to $\{(z(1, R'), R')\}$. It follows that $Q(R, R') \subseteq \{(z(1, R'), R')\}$. Again, $Q(R, R') = \varnothing$ if there exists a finite myopic improvement path from $(z(1, R'), R')$ to $M(R) \cup U(R)$, otherwise, $Q(R, R') = \{(z(1, R'), R')\}$. In either case, we have that $h \circ Q(R, R') \subseteq F(R)$ and that $Q(R, R')$ satisfies the property of deterrence of external deviations. Since the choice of $R' \in \mathscr{R}$ is arbitrary, it follows that Lemma 2 holds. Since Properties 1-2 implies that Lemmata 1-7 hold, it follows that $F(R) = h \circ MSS(\Gamma, R)$ and that $MSS(\Gamma, R) = M(R) \cup U(R) \cup Q(R)$.

To show that Γ partitions $MSS(\Gamma, R)$ in rotation programs, we proceed according to whether $\#F(R) = 1$ or not. We have already shown above that $M(R)$ is a rotation program.

Case 1: $\#F(R) \neq 1$. The set $U(R) = \varnothing$. To see it, suppose that there exists $x \in U(R)$. Since F is efficient and since, moreover, $\mathscr{R}$ satisfies the restriction in (1), it follows that $F(R) = \{x\}$, which is a contradiction. Thus, $MSS(\Gamma, R) = M(R) \cup Q(R)$. We have already shown above that $M(R)$ is a rotation program. Moreover, by its definition, it follows that $F(R) = h \circ M(R)$.

Fix any $R' \in \mathscr{R}$ such that $F(R') \neq F(R)$. We show that $Q(R, R') = \varnothing$. Fix any $z(i, R') \in F(R')$. *Rotation monotonicity* implies that there exist $x \in Z$ and a sequence of agents $i_1, ..., i_h$, with $1 \leqslant h \leqslant m$, such that:

$$z(i+\ell+1, R')\, P_{i_{\ell+1}} z(i+\ell, R') \text{ for all } \ell \in \{0, ..., h-1\} \text{ and}$$

$$z(i+h, R')\, R'_{i_h} x \text{ and } x P_{i_h} z(i+h, R').$$

Since, by Rule 1, for each $\ell \in \{0, ..., h-1\}$, $\{i_{\ell+1}\} \in \gamma(z(i+\ell, R'), z(i+\ell+1, R'))$ and since, moreover, by Rule 2, $\{i_h\} \in \gamma(z(i+h, R'), x)$, it follows that there exists a finite myopic improvement path from $(z(i, R'), R')$ to x. Since $U(R) = \varnothing$, Lemma 1 implies that there exists a finite myopic improvement path from x to $M(R)$. Therefore, we have established that there exists a finite myopic improvement path from $(z(i, R'), R')$ to $M(R)$, and so $(z(i, R'), R') \notin Q(R, R')$. Since the choice of $z(i, R') \in F(R')$ is arbitrary, we have that $Q(R, R') = \varnothing$.

Fix any $R' \in \mathscr{R}$ such that $F(R') = F(R)$. Nothing has to be proved if $Q(R, R') = \varnothing$. Suppose that $Q(R, R') \neq \varnothing$. We show that $Q(R, R') = M(R')$ and that $Q(R, R')$ is a rotation program. Since F is efficient and since $\mathscr{R}$ satisfies the restriction in (1), it follows that for all $(x(k, R'), R'), (x(k+1, R'), R') \in M(R')$, there exists $j \in N$ such that $x(k+1, R')\, P_j x(k, R')$. By definition of Rule 1, it follows that for each $1 \leqslant k \leqslant m$, there exists $j \in N$ such that $\{j\} \in \gamma((x(k, R'), R'), (x(k+1, R'), R'))$ and $x(k+1, R')\, P_j x(k, R')$. If there exists a finite myopic improvement path from some $(x(i, R'), R') \in M(R') \backslash Q(R, R')$ to $M(R) \cup U(R)$, it follows that for each state in $M(R')$ there exists a finite myopic improvement path to $M(R) \cup U(R)$. This implies that $Q(R, R') = \varnothing$, which is a contradiction. Thus, $Q(R, R') = M(R')$. Since Lemma 2 implies that $Q(R, R')$ satisfies the property of deterrence of external deviations, it follows that $Q(R, R')$ is a rotation program. Since the choice of $R' \in \mathscr{R}$, with $F(R') = F(R)$, is arbitrary, it follows that $MSS(\Gamma, R)$ is the union of partitioned rotation programs because for all $R', R'' \in \mathscr{R}$ such that $F(R') = F(R'') = F(R)$, it holds that $h \circ M(R') = h \circ M(R'')$ and $M(R') \cap M(R'') = \varnothing$. Thus, F is rotationally programmatically implementable.

