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*An early version of the *stationary* configuration ($T^* = 0$ in Axiom A1) of this study appeared as Section 4 in Drugeon & Ha-Huy (2018). The authors would like to thank Gaetano Bloise, Alain Chateauneuf, Rose-Anne Dana and Joanna Franaszek for their insightful comments. Thai Ha-Huy would like to thank the Labex MME-DII (ANR-11-LBX-0023-01) for its support during the completion of this article.

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Abstract

This article completes an axiomatic approach of utilities streams. The approach is more precisely based upon the robust pre-orders that open the scope for α-MaxMin representations. A general T-steps Temporal Bias axiom is first introduced, that encapsulates stationarity and 1-step present bias, aka quasi-hyperbolic discounting, as special cases. A detailed characterisation of the sets of probabilities that represent the weights of the future values of the utilities stream is then completed. This is first achieved for the close future pre-order where a generalised picture of present biases in brought into evidence. This is complemented for the distant future pre-order where it proved that, under the same system of axioms, the weights of the tail of the utility stream now correspond to Banach limits, who, in the evaluation of distant future, can be considered as the counterpart of the geometric discount rates in the evaluation of close future. The whole result is eventually given in an explicit α-Maxmin representation.

Keywords: Axiomatisation, Myopia, Multiple Discounts, α−MaxMin Citeria, Temporal Biases, Banach Limits, Infinite Dimensional Topologies.

JEL Classification: D11, D15, D90.

Contents

1. Introduction

1.1 Motivation and Concerns

The introduction by Gilboa & Schmeidler (1989) of the *multiple priors* approach to choice under uncertainty and the relevance of the associated *worst scenario* case and Maxmin criteria has been at the very inception of the numerous contemporaneous developments of the *ambiguity* literature. While this approach at remains the cornerstone of most studies, one of its strongest limits was pointed out by Ghirardato, Maccheroni & Marinacci (2004) who emphasised the need for a generalised criterion that would distinguish *ambiguity* from *ambiguity attitude* and support as an alternative the use of an elaborated version labelled α-Maxmin. Such preferences generalise the well-known α-MaxMin rule of Hurwicz (1951) to settings of uncertainty where the subjective perception of ambiguity can be described by a set of probability measures and the attitude towards ambiguity by a parameter α which describes the relative weight put on pessimism versus optimism. Its empirical relevance in an experimental environment having been argued and been the object of numerous studies, these are surveyed in Trautmann & van de Kuilen (2015).

The central aim of this study is then to examine the scope for such a representation when one is concerned with discounted infinite utility streams instead of choice under uncertainty over a range of states of the world. While this somewhat echoes some of the concerns of Chambers & Echenique (2018), this study aims at broadening the scope of their analysis in three distinct regards.

First, and along the above concerns, by completing an alternative approach that enables for extending their MinMax one to the larger class of α-MaxMin criteria.

Second, and in contradistinction with their work and most of the axiomatic approaches of discounting, this study also ambitions at taking explicit account of the arbitrarily remote components of the utility streams, its purpose being to provide a picture of the different classes of representations that are required to deal with the distant components of the utilities stream in comparison with the more standardly considered near future ones. It is to be emphasised that this article aims at building another α-MaxMin class of criteria for characterising this distinction.

Finally, while the original purpose of Chambers & Echenique (2018) was more to analyse how regular discounting criteria could accommodate the existence of diverging opinions between several experts, a large part of the literature has been involved with the anomalies and temporal biases that resulted from experimental studies. This had led to a renewal of interest for the present biased *quasi-hyperbolic* discounting representation, first introduced by Phelps & Pollack (1968) but more recently brought to the fore by Laibson (1997). This does also correspond to the third regard through which this article aims at extending their study, *i.e.*, encompassing potentially general temporal biases within a multiple discounts representation.

1.2 The Approach and the Results

This contribution echoes the decision under uncertainty literature by focusing on the scope for α-MaxMin criteria and is anchored on an environment that indirectly relates to it. It is more directly aimed at completing an axiomatic approach to the evaluation of infinite utility streams, the whole argument being cast for discrete time sequences. It is based upon robust or unanimous orders, a given utility stream being *robustly better* than an alternative one if and only if such a comparison is unanimous among a set of linear orders that can be understood as a set of possible evaluations. Each sub-order is shown to constitutes upon two separate components: a first that belongs to the set of $\sigma-$additive measures on $\mathbb{N}$ and a second part that belongs to the set of *purely finitely additive* measures[1] and is usually called a *charges* set.

Elaborating upon the properties of the robust pre-order and asssuming that its embeded *degree of optimism* does not decrease with respect to some robustness comparisons, the order is proved to assume a so-called α-MaxMin criterion representation, the evaluation of a given utility stream depending only on its best and worst evaluations.

Under the extra assumption that every sub-order satisfies some impatience and con-

[1]For a detailed exposition, see Bhaskara Rao & Bhaskara Rao (1983).

sistency properties, the $\sigma-$additive part of the sub-orders satisfies a *delay*-stationary property, *i.e.,* the evaluation, beginning after a delay of a certain periods, does not depend upon the date of evaluation. Interestingly, when this assumption is extended to encompass the scope for temporal biases, while its first feature keeps on characterising some impatience property, its second one now features some T^*-*delay* stability. Otherwise stated, if a combination is robustly better than a constant sequence, it remains robustly better if it is moved forward into the future, the effect according to the robust order becoming lower over time.

A further T^*-*delay equivalence* assumption is considered, that postulates the existence of a delay such that a delayed sequence of an alternative sequence would be equivalent to the original one while hedging with another delayed sequence. This equivalence can be understood in the sense that the chance to improve *robustly* the situation through hedging would remain unmodified. While, for the stationary configuration $T^* = 0$, the sequence of discount factors is proved to assume a geometrical representation, that sequence is shown to be associated to a representation that features a generalised version of quasi hyperbolic discounting for an arbitrary value of T^*.

Focusing then on the finitely additive measures that feature the distant future robust pre-order and do correspond to the remaining part of the sub-orders representation, it is established, under assumptions that are remarkably close from the previous σ-additive components, that the weights of the tail of the utility stream build from *Banach limits.*[2] The evaluation of a utilities stream under a Banach limit does not change if it is shifted one (ou many) period(s) to the future. Interestingly, this property echoes the evaluation of close future under systems of *geometrical discount rates*, where the comparison between two sequences does not depend on the period of departure. This property of *stability*, or, in another word, *anonymity*, makes the Banach limits, in the evaluation of distant future, the counterpart of the geometrical discount rates in the evaluation of close future.

The whole information on the sub-orders representation is eventually given in an

[2]For intuition about *Banach limits*, one can have in mind the lower-bound, the infimum limit and the upper-bound, the supremum limit of utilites streams. These functions satisfy every properties of *Banach limits* charges, minus the linearity. For a careful definition, see page 55 in Becker & Boyd (1997).

explicit α-Maxmin representation. This eventually allows for a subtle decomposition of the robust orders and a precise description of α-MaxMin representations.

1.3 Related Literature

The early and most influential attempts towards the axiomatisation of α-Maxmin criteria that would replace the ambiguity averse decision maker of Gilboa & Schmeidler (1989) by a more plausible mix of ambiguity-averse and ambiguity-seeking tendencies were due to Kopylov (2003) and Ghirardato, Maccheroni & Marinacci (2004). This representation has become extremely successful and went on being used on a common basis in numerous experimental studies where the weight α and the set of beliefs were then often interpreted as simple parameterisations of the decision maker's ambiguity attitude and perception of ambiguity, that was presumably key to its widespread use in applied experimental studies.

Numerous topics have further recently been under study. To name just a few of some insightful contributions, the generic non-uniqueness of the weighted representation has led Frick, Iijima & Le Yaouanq (2021) and Chateauneuf, Qu, Vergopoulos & Ventura (2021) to be interested in the uniqueness of the α-MaxMin representation. While the first convincingly incorporates *objective rationality* into a α-MaxMin expected utility model of choice under ambiguity can overcome several challenges faced by the baseline model without objective rationality, the second completes a rigorous appraisal of the scope for *falsibiability* of the α-Maxmin representation. In separate regards, while Beißner, Lin & Riedel (2020) have been interested in their connection with time consistency, Beißner & Werner (2021) have completed an original appraisal of their differentiability properties.

As a result of the importance of this representation for inter-temporal analysis, the examination of the axiomatics of the underpinnings of the discounted forms of utility have been the object of numerous efforts. While the pioneer study Koopmans (1972) keeps on playing an irreplaceable in this regard and while Dolmas (1995) brought an interesting clarification and Fishburn & Rubinstein (1982) an englightning alternative, this line of research is conveniently summed up in the influential work of Bleichrodt, Rohde & Wakker (2008). Since the prominent work of

Laibson (1997), a parallel range of efforts have been aimed at identifying the underpinnings of associated temporal biases and anomalies, noticeable efforts in this direction being due Chakraborty (2017) and Montiel Olea & Strzalecki (2014).

