Bidding in Multi-Unit Auctions under Limited Information

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Abstract

We study multi-unit auctions in which bidders have limited knowledge of opponent strategies and values. We characterize optimal prior-free bids; these bids minimize the maximal loss in expected utility resulting from uncertainty surrounding opponent behavior. Optimal bids are simply computable despite bidders having multi-dimensional private information, and in certain cases admit closed-form solutions. In the pay-as-bid auction the minimax-loss bid is unique; in the uniform-price auction the minimax-loss bid is unique if the bidder is allowed to determine the quantities for which they bid, as in many practical applications. Payments to the seller may be higher in either auction format, but minimax-loss bids are never uniformly higher in the pay-as-bid auction.

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1 Introduction

Multi-unit auctions play a critical role in markets for government debt and electricity; they regulate the supply of power plants in electricity markets and determine the interest rates at which governments can issue new debt. Theoretical analyses of multi-unit auctions typically take a Bayesian equilibrium perspective, leaving room for studies that explore bidding when bidders cannot perfectly anticipate other bidders' behavior. Non-equilibrium outcomes may be expected in auctions that take place infrequently, with coarse feedback, or in rapidly changing environments. Indeed, despite taking place regularly, the feedback auctions provide is typically coarse: bidders learn their outcome and the market-clearing price but do not observe their competitors' bids and values. Moreover, shocks to the economy may alter the fundamentals of the good up for auction and contribute to uncertainty about competitors' strategies. Thus, learning equilibrium strategies or the competing bid distribution may be infeasible in practice, leading to strategic uncertainty that is never resolved.

We employ a non-Bayesian approach to study how to bid in the presence of strategic uncertainty. Instead of bidders knowing each others' strategies, we consider bidders who believe that their opponents' bids might follow any distribution.[1] Bidders deal with this *maximal uncertainty* by minimizing the worst-case loss of not knowing the other bidders' behavior.[2] A bid leading to low loss is desirable as its payoff is close to optimal. If the worst-case loss of a bid is low, then the payoff is near-optimal for any kind of competitors' behavior.[3] We discuss below why within-firm incentives may lead to worst-case-loss minimization.

We study the pay-as-bid and uniform-price auction formats. Both auction mechanisms are frequently used to allocate homogeneous goods, including government securities and electricity generation.[4] In these auctions bidders submit demand curves to the auctioneer. The auctioneer uses submitted demand curves to compute market-clearing prices and quantities. Each bidder receives their market-clearing quantity; in the pay-as-bid auction they pay their bid for each unit received, while in the uniform-price auction they pay the constant market-clearing price for each unit received. Which of these auction formats yields higher revenue or

[1]Level-k provides an alternative non-equilibrium approach that has been applied to multi-unit auctions [Hortaçsu et al., 2019]. However, Rasooly [2021] does not find support for the level-k model in an experiment designed to disentangle level-k from equilibrium behavior in single-unit auctions.

[2]Under maximal uncertainty, worst-case loss is the greatest gain in interim utility achievable if the bidder had optimally bid against the true distribution of opponent behavior, considering any feasible distribution of opponent bids.

[3]An alternative to minimizing worst-case loss in expected utility is to maximize worst-case expected utility. The latter gives a payoff guarantee, which can be quite low. Indeed, in auctions with private values the worst that can happen is to surely lose the auction. Any bid is then optimal. Pycia and Woodward [2017] propose selections in an equilibrium framework with uncertainty over opponent types.

[4]For government securities, see Brenner et al. [2009] and OECD [2021]. For electricity generation, see Maurer and Barroso [2011] and Del Río [2017].

greater efficiency is known to be ambiguous [Ausubel et al., 2014], and little is known about equilibrium behavior.[5] Indeed, Swinkels [2001] and Hortaçsu and Kastl [2012] note that it is computationally infeasible to compute equilibria in a generic multi-unit auction.

Our worst-case loss minimization approach yields tractable expressions for optimal bids, particularly in light of the general intractability of equilibrium behavior in Bayesian analyses. We provide characterizations of minimax-loss bids in both the pay-as-bid and uniform-price auction formats. In certain cases, such as when demand is relatively flat or when a small number of bid points is submitted, these bids can be solved in closed form; in general we provide straightforward methods for numerical computation. The tractability of optimal bids in our environment, compared to the intractability of optimal bids in the Bayesian environment, arises from the fact that Bayesian bidders respond to the equilibrium distribution of opponent bids, while loss-minimizing bidders respond to the worst-case distribution of opponent bids. We prove that the worst-case distribution has a convenient analytical representation in which bidders respond to ex post regret conditional on winning a given quantity. Bayesian analysis requires the computation of fixed points and the inversion of strategies, neither of which is necessary for our minimax-loss analysis. A side effect is that multidimensional private information does not overly complicate our analysis.

We consider three settings of feasible bids in each of the pay-as-bid and uniform-price auction formats, and tractable bid representations appear in each setting. The first two settings are the discrete multi-unit case and the continuous divisible-good case; these cases feature prominently in theoretical analyses of auctions for homogeneous goods. The third and empirically-relevant setting presumes that a large number of goods is available but that bidders are constrained to submit a relatively small number of bid points, but are free to choose the quantities at which bids are submitted. The implied bid function is a step function with a small number of steps, and the location and height of the steps are the bidders' choice variables. Although step functions are mathematically simple they are economically complex: when bids are constant over wide intervals bidders are almost always rationed. When rationing occurs with positive probability Bayesian equilibrium bids must take bidding incentives for non-local units into account, and the equilibrium first-order conditions imply a complicated non-local differential system [Kastl, 2012; Woodward, 2016]. We show, by contrast, that minimax-loss bids with a small number of steps have a tractable

[5] Equilibrium constructions in these auctions exist in many parameterized contexts. For example, Engelbrecht-Wiggans and Kahn [2002] describe equilibrium when demand barely exceeds supply; Back and Zender [1993] and Wang and Zender [2002] when the good is divisible and bidders have common values; Ausubel et al. [2014] when bidders demand two units; Burkett and Woodward [2020a] when bidders' values are defined by order statistics; and Pycia and Woodward [2021] when bidders have common, decreasing marginal values.

representation. In some cases this representation is solvable in closed form, and in all cases the bid may be efficiently computed numerically.

The distinct payment rules in the pay-as-bid and uniform-price auctions imply distinct approaches to loss minimization: in the pay-as-bid auction a bid for a given quantity is too high whenever a higher quantity is received, while in the uniform-price auction a bid is too high only when it sets the market-clearing price. In the pay-as-bid auction, the fact that the bid for a given quantity is too high whenever a larger quantity is received implies that there is a unique optimal bid in the pay-as-bid auction in all settings we study. The optimal bid trades off the loss in utility from bidding too high—paying the bids above the market-clearing price—and the loss in utility from bidding too low—winning too few units due to shading bids below the bidder's true value. The uniqueness of the optimal bid intuitively follows from the incentive to shade bids conditional on winning.[6]

In the uniform-price auction, there is a unique optimal bidding function in the practically important case when the bidder may choose only a finite number of bid points; in the unconstrained multi-unit and divisible-good cases there are typically multiple optimal bids. Intuitively, when the bidder receives a small quantity they do not leave a lot of money on the table due to overbidding, because they received a small number of units and their payment is low; they also do not miss out on significant utility from underbidding, because the market price will tend to be high and they will not desire many units at this price. Thus the main source of loss is bids on intermediate quantities, leaving bids on small (and very large) bids only partially specified. This stands in contrast to the constrained case where the locations of the bid steps are choice variables. The multiplicity of optimal bids is reminiscent of the multiplicity of Bayesian Nash equilibria [Klemperer and Meyer, 1989; Back and Zender, 1993; Ausubel et al., 2014; Burkett and Woodward, 2020b]. Another insight is that the optimal bids depend on the exact market-clearing price only with unconstrained bids for discrete units. We thus contribute to the debate on the effect of the exact market-clearing price on auction outcomes. Some argue that the selection of a market-clearing price should not matter much (cf. Swinkels [2001]) whereas others show that the selection of market-clearing price may have dramatic effects on bidder behavior (cf. Burkett and Woodward [2020a]). Our results imply that the selection matters only in the unconstrained multi-unit case (as in Burkett and Woodward [2020a]).[7]

[6] The uniqueness of the optimal bid corresponds to existing theoretical work studying Bayesian Nash equilibrium [Pycia and Woodward, 2021].

[7] When submitted bids are step functions there may be a range of prices which equate demand and supply. In the main text we focus on the *last accepted bid* pricing rule (i.e., the highest market-clearing price), which is frequently implemented in practice. In the appendix we consider the *first rejected bid* pricing rule (i.e., the lowest market-clearing price), which is frequently analyzed in the theoretical literature.

Our findings suggest that the equilibrium selection problem considered inherent to uniform-price auctions is diminished when bidders are constrained to few bid steps. We argue that this discrepancy is at least partially related to the use of divisible-good models in the study of uniform-price auctions: in the pay-as-bid auction we show that constrained minimax-loss bids converge to unconstrained minimax-loss bids as the number of available bid points grows large, but there is no unique minimax-loss bid in the unconstrained uniform-price auction. As a practical matter, one need not worry about (equilibrium) selection so long as the number of available bid points is small in comparison to the available market quantity and one faces loss-minimizing bidders.[8]

Under a natural selection of minimax-loss bids in the uniform-price auction, we show that optimal bids in the uniform-price auction are both higher and steeper than in the pay-as-bid auction; this is in line with previous theoretical and empirical work [Burkett and Woodward, 2020a; Pycia and Woodward, 2021]. Intuitively, this results from the fact that bids for small quantities are always paid in the pay-as-bid auction, implying significant bid-shading incentives for these quantities. In general, absent the selection we take, the bids in the uniform-price auction may not be uniformly higher than in a pay-as-bid auction. It can be optimal to bid 0 for high quantities, echoing low-revenue "collusive" equilibria of the uniform-price auction. However, there is no optimal bid in the uniform-price auction that is uniformly below the optimal bid in the pay-as-bid auction. Additionally, minimized maximal loss is lower in the uniform-price auction, suggesting that it is "easier to get it right" in the uniform-price auction.

Finally, ex post payments are not generally comparable between auction formats. For small quantities, the high bids of the uniform-price auction yield higher revenue than the low bids of the pay-as-bid auction, but for large quantities the low bids of the uniform-price auction yield lower revenue than the aggregate payment of both high and low bids in the pay-as-bid auction.[9] Our uniqueness results suggest that a seller interested in certainty over the distribution of revenue may prefer the pay-as-bid auction: in both auction formats the variance of ex post revenue depends on the distribution of private information, but in the uniform-price auction this distribution depends as well on the method bidders use to select

[8] We do not explicitly quantify "small." However, as an example, Song and Zhu [2018] describe a subset of the Federal Reserve's Quantitative Easing program, in which the average auction sold roughly \$400 million of a given security. Bids were in \$1 million increments, but bidders were allowed only nine bid points. Similarly, in 2020 the average Canadian nominal bond auction was for CA\$4.7 billion in CA\$1 million increments, and allowed four bid points. As an extreme example, in 2020 the average Czech government bond auction was for Kč6.4 billion in Kč1000 increments, and allowed for ten bid points.

[9] Payment ambiguity has been observed in Bayesian Nash equilibrium [Ausubel et al., 2014]. A full expected revenue comparison depends on the distribution of opponent strategies, which our model leaves unspecified.

among optimal bids.[10] On the other hand, our results also show that selection ambiguity can be disposed of by limiting bidders to a finite number of self-selected bid points, hence in practice the distribution of value-relevant private information is the key determinant of expected revenue.

We offer a descriptive and a prescriptive interpretation of minimax-loss bids. From a prescriptive perspective, a practical advantage of our non-equilibrium approach is that the bids are completely prior-free, i.e., they do not depend on the other bidders' value distribution. All a bidder needs to know is their willingness-to-pay. The bids are robust as the bidder need not worry about misspecified beliefs. Indeed, if any bid distribution is deemed possible, then in particular the actually faced.[11] Kasberger and Schlag [2020] illustrate empirically that loss-minimizing bids perform well in first-price auctions despite bidders having very coarse beliefs about competitors' behavior. From a more descriptive perspective, group decision making serves as a motivation for minimax loss. Suppose a corporation tasks a team with finding the right bid. Based on information learned after the auction, the executive board or a rival colleague might criticize the bidding team for having missed an opportunity, and the bidding team may want to preemptively defend against such a critique. By selecting a minimax-loss bid the bidding team can claim, "Your alternative bid would have been worse than our bid had there been this other bid distribution. This bid distribution was a real possibility." The minimax bid is then robust to complaints that appeal to the materialized bid distribution.[12] Minimax bids are a way to justify the choice as an (undisputed) counterfactual case can be presented so that the minimax bid was the compromise between the two cases.[13]

Savage [1951] introduced the minimax loss (regret) decision criterion for statistical decision problems. Since then it has been applied in econometrics [Manski, 2021], mechanism design [Bergemann and Schlag, 2008, 2011; Guo and Shmaya, 2019, 2021], operations research [Perakis and Roels, 2008; Besbes and Zeevi, 2011], and more generally in strategic settings. Our paper belongs to the latter category. A first paper on analyzing games with minimax regret as the players' decision criterion was Linhart and Radner [1989] who study the minimization of worst-case regret in bargaining. Parakhonyak and Sobolev [2015] con-

[10] Revenue volatility in the uniform-price auction with unconstrained bids has been observed experimentally by Sade et al. [2006] and Morales-Camargo et al. [2013].

[11] The bids are also robust to uncertainty about the number of bidders and supply.

[12] Savage [1951] also suggests group decision making as a justification for the minimax principle. In his story group members have different subjective probability assessments and the minimax principle seeks to keep the greatest "violence" done to anyone's opinion to a minimum. In contrast, we interpret the minimax as a way to defend against ex post complaints.

[13] If the bid was chosen to maximize the payoff guarantee, then many opportunities might indeed be missed. Thus, maximin expected utility is not robust to complaints about missed opportunities.

sider Bayesian firms best responding to consumers whose search rules for the lowest price are derived from worst-case regret minimization. Renou and Schlag [2010], Halpern and Pass [2012], Schlag and Zapechelnyuk [2019], and Kasberger [2020] propose solution concepts for loss (regret) minimizing players.

Applying the minimax loss (regret) decision criterion to strategic situations requires the specification of the player's perspective. A first possibility takes an ex post perspective and asks what the optimal action would be if the realized opponent actions were known; this is the ex post regret framework as in most of the existing literature [Stoye, 2011, Bergemann and Schlag, 2011]. An alternative approach takes an interim perspective. Players are uncertain about the distribution of actions (or states), and the loss of an action is the difference between the expected payoff of best responding to the distribution and the expected payoff from the chosen action. The interim perspective is also adopted in the Bayesian approach where players best respond to (their belief of) the distribution of competing actions. As not even the Bayesian approach delivers ex post optimality in games of incomplete information, we prefer the interim perspective. Moreover, note that the interim but not the ex post perspective allows to meaningfully incorporate belief restrictions as in Kasberger and Schlag [2020] and Kasberger [2020]. Following Schlag and Zapechelnyuk [2021] and Kasberger and Schlag [2020], we refer to the interim concept as loss and to the ex post equivalent as regret.

We introduce the model in the next section. Section 3 contains the main theoretical analysis of the pay-as-bid and the uniform-price auction for the multi-unit, constrained and unconstrained divisible case, respectively. Section 4 illustrates the findings of Section 3. Section 5 concludes. Proofs, calculations, and an analysis of the uniform-price auction with a first rejected bid pricing rule are provided in the appendix.

2 Model

We consider an auction for quantity $Q > 0$ of a perfectly divisible, homogeneous good. There are $n \geq 2$ bidders participating in the auction. Buyer i, $i \in \{1, \dots, n\}$, has marginal value $v^i : [0, Q] \to \mathbb{R}_+$, where $v^i(q)$ is their marginal value for quantity q. We assume that marginal values are weakly decreasing, so that $v^i(q) \geq v^i(q')$ whenever $q \leq q'$. For notational simplicity we assume that bidders have a strictly positive value for each unit, hence $v^i(Q) > 0$.[14]

Bidder i submits a weakly decreasing bid functions $b^i : [0, Q] \to \mathbb{R}_+$. After observing the

[14] Our results remain valid when bidders do not strictly demand all units, provided we replace aggregate supply Q with the supremum of all quantities for which marginal value is strictly positive, $\bar{Q}_i = \sup\{q \colon v^i(q) > 0\}$. Additionally, if $\bar{Q}_i < Q$, all results obtain in the limit with values $v^i(q) + \varepsilon$, letting $\varepsilon \searrow 0$.

bid profile $(b^j)_{j=1}^n$ the auctioneer computes a *market-clearing price* $p^\star$,

$$
\begin{aligned}
&p^\star \in \left\{p^{\mathrm{LAB}}, p^{\mathrm{FRB}}\right\};\\
&p^{\mathrm{LAB}} = \inf\left\{p \colon \exists q \in [0,Q]^n \text{ s.t. } \sum_{i=1}^n q_i < Q \text{ and } b^i\left(q_i\right) \geq p \; \forall i\right\},\\
&p^{\mathrm{FRB}} = \sup\left\{p \colon \nexists q \in [0,Q]^n \text{ s.t. } \sum_{i=1}^n q_i > Q \text{ and } b^i\left(q_i\right) \leq p \; \forall i\right\}.
\end{aligned}
$$

The prices p^{LAB} and p^{FRB} are, respectively, the last bid accepted and the first bid rejected.[15] All bids strictly above the market-clearing price $p^\star$ are awarded, and all bids strictly below the market-clearing price are rejected. When there are multiple bids placed at the market-clearing price ties are broken randomly.[16]

Bidders are risk neutral. If a bidder with value v^i receives q_i units and makes transfer t_i, their utility is

$$\hat{u}\left(q_i, t_i; v^i\right) = \int_0^{q_i} v^i\left(x\right) dx - t_i.$$

We consider two common auction formats. In a *pay-as-bid* (or *discriminatory*) auction, transfers are equal to the sum of bids for received units, $t_i^{\mathrm{PAB}} = \int_0^{q_i} b^i(x)dx$. In a *uniform-price* auction, transfers are equal to the market-clearing price times the number of units received, $t_i^{\mathrm{UPA}} = p^\star q_i$. If opponent bids b^{-i} are distributed according to B^{-i}, we write the bidder's interim utility as $u(b^i, B^{-i}; v^i) = \mathbb{E}_{B^{-i}}[\hat{u}(q^i(b), t^i(b); v^i)]$.

2.1 Loss and regret

Given a distribution of opponent bids B^{-i}, the *loss* in utility from bidding b^i instead of the interim optimal bid is

$$L\left(b^i; B^{-i}, v^i\right) = \sup_{\tilde{b}} \mathbb{E}_{B^{-i}}\left[\hat{u}\left(q^i(\tilde{b}, b^{-i}), t^i\left(\tilde{b}, b^{-i}\right); v^i\right) - \hat{u}\left(q^i\left(b^i, b^{-i}\right), t^i\left(b^i, b^{-i}\right); v^i\right)\right].$$

Loss measures the difference between expected utility given bid b^i and the utility obtainable by optimizing the submitted bid with respect to distribution B^{-i}. For example, when bid b^i is a best response to distribution B^{-i}, loss is zero.