Case 2: $\#F(R) = 1$. Recall that $MSS(\Gamma, R) = M(R) \cup U(R) \cup Q(R)$. Let $F(R) = \{z(1, R)\}$. Note that $M(R) = (z(1, R), R)$. Also, note that if $U(R) \neq \varnothing$, it follows from the efficiency of F and the restriction of $\mathscr{R}$ in (1) that $U(R) = \{z(1, R)\}$. Note that $M(R)$ and $U(R)$ are rotation programs such that $M(R) \cap U(R) = \varnothing$. To proof is complete if we show that for all $R' \in \mathscr{R}$, either $Q(R, R') = \varnothing$ or $Q(R, R') = \{(z(1, R), R')\}$. To this end, fix any $R' \in \mathscr{R}$. Suppose that $F(R) = \{z(1, R)\} \neq F(R')$. Let us proceed according whether $F(R) \in F(R')$ or not. Suppose that $F(R) \notin F(R')$. Fix any $z(i, R') \in F(R')$. By the same arguments provided in Case 1 above, it follows that there exists a finite myopic improvement path from $(z(i, R'), R')$ to x. If $U(R) \neq \varnothing$, then there exists a finite myopic improvement path from $(z(i, R'), R')$ to $z(1, R) \in U(R)$. Otherwise, if $U(R) = \varnothing$, Lemma 1 implies that there exists a finite myopic improvement path from x to $M(R)$. Therefore, there exists a finite myopic improvement path from $(z(i, R'), R')$ to $M(R) \cup U(R)$, and so $(z(i, R'), R') \notin Q(R, R')$. Since the choice of $z(i, R') \in F(R')$ is arbitrary, we have that $Q(R, R') = \varnothing$. Suppose that $F(R) \in F(R') = \{z(1, R'), ..., z(m, R')\}$. Without loss of generality, suppose that $z(1, R) = z(1, R')$. By arguing as we have done above in the completion of the proof of Lemma 2, we have that either $Q(R, R') = \varnothing$ or $Q(R, R') = \{(z(1, R'), R')\}$, as we sought. ■

Proof of Theorem 4. In light of Theorem 2, it suffices to show that F satisfies properties 1 and 2. Since $\#F(R) > 1$ for all $R \in \bar{\mathscr{R}}$, it follows that *Property M* is vacuously satisfied. Therefore, let us show that F satisfies *rotation monotonicity* as well. To this end, we need to introduce additional notation.

For all $R \in \bar{\mathscr{R}}$ and all $i \in N$, let $N_i(R)$ denote the set of Pareto efficient allocations at R that assign j_1^* to agent i, with $n_i(R)$ representing the number of elements in $N_i(R)$. Since J is a finite set, it follows that $N_i(R)$ is a finite set. For all $R \in \bar{\mathcal{R}}$ and all $i \in N$, let $\tau_2(i, R)$ denote the second top-ranked job of agent i at R_i. For all $x \in \bar{J}$ and all $R \in \bar{\mathscr{R}}$, let $\bar{x}(R)$ be a permutation of x such that (i) the agent who obtains j_1^* at x, let us say agent i, obtains his second top-ranked job $\tau_2(i, R)$ at $\bar{x}(R)$; (ii) the agent who obtains agent i's second top-ranked job at x obtains j_1^* at $\bar{x}(R)$; whereas (iii) all other agents obtain the same job both at x and at $\bar{x}(R)$. Formally, $\bar{x}_i(R) = \tau_2(i, R)$ if $x_i = j_1^*$, $\bar{x}_j(R) = j_1^*$ if $x_j = \tau_2(i, R)$, and $x_h = \bar{x}_h(R)$ for all $h \in N\backslash\{i, j\}$.

The proof that F satisfies *rotation monotonicity* relies on the following lemmata.