It is however worth pointing out that, while the *multiple priors* approach of Gilboa & Schmeidler (1989) has brought a complete renewal of the topics, methods and aims of choice under uncertainty, it was not before Wakai (2007) and, perhaps more prominently, Chambers & Echenique (2018), that it got adapted to intertemporal choice. In parallel with the α-MaxMin current approach, two recent studies have been interested in such representations : while the first, due to Bich, Dong & Wigniolle (2021), has extended the Chambers & Echenique (2018) system of axioms to the scope for 1-step present bias and quasi-hyperbolic discounting, the second, due to the same authors as the current one, Drugeon & Ha-Huy (2021b), follows an alternative, more abstract and less aimed at representation results, that focuses on recursive time-dependent orders and the scope for therein defining multiple time-varying discount functions.

Finally, and as a result of its technical complexity of infinite dimensional topologies, the role of arbitrarily remote components of the utilities sequences, that is here completed in a simplified form through the use of finitely additive measures and *charges*, has been the object of a limited number of contributions due to Brown & Lewis (1981), Sawyer (1988), Gilles (1989), Drugeon & Ha-Huy (2021a) and, very recently, de Andrade, Bastianello & Orrillo (2021).

1.4 Contents

Section 2 describes the basic axioms required for a decomposition between the close future and the remote future as well the robust pre-order $\succeq^*$ and then completes an early extended α-MaxMin representation. Section 3 strengthens these results by achieving an α-MaxMin representation for Temporally-Biased Multiple Discounts. The proofs are given in the Appendix.

2. Basic Axioms, the Robustness Pre-Order $\succeq^*$ & an Extended α-Maxmin Decomposition

2.1 Fundamentals, Elementary Axioms & Construction of the Index Function

This paper considers an axiomatic approach to the evaluation of infinite utility streams, the whole argument being cast for discrete time sequences. To avoid confusion, letters like x, y, z will be used for sequences (of utils) with values in $\mathbb{R}$; a notation $c\mathbb{1}$, $c^*\mathbb{1}$, $d\mathbb{1}$ will be used for constant sequences, where $\mathbb{1}$ denotes $(1, 1, \ldots,)$. A notation λ, η, μ will also be used for constant scalars.

For every $x \in \ell_\infty$ and $T \geq 0$, let $x_{[0,T]} = (x_0, x_1, \ldots, x_T)$ denote its *head* $T + 1$ first components and $x_{[T+1,\infty[} = (x_{T+1}, x_{T+2}, \ldots)$ its *tail* starting from date $T+1$. For eaxmpla, given constant c and a sequence x, $(x_{[0,T]}, y_{[T+1,+\infty[})$ denotes the sequence $(x_0, x_1, \ldots, x_T, y_{T+1}, y_{T+2}, \ldots)$.

The properties in the subsequent axiom **F** are well-known in the literature. A more detailed discussions about their significations can be found in Chambers & Echenique (2018), Bich, Dong & Wigniolle (2021), or Drugeon & Ha-Huy (2021a).

Axiom F. The order $\succeq$ satisfies the following properties:

(i) *Completeness* For every $x, y \in \ell_\infty$, either $x \succeq y$ or $y \succeq x$.

(ii) *Transitivity* For every $x, y, z \in \ell_\infty$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. Denote as $x \sim y$ the case where $x \succeq y$ and $y \succeq x$. Denote as $x \succ y$ the case where $x \succeq y$ and $y \not\succeq x$.

(iii) *Monotonicity* If $x, y \in \ell_\infty$ and $x_s \geq y_s$ for every $s \in \mathbb{N}$, then $x \succeq y$.

(iv) *Non-triviality* There exist $x, y \in \ell_\infty$ such that $x \succ y$.

(v) *Archimedeanity* For $x \in \ell_\infty$ and $b\mathbb{1} \succ x \succ b'\mathbb{1}$, there are $\lambda, \mu \in]0,1[$ such that $(1-\lambda)b\mathbb{1} + \lambda b'\mathbb{1} \succ x$ and $x \succ (1-\mu)b\mathbb{1} + \mu b'\mathbb{1}$.

(vi) *Weak convexity* For every $x, y, b\mathbb{1} \in \ell_\infty$, and $\lambda \in]0,1]$, $x \succeq y \Leftrightarrow (1-\lambda)x + \lambda b\mathbb{1} \succeq (1-\lambda)y + \lambda b\mathbb{1}$.

Under these conditions, the order $\succeq$ can be represented by an index function I which is homogeneous of degree 1 and constantly additive: For every constant $b \in \mathbb{R}$, $\lambda \geq 0$, $\mathrm{I}(\lambda x + b\mathbb{1}) = \lambda \mathrm{I}(x) + b$.

2.2 Sub-Orders and the Decomposition

As this is detailed in Drugeon & Ha-Huy (2021a, Ex. 1-2), the axiom **F** is general, and one can construct complicated orders satisfying it. To a better understanding of the properties, one should add additional structure on $\succeq$. The following axiom assumes that there exists an *independtly evaluation of the close future componants and the distant future components of the utility stream.*

Axiom G1. For any $x \in \ell_\infty$

(i) take any constant $d \in \mathbb{R}$, either, for any $\epsilon > 0$, there exists $\mathrm{T}_0(\epsilon)$ such that for any $z \in \ell_\infty$, for every $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$: $\left(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\right) \succeq \left(z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}$, or, for any $\epsilon > 0$, there exists $\mathrm{T}_0(\epsilon)$ such that for any $z \in \ell_\infty$, for every $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$: $\left(z_{[0,\mathrm{T}]}, d\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) \succeq \left(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}$.

(ii) take a constant $c \in \mathbb{R}$, either, for any $\epsilon > 0$, there exists $\mathrm{T}_0(\epsilon)$ such that, for any $z \in \ell_\infty$ and for every $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$, $\left(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) \succeq \left(c\mathbb{1}_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}$, or there exists $\mathrm{T}_0(\epsilon)$ such that, for any $z \in \ell_\infty$ and for every $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$, $\left(c\mathbb{1}_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) \succeq \left(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}$.

Under axiom **G1**, the evaluation of an utilites stream can be decomposed as the combination of two values that respectively relate to the *close future* and *distant future* values of this this stream.

Definition 2.1. *The* close future *order* $\succeq_c$ *and the* distant future *order* $\succeq_d$ *are respectively defined as:*

(i) *For any* $x, y \in \ell_\infty$, *the satisfaction of* $x \succeq_c y$ *if and only if for any* $\epsilon > 0$, *there exists* $\mathrm{T}_0(\epsilon)$ *such that, for any sequence* $z \in \ell_\infty$ *and for every date* $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$, $\left(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) \succeq \left(y_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}$.

(ii) *For any* $x, y \in \ell_\infty$, *the satisfaction of* $x \succeq_d y$ *if and only if, for any* $\epsilon > 0$, *there exists* $\mathrm{T}_0(\epsilon)$ *such that, for any* $z \in \ell_\infty$ *and for every* $\mathrm{T} \geq \mathrm{T}_0(\epsilon)$*:* $\left(z_{[0,\mathrm{T}]}, x_{[\mathrm{T}+1,\infty[}\right) \succeq \left(z_{[0,\mathrm{T}]}, y_{[\mathrm{T}+1,\infty[}\right) - \epsilon\mathbb{1}$.

In Drugeon & Ha-Huy (2021a), under the axiom **G1**, the sub-orders $\succeq_c$ and $\succeq_d$ are shown to satisfy the axiom **F**, and can thus be represented correspondingly by the index functions I_d and I_c satisfing the *constant additive* and *homogeneity of degree one* properties. While the function I_c indeed satisfies a weak version of *tail-insentivity* property, the function I_d does not change its value upon the mere modification of a finite number of components in the utilities stream.[3] The value of the overall index function finally emerges as a convex combination of the two values of sub-order index functions, with a convexity parameter that itself depends on the utilities stream. Precisely, there exists $0 \leq \underline{\lambda} \leq \overline{\lambda} \leq 1$ such that, for any $x \in \ell_\infty$, the index I assumes one of the two following decompositions :

(i) $\mathrm{I}(x) = \min_{\underline{\lambda} \leq \lambda \leq \overline{\lambda}} \left[(1-\lambda)\mathrm{I}_c(x) + \lambda \mathrm{I}_d(x)\right]$,

(ii) $\mathrm{I}(x) = \max_{\underline{\lambda} \leq \lambda \leq \overline{\lambda}} \left[(1-\lambda)\mathrm{I}_c(x) + \lambda \mathrm{I}_d(x)\right]$.

As an example, consider the order $\succeq$ being represented as follows, with $0 \leq \underline{\lambda} \leq \overline{\lambda} \leq 1$ and D a compact subset of $]0,1[$:

$$\mathrm{I}(x) = \min_{\underline{\lambda} \leq \lambda \leq \overline{\lambda}} \left[(1-\lambda) \min_{\delta \in \mathrm{D}} \left((1-\delta)\sum_{s=0}^{\infty} \delta^s x_s\right) + \lambda \liminf_{s \to \infty} x_s\right].$$

In such an example, the initial order $\succeq$ can be decomposed into to two sub-orders $\succeq_c$ and $\succeq_d$ with two associated index functions available as:

$$\mathrm{I}_c(x) = \min_{\delta \in \mathrm{D}} \left((1-\delta)\sum_{s=0}^{\infty} \delta^s x_s\right),$$
$$\mathrm{I}_d(x) = \liminf_{s \to \infty} x_s.$$

[3]For every $x, z \in \ell_\infty$, $\lim_{\mathrm{T}\to\infty} \mathrm{I}_c\left(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) = \mathrm{I}_c(x)$ and $\mathrm{I}_d\left(x_{[0,\mathrm{T}]}, z_{[\mathrm{T}+1,\infty[}\right) = \mathrm{I}_d(z)$.