Loss measures potential misoptimization of bids from an interim perspective: how much

[15] See Burkett and Woodward [2020a]. Treasury auctions frequently apply last-accepted-bid pricing (e.g., the United States and Switzerland) while theoretical analyses frequently study first-rejected-bid pricing [Ausubel et al., 2014].

[16] As long as all bids strictly above the market-clearing price are awarded, the precise tiebreaking rule does not affect our results.

more utility could the bidder have received, if they had known the true distribution of opponent bids when selecting their own bid. The equivalent ex post concept is *regret*,

$$R\left(b^i; b^{-i}, v^i\right) = \sup_{\tilde{b}} \hat{u}\left(q^i\left(\tilde{b}, b^{-i}\right), t^i\left(\tilde{b}, b^{-i}\right); v^i\right) - \hat{u}\left(q^i\left(b^i, b^{-i}\right), t^i\left(b^i, b^{-i}\right); v^i\right).$$

Regret measures how much additional utility the bidder could receive if they had known the bids their opponents submitted prior to choosing their own bid. A utility-maximizing bidder with perfect foreknowledge of their opponents' bids will have zero regret.

Conditional on the distribution B^{-i}, bidder i evaluates potential bids by standard expected utility. However, in our model bidders face ambiguity regarding the true distribution B^{-i} and know only that $B^{-i} \in \mathcal{B}$, where $\mathcal{B}$ is the set of feasible distributions over opponent bids. In the presence of this ambiguity, bidder i evaluates potential bids according to the maximum loss generated by any feasible distribution of opponent bids; the optimal bid $b^\star$ minimizes this loss,

$$b^\star \in \operatorname*{arginf}_{b^i} \sup_{B^{-i} \in \mathcal{B}} L\left(b^i; B^{-i}, v^i\right).$$

We refer to $b^\star$ as bidder i's minimax-loss or optimal bid. We focus on the case of *maximal uncertainty*, in which $\mathcal{B}$ contains all joint distributions on feasible bid functions; i.e., all distributions over $n-1$ weakly-decreasing functions mapping $[0, Q]$ to $\mathbb{R}_+$. Note that $\mathcal{B}$ is rich enough to include uncertainty about the number of bidders and supply. Our analysis is simplified by the following observation.

Observation 1 (Reduction to residual supply)**.** *When minimizing maximal loss, it is sufficient to consider the distribution of residual supply for each quantity $q \in [0, Q]$.*

Bidder i's ex post utility is unaffected by the specific bids submitted by their opponents, provided that the aggregate demand curve of their opponents remains fixed. Minimaxing loss over the set of feasible joint distributions of opponent bids can therefore be replaced by minimaxing loss over the set of feasible residual supply curves. Importantly, in our subsequent analysis we do not need to consider the number of opponents bidder i faces: it is sufficient to consider an arbitrary supply curve, independent of its source. For this reason our results depend on bidder i alone.

3 Minimax-Loss Bidding under Maximal Uncertainty

In the case of maximal uncertainty, bidders believe every possible distribution of opponent bids is feasible. Following Observation 1, this is equivalent to bidders believing that every distribution of residual supply curves is feasible. Because this allows the feasibility of degenerate distributions on particular residual supply curves, maximum loss is equivalent to maximum regret. This is a consequence of linearity of bidder preferences, and is not specific to the analysis of auctions or other features of our model.

Lemma 1 (Reduction to maximum regret)**.** *Under maximal uncertainty, maximizing loss is equivalent to maximizing regret. That is, for all values v^i and bids b^i,*

$$\sup_{B^{-i}\in\mathcal{B}} L\left(b^i; B^{-i}, v^i\right) = \sup_{b^{-i}} R\left(b^i; b^{-i}, v^i\right).$$

Proof. Winkler [1988] proves that the extreme points of $\mathcal{B}$ are distributions with a single point in the support. Since loss is linear in B^{-i}, maximum loss is attained at an extreme point. □

Following Lemma 1, bidder i's loss maximization problem can be identified with a regret maximization problem. As noted in Observation 1, bidder i's utility depends only on the aggregate demand curve submitted by bidders $-i$ and does not depend directly on any other bidder's specific bid. We therefore consider the set of feasible demand functions $\mathcal{S}$,

$$\mathcal{S} = \left\{S : [0, Q] \to \mathbb{R}_+ : S \text{ is decreasing}\right\}.^{17}$$

Abusing notation, let $q^i(b^i, S)$ be the quantity bidder i receives when they submit bid b_i and face aggregate demand curve S, and let $R(b^i; S, v^i)$ be bidder i's regret when submitting bid b^i against opponent aggregate demand S.

We now decompose the regret maximization problem to the related problem of maximizing *conditional regret*. Given any quantity $q \in [0, Q]$, bidder i's regret conditional on winning q units is

$$R_q\left(b^i; v^i\right) = \sup_{S:\, q^i(b^i,S)=q} R\left(b^i; S, v^i\right).$$

Maximum regret, and hence maximum loss, is the highest regret conditional on receiving any quantity, $\sup_{b^{-i}} R(b^i; b^{-i}, v^i) = \sup_q R_q(b^i; v^i)$.

Conditional regret forms the basis of our subsequent results on bidding in pay-as-bid and uniform-price auctions under maximal uncertainty. In the analysis of each auction we provide a general result regarding the structure of bids which minimax loss, then consider three models of submitted bids. First, we assume that the bidder may submit M evenly-spaced bids, as in an unconstrained multi-unit auction. Second, we assume that the bidder

[17] Each opponent $j \neq i$ submits a decreasing bid $b^j : [0, Q] \to \mathbb{R}_+$, so the aggregate demand of bidder i's opponents is a function mapping $[0, (n-1)Q]$ to $\mathbb{R}_+$. However, because there are only Q units available demand is only relevant for quantities $q \in [0, Q]$.

may submit M bids with any spacing, an approximation of practical applications where the available quantity is large and bidders are constrained to a small number of bid points.[18] Third, we consider the unconstrained problem were the bidder may submit as many bid points as they desire.

3.1 Pay-as-bid auctions

We first gain intuition by considering the sources of regret in a single-unit first-price auction. Ex post, bids in single-unit discriminatory auctions are frequently either too high—because the bidder strictly outbid the second-highest bidder—or too low–because the bidder underbid the highest bidder, whose bid was below the bidder's value. The same is true pointwise in multi-unit discriminatory auctions: the bidder frequently would prefer to increase their bid for large quantities and decrease their bid for small quantities. We use this observation to define conditional regret in the pay-as-bid auction.

If bidder i submits bid b and obtains quantity q, they know that the market-clearing price is $p^\star \in [b^i_+(q), b^i(q)]$, where $b^i_+(q) = \lim_{q' \searrow q} b^i(q')$.[19] Their regret is at least

$$\underline{R}_q^{\text{PAB}}\left(b^i; p^\star, v^i\right) = \int_0^q \left(b^i\left(x\right) - p^\star\right) dx + \int_q^Q \left(v^i\left(x\right) - p^\star\right)_+ dx.$$

That is, their regret is at least their overpayment for units they received, plus the utility foregone by underbidding for units they value above the market-clearing price. This expression is strictly decreasing in $p^\star$, hence bidder i's conditional regret is at least

$$\underline{R}_q^{\text{PAB}}\left(b^i; v^i\right) = \int_0^q \left(b^i\left(x\right) - b^i_+\left(q\right)\right) dx + \int_q^Q \left(v^i\left(x\right) - b^i_+\left(q\right)\right)_+ dx.$$

Because $\underline{R}_q^{\text{PAB}}$ is the regret the bidder has in the case in which they wish they had bid slightly higher for larger quantities, we refer to $\underline{R}_q^{\text{PAB}}$ as *underbidding regret.*

Alternatively, bidder i might be able to obtain the same allocation by bidding just above zero for all units. This will be the case when their opponents strictly demand $Q - q$ units and submit zero bids for all remaining units. In this case all nonzero payment is wasted, and regret is

$$\overline{R}_q^{\text{PAB}}\left(b^i; v^i\right) = \int_0^q b^i\left(x\right) dx.$$

In the special case $q = Q$, we have $\overline{R}_Q^{\text{PAB}} = \underline{R}_Q^{\text{PAB}}$. Because $\overline{R}_q^{\text{PAB}}$ is the regret the bidder has

[18] See our discussion of treasury auctions in the introduction.

[19] For notational simplicity we define $b^i_+(Q) = 0$.

in the case in which they wish they had bid nearly zero for all units, we refer to $\overline{R}_q^{\text{PAB}}$ as *overbidding regret.* We show in Lemma 3 (Appendix B) that overbidding and underbidding regret are sufficient to determine conditional regret,

$$R_q^{\text{PAB}}\left(b_i; v_i\right) = \max\left\{\underline{R}_q^{\text{PAB}}\left(b^i; v^i\right), \overline{R}_q^{\text{PAB}}\left(b^i; v^i\right)\right\}.$$

Since $\underline{R}_Q^{\text{PAB}}(b^i; v^i) = \overline{R}_Q^{\text{PAB}}(b^i; v^i)$, and $\overline{R}_q^{\text{PAB}}(b^i; v^i)$ is weakly increasing in q, Lemma 1 implies that maximum loss is the supremum of underbidding regret, taken over all quantities q,

$$\sup_{B^{-i}\in\mathcal{B}} L^{\text{PAB}}\left(b^i; B^{-i}, v^i\right) = \sup_q \underline{R}_q^{\text{PAB}}\left(b^i; v^i\right).$$

3.1.1 Multiple units

In most practical applications the homogeneous commodity up for auction is not perfectly divisible. We first consider the case in which bidder i can bid on M discrete units, and their value for unit k is $v_{ik} = \int_{(k-1)Q/M}^{kQ/M} v^i(x)dx$. Bidder i submits an M-dimensional bid vector, $(b_{ik})_{k=1}^M$, implying a bid function $\hat{b}^i$,

$$\hat{b}^i(q) = \begin{cases} b_{ik} & \text{if } \frac{k-1}{M}Q \leq q < \frac{k}{M}Q, \\ 0 & \text{if } q = Q. \end{cases}$$

For quantities $q \in ((k-1)Q/M, kQ/M)$, underbidding regret is weakly decreasing in q: increasing q does not affect the value of the integral $\int_0^q(\hat{b}^i(x) - \hat{b}^i(q))dx$ since $\hat{b}^i(x) = \hat{b}^i(q)$ for x near q; increasing q shrinks the bounds of integration of $\int_q^Q(v^i(x) - \hat{b}^i(q))_+dx$, and since the integrand is weakly positive it follows that $\underline{R}_q^{\text{PAB}}(b^i; v^i)$ is weakly decreasing on this range.

An immediate implication is that maximum underbidding regret is obtained at a quantity $q = kQ/M$, for some $k \in \{0, 1, \ldots, M\}$,

$$L^{\text{PAB}}\left(b^i; v^i\right) = \max_{k\in\{0,1,\ldots,M\}} \underline{R}_{kQ/M}^{\text{PAB}}\left(b^i; v^i\right).$$

Underbidding regret for quantity q increases in the bid for quantities $q' < q$ and is unaffected by the bid for quantities $q' > q$. It follows that if b_i is an optimal bid, underbidding regret must be constant at all quantities $q = kQ/M$, for $k \in \{0, 1, \ldots, M\}$.

Theorem 1 (Equal conditional regret in multi-unit pay-as-bid)**.** *If b_i is a minimax-loss bid vector in the multi-unit pay-as-bid auction, then $\underline{R}_{kQ/M}^{PAB}(b_i; v_i) = \underline{R}_{k'Q/M}^{PAB}(b_i; v_i)$ for all*

[19] For ease of exposition, we define $q_{i0} = 0$ and $b_{iM+1} = 0$.

$k, k' \in \{0, 1, \ldots, M\}$.

The proof of Theorem 1 may be found, as most other proofs for this section, in Appendix B.

Theorem 1 gives a straightforward method for computing minimax-loss bids: minimize conditional regret for any quantity, conditional on equal conditional regret across all quantities. Unfortunately, this does not have a general analytical form. The definition of conditional regret contains a summation over all units which are valued more than a given bid. The extent of this summation depends on both the bid in question and on values, and there is a potentially nonlinear relationship between bids and loss.

Corollary 1. *The unique minimax-loss bid vector b_i^{PAB} is such that $b_{iQ}^{PAB} = v_{iQ}/(Q+1)$, and for all $k < Q$, b_{ik}^{PAB} is defined by the implicit equation*

$$\left(b_{ik}^{PAB} - b_{ik+1}^{PAB}\right) k = \left(v_{ik} - b_{ik}^{PAB}\right) + \sum_{k'=k+1}^{Q} \left[\left(v_{ik'} - b_{ik}^{PAB}\right)_+ - \left(v_{ik'} - b_{ik+1}^{PAB}\right)_+\right]. \tag{1}$$

Observation 2. *The minimax-loss bid vector is strictly below marginal values wherever $v_{ik} > 0$. Let $q = \max\{k \colon v_{ik} > 0\}$. If $q = Q$, Corollary 1 shows $b_{iq}^{PAB} = v_{iq}/(Q+1) < v_{iq}$; otherwise, Corollary 1 shows $b_{iq}^{PAB} = v_{iq}/(q+1) < v_{iq}$. In either case, the bid for the last positively-valued unit is below the value for this unit. Then if there is k with $b_{ik} = v_{ik} > 0$, there is a maximal such quantity. In this case, $(b_{ik} - b_{ik+1}^{PAB}) = v_{ik} - b_{ik+1}^{PAB} > 0$ and the left-hand side of (1) is strictly positive; the right-hand side of (1) is zero. It follows that $b_{ik}^{PAB} < v_{ik}$.*

Observation 3. *The minimax-loss bid vector is strictly decreasing in quantity wherever $v_{ik} > 0$. Otherwise, there is k such that $b_{ik} = b_{ik+1}$ and $v_{ik} > b_{ik}$ (see Remark 2). In this case, the left-hand side of (1) is zero and the right-hand side is strictly positive. Increasing b_{ik}^{PAB} decreases the left-hand side of (1) and increases the right-hand side, and it follows that $b_{ik}^{PAB} > b_{ik+1}^{PAB}$ so long as $v_{ik} > 0$.*

Equation (1) provides an implicit definition for minimax-loss bids in the pay-as-bid auction. Without further structure on values, bids cannot be expressed in closed form due to the nonlinear terms in the right-hand summation: when $b_{ik} < v_{ik'}$ the summation is locally linear in b_{ik}, but when $b_{ik} > v_{ik'}$ the summation is locally constant in b_{ik}. The summation therefore induces potential nonlinearities in the loss-minimization problem. Nonetheless, the bid representation in Theorem 1 is straightforward to compute algorithmically: first compute b_{iQ}^{PAB}, then iteratively compute each b_{ik-1}^{PAB} from the already-computed $(b_{ik'}^{\text{PAB}})_{k'=k}^{Q}$.

In Section 4 we apply Theorem 1 to compute minimax-loss bids under different parametric specifications.

3.1.2 Constrained bids

As discussed in the introduction, in practice bidders are frequently constrained from submitting a distinct bid for each unit. For example, bidders can submit up to 10 bidpoints in Czech treasury auctions [Kastl, 2011] or 40 steps in the Texas electricity market [Hortaçsu et al., 2019]. We now consider the case in which bidder i can submit up to M bid points, $\{(q_{ik}, b_{ik})\}_{k=1}^{M}$, where $q_{ik} \leq q_{ik+1}$ and $b_{ik} \geq b_{ik+1}$ for all k. The implied bid function is as in the multi-unit case; the only distinction is that the quantities at which bids are submitted are now a choice variable for the bidder.

Conditional on a set of bid points the minimax-loss problem is exactly as in the multi-unit auction, replacing the subscripts kQ/M with q_k where relevant. The minimax-loss bid will equate underbidding regret across all bid points q_k, and will select the bid points to minimize underbidding regret.

Theorem 2 (Constrained minimax-loss bids in pay-as-bid). *The unique minimax-loss bid in the constrained pay-as-bid auction solves*

$$\begin{aligned}(q, b) \in \operatorname*{argmin}_{q', b'} &\int_0^Q \left(v^i(x) - b_{i1}\right)_+ dx, \\ s.t. \quad &\int_0^{q'_k} \left(\hat{b}'(x) - \hat{b}'(q'_k)\right) + \int_{q'_k}^{Q} \left(v^i(x) - \hat{b}'(q'_k)\right)_+ dx = \int_0^Q \left(v^i(x) - b_{i1}\right)_+ dx \;\; \forall k.\end{aligned}$$

Theorem 2 provides a constrained optimization problem for computing minimax-loss bids in the pay-as-bid auction when the bidder may submit at most M bid points. The optimization problem is stated in terms of divisible goods, and in a multi-unit setting with constrained bids it is possible that the minimax-loss bid has bid points which are away from integer quantities. Rounding the quantities at which bids are submitted down to the nearest available unit yields loss that is relatively close to minimax loss, especially when the number of units available is large.

Proposition 1 (Approximate minimax loss in constrained multi-unit pay-as-bid). *Suppose that (q_i, b_i) is a minimax-loss bid in the constrained pay-as-bid auction with M_b bid points, and $L^\star$ is minimax loss in the constrained multi-unit pay-as-bid auction with M_q and M_b bid points. Define a bid (q'_i, b'_i) so that $q'_{ik} = \lfloor M_q q_{ik}/Q \rfloor (Q/M_q)$ and $b'_{ik} = b_{ik}$. Then (q'_i, b'_i) is feasible in the multi-unit auction, and*

$$L^\star \leq L^{PAB}\left(\hat{b}'_i; v^i\right) \leq L^\star + \int_0^{\frac{Q}{M_q}} v^i(x)\, dx.$$

Importantly, when M_q is relatively large—that is, when the number of discrete units available is large—the right-hand integral will tend to be small relative to $L^{\star}$, as the integral is bounded above by $Qv^i(0)/M_q$.

3.1.3 Unconstrained bids

With unconstrained bids and divisible goods, the equal conditional regret condition from the multi-unit and constrained cases requires that the derivative of conditional regret is equal to zero,

$$v^i(q) - b^i(q) = v^{-1}\left(b^i(q)\right)\frac{db^i(q)}{dq}.$$

Regret conditional on receiving the maximum possible allocation is $\int_0^Q b^i(x)dx$, so long as $b^i(q) > 0$ everywhere $v^i(q) > 0$. The fundamental theorem of differential equations implies that solutions to the system cannot cross, hence the bid for quantity Q must be minimal, and optimal unconstrained bids may be computed as the solution to a differential equation.