Lemma 8. *For all $R \in \bar{\mathscr{R}}$ and all $i \in N$, $\sum\limits_{j \in N\backslash\{i\}} n_j(R) \geqslant n_i(R)$.*

Proof of Lemma 8: The statement follows if we show that for all $R \in \bar{\mathscr{R}}$ and all $i \in N$, there exists an injective function g_i^R from $N_i(R)$ to $\bigcup_{j \in N\backslash\{i\}} N_j(R)$, that is, if we show that for for all $R \in \bar{\mathscr{R}}$ and all $i \in N$, every two distinct elements of $N_i(R)$ have distinct images in $\bigcup_{j \in N\backslash\{i\}} N_j(R)$ under g_i^R. Let us define $g_i^R : N_i(R) \longrightarrow \bigcup_{j \in N\backslash\{i\}} N_j(R)$ by $g_i^R(x) = \bar{x}(R)$. Take any two distinct $x, y \in N_i(R)$. Then, $g_i^R(x) = \bar{x}(R)$ and $g_i^R(y) = \bar{y}(R)$. Suppose that $x_j = y_j = \tau_2(i, R)$ for some $j \in N\backslash\{i\}$. Since $x \neq y$, it follows that $x_h \neq y_h$ for some $h \in N\backslash\{i, j\}$. It follows that $\bar{x}(R) \neq \bar{y}(R)$. Suppose that $x_j = \tau_2(i, R)$ and $y_h = \tau_2(i, R)$ for some $h, j \in N\backslash\{i\}$ such that $h \neq j$. It follows that $\bar{x}(R) \neq \bar{y}(R)$. Thus, g_i^R is an injective function. □

Lemma 9. *For all $R \in \bar{\mathscr{R}}$, elements of $F(R)$ can be ordered as $x(1, R), \ldots, x(m, R)$, with $m = \sum_{i \in N} n_i(R) > 1$, such that for all $k = 1, \ldots, m$ (mod m), if $x_i(k, R) = j_1^*$ for some $i \in N$, then $x_i(k+1, R) \neq j_1^*$.*

Proof of Lemma 9: Fix any $R \in \bar{\mathscr{R}}$. Without loss of generality, let us assume that$n_1(R) \geqslant n_2(R) \geqslant \ldots \geqslant n_{n-1}(R) \geqslant n_n(R)$. Let us apply the following procedure to arrange allocations of $F(R)$ in a way that the statement holds:

Step 0: If $n_1(R) - n_2(R) = 0$, then go to Step 1. If $n_1(R) - n_2(R) = k_0 > 0$, then take any $A \subseteq N_1(R)$ such that $\#A = k_0$. By Lemma 8, there exists $3 \leqslant h \leqslant n$ such that $\sum_{i=h}^{n} n_i(R) \geqslant k_0$ and $\sum_{i=h+1}^{n} n_i(R) < k_0$. Then, select any $B \subseteq N_h(R)$ such that $\sum_{i=h+1}^{n} n_i(R) + \#B = k_0$. List elements of the set A and elements of the set $B \cup \left(\cup_{i=h+1}^{n} N_i(R)\right)$ in a way that no element of A stands next to another element of set A. Start the list with an element of $A \subseteq N_1(R)$. By construction, no two consecutive allocations of the list allocate j_1^* to the same agent.

Step 1: Then, $n_1(R) - k_0 - n_2(R) = 0$, with $k_0 - 0$ if $n_1(R) = n_2(R)$, and that $n_1(R) - k_0 = n_2(R) \geqslant ... \geqslant n_h(R) - \#B$, where $B = \varnothing$ and $h = n$ if $n_1(R) = n_2(R)$. Let $n_h(R) - \#B = k_1$. Construct a sequence $\{x_i\}_{i=1}^{h}$ of elements in $\bigcup_{i=1}^{h} N_i(R) \setminus (A \cup B)$ (of length equal to h) such that $x_i \in N_i(R)$ for all $i = 1, ..., h$. Thus, the sequence is constructed in a way that that no element of $N_i(R)$ stands next to another element of $N_i(R)$, and the last element of the sequence belongs to $N_h(R)$. Since there are k_1 sequences of this type, list these sequences one after the other. By construction, no two consecutive allocations of this arrangement allocate j_1^* to the same agent. Join this linear arrangement to the right end of the arrangement of Step 0. If $n_h(R) - \#B = n_1(R) - k_0$, then the derived linear arrangement can be transformed into a circular arrangement by joining its ends. Otherwise, move to Step 2. For each $i = 1, ..., h-1$, let A_{1i} denote the set of elements of $N_i(R)$ used to construct the sequences. Thus, for each $i = 1, ..., h-1$, $\#A_{1i} = k_1$ and $N_i(R) \setminus A_{1i}$ is the set of allocations that still needs to be arranged.