2.3 The Robustness Pre-Order $\succeq^*$

In order to reach more explicit properties for the index function I, consider a *pre-order,* as opposed to the earlier complete order, $\succeq^*$, featuring the robustness of the order $\succeq$: whatever the mixture with a common component, the comparison would not be modified. In the same spirit as Gilboa & Schmeidler (1989), this approach leads to a characterization of the order $\succeq$ by a set of probability charges belonging to $(\ell_\infty)^*$.

Definition 2.2. *Let the pre-order $\succeq^*$ be defined by*

$$x \succeq^* y \textit{ if, for every } 0 \leq \lambda \leq 1, z \in \ell_\infty, \lambda x + (1-\lambda) z \succeq \lambda y + (1-\lambda) z.$$

It is first to be noticed that, in the general case, the pre-order $\succeq^*$ is not complete. Lemma 2.1 presents the fundamental properties of the pre-order $\succeq^*$.

Lemma **2.1**. *Assume that axiom* **F** *is satisfied. For every x, y: $x \succeq^* y$ if and only if either of the two following assertions is satisfied:*

(i) *For every $z \in \ell_\infty, x + z \succeq y + z$.*

(ii) *There exists $z \in \ell_\infty$, $x + z \succeq^* y + z$.*

The understanding of the properties of the pre-order $\succeq^*$ is important in the analysis of the order $\succeq$ and proposition 2.1 will clarify its precise status. The initial order $\succeq$ can be considered as a family of linear sub-orders, the pre-order $\succeq^*$ featuring the particular one that deals with *robustness* or *unanimity*. This pre-order $\succeq^*$ can be considered as depicting an *unanimous class of preferences*: a given sequence x is *robustly preferred* to another sequence y if and only if any sub-preference to the order $\succeq$ prefers x to y. These sub-preferences are a convex set with a measure belonging to $(\ell_\infty)^*$, defined as the normalized positive polar cone of the set x such that $x \succeq^* 0\mathbb{1}$.

Recall that the dual space of ℓ_∞,[4] *i.e.*, the set of real sequences such that $\sup_s |x_s| < +\infty$, can be decomposed into the direct sum of two subspaces, ℓ_1 and ℓ_1^d: $(\ell_\infty)^* = \ell_1 \oplus$

[4] The set of real continuous linear functions on ℓ_∞.

ℓ_1^d. The subspace ℓ_1 satisfies σ-additivity. The subspace ℓ_1^d, the *disjoint complement* of ℓ_1, is the one of finitely additive measures defined on $\mathbb{N}$. More precisely, for each measure $\phi \in \ell_1^d$, for any $x \in \ell_\infty$, the value of $\phi \cdot x$ depends only on the distant behaviour of x, and does not change if there only occurs a change in a finite number of values x_s, $s \in \mathbb{N}$.

Proposition 2.1. *Assume that axiom* **F** *is satisfied. There exists a convex set* Ω *of weights* $\big((1-\lambda)\omega, \lambda\phi\big)$ *which can be considered as finitely additive probabilistic measures on* $\mathbb{N}$ *where:*

(i) $0 \leq \lambda \leq 1$,

(ii) $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ *is a probability measure, i.e., a sequence of weights, belonging to* ℓ^1,

$$\sum_{s=0}^{\infty} \omega_s = 1,$$

with $\omega_s \geq 0$ *for every* s.

(iii) ϕ *is a charge in* ℓ_d^1 *satisfying* $\phi(\mathbb{N}) = 1$, $\phi(A) \geq 0$ *for every* $A \subset \mathbb{N}$,

such that, for every $x, y \in \ell_\infty$, $x \succeq^* y$ *if and only if, for any* $\big((1-\lambda)\omega, \lambda\phi\big) \in \Omega$,

$$(1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x \geq (1-\lambda)\sum_{s=0}^{\infty} \omega_s y_s + \lambda\phi \cdot y.$$

It is worth emphasizing that the value λ can change between different measures and the charges ϕ can be considered as a *purely finitely additive measure* on $\mathbb{N}$: for every finite subset $A \subset \mathbb{N}$, $\phi(A) = 0$.

The property (iii) is the most important one, which establishes the characterization of Ω. The sequence x is ensuingly robustly better than the sequence y if and only if it is confirmed *unanimously* by every probability charges in Ω.

2.4 MaxMin and α−MaxMin Representations

Define, for each $x \in \ell_\infty$, define b^*_x, b^{*x} as the best and worst evaluation of sequence x:

$$\begin{aligned} b^*_x &= \sup\Big\{b \in \mathbb{R} \text{ such that } x \succeq^* b\mathbb{1}\Big\} \\ &= \inf_{\big((1-\lambda)\omega,\lambda\phi\big)\in\Omega} \Big((1-\lambda)\omega \cdot x + \lambda\phi \cdot x\Big), \\ b^{*x} &= \inf\Big\{b \in \mathbb{R} \text{ such that } b\mathbb{1} \succeq^* x\Big\} \\ &= \sup_{\big((1-\lambda)\omega,\lambda\phi\big)\in\Omega} \Big((1-\lambda)\omega \cdot x + \lambda\phi \cdot x\Big). \end{aligned}$$

For any $x \in \ell_\infty$, define the degree of pessimism in distant future associated with x: the value a_x satisfying

$$\mathrm{I}(x) = a_x b^*_x + (1 - a_x) b^{*x}.$$

The value a_x is unique if $b^*_x < b^{*x}$. The value a_x can be considered as the pessimism degree associated with the sequence $x \in \ell_\infty$, as $1-a_x$ the optimism degree associated with x. It is natural to study the case where the optimism degree does not decrease in respect to the robustness order $\succeq^*$.

Axiom G 2. Consider $x, y \in \ell_\infty$ satisfying $b^*_x < b^{*x}$, and $b^*_x < b^*_y$. If $y \succeq^* x$ then $a_x \geq a_y$.

Along Proposition 2.2, under the assumption that the degree of pessimism cannot increase with respect to the robust pre-order $\succeq^*$, the index of distant future order assumes a α−MaxMin representation:

Proposition 2.2. *Assume that axioms* **F**, **G1** *and* **G2** *are satisfied. For any* $x \in \ell_\infty$ *such that* $b^*_x < b^{*x}$, a_x *is equal to a constant* a^*. *For any* x, *the distant index assumes the following representation:*

$$\begin{aligned} \mathrm{I}(x) &= a^* b^*_x + (1 - a^*) b^{*x} \\ &= a^* \sup_{\Omega}\Big((1-\lambda)\omega \cdot x + \lambda\phi \cdot x\Big) + (1 - a^*) \inf_{\Omega}\Big((1-\lambda)\omega \cdot x + \lambda\phi \cdot x\Big). \end{aligned}$$

Remark 2.1. Following two distinct approaches, Frick, Iijima & Le Yaouanq (2021) and Chateauneuf, Qu, Vergopoulos & Ventura (2021) have been recently interested in the uniqueness of the α-MaxMin representation. In this regard, it is to be emphasised that the current representation unambiguously avoids this *falsifiability* line of criticism in that all of the sets being determined through the orders $\succeq$, $\succeq_c$ and $\succeq_d$ as the positive polar cones with respect to the robust orders, they are uniquely determined, this uniqueness result extending to the parameter a^* in the above (as well as to the subsequent coefficients a_c^* and a_d^* in Proposition 3.3).

3. An α-MaxMin Representation for Temporally-Biased Multiple Discounts & the Role of Banach Limits

3.1 Temporal Bias Axiom

In order to better characterize the set Ω, consider axiom **G1** which characterizes the impatience and the stability properties of the pre-order $\succeq^*$.

Axiom A 1. *Impatience and* T**–delay stationarity* Fix $T^* \geq 0$. Given $x \in \ell_\infty$ and a constant b.

$$(b\mathbb{1}_{[0,T^*]}, x) \succeq^* b\mathbb{1} \Rightarrow (b\mathbb{1}_{[0,T^*]}, x) \succeq^* (b\mathbb{1}_{[0,T^*+1]}, x) \succeq^* b\mathbb{1}.$$

More precisely, Axiom A1 states that

(i) The case $T^* = 0$ corresponds to the *Stationarity* property:

$$x \succeq^* b\mathbb{1} \Rightarrow x \succeq^* (b, x) \succeq^* b\mathbb{1}.$$

(ii) The case $T^* = 1$ corresponds to the *Quasi-hyperbolic discounting* property:

$$(b, x) \succeq^* b\mathbb{1} \Rightarrow (b, x) \succeq^* (b, b, x) \succeq^* b\mathbb{1}.$$

(iii) The case $T^* \geq 1$ can be considered as a T^**-steps quasi-hyperbolic discounting* property:

$$(b\mathbb{1}_{[0,T^*]}, x) \succeq^* b\mathbb{1} \Rightarrow (b\mathbb{1}_{[0,T^*]}, x) \succeq^* (b\mathbb{1}_{[0,T^*+1]}, x) \succeq^* b\mathbb{1}.$$

In axiom **A1**, the first $\succeq^*$ characterizes impatience whereas the second one features T^*-*delay* stability. Otherwise stated, if a combination is robustly better than a constant sequence, it remains robustly better if it is moved forward into the future, the effect according to the order $\succeq^*$ becoming lower over time.