Proposition 2 (Unconstrained pay-as-bid bids)**.** *The unique minimax-loss bid in the unconstrained divisible-good pay-as-bid auction solves*

$$v^i(q) - b^i(q) = -v^{-1}\left(b^i(q)\right)\frac{db^i}{dq}(q), \text{ s.t. } b^i(Q) = 0.$$

The differential equation defining minimax-loss bids in the pay-as-bid auction is similar to the first-order condition defining best responses in a standard Bayesian Nash equilibrium; see, e.g., Hortacsu and McAdams [2010], Pycia and Woodward [2021], and Woodward [2021]. The distinction is that in Bayesian Nash equilibrium the first-order condition contains probabilistic effects—increasing the bid for a particular quantity increases the probability that this quantity is received—while the differential equation in Proposition 2 does not. Intuitively, this is because loss is maximized conditional on receiving any particular quantity, and hence the loss-maximizing probability a quantity is won is constant across all quantities.

Because any bid which is feasible when M bid points are allowed is also feasible when $M' > M$ bid points are allowed, minimax loss decreases as the constraint on the number of bid points is increased. Since the unconstrained-optimal bid b^i may be arbitrarily approximated by step functions with small step widths, it follows that minimax loss in the multi-unit and constrained pay-as-bid auctions converges to minimax loss in the unconstrained pay-as-bid auction. Because the minimax-loss bid is unique in the pay-as-bid auction, minimax-loss bids in the multi-unit and constrained pay-as-bid auctions converge to the minimax-loss bid in the unconstrained pay-as-bid auction.

Proposition 3 (Convergence to unconstrained minimax-loss bid)**.** *Let L^{M_q} and b^{M_q} be minimax loss and the minimax-loss bid (respectively) in the multi-unit pay-as-bid auction with M_q units, and let L^{M_b} and b^{M_b} be minimax loss and the minimax-loss bid (respectively) in the constrained pay-as-bid auction with M_b bid points. Let $L^\star$ and $b^\star$ be minimax loss and the minimax-loss bid (respectively) in the unconstrained pay-as-bid auction. Then*

$$\lim_{M_q \nearrow \infty} L^{M_q} = L^\star, \ \lim_{M_q \nearrow \infty} \left\| \hat{b}^{M_q} - b^\star \right\|_1 = 0, \ \lim_{M_b \nearrow \infty} L^{M_b} = L^\star, \ \text{and} \ \lim_{M_b \nearrow \infty} \left\| \hat{b}^{M_b} - b^\star \right\|_1 = 0,$$

where $\| \cdot \|_1$ represents the L_1 norm.

To wrap up the analysis of the pay-as-bid auction, we note that there is a unique and computationally tractable minimax loss bid in each of the three settings: multi-unit, constrained divisible and unconstrained divisible. Moreover, we show that the loss optimization problem is well behaved as the multi-unit and the constrained bid functions converge to the unconstrained. We illustrate the theorems in Section 4.

3.2 Uniform-price auctions

In the uniform-price auction, bids above the market-clearing price are relevant only to the extent that they guarantee a unit is awarded; they do not otherwise affect the bidder's utility. This is in contrast to the pay-as-bid auction, where bids above the market-clearing price are paid whenever the unit is awarded. We first establish expressions for underbidding and overbidding regret in the last accepted bid uniform-price auction; the market-clearing price is $p^\star = p^{\text{LAB}}$. An analysis of the first rejected bid uniform-price auction can be found in Appendix A. The analyses differ only in the multi-unit case.

When the bidder receives quantity q, the loss-maximizing market-clearing price is $b^i_+(q) = \lim_{q' \searrow q} b^i(q')$. Their underbidding regret is

$$\underline{R}_q^{\text{LAB}} \left(b^i; v^i\right) = \int_q^Q \left(v^i\left(x\right) - b^i_+\left(q\right)\right)_+ dx.$$

As in the pay-as-bid auction, underbidding regret accounts not only for the fact that the bidder might regret not bidding just above the market-clearing price, but also for the fact that the bidder might affect their own transfer. In particular, if the bidder is at the right of a step of their bid function and their bid is the last accepted, they can potentially (slightly) reduce their bid and also the market-clearing price, without affecting their allocation.

Alternatively, bidder i might be able to obtain the same allocation by bidding just above zero for all units. This will be the case when their opponents strictly demand $Q - q$ units and submit zero bids for all remaining units. In this case all nonzero bids are wasted (in the limit), and overbidding regret is

$$\overline{R}_q^{\text{LAB}}\left(b^i; v^i\right) = q b^i\left(q\right).$$

This differs from overbidding regret in the pay-as-bid auction, $\overline{R}_q^{\text{PAB}}$, since in the uniform-price auction only the marginal bid is relevant.

The conditional regret for any quantity q is

$$R_q^{\text{LAB}}\left(b^i; v^i\right) = \max\left\{\underline{R}_q^{\text{LAB}}\left(b^i; v^i\right), \overline{R}_q^{\text{LAB}}\left(b^i; v^i\right)\right\}.$$

We show in Lemma 4 (Appendix B) that maximizing conditional regret is sufficient to maximize loss.

3.2.1 Multiple units

As in our analysis of the pay-as-bid auction, we first consider the case in which bidder i can bid on M discrete units, and their value for unit k is $v_{ik} = \int_{(k-1)Q/M}^{kQ/M} v^i(x)dx$. Following Lemma 4, maximum loss is

$$\sup_{B^{-i}\in\mathcal{B}} L^{\text{LAB}}(b^i; B^{-i}, v^i) = \sup_q R_q^{\text{LAB}}\left(b^i; v^i\right) = \max_{k\in\{0,1,\ldots,M\}} R_{q_k}^{\text{LAB}}\left(b^i; v^i\right).$$

That is, maximum loss is a maximum over conditional regrets, which are defined as the higher of overbidding and underbidding regrets for quantity q_k. This can be written equivalently as

$$\sup_{B^{-i}\in\mathcal{B}} L^{\text{LAB}}(b^i; B^{-i}, v^i) = \max_{k\in\{0,1,\ldots,M\}} \max\left\{\underline{R}_{q_{k-1}}^{\text{LAB}}\left(b^i; v^i\right), \overline{R}_{q_k}^{\text{LAB}}\left(b^i; v^i\right)\right\}.$$

The terms $\underline{R}_{q_{k-1}}^{\text{LAB}}$ and $\overline{R}_{q_k}^{\text{LAB}}$ both depend on b_{ik} and not on $b_{ik'}$ for any $k' \neq k$. Since $\underline{R}_{q_{ik-1}}^{\text{LAB}}$ is decreasing in b_{ik} and $\overline{R}_{q_{ik}}^{\text{LAB}}$ is increasing in b_{ik}, if b^i is a minimax-loss bid there must be some quantity q_{ik} so that $\underline{R}_{q_{ik-1}}^{\text{LAB}}(b^i; v^i) = \overline{R}_{q_{ik}}^{\text{LAB}}(b^i; v^i)$. The following Theorem is immediate.

Theorem 3 (No unique optimal bid in uniform-price)**.** *Generically, there is not a unique minimax-loss bid in the multi-unit uniform-price auction unless $M = 1$.*

If there is a unique minimax-loss bid, then $\overline{R}_{q_k}^{\text{LAB}}(b_i; v_i) = \underline{R}_{q_{k-1}}^{\text{LAB}}(b_i; v_i)$ for all units k. By definition of overbidding regret $\overline{R}_{q_k}^{\text{LAB}}$, this implies that there is a constant c such that $q_{ik}b_{ik} = c$ for all units k. This in turn implies $\sum_{k'=k}^{Q}(v_{ik'} + c/q_k)_+ = c$ for all units k, and this equation cannot generically be solved simultaneously for all units.

If the bidder may submit only a single bid, $M = 1$, then the incentives are as in a first-price auction. Bidding half of the value v_1 is the unique minimax bid [Kasberger and Schlag, 2020].

To obtain sharp predictions for optimal strategies, we introduce a selection from the set of minimax-loss bids. We define a *cross-conditional regret-minimizing* strategy to be one which minimizes the larger of overbidding regret for unit q_k and underbidding regret for unit q_{k-1}, which we term *cross-conditional regret.*[20] By construction cross-conditional regret is independent across bid points; since regret is maximized by cross-conditional regret for some quantity, a cross-conditional regret-minimaxing bid is a minimax-loss bid.

Definition 1. *The bid vector b_i is a* cross-conditional regret minimizing *bid if $\underline{R}^{LAB}_{q_{k-1}}(b_i; v_i) = \overline{R}^{LAB}_{q_k}(b_i; v_i)$ for all units $k \in \{1, \dots, M\}$.*

The appeal of cross-conditional regret minimizing bids is that any bid b_{ik} is justifiable ex post. Suppose another minimax bid was chosen with the bid for the k^{th} unit below the respective cross-conditional regret minimizing bid for that unit. Then after winning q_{k-1} units, the case can be made that this bid was too low as it would have been profitable to win more units. Only the cross-conditional regret minimizing bid does not allow such complaints as the regret of paying too much for q_k units serves as a defense.

Theorem 4 (Cross-conditional regret minimizing bids in uniform-price)**.** *The unique cross-conditional regret minimizing bid b_i^{LAB} is such that for all units k,*

$$b_{ik}^{LAB} = \frac{1}{q_k} \sum_{q=k}^{M} \left(v_{iq} - b_{ik}^{LAB}\right)_+ . \tag{2}$$

Theorem 4 illustrates the nonuniqueness of minimax-loss bids in the uniform-price auction. If b_i^{LAB} is a cross-conditional regret minimizing bid, equation (2) implies

$$b_{iQ}^{\text{LAB}} = \frac{1}{Q} v_{iQ}, \text{ and } b_{i1}^{\text{LAB}} \geq \frac{1}{2} v_{i1}.$$

Overbidding regret at quantity $q = Q$ is $\overline{R}^{\text{LAB}}_Q(b_i; v_i) = v_{iQ}$ and underbidding regret at quantity 0 is $\underline{R}^{\text{LAB}}_0(b_i; v_i) \geq v_{i1}/2$. These are unequal if $v_{i1} > 2v_{iQ}$, in which case there cannot be a unique minimax-loss bid.

Although the bidding function of the theorem cannot be compared to all Bayes-Nash equilibria of the uniform-price auction, it is apparent that it does not resemble "collusive" low-revenue equilibria that are frequently discussed in the literature [Ausubel et al., 2014].

[20] We consider an alternative bid selection based on simplicity in Section 3.2.2.

Indeed, the bid on the last unit is positive in a cross-conditional regret minimizing strategy under maximal uncertainty, while it is zero in a low-revenue Bayes-Nash equilibrium.

3.2.2 Constrained bids

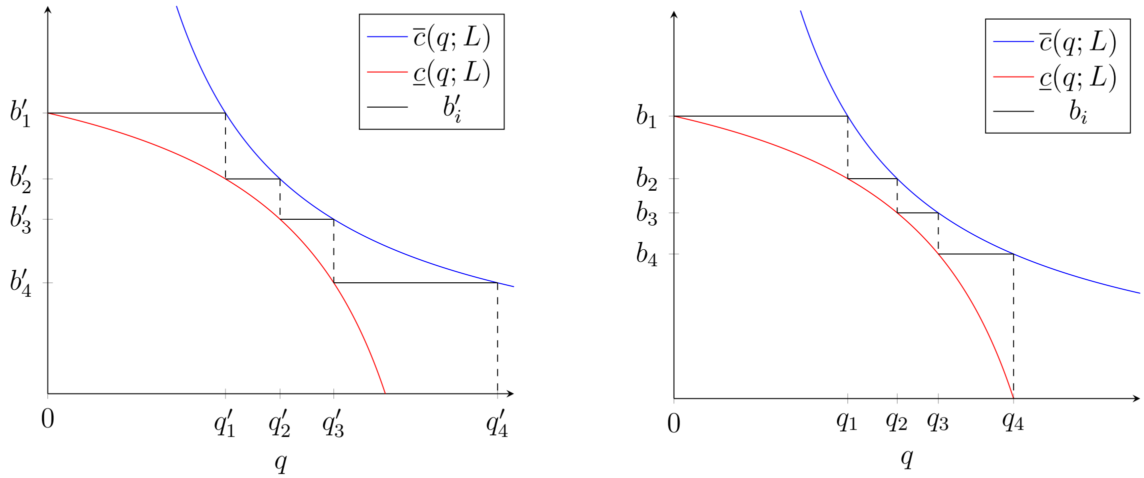
We now argue that there is a unique minimax-loss bid when bidders are constrained to submit M bid points but are free to choose the quantities at which bids are submitted. This stands in contrast to the multi-unit uniform-price auction where there is more than one minimax-loss bid. As there is a unique minimax bid when the location of the bid steps can be chosen, the nonuniqueness derives from the prespecified location of bid points in the multi-unit auction.

Theorem 5 (Minimax-loss bids in constrained uniform-price auction)**.** *In the constrained uniform-price auction with M bid points, the unique minimax-loss bid solves*

$$\left(q^{UPA}, b^{UPA}\right) \in \min_{q',b'} R,$$
$$s.t.\ q'_k b'_k = R \quad \forall k \in \{1, \ldots, M\},$$
$$and \int_{q'_{k-1}}^{Q} \left(v^i(x) - b'_k\right) dx = R \quad \forall k \in \{1, \ldots, M\}.$$

The chief distinction between the multi-unit and constrained-bid cases is that in the constrained-bid case the spacing of bid points is an additional tool for reducing ex post regret. The construction of the minimax-loss bid in the constrained uniform-price auction follows from observing that steps in the implied bid function extend between two iso-loss curves. Given loss L, the upper iso-loss curve is $\overline{c}(\cdot; L)$ such that $q\overline{c}(q; L) = L$, and the lower iso-loss curve is $\underline{c}(\cdot; L)$ such that $\int_q^Q (v^i(x) - \underline{c}(q; L))_+ dx = L$. The bid $b(q) = \overline{c}(q; L)$ induces overbidding loss which is constant in quantity, and the bid $b(q) = \underline{c}(q; L)$ induces underbidding loss which is constant in quantity. Figure 1 illustrates the two iso-loss curves. The upper iso-loss curve is always a hyperbola; the lower loss curve depends on marginal values.

Bids above the upper iso-loss curve induce loss above L by inducing overbidding regret above L, and bids below the lower iso-loss curve induce loss above L by inducing underbidding regret above L. It follows that the minimax-loss bid must lie entirely between the upper and lower iso-loss curves. Because the constrained bid must minimize steps for a particular level of loss—otherwise, the bidder could add a step and decrease maximum loss—the minimax-loss bid in the constrained uniform-price auction extends from the lower iso-loss curve to the upper iso-loss curve, then jumps down to the lower iso-loss curve, and extends again to the



(a) Loss L above constrained minimax loss

(b) Loss L equal to constrained minimax loss

Figure 1: Iso-loss curves of conditional underbidding and overbidding regret in the uniform-price auction.

upper iso-loss curve; this continues until a bid of zero is reached. Figure 1b illustrates this construction for $M = 4$.

Constructing constrained minimax-loss bids is straightforward. For loss L such that $\overline{c}(\cdot;L) \geq \underline{c}(\cdot;L)$, let $q_0 = 0$ and for all $k \in \{1,\ldots,M\}$ let $b_k = \underline{c}(q_{k-1};L)$ and let q_k be such that $\overline{c}(q_k;L) = b_k$.[21] If $\underline{c}(q_M;L) > 0$ constrained minimax loss is above L, and if $\underline{c}(q_M;L) < 0$ constrained minimax loss is below L. In either case, a new level of loss L' may be proposed, and the procedure continues until $\underline{c}(q_M;L) = 0$ (or is within numerical tolerance). Figure 1a illustrates the case when the level of loss is above the minimax loss. In the Figure, the final step q'_4 is too high, and loss can be decreased.

The construction of minimax-loss bids between the upper and lower iso-loss curves provides an intuitive argument for the uniqueness of minimax-loss bids in the uniform-price auction. Given a level of loss and associated iso-loss curves, either there is no M_b-step step function between them, or there is a single M_b-step step function between them, or there are multiple such step functions between them. If there is no feasible step function between the iso-loss curves, this level of loss is not feasible and minimax loss is above the assumed loss. On the other hand, if there are multiple feasible step functions between the iso-loss curves the iso-loss curves can be brought closer together (by reducing assumed loss) while still allowing for a feasible step function between them. This improvement in loss is infeasible only when there is a unique step function between the iso-loss curves, and at that point maximum loss is minimized.

[21]In the event that $\overline{c}(Q;L) > b_k$, define $q_k = Q$.

As is the case in the constrained pay-as-bid auction, the minimax-loss bid in the constrained uniform-price auction may be rounded to an approximate minimax-loss bid in the constrained multi-unit uniform-price auction. In the uniform-price auction this approximation is guaranteed by rounding bid points upward to the nearest feasible quantity, which differs from the pay-as-bid auction in which quantities are rounded down.

Proposition 4 (Approximate minimax loss in constrained multi-unit uniform-price)**.** *Suppose that (q_i, b_i) is a minimax-loss bid in the constrained uniform-price auction with M_b bid points, and $L^\star$ is minimax loss in the constrained multi-unit uniform-price auction with M_q units and M_b bid points. Define a constrained bid (q_i', b_i') so that $q_{ik}' = \lceil M_q q_{ik}/Q \rceil (Q/M_q)$ and $b_{ik}' = b_{ik}$. Then (q_i', b_i') is feasible in the constrained multi-unit auction, and*

$$L^\star \le L^{LAB}\left(\hat{b}_i'; v^i\right) \le L^\star + \int_0^{\frac{Q}{M_q}} v^i\left(x\right) dx.$$

As is the case in the pay-as-bid auction, the minimax loss approximation will be close when the number of available units is large.

3.2.3 Unconstrained bids

When bids are completely unconstrained, cross-conditional regret minimization requires

$$q b^{\text{LAB}}\left(q\right) = \int_q^Q \left(v^i\left(x\right) - b^{\text{LAB}}\left(q\right)\right)_+ dx$$

for all $q \in [0, Q]$. The cross-conditional regret minimizing bid is unique because overbidding regret increases in bid while underbidding regret decreases in bid. Note that the divisibility of the auctioned good turns cross-conditional regret into conditional regret.

Proposition 5 (Cross-conditional regret minimizing bid in unconstrained uniform-price auction)**.** *In the unconstrained uniform-price auction, the unique cross-conditional regret minimizing bid b^{LAB} solves*

$$q b^{LAB}\left(q\right) = \int_q^Q \left(v^i\left(x\right) - b^{LAB}\left(q\right)\right)_+ dx,\ \forall q.$$

The proposition implies that $b^{\text{LAB}}(0) = v^i(0)$, i.e., it is optimal to bid value for the "first unit." Moreover, it is optimal to bid 0 for the Q, $b^{\text{LAB}}(Q) = 0$.