Step 2: Then, $n_1(R) - k_0 - k_1 = n_2(R) - k_1 \geqslant ... \geqslant n_{h-1}(R) - k_1$. Let $n_{h-1}(R) - k_1 = k_2$. Construct a sequence $\{x_i\}_{i=1}^{h-1}$ of elements in

$$\bigcup_{i=1}^{h} N_i(R) \setminus \left(A \cup B \cup \left(\bigcup_{i=1}^{h-1} A_{1i}\right)\right)$$

(of length equal to $h-1$) such that $x_i \in N_i(R)$ for all $i = 1, ..., h-1$. Thus, the sequence is constructed in a way that that no element of $N_i(R)$ stands next to another element of $N_i(R)$, and the last element of the sequence belongs to

$N_{h-1}(R)$. Since there are k_2 sequences of this type, list these sequences one after the other. By construction, no two consecutive allocations of this arrangement allocate j_1^* to the same agent. Join this linear arrangement to the right end of the arrangement of Step 1. If $n_{h-1}(R) - k_1 - k_2 = n_1(R) - k_0 - k_1 - k_2$, then the derived linear arrangement can be transformed into a circular arrangement by joining its ends. Otherwise, move to Step 4. For each $i = 1, ..., h-2$, Let A_{2i} denote the set of elements of $N_i(R)$ used to construct the sequences. Thus, for each $i = 1, ..., h-2$, $\#A_{2i} = k_2$ and $N_i(R) \setminus (A_{1i} \cup A_{2i})$ is the set of allocations that still needs to be arranged.

$\vdots$

Step ℓ: Then, $n_1(R) - \sum_{i=0}^{\ell-1} k_i = n_2(R) - \sum_{i=1}^{\ell-1} k_i \geqslant ... \geqslant n_{h-(\ell-1)}(R) - \sum_{i=1}^{\ell-1} k_i$. Let $n_{h-(\ell-1)}(R) - \sum_{i=1}^{\ell-1} k_i = k_\ell$. Construct a sequence $\{x_i\}_{i=1}^{h-(\ell-1)}$ of elements in $\bigcup_{i=1}^{h-(\ell-1)} N_i(R) \setminus \left(A \cup B \cup \left(\bigcup_{i=1}^{h-(\ell-1)} \bigcup_{j=1}^{\ell-1} A_{ji}\right)\right)$ (of length equal to $h-(\ell-1)$) such that $x_i \in N_i(R)$ for all $i = 1, ..., h-(\ell-1)$. Thus, the sequence is constructed in a way that that no element of $N_i(R)$ stands next to another element of $N_i(R)$, and the last element of the sequence belongs to $N_{h-1}(R)$. Since there are k_ℓ sequences of this type, list these sequences one after the other. By construction, no two consecutive allocations of this arrangement allocate j_1^* to the same agent. Join this linear arrangement to the right end of the arrangement of Step $\ell - 1$. If $n_{h-(\ell-1)}(R) - \sum_{i=1}^{\ell-1} k_i = n_1(R) - \sum_{i=0}^{\ell-1} k_i$, then the derived linear arrangement can be transformed into a circular arrangement by joining its ends. Otherwise, move to Step $\ell + 1$. For each $i = 1, ..., h-\ell$, Let $A_{\ell i}$ denote the set of elements of $N_i(R)$ used to construct the sequences. Thus, for each $i = 1, ..., h-\ell$, $\#A_{\ell i} = k_\ell$ and $N_i(R) \setminus \left(\bigcup_{j=1}^{\ell} A_{ji}\right)$ is the set of allocations that still needs to be arranged.

$\vdots$

Since the set of allocations is finite, the above procedure is finite and it produces a circular arrangement of elements of $F(R)$ such that no two consecutive allocations allocate j_1^* to the same agent. □

For each $R \in \bar{\mathscr{R}}$, Lemma 9 implies that elements of $F(R)$ can be ordered as $x(1,R), ..., x(m,R)$,with $m = \sum_{i \in N} n_i(R) > 1$, such that for all $k = 1, ..., m$