3.2 Representation of the close future pre-order $\succeq_c^*$

3.2.1 Fundamental Properties

Definition 3.1. *Let* $\succeq_c^*$ *be defined as*

$$x \succeq_c^* y \textit{ if and only if, for every } \lambda \in [0,1], z \in \ell_\infty, \lambda x + (1-\lambda)z \succeq_c \lambda y + (1-\lambda)z.$$

Using the same arguments as the ones developed for the proof of Lemma 2.1, the following characterization of the robustness order $\succeq^*$ becomes available: For every $x, y \in \ell_\infty$, $x \succeq_c^* y$ if and only if, for every $z \in \ell_\infty$, $x + z \succeq_c y + z$.

In Drugeon & Ha-Huy (2021a), for any sequences x and z, the value $I(x_{[0,T]}, z_{[T+1,\infty[})$ converges to $I(x)$ when T tends to infinity. However, this convergence is not uniform: indeed, even-though the distant order of $\succeq_c$ is trivial, the order $\succeq_c$ does not necessarily satisfy the usual *tail-insensitivity* condition of the literature. To ensure this property, the article considers axiom **A2**. Axiom **A2** is the *close future* version of well-known axioms—the *continuity at infinity* axiom of Chambers & Echenique (2018) or other axioms in the literature—ensuring a strong version of myopia and, moreover, the compactness of the weights set Ω_c when it belongs to ℓ_1.

Axiom A2. For every $0 < c < 1$, $T_0 \geq 1$, there exists $\hat{T}(c, T_0)$ such that, for every $T \geq \hat{T}(c)$,

$$\left(0\mathbb{1}_{[0,T_0-1]}, \mathbb{1}_{[T_0,T_0+\hat{T}]}, 0\mathbb{1}_{[T_0+\hat{T}+1,\infty[}\right) \succeq^* \left(0\mathbb{1}_{[0,T_0-1]}, c\mathbb{1}_{[T_0,T]}, 0\mathbb{1}_{[T+1,\infty[}\right).$$

Under axiom **A2**, the robust order $\succeq_c$ satisfies a *tail-insensitivy* property and the weights set Ω_c is *tight*, or *weakly compact* in ℓ_1.

LEMMA **3.1**. *Assume that axioms* **F**, **G1**, *and* **A2** *are satisfied. There exists a set* $\Omega_c \subset \ell_1$ *that is weakly compact and such that, for* $x, y \in \ell_\infty$, $x \succeq_c^* y$ *if and only if, for every* $\omega \in \Omega_c$,

$$\sum_{s=0}^{\infty} \omega_s x_s \geq \sum_{s=0}^{\infty} \omega_s y_s.$$

3.2.2 The Temporal Bias Representation of $\succeq_c^*$

Under **A1**, one can obtain the detailed characterization for the sets of probabilities Ω. We can begin with the order close future order $\succ_c$.

Axiom **A1** provides a characterization of the exposed points of the set Ω_c. From Theorem 4 in Amir & Lindentrauss (1968), a weakly compact convex set is indeed the convex hull of its exposed points.

For $x \in \ell_\infty$, define $\mathscr{C}(x)$ as the supremum of value $c \in \mathbb{R}$ such that $x \succeq_c^* c\mathbb{1}$.

AXIOM A3. Consider T^* in Axiom **A1** and $x \in \ell_\infty$. Let $c = \mathscr{C}(x)$. There exists $y \in \ell_\infty$ such that $x \succeq_c^* (c\mathbb{1}_{[0,T^*]}, y) \succeq_c^* c\mathbb{1}$ and for every $\hat{y}$ satisfying $\mathscr{C}\left((c\mathbb{1}_{[0,T^*]}, \hat{y})\right) = c$, one has

$$\mathscr{C}\left(\frac{1}{2}x + \frac{1}{2}(c\mathbb{1}_{[0,T^*]}, \hat{y})\right) > c \text{ if and only if } \mathscr{C}\left(\frac{1}{2}(c\mathbb{1}_{[0,T^*]}, y) + \frac{1}{2}(c\mathbb{1}_{[0,T^*]}, \hat{y})\right) > c.$$

This axiom states the existence of a T^*-*delay equivalence.* For any sequence x, there exists a *delay* sequence of y that is equivalent to x in hedging with another delay sequence. The chance to improve *robustly* the situation by hedging with delayed $\hat{y}$ is the same for x and for delayed y.

PROPOSITION 3.1. *Assume axioms* **F**, **G1**, *and* **A1**, **A2**.

(i) Stationarity *If* $T^* = 0$, *then there exists* $\mathscr{D} \in]0,1[$ *such that* Ω_c *is the convex hull of*

$$\left\{(1-\delta, (1-\delta)\delta, \ldots, (1-\delta)\delta^s, \ldots)\right\}_{\delta \in \mathscr{D}}.$$

(ii) Quasi-hyperbolic discounting *Consider the case* $T^* = 1$. *By adding axiom* **A3**, *then there exists* $\mathcal{D} \in]0,1[^2$ *such that* Ω_c *is the convex hull of*

$$\left\{(1-\delta_0, \delta_0(1-\delta), \delta_0\delta(1-\delta), \delta_0\delta^2(1-\delta), \ldots, \delta_0(1-\delta)\delta^s, \ldots)\right\}_{(\delta_0,\delta)\in\mathcal{D}}.$$

(iii) T^*-steps quasi hyperbolic discounting *Consider the general case for* T^*. *By adding axiom* **A3**, *there exists* $\mathcal{D} \in]0,1[^{T^*+1}$ *such that* Ω_c *is the convex hull of the probabilities:*

$$\left\{(1-\delta_0, \delta_0(1-\delta_1), \delta_0\delta_1(1-\delta_2), \ldots, \delta_0\delta_1\cdots\delta_{T^*-1}(1-\delta), \ldots \right.$$
$$\left. \ldots, \delta_0\delta_1\ldots\delta_{T^*-1}\delta^s(1-\delta), \ldots)\right\}_{(\delta_0,\delta_1,\ldots\delta_{T^*-1},\delta)\in\mathcal{D}}.$$

Chambers & Echenique (2018) instead impose an *indifference stationarity* axiom, which supposes that any x which is equivalent to a constant sequence $c\mathbb{1}$, x is equivalent to any convex combination between x and $(c\mathbb{1}_{[0,T]}, x)$, for any T. In a recent work, dealing with multiple temporal biased discount rates, Bich, Dong & Wigniolle (2021), dealing with the axiomatic system configuration as Chambers & Echenique (2018), generalise the *Invariance to stationary relabelling* to a one period *delay* ISTAT condition, and obtain a multiple quasi-hyperbolic discounting representation.

This article supposes another property, namely the axiom **A1**. The difference between the two mentioned articles and this one essentially springs from the fact that, while Chambers & Echenique (2018) and Bich, Dong & Wigniolle (2021) work on a complete order $\succeq$ and complete a Min representation of the index function, this article works on a partial order $\succeq^*$, corresponding to a larger family of possible orders and index functions, for example the $\alpha-$MaxMin representation. Unsurprisingly, the two different approaches involve two rather different systems of axioms.

3.3 Banach Limits and the Representation of the Distant Future Pre-Order $\succeq_d^*$

Following the same idea about the robustness order, one can define the robustness order $\succeq_d^*$ for the order $\succeq_d$. Since the order $\succeq_d$ does not take into account the present and the close future, the pre-order $\succeq_d^*$ satisfies the same property.

Definition 3.2. *Let $\succeq_d^*$ be defined as*

$$x \succeq_d^* y \quad \textit{if and only if} \, \forall \, \lambda \in [0,1], z \in \ell_\infty, \lambda x + (1-\lambda)z \succeq_d \lambda y + (1-\lambda)z.$$

Lemma **3.2**. *Assume axioms* **F**, **G1**. *There exists a weights set $\Omega_d \subset \ell_d^1$ such that $x \succeq_d^* y$ if and only if $\phi \cdot x \geq \phi \cdot y$ for every $\phi \in \Omega_d$.*

Under **G1**, one can also obtain important properties of the set of *charges* characterizing the pre-orders $\succeq_d^*$. It states that the set Ω_d builds from *Banach limits*. This recalls a similar property of *geometrical discounting* that, the comparison between two sequences does not depend on the chosen date.

Proposition 3.2. *Assume axioms* **F**, **G1** *and* **A1**. *Then any charge $\phi \in \Omega_d$ is Banach limit: for every $x \in \ell_\infty$,*

$$\phi \cdot x = \phi \cdot (0, x).$$

3.4 α–MaxMin Representation and the Robust Sub-Preorders

It is natural to assume that axiom **G2** applies separately on the sub orders and the optimism parameter may differ in the case of close future and distant future ones.

Axiom A4. *Prudence in the distant future* For any $x \in \ell_\infty$ and $d \in \mathbb{R}$,

(i) If there exist an infinite periods s such that $d > x_s$, then $x \not\succeq_d^* d\mathbb{1}$.