Figure 2 illustrates the upper and lower iso-loss curves for a loss-level equal to minimax loss. In the unconstrained case the upper and lower iso-loss curves are tangent to each

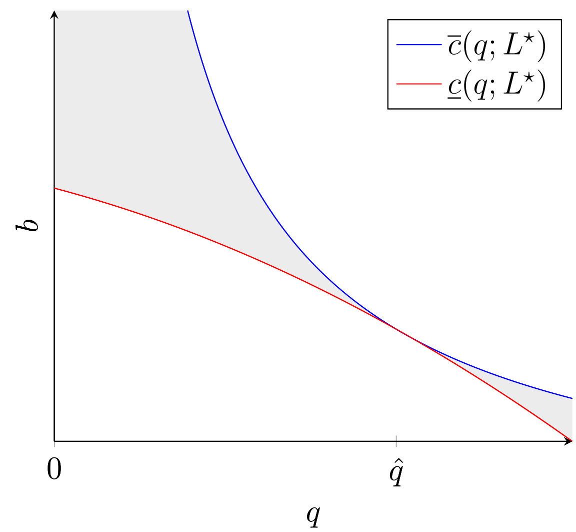


Figure 2: Iso-loss curves of unconstrained underbidding and overbidding regret in the uniform-price auction.

other. The bids at the points of tangency are uniquely determined and equal to the cross-conditional regret minimizing bids. In the example depicted in the figure, there is a single point of tangency $\hat{q}$. Other bids are only partially determined; any bid must be below the upper iso-loss curve and above the lower iso-loss curve. In the figure any decreasing bidding function in the shaded area is a minimax bid. All minimax bidding functions agree on $\hat{q}$.

The figure illustrates the nonuniqueness of the minimax bid. Indeed, as in the multi-unit uniform-price auction, there is not a unique minimax-loss bid in the unconstrained uniform-price auction. If there is, overbidding regret must be equal across all quantities, giving $qb(q) = L$ for all quantities q. This would imply that high bids for small quantities give zero underbidding regret; these bids can be reduced without affecting maximum loss, and the minimax-loss bid is nonunique.

The following theorem formally states that any weakly decreasing bid below marginal values and between the upper and lower iso-loss curves minimizes worst-case loss.

Theorem 6 (Minimax-loss bid in unconstrained uniform-price auction)**.** *Let $L^\star$ be such that $\overline{c}(\cdot; L^\star) \geq \underline{c}(\cdot; L^\star)$ and there exists q with $\underline{c}(q; L^\star) = \overline{c}(q; L^\star)$. The bid b^{LAB} minimizes maximal loss if and only if $\underline{c}(\cdot; L^\star) \leq b^{LAB} \leq \overline{c}(\cdot; L^\star)$.*

Finally, as the number of available bid points becomes large—either because the commodity becomes divisible, or because the limited-bid-step constraint is weakend—constrained bids can arbitrarily approximate an unconstrained minimax-loss bid. Since loss is continuous in bid, minimax loss will converge to unconstrained minimax loss; and, moreover, the limit of a sequence of constrained bids will be a minimax-loss bid in the unconstrained model.

Proposition 6 (Convergence to unconstrained minimax-loss bid)**.** *Let L^{M_q} be minimax loss and a minimax-loss bid (respectively) in the multi-unit uniform-price auction with M_q units, and let L^{M_b} and b^{M_b} be minimax loss and the minimax-loss bid (respectively) in the constrained uniform-price auction with M_b bid points. Let $L^\star$ be minimax loss in the unconstrained uniform-price auction. Then along any convergent sequence $\langle b^{M_q}\rangle \to b^{q\star}$ and any convergent sequence $\langle b^{M_b}\rangle \to b^{b\star}$,*

$$\lim_{M_q \nearrow \infty} L^{M_q} = L^\star, \qquad L^{LAB}\left(\lim_{M_q \nearrow \infty} \hat{b}^{M_q}; v^i\right) = L^\star,$$

$$\lim_{M_b \nearrow \infty} L^{M_b} = L^\star, \qquad \text{and } L^{LAB}\left(\lim_{M_b \nearrow \infty} \hat{b}^{M_b}; v^i\right) = L^\star.$$

3.3 Comparison of auction formats

Bids in the uniform-price auction may be higher or lower than in the pay-as-bid auction, and the revenue comparison of the two formats is inherently ambiguous. The bidder's analysis is independent of the actual distribution of their allocation, and thus the transfer to the auctioneer must depend on the likelihood that bidder i receives any particular quantity q_i.[22] To show revenue ambiguity, we focus attention on the cross-conditional regret maximizing bid in the uniform-price auction.

Comparison 1 (Uniform-price bids above pay-as-bid bids)**.** *Cross-conditional regret minimizing bids in the multi-unit and unconstrained uniform-price auctions are higher than the unique minimax-loss bid in the pay-as-bid auction: $b^{LAB} \geq b^{PAB}$.*

Although cross-conditional regret minimizing bids in the uniform-price auction are above the unique minimax-loss bid in the pay-as-bid auction, this is not the case for all selections of minimax-loss bids in the uniform-price auction. In particular, at large quantities a bid which is equal to the lower iso-loss curve in the uniform-price auction will fall below the minimax-loss bid in the pay-as-bid auction. Note that Comparison 1 does not apply to the constrained-bid setting, where cross-conditional regret minimization is ill-defined.[23]

[22] In other contexts the transfer to the auctioneer depends on the distribution of randomness in the auction both through its direct effect on the probability of allocation and on its indirect effect through bid levels. Here, the distribution of randomness affects only allocations.

[23] At any submitted bid point optimal bids in the uniform-price auction with constrained bids minimize cross-conditional regret, but away from bid points overbidding regret and underbidding regret are not equal. Since optimal bid points in the uniform-price auction differ from those in the pay-as-bid auction, optimal bids in the pay-as-bid auction with constrained bids may occasionally lie above optimal bids in the uniform-price auction with constrained bids.

Comparison 2 (Semi-comparability of optimal bids)**.** *There is a bid b^{LAB} that is optimal in the uniform-price auction, and is such that $b^{LAB} \not\geq b^{PAB}$. Furthermore, if b^{LAB} is optimal in the uniform-price auction, then $b^{LAB} \not\leq b^{PAB}$.*

While there is a minimax-loss bid in the uniform-price auction which is not everywhere greater than the minimax-loss bid in the pay-as-bid auction, there is no minimax-loss bid in the uniform-price auction which is everywhere below the minimax-loss bid in the pay-as-bid auction. In the unconstrained uniform-price auction, bids minimax loss if and only if they fall between the upper and lower iso-loss curves, and these curves must be equal somewhere. At this point of intersection all minimax-loss bids in the uniform-price auction are equal. Then following Comparison 1, at this point the uniform-price bid will exceed the pay-as-bid bid.

Previous theoretical work has identified uniform-price bids as more elastic (i.e., steeper) than pay-as-bid bids [Malvey and Archibald, 1998; Ausubel et al., 2014; Pycia and Woodward, 2021]. This results from the significant demand-shading incentives for small quantities in the pay-as-bid auction—where bids for small quantities are paid for all larger quantities—and the significant demand-shading incentives for large quantities in the uniform-price auction—where bids are paid times the quantity for which they are offered. Uniform-price bids remain steep in comparison to pay-as-bid bids, when restriction is made to cross-conditional regret minimization.

Comparison 3 (Uniform-price bids steeper than pay-as-bid bids)**.** *Define the average slope of the bid b to be $\alpha = (b_1 - b_Q)/Q$. Cross-conditional regret-minimizing bids in the multi-unit and unconstrained uniform-price auction are on average steeper than the unique minimax-loss bid in the pay-as-bid auction: $\alpha^{LAB} \geq \alpha^{PAB}$.*

A direct implication of Comparison 1 is that revenues cannot be generically compared across the two auction formats.

Comparison 4 (Ambiguous revenue)**.** *Both ex post and expected revenues are generically incomparable across auction formats.*

Minimax-loss bids do not depend on the distribution of opponent values, which is necessary to compute expected revenue. If the distribution places significant probability on each bidder demanding exactly one unit, the uniform-price auction may yield higher revenue; following Comparison 1, the cross-conditional regret minimizing bid in the uniform-price auction is higher than the unique minimax-loss bid in the pay-as-bid auction, and therefore the ex post transfer to the auctioneer can be higher in the uniform-price auction than in the pay-as-bid auction. Similarly, although the uniform-price bid for quantity Q may be above the pay-as-bid bid for quantity Q, price discrimination in the pay-as-bid auction will yield a higher transfer to the auctioneer than the uniform payment given cross-conditional regret minimizing bids. In total, when the distribution places significant probability on bidders having zero value, with small probability on demanding the entire market, the pay-as-bid auction will yield higher revenue; and, when the distribution places significant probability on bidders having comparable values, the uniform-price auction may yield higher revenue.

A final comparison relates to the levels of minimax loss. The following comparison proves that minimax loss is higher in the pay-as-bid auction than in the uniform-price auction. This holds for any bidding language we consider.

Comparison 5 (Minimax loss). *Optimal loss is lower in the uniform-price than in the pay-as-bid auction, i.e.,*

$$\sup_{B^{-i}\in\mathcal{B}} L^{UPA}(b^{UPA}; B^{-i}, v^i) \leq \sup_{B^{-i}\in\mathcal{B}} L^{PAB}(b^{PAB}; B^{-i}, v^i).$$

This comparison is strict except in the multi-unit case with a single unit, $M_q = 1$.

What does it mean that a mechanism has lower minimax loss than another? Suppose a bidder can obtain costly information about the other bidders' behavior; this information shrinks the set of possible bid distributions $\mathcal{B}$. The bidder stops with the information acquisition if the level of minimax loss is below some threshold (satisficing) or if the information is too costly. The theorem implies that there are threshold levels in the first type of behavior that obtain information in the pay-as-bid auction but not in the uniform-price auction. Instead of information acquisition we can think of the cost of adding another bid point. We would then expect to see fewer bid points in the uniform-price auction through the extensive margin.

Our algorithm for computing minimax-loss bids in the uniform-price auction with constrained bids implies that reducing loss in the uniform-price auction is no more difficult than in the pay-as-bid auction. In the multi-unit setting, finding an optimal bid in either auction format requires solving a single-variable equation at each unit.[24] In the constrained-bid setting, the bidder can use the minimax loss from the pay-as-bid auction to compute upper and lower iso-loss curves. Our bid-computation algorithm can be efficiently applied to obtain a bid in the uniform-price auction with loss no greater than that in the pay-as-bid auction, hence it is not substantially more difficult to obtain a level of loss, feasible in the pay-as-bid auction, in the uniform-price auction. Comparison 5 implies that further reduction of loss will typically be possible, should the bidder find it worthwhile.

[24]In the pay-as-bid auction this solution must be found recursively, working backward from the final unit. In the uniform-price auction the solution can be found in parallel.

4 Applications

We now illustrate the results of the previous section. We first represent the optimal multi-unit bidding functions in closed form when the bidder has demand for two units or when the marginal values are (relatively) flat. At the end of the section, we study constrained bidding functions with few permitted bid points.

4.1 Multi-unit auctions

4.1.1 Demand for two units

Suppose that bidder i demands two units in a multi-unit auction. We model this as a bidder who can submit two bids (b_{i1}, b_{i2}) at bid points $q_1 = Q/2$ and $q_2 = Q$.

Example 1 (Two-unit demand in pay-as-bid). In the pay-as-bid auction with demand for two units, the minimax-loss bid vector b_i^{PAB} is such that

$$b_{i1}^{\text{PAB}} = \begin{cases} \frac{1}{9}\left(3v_{i1} + 2v_{i2}\right) & \text{if } 7v_{i2} \geq 3v_{i1}, \\ \frac{1}{6}\left(3v_{i1} - v_{i2}\right) & \text{if } 7v_{i2} < 3v_{i1}; \end{cases} \quad \text{and } b_{i2}^{\text{PAB}} = \frac{v_{i2}}{3}.$$

These bids follow directly from Theorem 1. A direct derivation balances three types of conditional regret: underbidding regret conditional on losing the auction (winning zero units), overbidding regret conditional on winning one unit, and overbidding regret conditional on winning two units. First, overbidding regret conditional on winning two units is $b_{i1} + b_{i2}$; bidder i could also have won two units by bidding almost 0 for both units. Second, overbidding regret conditional on winning one unit is $b_{i1} + v_{i2} - 2b_{i2}$; the bidder could have won both units (and not just one) by bidding marginally above b_{i2}. Finally, underbidding regret conditional on losing the auction is $v_{i1} - b_{i1} + (v_{i2} - b_{i1})_+$; both units could be won at a bid marginally above b_{i1}, the value v_{i2} needs to be sufficiently high so that bidder i actually wants to win two units at this price.

Equalizing the first two expressions leads to $b_{i2} = v_{i2}/3$. Equalizing the third with the first leads to the formula for b_{i1}. The case distinction is due to the value for the second good being below or above the bid for the first. If the second marginal value is sufficiently high, then the potential nonlinearities are irrelevant.

Figure 3 illustrates the bidding functions as a function of v_{i2}, $v_{i2} \in [0, 1]$. If $v_{i2} = 0$, then the minimax bid is $b_{i1}^{\text{PAB}} = 1/2$, which is as in the first-price auction for a single good [Kasberger and Schlag, 2020]. The bid b_{i1}^{PAB} decreases in v_{i2} for $v_{i2} \leq 3/7$. Marginally raising the value for the second unit decreases the spread between the two bids. For values above

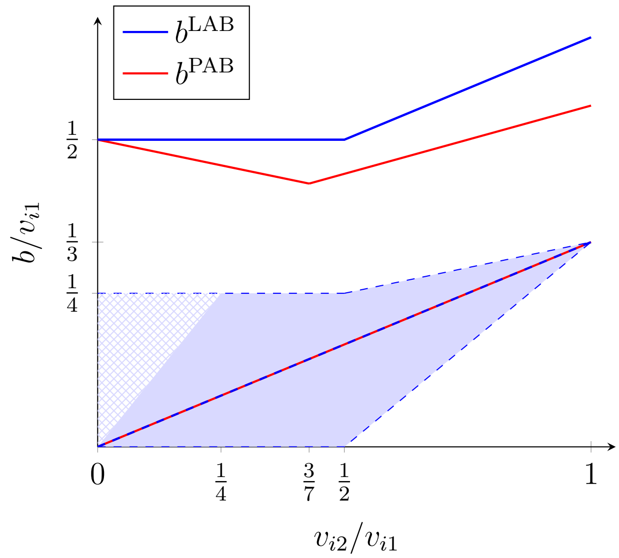


Figure 3: First- and second-unit bids in the pay-as-bid and uniform-price auctions, when the bidder demands two units.

3/7, both bids increase in v_{i2}. The second bid b_{i2}^{PAB} increases stronger in v_{i2} than the first bid so that the spread between the two bid points decreases in v_{i2} over the entire domain.

Example 2 (Two-unit demand in uniform-price)*.* In the last accepted bid uniform-price auction with demand for two units, the cross-conditional regret minimizing bid vector b_i^{LAB} is such that

$$b_{i1}^{\text{LAB}} = \begin{cases} \frac{1}{3}\left(v_{i1}+v_{i2}\right) & \text{if } v_{i1} \leq 2v_{i2}, \\ \frac{1}{2}v_{i1} & \text{otherwise;} \end{cases} \quad \text{and } b_{i2}^{\text{LAB}} = \frac{v_{i2}}{3}.$$

This follows immediately from Theorem 4. The first bid can be found by equalizing the underbidding regret conditional on losing the auction $v_{i1} - b_{i1} + (v_{i2} - b_{i1})_+$ with the overbidding regret conditional on winning one unit b_{i1}. The case distinction is due to the value for the second unit being below or above the bid for the first unit. The second bid can be found by equalizing the underbidding regret conditional on winning one unit $v_{i2} - b_{i2}$ and the overbidding regret conditional on winning two units $2b_{i2}$.

While minimax-loss bids must minimize cross-conditional regret for some unit, this will not in general determine the minimax-loss bid for all units. With demand for two units, loss minimaximization uniquely determines the bid for the first unit, but the bid for the second unit need only lie within the bounds $v_{i2} - L^{\text{LAB}} \leq b_{i2} \leq L^{\text{LAB}}/2$, where L^{LAB} is minimax loss in the uniform-price auction. The range of feasible minimax-loss bids in the uniform-price auction is depicted in Figure 3.

4.1.2 Flat marginal values

For simplicity, in this section we assume that there are $M_q = Q$ units available. The implicit definitions of minimax-loss bids given in Section 3 are intractable because the range of summation (or integration) depends on the submitted bid. In particular, the integral $\int_q^Q (v^i(x) - p)_+ dx$ may have a nontrivial dependence on p. In this section we compute bids in closed form for the case in which $v^i(Q)$ is relatively high, so that $v^i(Q) \geq b(0)$ for all relevant bids. When marginal values are relatively flat—that is, when the value for the Q^{th} unit is relatively high—the bounds of integration are independent of the submitted bid, and minimax-loss bids may be computed in closed form.

Example 3 (Multi-unit pay-as-bid with flat marginal values). Suppose that marginal values are relatively flat, so that $v_{iQ} \geq (1 - 1/\exp(1))v_{i1}$. The optimal bid vector b_i^{PAB} is such that

$$b_{ik}^{\text{PAB}} = \frac{1}{Q+1} \sum_{k'=k}^{Q} \left(\frac{Q}{Q+1} \right)^{k'-k} v_{ik'}.$$

First, assume that the bounds of integration do not depend on b. Then equation (1) becomes

$$(Q+1)\, b_{ik}^{\text{PAB}} = v_{ik} + Q b_{ik+1}^{\text{PAB}} \iff b_{ik}^{\text{PAB}} = \frac{1}{Q+1} \left(v_{ik} + Q b_{ik+1}^{\text{PAB}} \right).$$

Expanding the right-hand b_{ik+1}^{PAB} recursively gives

$$b_{ik}^{\text{PAB}} = \frac{1}{Q+1} \sum_{k'=k}^{Q-1} \left(\frac{Q}{Q+1} \right)^{k'-k} v_{ik'} + \left(\frac{Q}{Q+1} \right)^{Q-k} b_{iQ}^{\text{PAB}}.$$

Substituting in for $b_{iQ}^{\text{PAB}} = v_{iQ}/(Q+1)$ gives the claimed expression for regret minimizing bids.

We now confirm that the bounds of integration are independent of the submitted bid. Note that $b_{i1}^{\text{PAB}} \leq \sum_{k'=1}^{Q} [Q/(Q+1)]^{k'} v_{i1}/Q$. For potential nonlinearities to be irrelevant, it must be that $b_{i1}^{\text{PAB}} \leq v_{iQ}$, which is implied by $Q v_{iQ} \geq \sum_{k'=1}^{Q} [Q/(Q+1)]^{k'} v_{i1}$. Straightforward computation gives

$$\sum_{k'=1}^{Q} \left(\frac{Q}{Q+1} \right)^{k'} = \left(1 - \left(\frac{Q}{Q+1} \right)^{Q} \right) Q.$$

Then potential nonlinearities are irrelevant if $(1 - [Q/(Q+1)]^Q) v_{i1} \leq v_{iQ}$. Since $[Q/(Q+1)]^Q$ converges to $1/\exp(1)$ from above, the stated condition is sufficient.