(mod m), if $x_i(k,R) = j_1^*$ for some $i \in N$, then $x_i(k+1,R) \neq j_1^*$. Fix any $R' \in \bar{\mathscr{R}}$ such that $F(R) \neq F(R')$. We need to consider only the case that $\#F(R') > 1$. Suppose that for all $x(i,R) \in F(R)$, there do not exist any agent ℓ and any allocation $z \in \bar{J}$ such that $zP'_\ell x(i,R)$ and $x(i,R)R_\ell z$. This implies that for all $x(i,R) \in F(R)$, $L_\ell(x(i,R),R) \subseteq L_\ell(x(i,R),R')$ for all $\ell \in N$. Since F is (Maskin) monotonic, it follows that $F(R) = F(R')$, which is a contradiction. Thus, for some $x(i,R) \in F(R)$, there exist an agent ℓ and an allocation $z \in \bar{J}$ such that $zP'_\ell x(i,R)$ and $x(i,R)R_\ell z$. Fix any of such $x(i,R) \in F(R)$. Since by construction of the set $\{x(1,R),...,x(m,R)\}$ we have that for all $k = 1,...m$, with $k \neq i$, it holds that $x(k+1,R)P'_j x(k,R)$ for some j, it follows that $x(i,R)$ can be reached via a myopic improvement path at R' by any outcome in $x(k,R) \in \{x(1,R),...,x(m,R)\} \setminus \{x(i,R)\}$. Thus, F satisfies *rotation monotonicity*.

Proof of Theorem 5. Observe that $\#\phi(R) = 2m$, where m is the number of such allocations at R where all jobs except $\tau(R_1)$ are assigned to agents $N\backslash\{1,2\}$ in an efficient way (agent 2 getting the leftover). It follows that *Property M* is always satisfied by ϕ and Corollary 3 applies. Thus it suffices to prove that *rotation monotonicity* is satisfied.

Fix any $R \in \hat{\mathscr{R}}$ and any $x \in \phi(R)$. Let $\hat{x}$ be the allocation obtained from x in which the job assigned to agent 1 under x is assigned to agent 2 under $\hat{x}$, the job assigned to agent 2 under x is assigned to agent 1 under $\hat{x}$, whereas all other assignments are unchanged. That is, $\hat{x}_1 = x_2$, $\hat{x}_2 = x_1$, and $\hat{x}_i = x_i$ for every agent $i \neq 1,2$. Observe that $\hat{x} \in \phi(R)$ if and only if $x \in \phi(R)$. The next result show that the efficient solution ϕ is implementable in rotation programs. This result is obtained by requiring that the ordered set

$$\phi(R) = \{x(1,R), x(2,R), ..., x(2n-1,R), x(2m,R)\}$$

satisfies the following properties for all $i \in \{1,...,2m\}$: (1) If i is odd, then $x_1(i,R) = \tau(R_1)$. (2) If i is even, then $x_2(i,R) = \tau(R_2)$. (3) If $x(i,R) = x$ and i is odd, then $x(i+1,R) = \hat{x}$. $\phi(R)$ is implementable in rotation programs because we can devise a rights structure that allows agent 1 (agent 2) to be effective in moving from the outcome $x(i, R)$ to $x(i+1, R)$ provided that i is even (odd). The reason is that agent 1 (agent 2) has incentive to move from $x(i, R)$ to his top-ranked outcome $x(i+1, R)$ when i is odd (even). To see that *rotation monotonicity* is satisfied, fix any R' such that $\phi(R) \neq \phi(R')$. This implies that at least one allocation $x(i, R) \in \phi(R)$ is Pareto dominated at R', that is, there exists an allocation z such that $zR'_j x(i, R)$ for each agent $j \in N$ and $zP'_j x(i, R)$ for some agent $j \in N$. We can proceed according to whether $\tau(R_1) \neq \tau(R'_1)$. Suppose that $\tau(R_1) \neq \tau(R'_1)$. This implies that $\tau(R_1) = \tau(R_2)$ has fallen strictly in agent $j = 1, 2$'s ranking when the profile moves from R to R'. This preference reversal both agent 1 and agent 2 guarantees that *rotation monotonicity* is satisfied for every $x(i, R) \in \phi(R)$. Suppose that $\tau(R_1) = \tau(R'_1)$. We have already observed that at R, it holds that $x(i+1, R) P_2 x(i, R)$ if i is odd, and that $x(i+1, R) P_1 x(i, R)$ if i is even. In other words, there is the following cycle among outcomes in $\phi(R)$:

$$x(1, R) P_1 x(2m, R) P_2 x(2n-1, R) \cdots x(3, R) P_1 x(2, R) P_2 x(1, R)$$

Since $\tau(R_j) = \tau\left(R'_j\right)$ for $j = 1, 2$, it follows that the above cycle also exists at R'. Since $\phi(R) \neq \phi(R')$, we already know that there is at least one allocation $x(i, R) \in \phi(R)$ that is Pareto dominated at R'. Since $x(i, R)$ is efficient at R, it follows that $x(i, R) \in \phi(R)$ has strictly fallen in the preference ranking of at least one agent $j \neq 1, 2$ when the profile moves from R to R'. It follows that *rotation monotonicity* is satisfied. □