(ii) If there exist an infinite periods s such that $x_s > d$, then $d\mathbb{1} \not\succeq_d^* x$.

Axiom **A4** establishes a *prudence* property for the distant future order $\succeq_d$. The utilites stream dominates (or is dominated) by a constant one if and only if its utility values are all greater (or worse) in a sufficiently long future.

Proposition 3.3 summerizes the main results of this article. While axiom **G2** leads to an $\alpha-$MaxMin representation, axioms **A1** to **A4** complete the description of close and distant future orders.

PROPOSITION 3.3. *Assume that axioms* **F**, **G1** *are satisfied. Assume also that the property about optimism degree in axiom* **G2** *is satisfied for the close and future orders* $\succeq_c$ *and* $\succeq_d$.

(i) *There exist the constants* $a_c^*, a_d^* \in [0,1]$ *such that*

$$\mathrm{I}_c(x) = a_c^* \sup_{\omega\in\Omega_c} \sum_{s=0}^{\infty} \omega_s x_s + (1-a_c^*) \inf_{\omega\in\Omega_c} \sum_{s=0}^{\infty} \omega_s x_s,$$

$$\mathrm{I}_d(x) = a_d^* \sup_{\phi\in\Omega_d} \phi\cdot x + (1-a_d^*) \inf_{\phi\in\Omega_d} \phi\cdot x.$$

(ii) *Adding axioms* **A1**, **A2** *and* **A3**, *the close future index function can be represented as follows, with* $\mathcal{D} \subset]0,1[^{\mathrm{T}^*+1}$:

$$\begin{aligned}\mathrm{I}_c(x) = a_c^* &\max_{(\delta_0,\delta_1,\cdots,\delta_{\mathrm{T}^*-1},\delta)\in\mathcal{D}} \Bigg[(1-\delta_0)x_0 + \delta_0(1-\delta_1)x_1 + \cdots + \prod_{i=0}^{\mathrm{T}^*-2}\delta_i(1-\delta_{\mathrm{T}^*-1})x_{\mathrm{T}^*-1} \\ &\qquad + \prod_{i=0}^{\mathrm{T}^*-1}\delta_i(1-\delta)x_{\mathrm{T}^*} + \prod_{i=0}^{\mathrm{T}^*-1}\delta_i\delta(1-\delta)\sum_{s=0}^{\infty}\delta^s x_{\mathrm{T}^*+s}\Bigg] \\ + (1-a_c^*) &\min_{(\delta_0,\delta_1,\cdots,\delta_{\mathrm{T}^*-1},\delta)\in\mathcal{D}} \Bigg[(1-\delta_0)x_0 + \delta_0(1-\delta_1)x_1 + \cdots + \prod_{i=0}^{\mathrm{T}^*-2}\delta_i(1-\delta_{\mathrm{T}^*-1})x_{\mathrm{T}^*-1} \\ &\qquad + \prod_{i=0}^{\mathrm{T}^*-1}\delta_i(1-\delta)x_{\mathrm{T}^*} + \prod_{i=0}^{\mathrm{T}^*-1}\delta_i\delta(1-\delta)\sum_{s=0}^{\infty}\delta^s x_{\mathrm{T}^*+s}\Bigg].\end{aligned}$$

(iii) *Adding axiom* **A4**, *the distant future index function can be represented as:*

$$\mathrm{I}_d(x) = a_d^* \limsup_{s\to\infty} x_s + (1-a_d^*) \liminf_{s\to\infty} x_s.$$

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A. Proof of Lemma 2.1

Suppose that $x \geq^* y$, then and for every z, $(1/2)x+(1/2)z \geq (1/2)y+(1/2)z$. Recall that this is equivalent to $x+z \geq y+z$. Suppose that for every z, $x+z \geq y+z$. Fix any $0 \leq \lambda < 1$. Fix any $z \in \ell_\infty$. One has

$$x + \frac{\lambda}{1-\lambda} z \geq y + \frac{\lambda}{1-\lambda} z,$$

which implies the holding of $(1-\lambda)x + \lambda z \geq (1-\lambda y) + \lambda z$, whence the one of $x \geq^* y$. QED

B. Proof of Proposition 2.1

Define $\mathscr{P}^*$ as the positive polar cone of $\mathscr{P} = \left\{x \in \ell_\infty \text{ such that } x \geq^* 0\mathbb{1}\right\}$ in the dual space $\left(\ell_\infty\right)^*$:

$$\mathscr{P}^* = \left\{\mathrm{P} \in (\ell_\infty)^* \text{ such that } \mathrm{P} \cdot x \geq 0 \text{ for every } x \geq^* 0\right\}.$$

Observe that by the very definition of the order $\geq^*$, $\mathscr{P}$ is convex and separable by the vector $-\mathbb{1}$, the cone $\mathscr{P}^*$ does not degenerate to $\{0\}$.

For each $\mathrm{P} \in \mathscr{P}^*$, define

$$\pi(\mathrm{P}) = \frac{1}{\mathrm{P} \cdot \mathbb{1}} \mathrm{P}.$$

Since $x \geq^* 0\mathbb{1}$ for every $x \in \ell_\infty$ satisfying $x_s \geq 0$ for all s, it follows that $\mathrm{P} \cdot x \geq 0$ for every x such that $x_s \geq 0$ for every s. Let then $\Omega = \pi(\mathscr{P})$. As $\mathrm{P} \cdot x \geq 0$ if and only if $\pi(\mathrm{P}) \cdot x \geq 0$, $x \geq^* 0\mathbb{1}$ is equivalent to $\pi(\mathrm{P}) \cdot x \geq 0$ for every $\mathrm{P} \in \mathscr{P}$. For every P, $\pi(\mathrm{P})$ can be decomposed as $\pi(\mathrm{P}) = \lambda_c \omega + \lambda_d \phi$, where $\omega = (\omega_0, \omega_1, \cdots, \omega_s, \cdots) \in \ell_1$ and $\phi \in \ell_1^d$ is a finite additive measure: considering ϕ as a measure on $\mathbb{N}$, $\phi(A) = 0$ for every finite subset of $\mathbb{N}$. From the definition of Ω, for every $\left(\lambda_c \omega, \lambda_d \phi\right) \in \Omega$, $\lambda_c \sum_{s=0}^{\infty} \omega_s + \lambda_d \phi \cdot \mathbb{1} = 1$. The set Ω can be considered as a set of finite additive probabilities on $\mathbb{N}$. QED

C. Proof of Proposition 2.2

Consider $x, y \in \ell_\infty$ satisfying $b^*_x < b^{*x}$, $b^*_y < b^{*y}$: it is then to proved that $a_x = a_y$. Take a constant b sufficiently big such that $x + b\mathbb{1} \geq^* y$. One gets $b^{*,x+b\mathbb{1}} = b^*_x +$, $b^*_{x+b\mathbb{1}} = b^{*x} + b$ and $\mathrm{I}(x + b\mathbb{1}) = \mathrm{I}(x) + b$. This implies $a_{x+b\mathbb{1}} = a_x$, whence $a_x = a_{x+b\mathbb{1}} \leq a_y$. Take then a constant b' such that $y + b'\mathbb{1} \geq^* x$. Relying to the same arguments, $a_y = a_{y+b'\mathbb{1}} \leq a_x$, whence for every $x, y \in \ell_\infty$ such that $b^*_x < b^{*x}$ and $b^*_y < b^{*y}$, the satisfaction of $a_x = a_y$. QED

D. Proof of Lemma 3.1

Relying upon the same arguments as in the proof of Proposition 2.1, there exists a probability set $\Omega_c \subset \ell_1 \oplus \ell^1_d$ such that

$$x \geq^*_c y \Leftrightarrow (1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x \geq (1-\lambda)\sum_{s=0}^{\infty} \omega_s x_s + \lambda\phi \cdot x,$$

for every $((1-\lambda)\omega, \lambda\phi) \in \Omega_c$. Suppose that there exists $((1-\lambda)\omega, \lambda\phi) \in \Omega_c$ satisfying $\lambda\phi \neq 0$, or equivalently $\lambda\phi \cdot \mathbb{1} > 0$. Fix c such that $1 - \lambda < c < 1$.

From Lemma **A2**, applying with $\mathrm{T}_0 = 0$, there exists a large enough $\hat{\mathrm{T}}$ such that for $\mathrm{T} \geq \hat{\mathrm{T}}$,

$$\left(\mathbb{1}_{[0,\hat{\mathrm{T}}]}, 0\mathbb{1}_{[\hat{\mathrm{T}}+1,\infty[}\right) \geq^* \left(c\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right).$$

Whence, for every $z \in \ell_\infty$ and from the definition of the pre-order $\geq^*$,

$$\left(\mathbb{1}_{[0,\hat{\mathrm{T}}]}, 0\mathbb{1}_{[\hat{\mathrm{T}}+1,\infty[}\right) + \left(z_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) \geq \left(c\mathbb{1}_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right) + \left(z_{[0,\mathrm{T}]}, 0\mathbb{1}_{[\mathrm{T}+1,\infty[}\right).$$

This is true for every $\mathrm{T} \geq \hat{\mathrm{T}}$, so that, for every z,

$$\left(\mathbb{1}_{[0,\hat{\mathrm{T}}]}, 0\mathbb{1}_{[\hat{\mathrm{T}}+1,\infty[}\right) + z \geq_c c\mathbb{1} + z.$$

This implies that:

$$\left(\mathbb{1}_{[0,\hat{T}]}, 0\mathbb{1}_{[\hat{T}+1,\infty[}\right) \succeq_c^* c\mathbb{1}.$$

Hence,

$$\left((1-\lambda)\omega, \lambda\phi\right) \cdot \left(\mathbb{1}_{[0,\hat{T}]}, 0\mathbb{1}_{[\hat{T}+1,\infty[}\right) \geq c,$$

with a direct consequence that $1-\lambda \geq c$, a contradiction.