Example 4 (Multi-unit uniform-price with flat marginal values). Suppose that marginal values are relatively flat, so that $(Q+1)v_{iQ} \geq Qv_{i1}$. The conditionally regret-minimizing bid vector b_i^{LAB} is such that

$$b_{ik}^{\text{LAB}} = \frac{1}{Q+1} \sum_{k'=k}^{Q} v_{ik'}.$$

Bids are decreasing in quantity. If potential nonlinearities are irrelevant, then Theorem 4 implies the bids as stated. We then check

$$b_{i1}^{\text{LAB}} \leq v_{iQ} \quad \Longleftrightarrow \quad (Q+1)\, v_{iQ} \geq \sum_{k'=1}^{Q} v_{ik'}.$$

Since $v_{ik'} \leq v_{i1}$ for all k', algebraic rearrangement gives the stated condition.

The conditions defining flat marginal values in Examples 3 and 4 are algebraically simple and sufficient, but leave some room for tighter bounds. For the bounds of integration to be independent of the bid submitted, it must be that $b_{i1} \leq v_{iQ}$; this gives

$$b_{i1}^{\text{PAB}} = \frac{1}{Q+1} \sum_{k'=1}^{Q} \left(\frac{Q}{Q+1}\right)^{k'-1} v_{ik'} \leq v_{iQ}, \text{ and } b_{i1}^{\text{LAB}} = \frac{1}{Q+1} \sum_{k'=1}^{Q} v_{ik'} \leq v_{iQ}.$$

The condition on in the pay-as-bid auction is weaker, because $b_{i1}^{\text{PAB}} < b_{i1}^{\text{LAB}}$. In the uniform-price auction the bid b_{ik} is relevant to cross-conditional regret only for unit k, while in the discriminatory auction the bid for unit k is relevant when any unit $k' \geq k$ is won. This implies a downward pressure on bids in the discriminatory auction beyond that in the uniform-price auction. Because bids are initially higher in the uniform-price auction, a stronger condition is necessary to rule out quantity-dependent bounds of integration.

4.2 Constrained bids

In practice, bidders cannot submit a distinct point bid for each quantity demanded. We now consider the constrained auction where a bidder may submit at most M bids, but is free to choose the location of the bid points.

4.2.1 Constant marginal values

As a special case of marginal values, suppose that bidder i's marginal value is constant, $v^i(q) = v$ for all q.

Example 5 (Pay-as-bid with constant marginal values). The constrained loss optimization problem is

$$\min_{q',b'} \left(v - b_1'\right) Q, \text{ s.t. } \left(Q - q_{k-1}'\right)\left(v - b_k'\right) + \sum_{k'=1}^{k} \left(q_{k'}' - q_{k'-1}'\right)\left(b_{k'} - b_k\right) = \left(v - b_1'\right) Q.$$

Equating conditional loss across units requires $R_{k+1}^{\text{PAB}} - R_k^{\text{PAB}} = 0$, or

$$0 = -Qb_{k+2} - \left(q_{k+1} - q_k\right) v + \left(Q + \left(q_{k+1} - q_k\right)\right) b_{k+1}.$$

Solving this equation recursively, backwards from $b_{k+1} = 0$, gives a closed-form expression for optimal bids conditional on quantities,

$$b_k = \sum_{k'=k}^{M} \frac{Q^{k'-k}\left(q_{k'} - q_{k'-1}\right)}{\prod_{j=k}^{k'}\left[Q + \left(q_j - q_{j-1}\right)\right]} v.$$

Minimizing loss then implies

$$q_k = \frac{k}{M} Q, \text{ and } b_k = \frac{v}{M} \sum_{k'=k}^{M} \left[\frac{M}{M+1}\right]^{k'-k+1}.$$

Example 6 (Uniform-price with constant marginal values). Suppose that bidder i's marginal value v^i is constant, $v^i(q) = v$ for all q. The constrained loss optimization problem is

$$\min_{q',b'} b_1' q_1', \text{ s.t. } b_k' q_k' = \left(v - b_k'\right)\left(Q - q_k'\right) \quad \forall k.$$

The minimax-loss bid induces loss $C_M Q v$, and solves

$$q_0 = 0, \quad q_k = \left(C_M - \frac{C_M^2}{q_{k-1} - \left(1 - C_M\right)}\right) Q, \quad q_M = \left(1 - C_M\right) Q, \quad \text{and } b_k = \frac{C_M v}{q_k}.$$

The solution to this expression is unique: the recursive equation for q_k increases in C_M, while the endpoint condition for q_M decreases in C_M.

When the bidder is allowed a single bid point, $M = 1$, the unique minimax-loss bid solves

$$q_1 = \frac{C_1}{1 - C_1} Q, \text{ and } q_1 = \left(1 - C_1\right) Q \quad \Longleftrightarrow \quad q_1 = \left(\phi - 1\right) Q, \; b_1 = \left(\phi - 1\right) v,$$

where $\phi \approx 1.61803$ is the golden ratio. Regardless of M, the conditional regret-minimizing bid in the unconstrained model will solve $qb(q) = (v - b(q))(Q - q)$, hence $b(q) = (1 - q/Q)v$, and maximum loss under this (minimax-loss) bid is $vQ/4$. Since loss is higher when bids are

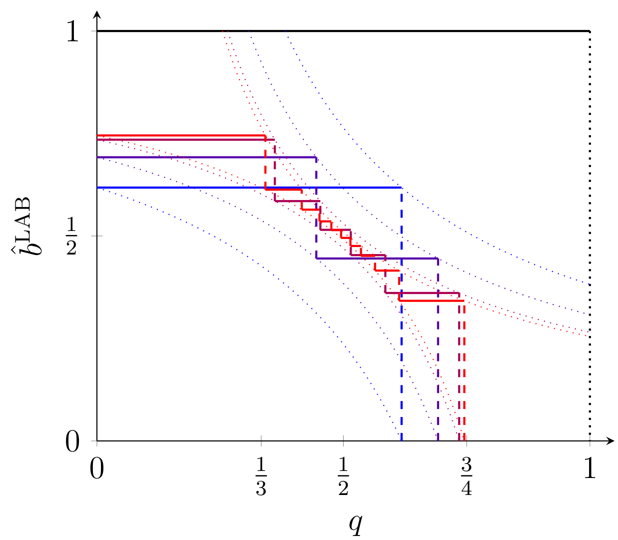


Figure 4: Minimax-loss bids in the constrained uniform-price auction with $Q = 1$, $v = 1$, and varying numbers of bid points.[25] Submitted bids lie on upper and lower iso-loss curves, and more-central iso-loss curves (a lower upper iso-loss curve and a higher lower iso-loss curve) correspond to lower loss.

constrained, it follows that $q_1 \geq Q/3$ and $q_M \leq 3Q/4$. That is, minimax-loss bid points are all for interior quantities.

4.2.2 A single bid point

As a special case of constrained bidding, suppose that bidders may submit at most a single bid point, $M = 1$. Note that this is not a single-unit auction as the bidders may choose the location q of the bid.

Example 7 (Pay-as-bid with a single bid point). Following Theorem 2, the constrained loss minimaximization problem is

$$\min_{q',b'} \int_0^Q \left(v^i\left(x\right) - b'\right) dx, \text{ s.t. } \int_0^Q \left(v^i\left(x\right) - b'\right)_+ dx = \int_0^{q'} b' dx + \int_{q'}^Q v^i\left(x\right) dx.$$

A standard Lagrangian analysis implies that $q_{i1} = v^{-1}(b_{i1})$, where b_{i1} is the unique solution to

$$\int_0^Q \left(v^i\left(x\right) - b_{i1}\right)_+ dx = b_{i1} v^{-1}\left(b_{i1}\right) + \int_{v^{-1}(b_{i1})}^Q v^i\left(x\right) dx.$$

The case of affine marginal values, $v^i(q;\alpha) = 1 - \alpha q$ with aggregate quantity $Q = 1$,

[25]The choice of Q and v is a normalization, as bid levels scale linearly with v and are constant in Q, and bid points scale linearly with Q and are constant in v.

is examined in Figure 5. Provided marginal values are not too steep—that is, when α is small—the bidder bids for the entire available quantity, $q_1^{\text{PAB}} = Q$. The bid level is equal to half the average marginal value on this interval, $b_1^{\text{PAB}} = \int_0^Q v^i(x)dx/2$. This corresponds directly to the results of Kasberger and Schlag [2020], where bidders in a first price auction bid half their values: because the bidder bids for the entire quantity and their value is the integral of their marginal value, the pay-as-bid auction is equivalent to a first price auction for a single unit.

Example 8 (Uniform-price with a single bid point). Following Theorem 5, the constrained loss minimaximization problem is

$$\min_{q',b'} b'q', \text{ s.t. } b'q' \geq \int_{q'}^{Q} v^i\left(x\right)dx \text{ and } b'q' \geq \int_0^Q \left(v^i\left(x\right) - b'\right)_+ dx.$$

Note that both constraints must bind. If only the first constraint bids, there is a trivial solution $(q', b') = (Q, 0)$ and the second constraint is violated. If only the second constraint bids, there is the trivial solution $(q', b') = (0, v^i(0))$ and the first constraint is violated. Since both constraints bind and there are two degrees of freedom, it follows that the minimax-loss bid is (q', b'), where

$$\int_0^Q \left(v^i\left(x\right) - \frac{1}{q'}\int_{q'}^Q v^i\left(y\right)dy\right)_+ dx = \int_{q'}^Q v^i\left(x\right)dx, \text{ and } b' = \frac{1}{q'}\int_{q'}^Q v^i\left(x\right)dx.$$

The solution is unique, since the quantity q' uniquely determines the bid b' and the implicit equation for quantity has a unique solution (the left-hand side is strictly increasing in q' while the right-hand side is decreasing).

Bids have ambiguous comparative statics with regard to changes in value. Higher marginal values lead unambiguously to higher bids. But, for example, increasing marginal values above q' but leaving them unaffected below q' will cause q' to rise, while increasing marginal values below q' but leaving them unaffected above q' will cause q' to fall. The effect of a general change of marginal values will depend on the relative increase of marginal values for low quantities and high quantities. The specific case of affine marginal values, $v^i(q;\alpha) = 1 - \alpha q$ with aggregate quantity $Q = 1$, is examined in Figure 5.

4.2.3 Simulation

Example 9. We simulate auction outcomes for different choices of the number of allowed bid points M. In the simulated auctions there are 100 divisible units for sale ($Q = 100$), and the number of bidders varies from $n = 2$ to $n = 10$. Bidders' marginal values are constant,

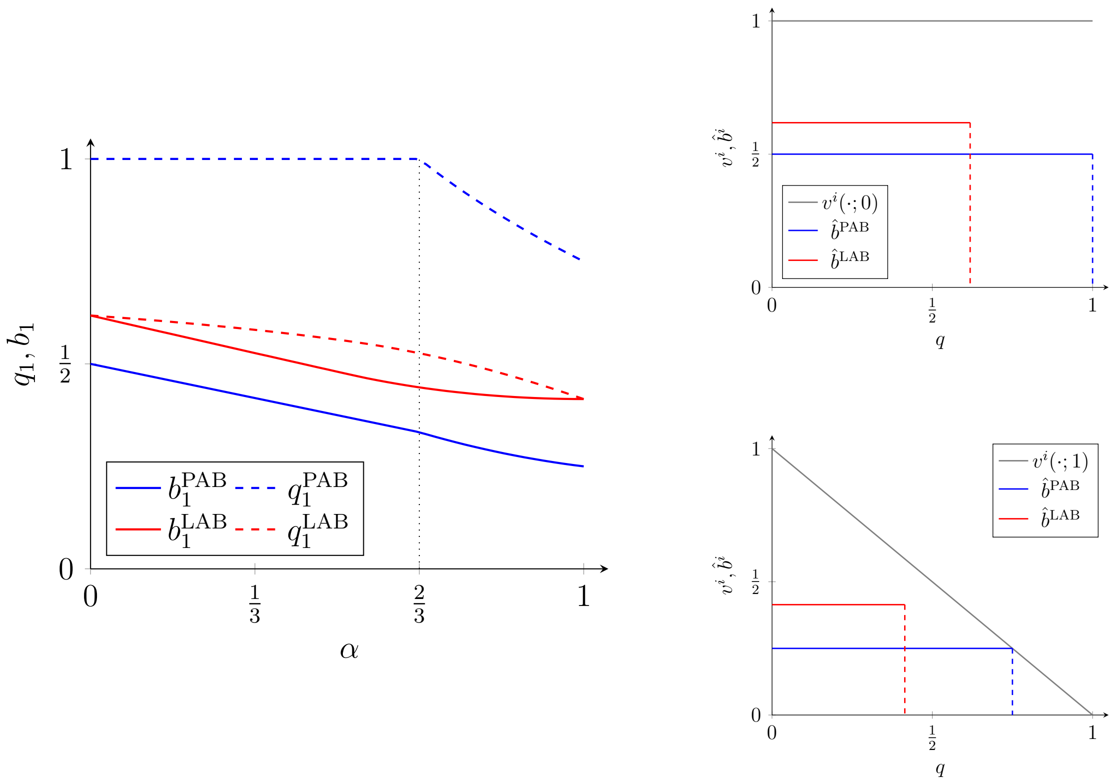


Figure 5: Bid points and levels when bidders may submit a single bid point (left panel), and bids for $\alpha = 0$ and $\alpha = 1$ (right panel).

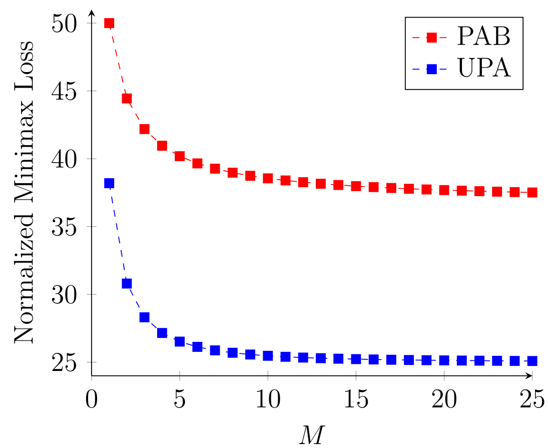


Figure 6: Normalized constrained minimax loss as a function of the number of bid points M

$v(q) = v_0$, where v_0 follows a truncated lognormal distribution with support $v_0 \in [0.5, 2]$ and mean 1. For each number of allowed bid points, M, we first compute constrained minimax-loss bids in both the pay-as-bid and uniform-price auctions. In the pay-as-bid auction bids are obtained from the expressions in Example 5; in the uniform-price auction bids are obtained from the simple search procedure outlined in Section 3.2.2. After simulating bids we compute revenue, welfare, and loss.

Figure 6 reports the normalized minimax loss, computed as realized loss divided by the constant marginal value v_{0i}. As predicted by Comparison 5, the level of minimax loss is higher in the pay-as-bid auction. The figure also shows decreasing gains from adding another bid point and a relatively fast convergence to the unconstrained level of minimax loss. Indeed, minimax loss with four bid points is less than 10% higher than with 25 bid points in both auctions formats.

Figure 7 plots average auction revenue as a function of the number of bid points M. The plot nicely illustrates that increasing the number of bidders increases the seller's expected revenue. This is expected as the highest value of n independent draws increases in n in expectation. In general, revenue is ambiguous in the auction format and the number of bid points M. As observed in Examples 5 and 6, bidders in a pay-as-bid auction with a single bid point will bid half their value for the full market quantity, and bidders in a uniform-price auction with a single bid point will bid more than half their value for less than the full market quantity. Revenue in the pay-as-bid auction is therefore half the highest marginal value, while revenue in the uniform-price auction is more than half the second-highest marginal value. It follows that expected revenue will be higher in the pay-as-bid auction when the both the number of bid points and the number of bidders are small.

Although average revenues may be ranked, reverse rankings can be observed ex post.

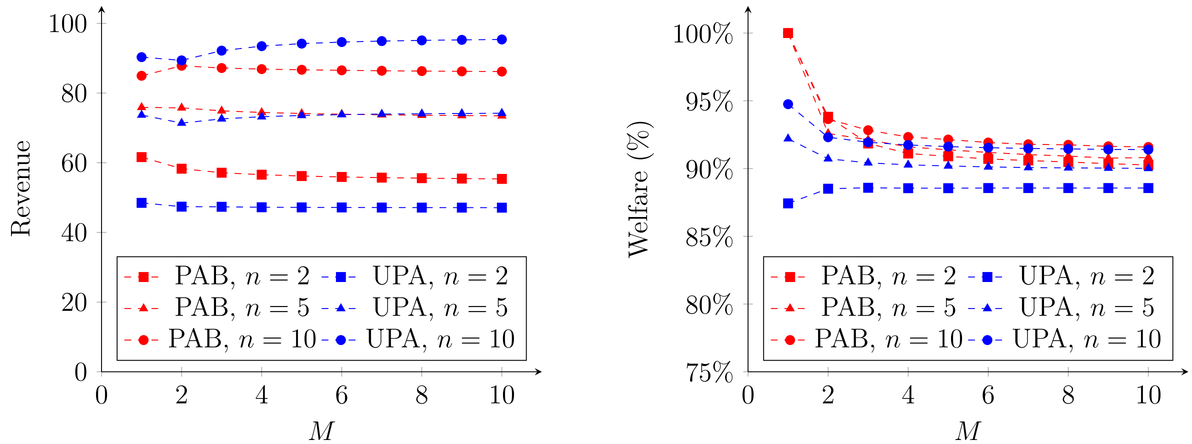


Figure 7: Average revenue (left) and welfare (right) as a function of number of bid points M.

Figure 8 compares ex post revenues and depicts the share of simulated auctions in which uniform-price revenue was higher than pay-as-bid revenue. As the number of bidders increases, the share of auctions in which revenue is higher in the uniform-price auction increases. Low-revenue outcomes mainly appear in uniform-price auctions with two bidders, and these "collusive" outcomes are less likely when there are many bidders. The uniform-price auction dominates the pay-as-bid auction with ten bidders in terms of revenue in expectation and ex post in the majority of auctions. The ambiguous revenue ranking is in line with empirical results on multi-unit auctions.[26]

Figure 7 also shows average welfare, normalized by maximum attainable welfare, as a function of the number of bid points M. Welfare tends to be higher in the pay-as-bid auction. The pay-as-bid auction with $M = 1$ is efficient as it is strategically equivalent to a first-price auction with monotone strategies. Both auction formats tend to become less efficient as the number of bid points increases: more bid points allow finer control over bid shading, increasing the chance that low-value bidders win small quantities that would be more-efficiently allocated to high-value bidders. Welfare tends to increase in the number of bidder n as the distance between the highest and second-highest value decreases in n. As with our simulated revenue comparisons, ex post outcomes can be different than expected welfare rankings (provided more than a single bid point). Figure 8 also shows the share of uniform-price auctions in which welfare is higher than in the corresponding pay-as-bid auction. For most combinations of bid points and number of bidders, welfare is higher in the pay-as-bid auction in more than 50% of the auctions. The share of more efficient uniform-price auctions tends to increase in the number of bidders: inefficiency results from

[26]See Pycia and Woodward [2021] for a summary of the ambiguous revenue rankings obtained in the empirical literature.

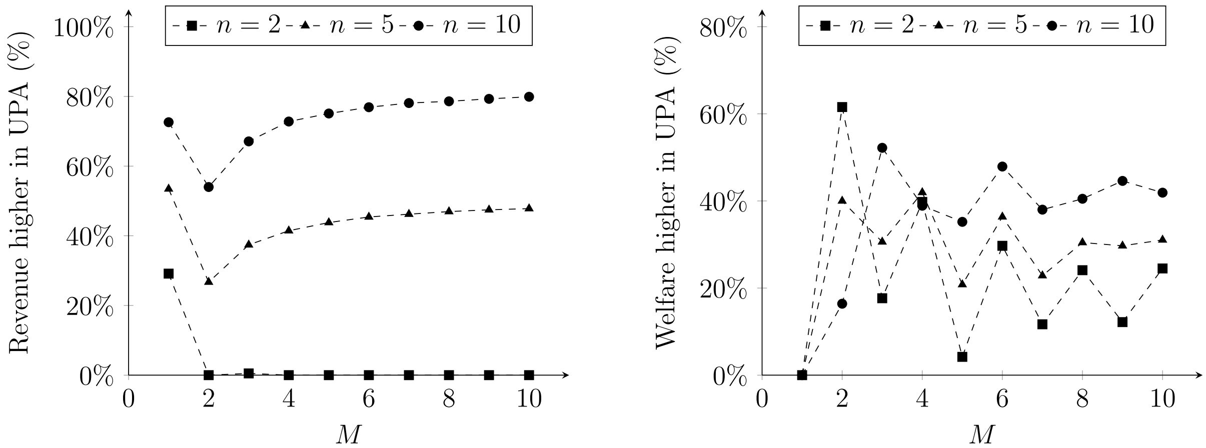


Figure 8: Ex post comparison of auctions

bid shading, but as the number of bidders increases typical allocations become small and are focused on intervals on which bids are ordered the same as bidders' values.

5 Conclusion

The pay-as-bid and the uniform-price auction are two leading auction formats for allocating homogeneous goods such as electricity and government debt. In this paper we characterize optimal prior-free bids for these auction formats. In particular, we consider optimal unconstrained bids and optimal bids when the bidder is constrained to a few bid steps as in many practical applications. The two pricing rules create different incentives for the bidders; our analysis shows that the worst-case loss approach teases their implications out in a tractable way. Remarkably, the analysis is tractable even with multi-dimensional private information because we do not require the inversion of strategies as in the Bayes-Nash equilibrium approach. Hence, the worst-case loss approach may also be fruitfully applied to other complex strategic interactions.

The bids are derived under maximal uncertainty, which makes them robust to misspecified perceptions about the faced bidding environment. The bids can be readily put into practice; they do not have any free parameters. It would be interesting to assess their empirical performance. Another interesting avenue for research is to explore bidding with beliefs about opponent behavior that are not as extreme as maximal uncertainty or Bayes-Nash equilibrium.

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A Uniform-price auctions with a first rejected bid price

The pricing rule of the uniform-price auction impacts the strategic incentives if the number of auctioned units is discrete—our multi-unit case. We thus analyze the uniform-price auction with the first rejected bid pricing rule in the multi-unit setting in this appendix. For notational simplicity, we assume that Q is a positive integer and that bidders may demand each unit, so $M = Q$.

In the uniform-price auction with the first rejected bid pricing rule, winning bids can never be too high as they do not determine the market-clearing price. In particular, the bid for the first unit, b_{i1}, can be too high only if it is above value; that is, overbidding regret for the first unit corresponds to underbidding regret for all units $k > 1$. Conditional on bidder i winning k units, $k \geq 1$, the only bid that may be too high is bidder i's first rejected bid b_{ik+1}. For this case, we define *overbidding regret* as

$$\overline{R}_k^{\mathrm{FRB}}(b_i; v_i) = k b_{ik+1}.$$

This is the additional utility the bidder could have received if they reduced their bid b_{ik+1} to zero. The case occurs if the other bidders bid a strictly positive amount for only the $Q - k$ units they received. The overbidding regret of winning 0 units is 0. Similar to the other pricing rules, bids are too low if one wants to win more units given the market-clearing price. Maximal regret arises if the opponents all bid just above b_{ik+1} for the units they received. The resulting *underbidding regret* is

$$\underline{R}_k^{\mathrm{FRB}}(b_i; v_i) = \sup_{p \in [b_{ik+1}, b_{ik}]} \sum_{k'=k+1}^{Q} (v_{ik'} - p)_+ = \sum_{k'=k+1}^{Q} (v_{ik'} - b_{ik+1})_+ .$$

This is the additional utility the bidder could have received if they bid just above b_{ik+1} for all units for which it is profitable to do so. Note that these regret terms correspond to those in the pay-as-bid auction, except that underbidding regret in the uniform-price auction does not consider bids for submarginal quantities.

The conditional regret for unit k, $k \in \{0, 1, \ldots, Q-1\}$, is the maximum of overbidding and underbidding regret,

$$R_k^{\text{FRB}}(b_i; v_i) = \max\left\{\overline{R}_k^{\text{FRB}}(b_i; v_i), \underline{R}_k^{\text{FRB}}(b_i; v_i)\right\}.$$

Observation 4. *Note that if $b_{ik} > v_{ik}$, then overbidding regret is $(k-1)b_{ik} > 0$ and underbidding regret is $\sum_{k'=k}^{Q}(v_{ik'} - b_{ik})_+ = 0$. That is, bidding above value equates overbidding and underbidding regret only when $k = 1$. Since the minimax-loss bid must equate overbidding and underbidding regret for some unit (see Lemma 2 below), there is a minimax-loss bid that is weakly below the bidder's value vector.*

Lemma 2 (Maximal loss in first rejected bid uniform-price auction)**.** *In the first rejected bid uniform-price auction, the maximal loss given bid $b_i \le v_i$ is*

$$\max_{k \in \{1, \ldots, M\}} \left[\max\left\{(k-1) b_{ik}, \sum_{k'=k}^{M} (v_{ik'} - b_{ik})_+\right\}\right].$$

Proof. We first consider the augmented problem in which the bidder receives q units at market price $p^\star \in [b_{iq+1}, b_{iq}]$. In a uniform-price auction, a bidder facing a known residual supply curve should pick a point on the supply curve to maximize their own utility. The bidder's utility from this optimiziation increases as the residual supply curve falls, hence the loss-maximizing supply curve must be as low as possible. When receiving q units, the bidder knows that either their opponents demanded $Q - q$ units with bids weakly above $p^\star$ and the $Q - q + 1^{\text{th}}$ unit at $p^\star$, or their opponents demanded $Q - q$ units with bids weakly above $p^\star$ and the market-clearing price is $p^\star = b_{iq+1}$. The loss-maximizing supply curve S is given by

$$S(q'; p^\star) = \begin{cases} p^\star & \text{if } q' \in \{1, \ldots, Q-q\}, \\ p^\star & \text{if } q' = Q - q + 1 \text{ and } p^\star > b_{iq+1}, \\ 0 & \text{if } q' = Q - q + 1 \text{ and } p^\star = b_{iq+1}, \\ 0 & \text{if } q' > Q - q + 1. \end{cases}$$

The loss problem is then

$$\max_{\tilde{q}} \sum_{k=1}^{\tilde{q}} (v_{ik} - S(Q - \tilde{q} + 1; p^\star)) - \sum_{k=1}^{q} (v_{ik} - p^\star).$$

Note that $S(Q-\tilde{q})$ is increasing and locally constant in $\tilde{q}$; then loss is obtained at $q' \in \{q-1, q, v^{-1}(p^\star)\}$, where $v^{-1}(p^\star) = \max\{q \colon v_{iq} > p^\star\}$. It follows that loss, conditional on market-clearing price $p^\star$, is

$$R_q^{\mathrm{FRB}}\left(b^i; v^i\right) = \max\left\{(q-1)\, p^\star - (v_{iq} - p^\star)\,, q b_{iq+1}, \sum_{k=q+1}^{Q} (v_{ik} - p^\star)_+\right\}.$$

By construction, $p^\star \leq b_{iq}$; then

$$(q-1)\, p^\star - (v_{iq} - p^\star) \leq q b_{iq} - v_{iq} \leq (q-1)\, b_{iq}.$$

The right-hand inequality follows by the assumption that $b_i \leq v_i$. Then the leftmost in the maximization expression for R_q^{FRB} is bounded above by the middle term in R_{q-1}^{FRB}, and hence

$$\max_q R_q^{\mathrm{FRB}}\left(b_i; v_i\right) = \max_k \left[\max\left\{(q-1)\, b_{iq}, \sum_{q'=q}^{Q} (v_{iq'} - b_{iq})_+\right\}\right].$$

□

Similar to the analysis of cross-conditional regret minimizing bids in the last accepted bid uniform-price auction, note that $\overline{R}_k^{\mathrm{FRB}}$ is increasing in b_{ik+1} while $\underline{R}_k^{\mathrm{FRB}}$ is decreasing in b_{ik+1}, and both terms are independent of $b_{ik'}$ for $k' \neq k+1$. Then if maximum loss is determined by conditional regret for unit k, it must be that $\overline{R}_k^{\mathrm{FRB}}(b_i; v_i) = \underline{R}_k^{\mathrm{FRB}}(b_i; v_i)$. There is, however, no unique optimal bid.

Theorem 7 (No unique minimax-loss bid)**.** *If $Q > 1$, then there is not a unique minimax-loss bid in the uniform-price auction with the first rejected bid pricing rule.*

Proof. It is sufficient to consider b_{i1}. When the bidder receives 0 units, overbidding regret is 0 and underbidding regret $\underline{R}_0^{\mathrm{FRB}}$ is non-negative but arbitrarily close to 0 when b_{i1} is close to v_{i1}. As overall minimax loss L^{FRB} is strictly positive, any choice of b_{i1} such that $\max\{b_{i2}, v_{i1} - L^{\mathrm{FRB}}\} \leq b_{i1} \leq v_{i1}$ minimaxes loss. □

In the specific case of a single unit, $Q = 1$, Lemma 2 gives maximum loss as

$$\max\left\{0, (v_{i1} - b_{i1})_+\right\}.$$

Then the unique minimax-loss bid is $b_{i1} = v_{i1}$.

In light of Theorem 7, minimax-loss bids are not uniquely defined in the uniform-price auction for quantity $Q > 1$. To obtain sharp predictions for minimax-loss bids, we define a conditional regret minimizing strategy as one which minimizes conditional regret for each unit. Because conditional regret is independent across units, and regret is minimized by conditional regret for some unit, a conditional regret minimizing strategy is a regret minimizing strategy.

Definition 2. *The bid vector b_i is* conditionally regret minimizing *if $b_i \in \operatorname{argmin}_{b'} R_k^{FRB}(b'; v_i)$ for all units $k \in \{0, 1 \ldots, Q-1\}$.*

The following theorem characterizes the unique conditional regret-minimizing bid vector.

Theorem 8 (Conditional regret-minimizing bids)**.** *The unique conditional regret-minimizing bid vector b_i^{FRB} is such that $b_{i1}^{FRB} = v_{i1}$ and for all $k \in \{1, \ldots, Q-1\}$,*

$$b_{ik+1}^{FRB} = \frac{1}{k} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik+1}^{FRB}\right)_+ .$$

Proof. The claim follows immediately from earlier arguments. It remains to be shown that b_i^{FRB} is a valid bid (that is, monotone). Suppose to the contrary that there is k such that $b_{ik}^{\mathrm{FRB}} < b_{ik+1}^{\mathrm{FRB}}$. Then

$$\begin{aligned} b_{ik}^{\mathrm{FRB}} &= \frac{1}{k-1} \sum_{k'=k}^{Q} \left(v_{ik'} - b_{ik}^{\mathrm{FRB}}\right)_+ \geq \frac{1}{k-1} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik}^{\mathrm{FRB}}\right)_+ \\ &\geq \frac{1}{k} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik}^{\mathrm{FRB}}\right)_+ \geq \frac{1}{k} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik+1}^{\mathrm{FRB}}\right)_+ = b_{ik+1}^{\mathrm{FRB}}. \end{aligned}$$

This is a contradiction, so it cannot be that $b_{ik}^{\mathrm{FRB}} < b_{ik+1}^{\mathrm{FRB}}$. □

Similar to minimax-loss bids in the auction formats analyzed in the main text, conditional regret minimizing strategies in the first rejected bid uniform-price auction are straightforward to compute but potentially infeasible to represent in closed form. In particular, determination of b_{ik}^{FRB} still faces issues of potential nonlinearities in $\underline{R}_k^{\mathrm{FRB}}(\cdot; v_i)$. We consider two examples.

Example 10 (Two-unit demand in first rejected bid uniform-price). In the first rejected bid price auction with demand for two units, the conditional regret minimizing bid vector $b_i^\star$ is such that

$$b_{i1}^{\mathrm{FRB}} = v_{i1}; \quad \text{and } b_{i2}^{\mathrm{FRB}} = \frac{v_{i2}}{2}.$$

This follows immediately from Theorem 8. The first bid b_{i1} cannot be too high, provided that it is below value. Thus, the overbidding regret conditional on losing is 0. The bid is too low if one could win more units by marginally raising it, leading to a worst-case regret conditional on losing the auction of $v_{i1} - b_{i1} + (v_{i2} - b_{i1})_+$. These two types of conditional regret are equalized by bidding value on the first unit. The bid b_{i2} is found by equalizing the overbidding regret conditional on winning one unit b_{i2} with the underbidding regret conditional on winning one unit $v_{i2} - b_{i2}$.

Example 11 (Multi-unit FRB with flat marginal values). Suppose that marginal values are relatively flat, so that $(Q-1)v_{iQ} \geq (Q-2)v_{i2}$. The conditionally regret-minimizing bid vector b_i^{FRB} is such that

$$b_{i1}^{\text{FRB}} = v_{i1}, \text{ and } b_{ik}^{\text{FRB}} = \frac{1}{Q}\sum_{k'=k}^{Q} v_{ik'}.$$

Bids are decreasing in quantity. Then following Theorem 8, potential nonlinearities are irrelevant when $b_{i2}^{\text{FRB}} \leq v_{iQ}$. When this is true, bids are as given as stated. We then check

$$b_{i2}^{\text{FRB}} \leq v_{iQ} \iff (Q-1)\, v_{iQ} \geq \sum_{k'=2}^{Q-1} v_{ik'}.$$

Since $v_{ik'} \leq v_{i2}$ for all $k' \geq 2$, the condition in the proposition is immediate.

B Proofs for Section 3

B.1 Pay-as-bid

Lemma 3 (Maximal loss in pay-as-bid)**.** *In the pay-as-bid auction, maximal loss given bid* b_i *is*

$$\sup_q \left[\max\left\{\int_0^q b^i(x)dx, \int_0^q \left(b^i(x) - b_+^i(q)\right)_+ dx + \int_q^Q \left(v^i(x) - b_+^i(q)\right)_+ dx\right\}\right].$$

Proof. In a pay-as-bid auction, a bidder facing a known residual supply curve should bid a constant amount for all units they desire: because bids are paid, a bid above the resulting market-clearing price can be reduced to save payment without affecting allocation. Since maximizing loss is equivalent to finding an ex post residual supply curve that maximizes regret, the loss-maximization problem is equivalent to solving

$$R_q\left(b^i; v^i\right) \approx \sup_{S\colon q^i(b^i,S)=q} U^\star\left(S; v^i\right) = \sup_{S\colon q^i(b^i,S)=q}\ \sup_{\tilde{q}} \int_0^{\tilde{q}} v^i(x) - S\left(Q-\tilde{q}\right) dx.$$ [27]

[27] To beat the opponent bid for unit $Q-q$ with certainty, bidder i must bid strictly above $S(Q-q)$, or $b_i(q) = S(Q-q)+\varepsilon$ for any $\varepsilon > 0$. Since regret is defined by a supremum, we let $\varepsilon = 0$ while retaining the assumption that bidder i wins unit q for sure.

Note that $U^\star$ is decreasing in S. Let $\tilde{S} < S$, and consider

$$q^\star \in \arg\sup_{\tilde{q}} \int_0^{\tilde{q}} v^i(x) - S\left(Q - \tilde{q}\right), \text{ and } \tilde{q}^\star \in \arg\sup_{\tilde{q}} \int_0^{\tilde{q}} v^i(x) - \tilde{S}\left(Q - \tilde{q}\right).$$

If $q^\star = \tilde{q}^\star$, then $U^\star(S; v^i) \leq U^\star(\tilde{S}; v^i)$ since the required bid under $\tilde{S}$ is lower than the required bid under S. If $q^\star \neq \tilde{q}^\star$, then $U^\star(S; v^i) \leq U^\star(\tilde{S}; v^i)$ since loss is higher under $\tilde{S}$ with selected quantity $\tilde{q}^\star$ than with selected quantity $q^\star$, and the required bid is lower even with selected quantity $q^\star$.

Then when considering maximum loss, it is sufficient to consider residual supply curves which are as low as possible. Conditional on bidder i receiving share q, the only constraint on the residual supply curve is $S(Q - q) \geq b^i_+(q)$; that is, bidder i's opponents bid more for their aggregate $Q - q$ unit than bidder i bid for their "next" unit. Because bids are monotone, the lowest residual supply curve satisfying this constraint is

$$S_q\left(\tilde{q}; b^i i\right) = \begin{cases} 0 & \text{if } \tilde{q} < Q - q, \\ b^i_+(q) & \text{if } \tilde{q} \geq Q - q. \end{cases}$$

Given this residual supply curve, bidder i's optimal bid will either win q units at a price of 0, or will win as many units as desired at a price of $b^i_+(q)$. In light of Lemma 1, which shows that maximum loss is equivalent to maximum regret, the result follows from evaluating the ex post utility of this decision. □

B.1.1 Multiple units

Proof of Theorem 1. We show that $\underline{R}_k^{\text{PAB}}(b_i; v_i) = \sum_{k'=1}^{Q} b_{ik'}$ for all k. First, since $\sum_{k'=1}^{k} b_{ik'}$ is increasing in k, Lemma 3 implies that maximum loss is

$$\max\left\{\max_k \underline{R}_k^{\text{PAB}}\left(b_i; v_i\right), \sum_{k=1}^{Q} b_{ik}\right\}.$$

Importantly, loss is continuous in bid. Note that increasing all bids by $\varepsilon > 0$ will weakly decrease $R_k(b_i; v_i)$ for all k and strictly increase $\sum_{k=1}^{Q} b_{ik}$; then if b_i is loss-minimizing, it must be that $\sum_{k=1}^{Q} b_{ik} \geq \max_k \underline{R}_k^{\text{PAB}}(b_i; v_i)$. Similarly, decreasing all bids by $\varepsilon > 0$ strictly decreases $\sum_{k'=1}^{Q} b_{ik'}$ and continuously affects $\underline{R}_k^{\text{PAB}}(b_i; v_i)$, thus $\sum_{k=1}^{Q} b_{ik} = \max_k \underline{R}_k^{\text{PAB}}(b_i; v_i)$.[28]

[28] A bid vector which is not strictly positive—i.e., for which there exists k with $b_{ik} = 0$—cannot be uniformly decreased by ε. Nonetheless, decreasing the bid by ε where possible will decrease $\sum_{k=1}^{Q} b_{ik}$ and will continuously affect $\max_k \underline{R}_k^{\text{PAB}}(b_i; v_i)$.