To sum up and for every $\left((1-\lambda), \lambda\phi\right) \in \Omega_c$, $\lambda\phi = 0$. Since Ω_c is a set of probabilities, this implies that $\lambda = 0$, and Ω_c can be considered as a subset of probabilites that is included in ℓ_1. With axiom **A2**, the set Ω_c can be considered as a set of tight measures, it is therefore compact in the weak topology. Otherwise stated, Ω_c is weakly compact in ℓ_1. QED

E. Proof of Proposition 3.1

The proof of this Proposition begins by a preparation Lemma. In Lemma E.1, under axiom **G1**, for each sequence $x \in \ell_\infty$, the value of the worst scenario corresponding to $(c^*\mathbb{1}_{[0,T^*]}, x)$, evaluated under the order $\succeq_c$, does neither change with the shift of the sequence to the future nor with a convex combination with this shift. In another words, beginning from T^*, the robust order satisfies a version of *stability* property.

For any $x \in \ell_\infty$, recall that $\mathscr{C}(x)$ is the supremum value c such that $x \succeq_c^* c\mathbb{1}$.

Lemma **E.1**. *Assume that axioms **F**, **G1**, and **A1**, **A2** are satisfied.*

(i) *for any constant c, $(c\mathbb{1}_{[0,T^*]}, x) \succeq_c^* c\mathbb{1}$ implies:*

$$(c\mathbb{1}_{[0,T^*]}, x) \succeq_c^* (c\mathbb{1}_{[0,T^*+1]}, x) \succeq_c^* (c\mathbb{1}_{[0,T^*+2]}, x) \succeq_c^* \ldots \succeq_c^* c\mathbb{1};$$

(ii) *let $c^* = \mathscr{C}((c^*\mathbb{1}_{[0,T^*]}, x))$, for any $T \geq T^*$, $\mathscr{C}\left(c^*\mathbb{1}_{[0,T]}, x\right) = c^*$;*

(iii) *for any $T \geq T^*$, $\mathscr{C}\left(\frac{1}{2}(c^*\mathbb{1}_{[0,T^*]}, x) + \frac{1}{2}\left(c^*\mathbb{1}_{[0,T]}, x\right)\right) = c^*$.*

Proof. (i) Consider $x \in \ell_\infty$, a constant c such that $(c\mathbb{1}_{[0,T^*]}, x) \succeq_c^* c\mathbb{1}$. First, observe that for $c' \leq c$, one has $(c'\mathbb{1}_{[0,T^*]}, x) \succeq_c^* c'\mathbb{1}$.

Indeed, for any $\omega \in \Omega_c$, we have $c\sum_{s=0}^{T^*}\omega_s + \sum_{s=T^*+1}^{\infty}\omega_s x_{s-T^*-1} \geq c$. This implies

$$\begin{aligned}\sum_{s=T^*+1}^{\infty}\omega_s x_{s-T^*-1} &\geq c\sum_{s=T^*+1}^{\infty}\omega_s \\ &\geq c'\sum_{s=T^*+1}^{\infty}\omega_s,\end{aligned}$$

which is equivalent to

$$c'\sum_{s=0}^{T^*}\omega_s + \sum_{s=T^*+1}^{\infty}\omega_s x_{s-T^*-1} \geq c'.$$

Since this inequality is verified for any $\omega \in \Omega_c$, the claim is proven.

Fix any $c' < c$ and $\epsilon > 0$ such that $c' + \epsilon < c$. Hence, $((c'+\epsilon)\mathbb{1}_{[0,T^*]}, x) \succeq_c^* (c'+\epsilon)\mathbb{1}$. For any $\omega \in \Omega_c$, any $T \geq T^*$, one has

$$c'\sum_{s=0}^{T^*}\omega_s + \epsilon\sum_{s=0}^{T^*}\omega_s + \sum_{s=T^*+1}^{T}\omega_s x_{s-T^*-1} + \sum_{s=T+1}^{\infty}\omega_s x_{s-T^*-1} \geq c' + \epsilon.$$

This is equivalent to

$$c'\sum_{s=0}^{T^*}\omega_s + \sum_{s=T^*+1}^{T}\omega_s x_{s-T^*-1} + \sum_{s=T+1}^{\infty}\omega_s x_{s-T^*-1} \geq c' + \epsilon\sum_{s=T^*+1}^{\infty}\omega_s.$$

From the compactness of Ω_c in Lemma 3.1, there exists $\hat{T}$ sufficiently large enough, such that for every $T \geq \hat{T}$, for every $\omega \in \Omega_c$,

$$\sum_{s=T+1}^{\infty}\omega_s x_{s-T^*-1} - c'\sum_{T+1}^{\infty}\omega_s < \epsilon\sum_{s=T^*+1}^{\infty}\omega_s.$$

Hence, for every $\omega \in \Omega_c$:

$$c'\sum_{s=0}^{T^*}\omega_s + \sum_{s=T^*+1}^{T}\omega_s x_{s-T^*-1} + \sum_{s=T+1}^{\infty}\omega_s x_{s-T^*-1} \geq c' + \sum_{s=T+1}^{\infty}\omega_s x_{s-T^*-1} - c'\sum_{T+1}^{\infty}\omega_s,$$

which is equivalent to

$$c' \sum_{s=0}^{T^*} \omega_s + \sum_{s=T^*+1}^{T} \omega_s x_{s-T^*-1} + c' \sum_{T+1}^{\infty} \omega_s \geq c'$$

We have, for $T \geq \hat{T}$, with the observation that these two sequences have the same *tail*,

$$(c'\mathbb{1}_{[0,T^*]}, x_{[0,T]}, c'\mathbb{1}) \succeq^* c'\mathbb{1}.$$

By axiom **A1**,

$$(c'\mathbb{1}_{[0,T^*]}, x_{[0,T]}, c'\mathbb{1}) \succeq^* (c'\mathbb{1}_{[0,T^*+1]}, x_{[0,T]}, c'\mathbb{1}) \succeq^* c'\mathbb{1}.$$

Once again, being based on the fact that the two sequences have the same *tail*, it derives that:

$$(c'\mathbb{1}_{[0,T^*]}, x_{[0,T]}, c'\mathbb{1}) \succeq_c^* (c'\mathbb{1}_{[0,T^*+1]}, x_{[0,T]}, c'\mathbb{1}) \succeq_c^* c'\mathbb{1}.$$

Let T converges to infinity:

$$(c'\mathbb{1}_{[0,T^*]}, x) \succeq_c^* (c'\mathbb{1}_{[0,T^*+1]}, x) \succeq_c^* c'\mathbb{1}.$$

Finally, and as c' was arbitrarily selected to be strictly smaller than c, by continuity, it derives that:

$$(c\mathbb{1}_{[0,T^*]}, x) \succeq_c^* (c\mathbb{1}_{[0,T^*+1]}, x) \succeq_c^* c\mathbb{1}.$$

(ii) Let $c^* = \mathscr{C}((c^*\mathbb{1}_{[0,T^*]}, x))$. From part (i), for $T \geq T^*$,

$$(c^*\mathbb{1}_{[0,T^*]}, x) \succeq_c^* (c^*\mathbb{1}_{[0,T]}, x) \succeq_c^* c^*\mathbb{1}.$$

This implies $c^* = \mathscr{C}\left(c^*\mathbb{1}_{[0,T^*]}, x\right) \geq \mathscr{C}\left(c^*\mathbb{1}_{[0,T]}, x\right) \geq c^*$.