Now, suppose that there is k with $\underline{R}_k^{\text{PAB}}(b_i;v_i) < \sum_{k'=1}^{Q} b_{ik'}$. If $b_{ik+1}=0$, then

$$\underline{R}_k^{\text{PAB}}(b_i;v_i) = \sum_{k'=1}^{k}(b_{ik'}-b_{ik+1}) + \sum_{k'=k+1}^{Q}(v_{ik'}-b_{ik+1})_+ = \sum_{k'=1}^{k} b_{ik'} + \sum_{k'=k+1}^{Q} v_{ik'} \geq \sum_{k'=1}^{Q} b_{ik'}.$$

This is a contradiction, and it must be that $b_{ik+1}>0$. In this case, reducing b_{ik+1} will weakly increase $\underline{R}_k^{\text{PAB}}(b_i;v_i)$, strictly decrease $\underline{R}_{k'}^{\text{PAB}}(b_i;v_i)$ for all $k'>k$, and will not affect $\underline{R}_{k'}^{\text{PAB}}(b_i;v_i)$ for $k'<k$; reducing b_{ik+1} also reduces $\sum_{k'=1}^{Q} b_{ik'}$, and the arguments above show that increasing all bids by some small amount will strictly reduce loss.

It follows that $\underline{R}_k^{\text{PAB}}(b_i;v_i) = \sum_{k'=1}^{Q} b_{ik'}$ for all k, and the result is immediate. □

Proof of Corollary 1. Following Theorem 1, conditional regret is equalized across all units. Then for all units k, $1\leq k\leq Q$,

$$\begin{aligned} & R_{k-1}^{\text{PAB}}(b_i;v_i) - R_k^{\text{PAB}}(b_i;v_i) = 0 \\ \iff \quad & [kb_{ik+1}-kb_{ik}] + (v_{ik}-b_{ik}) + \sum_{k'=k+1}^{Q}\left[(v_{ik'}-b_{ik})_+ - (v_{ik'}-b_{ik+1})_+\right] = 0. \end{aligned} \tag{3}$$

From this, it immediately follows that $b_{iQ}^{\text{PAB}} = v_{iQ}/(Q+1)$. Fixing b_{ik+1}^{PAB}, the left-hand side of (3) is strictly positive when $b_{ik}=b_{ik+1}^{\text{PAB}}$, strictly negative when $b_{ik}=v_{ik}$, and continuous and monotone in b_{ik}. Then there is a unique b_{ik} that solves equation (3) conditional on b_{ik+1}^{PAB}. □

B.1.2 Constrained bids

Proof of Theorem 2. This proof is substantially similar to proof of the equivalent result for the multi-unit pay-as-bid auction (Theorem 1). As in the proof of Theorem 1, Lemma 3 implies that the loss minimization problem is

$$(q^\star,b^\star) \in \operatorname*{argmin}_{(q',b')}\left[\max_{k\in\{0,1,\ldots,M\}}\left[\max\left\{\overline{R}_{q'_k}(b';v_i),\underline{R}_{q'_k}(b';v_i)\right\}\right]\right].$$

By definition, $\underline{R}_M^{\text{PAB}}(b;v_i) \geq \overline{R}_k^{\text{PAB}}(b;v_i)$ for all k. Then the loss optimization problem in the pay-as-bid auction can be written

$$(q^\star,b^\star) \in \operatorname*{argmin}_{(q',b')}\left[\max_{k\in\{0,1,\ldots,M\}}\underline{R}_k^{\text{PAB}}(b';v_i)\right].$$

Recall that

$$\underline{R}_k^{\text{PAB}}\left(b';v_i\right)=\int_0^{q_k}\left(\hat{b}'\left(x\right)-\hat{b}'\left(q_k\right)\right)dx+\int_{q_k}^{Q}\left(v^i\left(x\right)-\hat{b}'\left(q_k\right)\right)_+dx.$$

Note that $\underline{R}_k^{\text{PAB}}$ decreases as q_k increases while, for all $k'>k$, $\underline{R}_{k'}^{\text{PAB}}$ increases as q_k increases. It follows that if $(q^\star,b^\star)$ is optimal, then $\underline{R}_k^{\text{PAB}}(b^\star;v_i)=\underline{R}_{k'}^{\text{PAB}}(b^\star;v_i)$ for all k,k'. □

Proof of Proposition 1. Let (q,b) minimax loss in the constrained pay-as-bid auction. Loss is

$$\max_k\left[\max\left\{\int_0^{q_k}b\left(x\right)dx,\int_0^{q_k}\left(b\left(x\right)-b_+\left(q_k\right)\right)+\int_{q_k}^{Q}\left(v^i\left(x\right)-b_+\left(q_k\right)\right)+dx\right\}\right].$$

Now consider the set of bid points q', where

$$q'_k=\left\lfloor\frac{q_k}{Q}M_q\right\rfloor\frac{Q}{M_q}.$$

That is, q'_k is the feasible bid point nearest to (but below) q_k. Define the bid vector b' so that $b'_k=b(q_k)$. By construction, (q',b') is feasible in the multi-unit auction. Since (q,b) is optimal, loss is higher under (q',b'), and since $b'\le b$ the loss is bounded above by

$$\max_k\int_{q'_k}^{q_k}\left(v^i\left(x\right)-\hat{b}'_+\left(q_k\right)\right)_+dx\le\int_{q'_k}^{q_k}v^i\left(x\right)dx\le\int_0^{Q/M_q}v^i\left(x\right)dx.$$

Then there is a feasible bid in the multi-unit auction with loss no more than $\int_0^{Q/M_q}v^i\left(x\right)dx$ higher than the optimal bid in the constrained auction. □

B.1.3 Unconstrained bids

Proof of Proposition 2. Arguments in the main text (preceding the statement of Proposition 2) establish the basic differentiable form. It remains to establish the initial condition. Because $b^i(Q)\ge 0$ by constraint, it is sufficient to show that $b^i(Q)$ cannot be strictly positive. By the fundamental theorem of differential equations (the Picard–Lindelöf theorem), if there are solutions b^i and $\tilde{b}^i$ with $b^i(Q)=0<\tilde{b}^i(Q)$, then $b^i\le\tilde{b}^i$. The differential form ensures equal conditional regret for all units, and conditional regret for unit $q=Q$ under bid $\tilde{b}^i$ is $\int_0^Q\tilde{b}^i(x)dx>\int_0^Q b^i(x)dx$. Then maximum loss is lower under bid b^i than under bid $\tilde{b}^i$, and $\tilde{b}^i$ is not a minimax-loss bid. Then $b^i(Q)=0$ for any minimax-loss bid, and uniqueness follows from the fundamental theorem of differential equations. □

Proof of Proposition 3. Because $L^\star$ is minimax loss when bids are unconstrained, $L^\star\le L^{M_q}$ and $L^\star\le L^{M_b}$ for all M_q and M_b. Since maximum loss is continuous in bid and the minimax-loss bid $b^\star$ can be arbitrarily approximated by a step function (when the number of steps grows large), it follows that $\lim_{M_q\nearrow\infty} L^{M_q} = L^\star$ and $\lim_{M_b\nearrow\infty} L^{M_b} = L^\star$.

Now suppose that $|\hat{b}^{M_q} - b^\star|$ does not converge to 0 as M_q grows large. Then there is a $\varepsilon > 0$ such that for all $\bar{M}$, there is $M_q > \bar{M}$ with $|b^{M_q} - b^\star| > \varepsilon$. Let $\langle b^{M_q k}\rangle_{k=1}^{\infty}$ be a sequence of such minimax-loss bids, where $M_{qk} M_{qk'}$ whenever $k < k'$. Bids are decreasing in quantity, hence by Helly's selection theorem it is without loss of generality to assume that $\hat{b}^{M_{qk}} \to \tilde{b}^\star$ in the L_1 norm, and since minimax loss converges the maximum loss associated with bid $\tilde{b}^\star$ is $L^\star$, the maximum loss associated with bid $b^\star$. It follows that $\tilde{b}^\star$ is a minimax-loss bid in the unconstrained pay-as-bid auction. Since there is a unique minimax-loss bid in the pay-as-bid auction (Proposition 2), this contradicts the assumption that $\lim_{k\nearrow\infty} b^{M_{qk}} \neq b^\star$.

Showing that $|\hat{b}^{M_b} - b^\star|$ converges to 0 is essentially identical to the argument above, and is omitted. □

B.2 Uniform-price

Lemma 4 (Maximal loss in uniform-price)**.** *In the uniform-price auction, maximal loss given bid b_i is*

$$\sup_{B^{-i}\in\mathcal{B}} L\left(b^i; B^{-i}, v^i\right) = \sup_{q\in[0,Q]} \left[\max\left\{\int_q^Q (v^i(x) - b^i_+(q))_+ dx,\, q b^i(q)\right\}\right].$$

Proof. The proof of this claim is substantially similar to the proof of the equivalent result for the pay-as-bid auction, Lemma 3, and is omitted here. □

B.2.1 Multiple units

Proof of Theorem 3. If there is a unique minimax-loss bid b_i, then $\underline{R}_k^{\mathrm{LAB}}(b_i; v_i) = \overline{R}_{k+1}^{\mathrm{LAB}}(b_i; v_i)$ for all $k \in \{1, \ldots, Q-1\}$. Otherwise, increasing b_{ik+1} weakly decreases $\underline{R}_k^{\mathrm{LAB}}(b_i; v_i)$ and weakly increases $\overline{R}_{k+1}^{\mathrm{LAB}}(b_i; v_i)$, and if these terms are nonequal b_{ik+1} can be adjusted without affecting loss, since b_i is optimal. The same argument is sufficient to show that $\underline{R}_k^{\mathrm{LAB}}(b_i; v_i) = \overline{R}_k^{\mathrm{LAB}}(b_i; v_i)$ for all $k \in \{1, \ldots, Q-1\}$.

Then if there is a unique minimax-loss bid b_i, $\overline{R}_k^{\mathrm{LAB}}(b_i; v_i) = k b_{ik+1} = c$ is constant. By corollary, for all $k \in \{1, \ldots, Q-1\}$,

$$\underline{R}_k^{\mathrm{LAB}}(b_i; v_i) = \sum_{k'=k+1}^{Q} (v_{ik'} - b_{ik+1})_+ = \sum_{k'=k+1}^{Q} \left(v_{ik'} - \frac{c}{k}\right)_+ = c.$$

Except in special cases, this equality cannot simultaneously hold for all k. □

Proof of Theorem 4. The claim follows immediately from earlier arguments. Bids are weakly below values, since $\underline{R}_k^{\text{LAB}}$ is zero and $\overline{R}_k^{\text{LAB}}$ is strictly positive when $b_{ik} > v_{ik}$. We now show that the proposed bid is monotone. Suppose to the contrary that there is k such that $b_{ik}^{\text{LAB}} < b_{ik+1}^{\text{LAB}}$. Then

$$\begin{aligned} b_{ik}^{\text{LAB}} = \frac{1}{k} \sum_{k'=k}^{Q} \left(v_{ik'} - b_{ik}^{\text{LAB}}\right)_+ &\geq \frac{1}{k} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik}^{\text{LAB}}\right)_+ \\ &\geq \frac{1}{k+1} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik}^{\text{LAB}}\right)_+ \geq \frac{1}{k+1} \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik+1}^{\text{LAB}}\right)_+ = b_{ik+1}^{\text{LAB}}. \end{aligned}$$

The penultimate inequality follows since we have assumed, by way of contradiction, that $b_{ik}^{\text{LAB}} < b_{ik+1}^{\text{LAB}}$; but this assumption implies $b_{ik}^{\text{LAB}} \geq b_{ik+1}^{\text{LAB}}$, a contradiction, thus it cannot be that $b_{ik}^{\text{LAB}} < b_{ik+1}^{\text{LAB}}$. □

B.2.2 Constrained bids

Proof of Theorem 5. We first prove that the minimax bid (b_i, q_i) must solve

$$\begin{aligned} b_1 q_1 &= b_k q_k && \text{for } k = 1, 2, \ldots, M \\ b_1 q_1 &= \int_{q_{k-1}}^{v^{-1}(b_k)} v(x) - b_k dx && \text{for } k = 1, 2, \ldots, M+1. \end{aligned}$$

Let k denote the largest index for which maximal loss is attained, i.e, either $k = M+1$ if $\sup_{B^{-i} \in \mathcal{B}} L(b_i; B^{-i}, v^i) = \int_{q_M}^{Q} v(x)dx$ or

$$k = \max\{k' = 1, 2, \ldots, M : \sup_{B^{-i} \in \mathcal{B}} L(b_i; B^{-i}, v^i) = \max\{\underline{R}_{q_{k'-1}}, \overline{R}_{q_{k'}}\}\}.$$

Let $k < M+1$. We show that $\underline{R}_{q_{k-1}} = \overline{R}_{q_k}$. Suppose $\underline{R}_{q_{k-1}} > \overline{R}_{q_k}$. As b_k only appears in these two expression, raising b_k only decreases $\underline{R}_{q_{k-1}}$ and increases $\overline{R}_{q_k}$. Suppose $\underline{R}_{q_{k-1}} < \overline{R}_{q_k}$. Decreasing b_k decreases $\overline{R}_{q_k}$ and increases $\underline{R}_{q_{k-1}}$. We do not have to worry about the effect on $\underline{R}_{q_k}$ as $\underline{R}_{q_k} < \overline{R}_{q_k}$.

Let $k = M+1$. Observe that $\int_{q_M}^{Q} v(x)dx = \underline{R}_{q_M} \leq \underline{R}_{q_{M-1}}$ as underbidding regret decreases in b_k and q_{k-1}. As regret is maximized by $M+1$, the inequality must hold with equality. The argument of the previous paragraph implies $\underline{R}_{q_{M-1}} = \overline{R}_{q_M}$. The result follows.

We now prove that a unique solution exists. To do so, note that we can express b_k as a function of q_{k-1} and q_k by solving

$$b_k q_k = \int_{q_{k-1}}^{v^{-1}(b_k)} v(x) - b_k dx$$

for b_k. The left-hand side increases in b_k and is 0 at $b_k = 0$. The right-hand side decreases in b_k, is positive for $b_k = 0$, and tends to 0 as b_k increases. Thus, there is a unique $b_k(q_{k-1}, q_k)$ that solves the equation. The bid $b_k(q_{k-1}, q_k)$ decreases in q_{k-1} and q_k.

We then proceed by expressing $q_{k'}$ as a function of q_1 by solving $b_1(q_0, q_1)q_1 = b_{k'}(q_{k'-1}, q_{k'})q_{k'}$ iteratively for $q_{k'}$, $k' = 2, 3, \ldots, M$. There is a unique $q_{k'}$ for each q_1. Finally, the condition $b_M(q_{M-1}(q_1), q_M(q_1))q_M(q_1) = \int_{q_M(q_1)}^{v^{-1}(0)} v(x) - 0dx$ pins down q_1. □

Proof of Proposition 4. Let (q, b) be optimal in the constrained uniform-price auction. Loss is

$$\max_k \left[\max\left\{q_k b\left(q_k\right), \int_{q_k}^{Q} \left(v^i\left(x\right) - b_+\left(q\right)\right)_+ dx\right\}\right].$$

Here, $b_+(q) = \lim_{\varepsilon \searrow 0} b(q + \varepsilon)$. Now consider the set of bid points q', where

$$q'_k = \left\lceil \frac{q_k}{Q} M_q \right\rceil \frac{Q}{M_q}.$$

That is, q'_k is the feasible bid point nearest to (but above) q_k. Define the bid vector b' so that $b'_k = b(q_k)$. By construction, (q', b') is feasible in the multi-unit auction. Since (q, b) is optimal, loss is higher under (q', b'), and the difference for a given quantity q_k is

$$\begin{aligned}
&\max\left\{q'_k b'_k, \int_{q'_k}^{Q} \left(v^i\left(x\right) - \hat{b}'_+\left(q'_k\right)\right)_+ dx\right\} - \max\left\{q_k b_k, \int_{q_k}^{Q} \left(v^i\left(x\right) - \hat{b}_+\left(q\right)\right)_+ dx\right\} \\
&\leq \left(q'_k - q_k\right)\hat{b}\left(q_k\right) = \int_{q_k}^{q'_k} \hat{b}\left(q_k\right) dx \leq \int_0^{Q/M_q} v^i\left(x\right) dx.
\end{aligned}$$

Then there is a feasible bid in the multi-unit auction with loss no more than $\int_0^{Q/M_q} v^i\left(x\right) dx$ higher than the optimal constrained bid in the divisible-good auction. □

B.2.3 Unconstrained bids

Proof of Proposition 5. This proof is provided in the main text, preceding Proposition 5. □

Proof of Theorem 6. Suppose that $\underline{c}(\cdot; L) \leq b \leq \overline{c}(\cdot; L)$. At any quantity q, overbidding regret is $qb(q) \leq q\bar{c}(\cdot; L) = L$; and at any quantity q, underbidding regret is

$$\int_q^Q \left(v^i(x) - b_+(q)\right)_+ dx \leq \int_q^Q \left(v^i(x) - \underline{c}(x; L)\right)_+ dx = L.$$

The left-hand inequality follows from the fact that $\underline{c}(\cdot; L)$ is continuous. Then conditional regret at quantity q is such that $R_q^{\mathrm{LAB}}(b; v^i) \leq L$, and it follows that the loss of bid b is weakly below L. □

Proof of Proposition 6. The convergence of optimal loss follows from the fact that, as M_q and M_b tend toward infinity, multi-unit and constrained-bid bids can arbitrarily approximate the unconstrained cross-conditional regret minimizing bid b^{LAB}. Since loss is converging, in the limit bids must lie between the limiting upper and lower iso-loss curves, which are continuous in loss. Theorem 6 implies the desired result. □

B.3 Comparison of auction formats

Proof of Comparison 1. We prove the claim for multiple-unit case; the claim for unconstrained case follows by taking limits where appropriate. We show that when b_i^{LAB} is cross-conditionally regret minimizing and b_i^{PAB} minimizes loss in the pay-as-bid auction, $b_{ik}^{\mathrm{LAB}} \geq b_{ik}^{\mathrm{PAB}}$ for all k. Note first that $b_{iQ}^{\mathrm{PAB}} = v_{iQ}/(Q+1) = b_{iQ}^{\mathrm{LAB}}$. Additionally, $b_{ik+1}^{\mathrm{PAB}} \leq b_{ik+1}^{\mathrm{LAB}}$ implies $b_{ik}^{\mathrm{PAB}} < b_{ik}^{\mathrm{LAB}}$. To see this, observe that loss minimization in the pay-as-bid auction requires

$$kb_{ik}^{\mathrm{PAB}} - \sum_{k'=k}^{Q} \left(v_{ik'} - b_{ik}^{\mathrm{PAB}}\right)_+ = kb_{ik+1}^{\mathrm{PAB}} - \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik+1}^{\mathrm{PAB}}\right)_+. \tag{4}$$

Cross-conditional regret minimization in the uniform-price auction requires $(k+1)b_{ik+1}^{\mathrm{LAB}} = \sum_{k'=k+1}^{Q}(v_{ik'} - b_{ik+1}^{\mathrm{LAB}})_+$; then under the assumption that $b_{ik+1}^{\mathrm{PAB}} \leq b_{ik+1}^{\mathrm{LAB}}$, it must be that

$$kb_{ik+1}^{\mathrm{PAB}} - \sum_{k'=k+1}^{Q} \left(v_{ik'} - b_{ik+1}^{\mathrm{PAB}}\right)_+ \leq 0. \tag{5}$$

This inequality will be strict whenever $b_{ik+1}^{\mathrm{PAB}} > 0$, which is true whenever $v_{ik+1} > 0$. Substituting inequality (5) into equation (4) gives

$$kb_{ik}^{\mathrm{PAB}} - \sum_{k'=k}^{Q} \left(v_{ik'} - b_{ik}^{\mathrm{PAB}}\right)_+ \leq 0 \quad \Longleftrightarrow \quad kb_{ik}^{\mathrm{PAB}} \leq \sum_{k'=k}^{Q} \left(v_{ik'} - b_{ik}^{\mathrm{PAB}}\right)_+.$$

Since the left-hand side of the above inequality is increasing in b_{ik}^{PAB} and the right-hand side is decreasing in b_{ik}^{PAB}, the fact that $kb_{ik}^{\mathrm{LAB}} = \sum_{k'=k}^{Q}(v_{ik'} - b_{ik}^{\mathrm{LAB}})_+$ implies that $b_{ik}^{\mathrm{LAB}} \geq b_{ik}^{\mathrm{PAB}}$. $\square$

Proof of Comparison 2. In light of Comparison 5, we show that the initial bid in the uniform-price auction must lie above the initial bid in the pay-as-bid auction, and that the lower iso-loss curve in the uniform-price auction reaches zero at some $q < Q$.