(iii) Consider the sequences $(c^*\mathbb{1}_{[0,T^*]}, x)$ and $\left(c^*\mathbb{1}_{[0,T]}, x\right)$. From (i), it follows that:

$$\begin{aligned}(c^*\mathbb{1}_{[0,T^*]}, x) + \left(c^*\mathbb{1}_{[0,T]}, x\right) &\succeq_c^* c^*\mathbb{1} + \left(c^*\mathbb{1}_{[0,T]}, x\right) \\ &\succeq_c^* 2c^*\mathbb{1},\end{aligned}$$

whence

$$\mathscr{C}\left(\frac{1}{2}(c^*\mathbb{1}_{[0,T^*]}, x) + \frac{1}{2}(c^*\mathbb{1}_{[0,T]}, x)\right) \geq c^*.$$

Fix $c > c^*$. From the definition of c^*, there exists $z \in \ell_\infty$ such that $c\mathbb{1} + z \succ_c (c^*\mathbb{1}_{[0,T^*]}, x) + z$. This in its turn implies that $2c\mathbb{1} + 2z \succ_c 2(c^*\mathbb{1}_{[0,T^*]}, x) + 2z$. Hence,

$$\begin{aligned}2c\mathbb{1} + 2z &\succ_c 2(c^*\mathbb{1}_{[0,T^*]}, x) + 2z \\ &= (c^*\mathbb{1}_{[0,T^*]}, x) + (c^*\mathbb{1}_{[0,T^*]}, x) + 2z \\ &\succeq_c \left(c^*\mathbb{1}_{[0,T]}, x\right) + (c^*\mathbb{1}_{[0,T^*]}, x) + 2z.\end{aligned}$$

This implies that $\frac{1}{2}(c^*\mathbb{1}_{[0,T^*]}, x) + \frac{1}{2}(c^*\mathbb{1}_{[0,T]}, x) \not\succeq_c^* c\mathbb{1}$. Precisely, $c > \mathscr{C}\left(\frac{1}{2}(c^*\mathbb{1}_{[0,T^*]}, x) + \frac{1}{2}(c^*\mathbb{1}_{[0,T]}, x)\right)$. Since c was chosen arbitrarily bigger than c^*, it finally holds that

$$\begin{aligned}c^* &\geq \mathscr{C}\left(\frac{1}{2}(c^*\mathbb{1}_{[0,T^*]}, x) + \frac{1}{2}(c^*\mathbb{1}_{[0,T]}, x)\right) \\ &\geq c^*.\end{aligned}$$

QED

Now, return to the main part of the proof. For each probability $\omega = (\omega_0, \omega_1, \cdots) \in \ell_1$ and $T \geq 0$, let ω^T be the probability defined as

$$\omega_s^T = \frac{\omega_{T+s}}{\sum_{s'=0}^{\infty} \omega_{T+s'}}.$$

Let $\Omega^{T^*} = \left\{\omega^{T^*} \text{ such that } \omega \in \Omega_c\right\}$. Take $\omega \in \Omega_c$ such that ω^{T^*} is an exposed point of $\Omega_c^{T^*}$. We will establish that $\omega^{T^*} = (\omega^{T^*})^T$ for all $T \geq 0$.

By the definition of ω, there exists $x \in \ell_\infty$ such that $\omega^{T^*} \cdot x < \tilde{\omega}^{T^*} \cdot x$ for every

$\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Let $c^* = \omega^{T^*} \cdot x$. It is obvious that the following inequality is verified:

$$c^* = \omega \cdot \left(c^* \mathbb{1}_{[0,T^*]}, x\right) < \tilde{\omega} \cdot \left(c^* \mathbb{1}_{[0,T^*]}, x\right).$$

This implies that $\mathscr{C}\left(\left(c^* \mathbb{1}_{[0,T^*]}, x\right)\right) = c^*$ and $\left(c^* \mathbb{1}_{[0,T^*]}, x\right) \succeq_c^* c^* \mathbb{1}$. Fix $T \geq 0$ and from Lemma E.1,

$$\mathscr{C}\left(\frac{1}{2}\left(c^* \mathbb{1}_{[0,T^*]}, x\right) + \frac{1}{2}\left(c^* \mathbb{1}_{[0,T^*+T]}, x\right)\right) = c^*.$$

This implies that there exists ω' such that

$$c^* = \omega' \cdot \left(\frac{1}{2}\left(c^* \mathbb{1}_{[0,T^*]}, x\right) + \frac{1}{2}\left(c^* \mathbb{1}_{[0,T^*+T]}, x\right)\right) = \min_{\omega \in \Omega_c} \omega \cdot \left(\frac{1}{2}\left(c^* \mathbb{1}_{[0,T^*]}, x\right) + \frac{1}{2}\left(c^* \mathbb{1}_{[0,T^*+T]}, x\right)\right).$$

By (i), $\omega' \cdot \left(c^* \mathbb{1}_{[0,T^*]}, x\right) \geq c^*$ and $\omega' \cdot \left(c^* \mathbb{1}_{[0,T^*+T]}, x^*)\right) \geq c^*$. It follows that

$$\omega' \cdot \left(c^* \mathbb{1}_{[0,T^*]}, x\right) = \omega' \cdot \left(c^* \mathbb{1}_{[0,T^*+T]}, x^*\right) = c^*.$$

Hence:

$$(\omega')^{T^*} \cdot x = c^*,$$
$$(\omega')^{T^*} \cdot \left(c^* \mathbb{1}_{[0,T^*+T]}, x\right) = c^*.$$

Since ω^{T^*} is an exposed point of $\Omega_c^{T^*}$, the first equality implies that $(\omega')^{T^*} = \omega^{T^*}$. Oserve that $\omega \cdot \left(c^* \mathbb{1}_{[0,T^*+T]}, x\right) = c^*$ is equivalent to $(\omega^{T*})^T \cdot x = c^*$. Moreover, $(\omega^{T*})^T$ belongs to $\Omega_c^{T^*}$. Indeed, suppose the contrary: from the weakly compactness of $\Omega_c^{T^*}$, there exists $\epsilon > 0$ such that the intersection between $\Omega_c^{T^*}$ and the open set $\left\{\tilde{\omega} \text{ such that } \left\|\tilde{\omega} - (\omega^{T*})^T\right\|_{\ell_1} < \epsilon\right\}$ is empty. By the Hahn-Banach theorem, there exists x' and a constant c such that $\tilde{\omega}^{T^*} \cdot x' > c > \omega^T \cdot x'$ for every $\omega \in \Omega_c$. This implies that $\left(c\mathbb{1}_{[0,T^*]}, x'\right) \succeq_c^* c\mathbb{1}$ and therefore that $\left(c\mathbb{1}_{[0,T^*]}, x'\right) \succeq_c^* \left(c\mathbb{1}_{[0,T^*+T]}, x'\right) \succeq_c^* c\mathbb{1}$, whence $\omega \cdot \left(c\mathbb{1}_{[0,T^*+T]}, x'\right) \geq c$, which is equivalent to $(\omega^{T*})^T \cdot x' \geq c$, a contradiction. The probability $(\omega^{T*})^T$ belongs to $\Omega_c^{T^*}$, and satisfies $(\omega^{T*})^T \cdot x = c^*$. From the definition of ω^{T^*} and x, $\omega^{T^*} = (\omega^{T^*})^T$, for every $T \geq 0$. It follows that

$$\omega_s^{T^*} = \frac{\omega_{T^*+T+s}}{\sum_{s'=0}^{\infty} \omega_{T^*+T+s'}} \text{ and } \omega_{s+1}^{T^*} = \frac{\omega_{T^*+T+s+1}}{\sum_{s'=0}^{\infty} \omega_{T^*+T+s'}}.$$

This implies, for every T, s, that:

$$\frac{\omega_{s+1}^{T^*}}{\omega_s^{T^*}} = \frac{\omega_{T^*+T+s+1}}{\omega_{T^*+T+s}}.$$

This is equivalent, for some $\delta > 0$ and for every $s \geq 0$, to

$$\frac{\omega_{s+1}^{T^*}}{\omega_s^{T^*}} = \delta,$$

or to $\omega_s^{T^*} = \delta^s \omega_0^{T^*}$ for every $s \geq 0$. Since $\sum_{s=0}^{\infty} \omega_s^{T^*} = 1$, it eventually follows that $0 < \delta < 1$ and $\omega_s = (1-\delta)\delta^s$ for $s \geq 0$.

To sum up, every exposed point of $\Omega_c^{T^*}$ has a geometrical representation. The set $\Omega_c^{T^*}$ being weakly compact, by Theorem 4 in Amis & Lindenstrauss Amir & Lindentrauss (1968), $\Omega_c^{T^*}$ is the convex hull of its exposed points. This implies the existence of a subset $D^* \in]0,1[$ such that

$$\Omega_c^{T^*} = \text{convex}\{(1-\delta, (1-\delta)\delta, \ldots, (1-\delta)\delta^s, \ldots)\}_{\delta \in D^*}.$$

The part (i), where $T^* = 0$ is proven.

Consider the case $T^* \geq 1$. Observe that if ω^{T^*} is an exposed point of $\Omega_c^{T^*}$, then ω is an exposed point of Ω_c. Indeed, in that case, there exists $x \in \ell_\infty$ such that $\omega^{T^*} \cdot x < \tilde{\omega}^{T^*} \cdot x$ for every $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Let $c = \omega^{T^*} \cdot x$. It is easy to verify that $c = \omega \cdot (c\mathbb{1}_{[0,T^*]}, x))$ and $c < \tilde{\omega} \cdot (c\mathbb{1}_{[0,T^*]}, x)$. Hence, ω is an exposed point of Ω_c.

Consider an exposed point ω of Ω_c. We will prove that ω^{T^*} is an exposed point of $\Omega_c^{T^*}$.

By the choice of ω, there exists $x \in \ell_\infty$ such that $\omega \cdot x < \tilde{\omega} \cdot x$, for every $\tilde{\omega} \in \Omega_c \setminus \{\omega\}$. Let $c = \mathscr{C}(x) = \omega \cdot x$. Consider the utilities stream y, which is a T−delay equivalence of x, being defined in the statement of axiom **A3**. Using the same arguments as in the proof of Lemma E.1, we obtain

$$\mathscr{C}\left(\frac{1}{2}x+\frac{1}{2}\left(c\mathbb{1}_{[0,\mathrm{T}^*]},y\right)\right)=c,\ \text{and}\ \omega\cdot x=\omega\cdot\left(c\mathbb{1}_{[0,\mathrm{T}^*]},y\right)=c.$$

Since $\mathscr{C}\left(\left(c\mathbb{1}_{[0,\mathrm{T}^*]},y\right)\right)=c$, for every $\tilde{\omega}\in\Omega_c$, $\tilde{\omega}\cdot\left(c\mathbb{1}_{[0,\mathrm{T}^*]},y\right)\geq c$, which is equivalent to $\tilde{\omega}^{\mathrm{T}^*}\cdot y\geq c$. We prove that for every exposed point $\hat{\omega}^{\mathrm{T}^*}$ of $\Omega_c^{\mathrm{T}^*}$ that is not ω^{T^*}, $\hat{\omega}^{\mathrm{T}^*}\cdot y>c$.

Assume the contrary, and consider such a point $\hat{\omega}^{\mathrm{T}^*}$, which is an exposed point and $\hat{\omega}^{\mathrm{T}^*}\cdot y=c$. There exists y' such that $\hat{\omega}^{\mathrm{T}^*}\cdot y'<\tilde{\omega}^{\mathrm{T}^*}\cdot y'$, for every $\tilde{\omega}^{\mathrm{T}^*}\in\Omega^{\mathrm{T}^*}\setminus\{\hat{\omega}^{\mathrm{T}^*}\}$, including $\omega^{\mathrm{T}*}$. Let $\hat{y}=y'+\left(c-\hat{\omega}^{\mathrm{T}^*}\cdot y'\right)\mathbb{1}$. The sequence $\hat{y}$ satisfies

$$c=\hat{\omega}^{\mathrm{T}^*}\cdot\hat{y}<\tilde{\omega}^{\mathrm{T}^*}\cdot\hat{y},$$

for every $\tilde{\omega}^{\mathrm{T}^*}\in\Omega^{\mathrm{T}^*}\setminus\{\hat{\omega}^{\mathrm{T}^*}\}$, including $\hat{\omega}^{\mathrm{T}^*}$. Moreover, for every $\tilde{\omega}^{\mathrm{T}^*}\in\Omega_c^{\mathrm{T}^*}$,

$$\tilde{\omega}^{\mathrm{T}*}\cdot\left(\frac{1}{2}y'+\frac{1}{2}\hat{y}\right)\geq c,$$

with the equality being obtained at $\tilde{\omega}^{\mathrm{T}^*}=\hat{\omega}^{\mathrm{T}^*}$. One has

$$\mathscr{C}\left(\frac{1}{2}\left(c\mathbb{1}_{[0,\mathrm{T}^*]},y\right)+\frac{1}{2}\left(c\mathbb{1}_{[0,\mathrm{T}^*]},\hat{y}\right)\right)=c.$$

As an opposition to this, the inequality $\omega^{\mathrm{T}^*}\cdot y'>c$ implies $\omega\cdot\left(c\mathbb{1}_{[0,\mathrm{T}^*]},y'\right)>c$, and

$$\omega\cdot\left(\frac{1}{2}x+\frac{1}{2}\left(c\mathbb{1}_{[0,\mathrm{T}^*]},\hat{y}\right)\right)>c.$$

For any $\tilde{\omega}\in\Omega_c\setminus\{\omega\}$, $\tilde{\omega}\cdot x>c$. Hence, the satisfaction of the same *strict* inequality

$$\begin{aligned}\tilde{\omega}\left(\frac{1}{2}x+\frac{1}{2}\left(c\mathbb{1}_{[0,\mathrm{T}^*]},\hat{y}\right)\right)&=\frac{1}{2}\tilde{\omega}\cdot x+\frac{1}{2}\tilde{\omega}\cdot\left(c\mathbb{1}_{[0,\mathrm{T}^*]},\hat{y}\right)\\&>c.\end{aligned}$$

The compactness of Ω_c implies that $\mathscr{C}\left(\frac{1}{2}x+\frac{1}{2}\left(c\mathbb{1}_{[0,\mathrm{T}^*]},\hat{y}\right)\right)>c$, a contradiction.

This contradiction ensures that for every $\hat{\omega}^{\mathrm{T}^*}\in\Omega_c^{\mathrm{T}^*}\setminus\{\omega^{\mathrm{T}^*}\}$, one has $\hat{\omega}^{\mathrm{T}^*}\cdot y>c$. This implies ω^{T^*} is an exposed point of $\Omega_c^{\mathrm{T}^*}$, and has a geometrical representation with some discount rate δ. It is easy to find $\delta_0, \delta_1, \ldots, \delta_{T^*-1}$ such that $\omega_0 = 1 - \delta_0, \omega_1 = \delta_0(1-\delta_1), \ldots, \omega_{T^*-1} = \delta_0\delta_1 \ldots \delta^{T^*-1}(1-\delta)$ and $\omega_{T+s} = \delta_0\delta_1 \ldots \delta_{T^*-1}\delta \times \delta^s(1-\delta)$, for $s \geq 0$.

The set Ω_c being the convex hull of its exposed points, the proof is completed.

QED

F. Proof of Lemma 3.2

Define $\mathscr{P}^d$ the set of $x \in \ell_\infty$ such that $x \succeq_d^* 0\mathbb{1}$, denote by $\mathscr{P}^{d*}$ its positive polar cone and let

$$\Omega_d = \left\{ \frac{1}{P \cdot \mathbb{1}} P \text{ with } P \in \mathscr{P}^{d*} \right\}.$$

It is first claimed that, for all $\big((1-\lambda)\omega, \lambda\phi\big) \in \Omega_d$, $(1-\lambda)\omega = 0$. Suppose the opposite. Then there exists T such that $\omega_T > 0$. Take a constant $c > 0$ such that $(1-\lambda)\omega_T c > \lambda$ and let $x = \big(-c\mathbb{1}_{[0,T]}, \mathbb{1}\big)$. For every $z \in \ell_\infty$ one has

$$\begin{aligned} I_d(x+z) &= I_d(\mathbb{1}+z) \\ &= 1 + I_d(z) \\ &> I_d(z), \end{aligned}$$

whence $x \succeq_d^* 0\mathbb{1}$. Then

$$(1-\lambda)\omega \cdot x + \lambda\phi \cdot x \geq 0,$$

which implies $-(1-\lambda)\omega_T c + \lambda \geq 0$, a contradiction. Whence the satisfaction of $(1-\lambda)\omega = 0$, which also implies the holding of $\lambda = 1$. To sum up, the weights set Ω_d can therefore be considered as a subset of *charges* belonging to ℓ_d^1. QED

G. Proof of Proposition 3.2

Fix $d \leq \inf_{s \geq 0} x_s$. Obviously, for every $T \geq 0$, $(d\mathbb{1}_{[0,T]}, x_{[T+1,\infty)}) \succeq^* d\mathbb{1}$. It follows that

$$\left(d\mathbb{1}_{[0,T]}, x_{[T+1,\infty)}\right) \succeq^* \left(d\mathbb{1}_{[0,T+1]}, x_{[T+1,\infty)}\right) \succeq^* d\mathbb{1}.$$

As a consequence of this, for every $z \in \ell_\infty$,

$$\left(d\mathbb{1}_{[0,T]}, x_{[T+1,\infty)}\right) + z \succeq \left(d\mathbb{1}_{[0,T+1]}, x_{[T+1,\infty)}\right) + z.$$

Rewritting this inequality, for every $T \geq 0$,

$$\begin{aligned}&\left((d+z)\mathbb{1}_{[0,T]}, x_{T+1} + z_{T+1}, x_{T+2} + z_{T+2}, \ldots\right)\\ &\succeq \left((d+z)\mathbb{1}_{[0,T]}, d + z_{T+1}, x_{T+1} + z_{T+2}, x_{T+2} + z_{T+3}, \ldots\right).\end{aligned}$$

From the very definition of the distant future order $\succeq_d$, this implies

$$x + z \succeq_d \left(d, x\right) + z.$$

The inequality being verified for every $z \in \ell_\infty$, it follows that

$$x \succeq_d^* \left(d, x\right).$$

Recall that for every charge ϕ belonging to Ω_d, $\phi \cdot \left(d, x\right) = \phi \cdot \left(0, x\right)$, one gets

$$\phi \cdot x \geq \phi \cdot \left(0, x\right).$$

By applying the same arguments with $-x$ in the place of x, and $d < -\sup_{s \geq 0} x_s$, it follows that $\phi \cdot (-x) \geq \phi \cdot \left(0, -x\right)$. Hence,

$$\phi \cdot x = \phi \cdot \left(0, x\right).$$

H. Proof of Proposition 3.3

To prove part (i), one can use the same arguments as in the proof of Proposition 2.2. Part (ii) is a direct consequence of Propositions 2.2, 3.1 and 3.2.

Consider part (iii). First, observe that for every charge $\phi \in \Omega_d$, $x \in \ell_\infty$, one has

$$\liminf_{s \to \infty} x_s \leq \phi \cdot x \leq \limsup_{s \to \infty} x_s.$$

Axiom **A4** implies that $\inf_{\phi \in \Omega_d} \phi \cdot x = \liminf_{s \to \infty} x_s$ and $\sup_{\phi \in \Omega_d} \phi \cdot x = \limsup_{s \to \infty} x_s$. Part (iii) is then a direct consequence of (i).