First, conditional regret in the pay-as-bid auction at quantity $q = 0$ is $\int_0^Q (v^i(x) - b^{\mathrm{PAB}}(0))_+ dx = L^{\mathrm{PAB}}$. Conditional regret in the uniform-price auction at quantity $q = 0$ is $\int_0^Q (v^i(x) - b^{\mathrm{LAB}}(0))_+ dx \leq L^{\mathrm{UPA}} \leq L^{\mathrm{PAB}}$. It follows that $b^{\mathrm{UPA}}(0) \geq b^{\mathrm{PAB}}(0)$, and thus it cannot be that $b^{\mathrm{UPA}} < b^{\mathrm{PAB}}$. Note that Comparison 5 implies $b^{\mathrm{UPA}}(0) > b^{\mathrm{PAB}}(0)$ except in the multi-unit case with a single bid point, $M_q = 1$.

Second, observe that for q close to Q underbidding loss becomes arbitrarily close to 0. Thus, the lower iso-loss curve must intersect the horizontal axis at some $q < Q$. $\square$

Proof of Comparison 3. We prove the claim for multiple-unit case; the claim for unconstrained case follows by taking limits where appropriate. We first show $\alpha^{\mathrm{FRB}} \geq \alpha^{\mathrm{LAB}}$. Note that $b_{iQ}^{\mathrm{FRB}} = v_{iQ}/Q$ and $b_{iQ}^{\mathrm{FRB}} = v_{iQ}/(Q+1)$. Furthermore, $b_{i1}^{\mathrm{FRB}} = v_{i1}$, and b_{i1}^{LAB} can be bounded,

$$b_{i1}^{\mathrm{LAB}} = \sum_{k'=1}^{Q} \left(v_{ik'} - b_{i1}^{\mathrm{LAB}}\right)_+ \leq \left(v_{i1} - b_{i1}^{\mathrm{LAB}}\right) Q \quad \Longrightarrow \quad b_{i1}^{\mathrm{LAB}} \leq \frac{Q.}{Q+1} v_{i1}.$$

Then to show $\alpha^{\mathrm{FRB}} \geq \alpha^{\mathrm{LAB}}$ it is sufficient to show

$$v_{i1} - \frac{1}{Q} v_{iQ} \geq \frac{Q}{Q+1} v_{i1} - \frac{1}{Q+1} v_{iQ} \quad \Longleftrightarrow \quad \frac{1}{Q+1} v_{i1} \geq \frac{1}{Q^2+Q} v_{iQ}.$$

Since $v_{i1} \geq v_{iQ}$ the result follows.

We now show that $\alpha^{\mathrm{LAB}} \geq \alpha^{\mathrm{PAB}}$. Since $b_{iQ}^{\mathrm{LAB}} = v_{iQ}/(Q+1) = b_{iQ}^{\mathrm{PAB}}$, Comparison 1 implies the desired result. $\square$

Proof of Comparison 4. When bidder i has unit demand, the ex post transfer to the auctioneer is identical in the last accepted bid uniform-price auction and the pay-as-bid auction. We therefore assume the bidder demands at least two units, $M_q > 1$.

We now show that when bidder i is awarded a small quantity, the ex post transfer to the auctioneer can be larger in the last accepted bid auction than in the pay-as-bid auction; and, when bidder i is awarded a large quantity, the ex post transfer to the auctioneer can be smaller in the last accepted bid auction than in the pay-as-bid auction. The former claim follows from Proposition 1, which implies that $b_{i1}^{\mathrm{LAB}} > b_{i1}^{\mathrm{PAB}}$ whenever $v_{i2} > 0$. The the transfer is higher in the last accepted bid auction when the market-clearing price (which is bounded above by b_{i1}^{LAB}) is relatively close to b_{i1}^{LAB}. The latter claim is immediate: since $b_{iQ}^{\text{LAB}} = b_{iQ}^{\text{PAB}}$ and bids are strictly decreasing in the pay-as-bid auction, $b_{i2}^{\text{PAB}} > 0$ implies that $\sum_{k=1}^{Q} b_{ik}^{\text{PAB}} > Q b_{iQ}^{\text{PAB}}$.

Because the comparison of ex post transfers is ambiguous and depends on the quantity allocated, expected transfers are also ambiguous: quantity distributions which place significant weight on quantities under which uniform-price revenue is higher will have higher expected revenue in the uniform-price auction, and quantity distributions which place significant weight on quantities under which pay-as-bid revenue is higher will have higher expected revenue in the pay-as-bid auction. □

Proof of Comparison 5. Given a bid function $\hat{b}$, for any quantity q conditional underbidding regret is identical in the pay-as-bid and uniform-price auctions, $\underline{R}_q^{\text{PAB}}(\hat{b}; v^i) = \underline{R}_q^{\text{LAB}}(\hat{b}; v^i)$. However, overbidding regret is weakly higher in the pay-as-bid auction, $\overline{R}_q^{\text{PAB}}(\hat{b}; v^i) \geq \overline{R}_q^{\text{LAB}}(\hat{b}; v^i)$. Since loss is the supremum of the higher of conditional overbidding and underbidding regrets, taken over all units, it follows that loss is weakly lower in the uniform-price auction.

In the multi-unit case with quantity $M_q > 1$ and the unconstrained cases the comparison is strict. The proof of Condition 1 shows that $b_{iq}^{\text{LAB}} > b_{iq}^{\text{PAB}}$ except at $q = M_q$ (in the multi-unit case) or $q = Q$ (in the unconstrained case). Let q be the quantity for which worst-case loss equals conditional regret in the uniform-price auction. Let b^{LAB} denote the cross-conditional regret minimizing bids of the uniform-price auction. Then we have that

$$
\begin{aligned}
\sup_{B^{-i} \in \mathcal{B}} L^{\text{LAB}}(b^{\text{LAB}}; B^{-i}, v^i) &= \int_q^Q \left(v(x) - b_+^{\text{LAB}}(q)\right)_+ dx \\
&\leq \int_q^Q \left(v(x) - b_+^{\text{PAB}}(q)\right)_+ dx \\
&\leq \int_q^Q \left(v(x) - b_+^{\text{PAB}}(q)\right)_+ dx + \int_0^q b^{\text{PAB}}(x) dx \\
&= \sup_{B^{-i} \in \mathcal{B}} L^{\text{PAB}}(b^{\text{PAB}}; B^{-i}, v^i),
\end{aligned}
$$

where we use that $b^{\text{PAB}} \leq b^{\text{LAB}}$ (Comparison 1) and the fact that underbidding regret involves lowering the bids on $[0, q]$.

Finally, we show that optimal loss in the constrained uniform-price auction with M_b bid points is lower than in the constrained pay-as-bid auction with M_b bid points. Consider the optimal loss L^{PAB} in the constrained pay-as-bid auction, generated by bid $(q^{\text{PAB}}, b^{\text{PAB}})$. This bid is feasible in the constrained uniform-price auction. Given the same bid, conditional regret is identical in the two auction formats for any quantity $q \in [0, q_1^{\text{PAB}}]$, and is strictly lower in the uniform-price auction for all quantities $q > q_1^{\mathrm{PAB}}$ since conditional regret in the pay-as-bid auction includes payments for lower units but conditional regret in the uniform-price auction does not. Since conditional regret is continuous in bid, it follows that a small deviation from $(q^{\mathrm{PAB}}, b^{\mathrm{PAB}})$—namely, increasing b_1^{PAB} slightly to $b_1^{\mathrm{UPA}} > b_1^{\mathrm{PAB}}$ and decreasing q_1^{PAB} slightly to $q_1^{\mathrm{UPA}} < q_1^{\mathrm{PAB}}$—will strictly lower conditional regret for quantities $q \in [0, \tilde{q}_1^{\mathrm{PAB}}]$ while keeping conditional regret for higher quantities below L^{PAB}. Since minimax loss is the maximum of conditional regret, taken over all bid points, it follows that optimal loss in the constrained uniform-price auction with M_b bid points is strictly below optimal loss in the constrained pay-as-bid auction with M_b bid points. □

C Calculations for Section 4

C.1 Multi-unit auctions

C.1.1 Demand for two units

Calculations for Example 2. Overbidding loss for the first unit is $Qb_{i1}^{\mathrm{LAB}}/2$, and overbidding loss for the second unit is Qb_{i2}^{LAB}. Since minimax loss is equal to overbidding loss for one of the units, we check

$$\begin{aligned}
& \frac{Q}{2} \cdot \begin{cases} \frac{1}{3}\left(v_{i1} + v_{i2}\right) & \text{if } v_{i1} \le 2v_{i2}, \\ \frac{1}{2} v_{i1} & \text{otherwise;} \end{cases} \gtrless \frac{Q}{3} v_{i2} \\
\iff \quad & \begin{cases} v_{i1} - v_{i2} & \text{if } v_{i1} \le 2v_{i2}, \\ \frac{3}{2} v_{i1} - 2v_{i2} & \text{otherwise;} \end{cases} \gtrless 0.
\end{aligned}$$

The first expression is always positive, since $v_{i1} \ge v_{i2}$, and the second expression is positive when $v_{i1} > 2v_{i2}$; then minimax loss is

$$L^{\mathrm{LAB}} = \frac{Q}{2} \cdot \begin{cases} \frac{1}{3}\left(v_{i1} + v_{i2}\right) & \text{if } v_{i1} \le 2v_{i2}, \\ \frac{1}{2} v_{i1} & \text{otherwise.} \end{cases}$$

Since minimax loss is always equal to overbidding loss for the first unit, the bid for the first unit must equal the cross-conditional regret minimizing bid for this unit, which is unique. On the other hand, the bid for the second unit must not induce cross-conditional regret above the minimax loss L^{LAB}. Underbidding regret for this unit is $v_{i2} - b_{i2}$. It follows that any bid b_{i2} such that $v_{i2} - b_{i2} \le L^{\mathrm{LAB}}$ and $2b_{i2} \le L^{\mathrm{LAB}}$ minimaxes loss; that is, $v_{i2} - L^{\mathrm{LAB}} \le b_{i2} \le L^{\mathrm{LAB}}/2$. □

C.2 Constrained bids

C.2.1 Constant marginal values

Calculations for Example 5. Equating conditional loss across units requires $R_{k+1} - R_k = 0$ for all k. This is

$$
\begin{aligned}
0 &= \left[\sum_{k'=0}^{k+1} \left(b_{k'} - b_{k+2}\right)\left(q_{k'} - q_{k'-1}\right) + \left(Q - q_{k+1}\right)\left(v - b_{k+2}\right)\right] - \\
&\qquad \left[\sum_{k'=0}^{k} \left(b_{k'} - b_{k+1}\right)\left(q_{k'} - q_{k'-1}\right) + \left(Q - q_{k}\right)\left(v - b_{k+1}\right)\right] \\
&= \left(b_{k+1} - b_{k+2}\right)\left(q_{k+1} - q_k\right) + \left(Q - q_{k+1}\right)\left(v - b_{k+2}\right) \\
&\qquad + \sum_{k'=0}^{k} \left(b_{k+1} - b_{k+2}\right)\left(q_{k'} - q_{k'-1}\right) - \left(Q - q_k\right)\left(v - b_{k+1}\right) \\
&= \left(b_{k+1} - b_{k+2}\right) q_{k+1} + \left(Q - q_{k+1}\right)\left(v - b_{k+2}\right) - \left(Q - q_k\right)\left(v - b_{k+1}\right) \\
&= -Q b_{k+2} - \left(q_{k+1} - q_k\right) v + \left(Q + \left(q_{k+1} - q_k\right)\right) b_{k+1}.
\end{aligned}
$$

Let $g_k \equiv q_k - q_{k-1}$ be the gap between the k^{th} and $k+1^{\text{th}}$ bid points. Then we have

$$
\begin{aligned}
\left(Q + g_{k+1}\right) b_{k+1} = g_{k+1} v + Q b_{k+2} \quad &\iff \quad b_{k+1} = \frac{g_{k+1}}{Q + g_{k+1}} v + \frac{Q}{Q + g_{k+1}} b_{k+2} \\
&\iff \quad b_k = \frac{g_k}{Q + g_k} v + \frac{Q}{Q + g_k} b_{k+1}.
\end{aligned}
$$

We now solve recursively for optimal bids, conditional on bid points. When $k = M$, we have $b_{k+1} = 0$ by assumption, and $b_M = \frac{g_M}{Q + g_M} v$. For $k < M$, we have

$$
b_k = \sum_{k'=k}^{M} \frac{Q^{k'-k} g_{k'}}{\prod_{j=k}^{k'} \left[Q + g_j\right]} v.
$$

Since $R_0 = (v - b_1)Q$, the loss-minimization problem is (dropping the irrelevant constants v and Q)

$$\begin{aligned}
&\min_g 1 - \sum_{k=1}^{M} \frac{Q^{k-1} g_k}{\prod_{k'=1}^{k} [Q + g_{k'}]} \\
&= \min_g 1 - \frac{\sum_{k=1}^{M} \frac{1}{Q+g_k} \prod_{k'=k}^{M} [Q + g_{k'}] \, Q^{k-1} g_k}{\prod_{k'=1}^{M} [Q + g_{k'}]} \\
&= \min_g \frac{\prod_{k'=1}^{M} [Q + g_{k'}] - \sum_{k=1}^{M} \frac{1}{Q+g_k} \prod_{k'=k}^{M} [Q + g_{k'}] \, Q^{k-1} g_k}{\prod_{k'=1}^{M} [Q + g_{k'}]}.
\end{aligned}$$

Denote the numerator by A_M. We show that $A_M = Q^M$. First, $A_1 = Q$:

$$A_1 = [Q + g_1] - \frac{1}{Q + g_1} [Q + g_1] \, g_1 = Q.$$

The result follows by induction on M; assuming $A_M = Q^M$, we have

$$\begin{aligned}
&\prod_{k'=1}^{M+1} [Q + g_{k'}] - \sum_{k=1}^{M+1} \frac{1}{Q + g_k} \prod_{k'=k}^{M+1} [Q + g_{k'}] \, Q^{k-1} g_k \\
&= [Q + g_{M+1}] \left[Q^M + \sum_{k=1}^{M} \frac{1}{Q + g_k} \prod_{k'=k}^{M} [Q + g_{k'}] \, Q^{k-1} g_k \right] - \sum_{k=1}^{M+1} \frac{1}{Q + g_k} \prod_{k'=k}^{M+1} [Q + g_{k'}] \, Q^{k-1} g_k \\
&= [Q + g_{M+1}] \, Q^M + \sum_{k=1}^{M} \frac{1}{Q + g_k} \prod_{k'=k}^{M+1} [Q + g_{k'}] \, Q^{k-1} g_k - \sum_{k=1}^{M+1} \frac{1}{Q + g_k} \prod_{k'=k}^{M+1} [Q + g_{k'}] \, Q^{k-1} g_k \\
&= [Q + g_{M+1}] \, Q^M - Q^M g_{M+1} = Q^{M+1}.
\end{aligned}$$

Then the loss minimization problem is

$$\min_g \frac{Q^M}{\prod_{k=1}^{K} [Q + g_k]}, \text{ s.t. } g_k \geq 0 \text{ and } \sum_{k=1}^{M} g_k \leq Q.$$

This is solved by $g_k = Q/M$. The resulting bids are

$$\begin{aligned}
b_{k|M} = \sum_{k'=k}^{M} \frac{Q^{k'-k} g_{k'}}{\prod_{j=k}^{k'} [Q + g_j]} v &= \sum_{k'=k}^{M} \frac{\frac{1}{M} Q^{k'-k+1}}{\prod_{j=k}^{k'} \left[\frac{M+1}{M} Q\right]} v \\
&= \sum_{k'=k}^{M} \frac{\frac{1}{M} Q^{k'-k+1}}{\left[\frac{M+1}{M} Q\right]^{k'-k+1}} v = \frac{v}{M} \sum_{k'=k}^{M} \left[\frac{M}{M+1}\right]^{k'-k+1}.
\end{aligned}$$

□

C.2.2 A single bid point

Calculations for Example 7. The first-order conditions of the problem are

$$\frac{d}{dq'}: \quad 0 = b' - v^i(q') - \left(b' - v^i(q')\right)\lambda,$$
$$\frac{d}{db'}: \quad 0 = q' + \left(-v^{-1}(b') - q'\right)\lambda.$$

Then $\lambda = q'/(q' + v^{-1}(b') \neq 1$, and to solve the first equation it must be that $q_{i1} = v^{-1}(b_{i1})$. It follows that b_{i1} is the unique solution to

$$\int_0^Q \left(v^i(x) - b_{i1}\right)_+ dx = b_{i1} v^{-1}(b_{i1}) + \int_{v^{-1}(b_{i1})}^Q v^i(x)\, dx.$$

□

[28]The domain of v^i is $[0, Q]$ and v^i is weakly decreasing, so we adopt the convention that $v^{-1}(p) = 0$ for all $p > v^i(0)$, and $v^{-1}(p) = Q$ for all $p < v^i(Q)$.