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Abstract

This study adresses time-dependent orders that are shown to lead to recursive representations based upon a Max-Min dichotomy and introduce a structure that is naturally based upon time-varying multiple discounts. It is argued that this setup naturally provides an enriched understanding of the much discussed present biases. It is established how a multiple discounts version of *present biais* becomes available and directly builds upon the features of the order defined from the head of the utilities sequence.

Keywords: Axiomatization, Time-Dependent Orders, Time-Varying Multiple Discounts, Multiple Present Biases.

JEL Classification: D11, D90.

Contents

1. Introduction

As a result from the strong evidence documented by numerous laboratory experiments, a large set of the recent literature on the evaluation of utility streams has been interested in the time inconsistencies and present biases that result from the consideration of time-dependencies in a preferences order, the *quasi-hyperbolic* acception of inter-temporal utility starting to increasingly question the relevance of the standard discounted criterion. It is the aim of this study to examine whether the consideration of a related time-dependent preference orders can be reconciled with the *multiple discounts* understanding that have gained increasing popularity in the recent literature[1].

The analysis proceeds by considering a list of standard axioms that result in the possibility of representing the preferences order by a constantly additive homogeneous of degree one utilities index function. This is supplemented by assuming that the evaluation of the head of the utility stream can be completed independently of its tail. Time-dependency is in turn introduced by retaining a variation of the classical limited dependence axiom for which the evaluation at a given date does not depend on the head of the utilities sequence. It is then shown that this evaluation consists of a *recursive convex sum* between the utility level at that date, and the evaluation at the subsequent date of the utility stream, a *multitude of choices* remaining however admissible for the weight parameters of this convex sum, or in another words, the possible discount rates chosen to evaluate the futre.

In contradistinction with the earlier contributions that hinged upon some form of stationarity from a given date, the approach of this article is specific in postulating that the components of two utility streams starting from a given date can be compared independently from their earlier past components, that happens to be core to the emergence of multiple discount rates.

Facing then with the scope for temporal biases, some behaviour shall be labelled as *present biased* when, according to the perception of a given agent, the *temporal distance* between two successive dates it decreasing over time. This more specif-

[1] *Vide* Chambers & Echenique [4] or Drugeon, Ha-Huy & Nguyen [6].

ically means that the *optimal* discount factor is increasing and this is shown to result from an specific axiom that further constrains the range of admissible time-dependent orders. Otherwise stated, the *temporal distance* that is perceived between two successive dates in a immediate future is larger that the one that is perceived between two successive dates in a more remote future.

The results of this study compare with the earlier literature as follows. First, and as for the multiple discounts dimension of this article, following parallel roads and the decision theory multiple priors axiomatizations of Gilboa & Schmeidler [10] but relying upon a different system of axioms based upon *time-variability aversion,* Wakai [15] has provided an insightful account of smoothing behaviours where the optimal discount assumes an maxmin recursive representation. Also related with the current study with an analysis completed over the set of bounded real sequences, Chambers & Echenique [4] have recently put forth an axiomatic approach to multiple discounts. Following the same axiomatic configuration, Bich *et al* [1] also come to a MaxMin representation of the index function, with a generalisation of *quasi-hyperbolic* discounting.

Second, and as for the temporal biases dimension of this study, the most influencial temporal inconsistencies tradition dates back Phelps & Pollack [14] and got ravitalized by the works of Laibson [12] and Frederick, Loewenstein & O'Donoghue [8] under the so-called *quasi-hyperbolic discounting hypothesis.* Numerous experiments having supported the accuracy of this formulation, Montiel Olea & Strzalecki [13] have completed an axiomatic approach to the *quasi-hyperbolic discounting* representation and, more generally, to *present biased* preferences. They suppose that, for any two equivalent future sequences, a patient one and an impatient one, pushing both of them towards the present will distort the preference towards the impatient choice.

In that respect, it is to be emphasized that this article assumes the *present bias* notion for every given date and not only for the initial one. The associated utilities index functions at that date are further determined from a set of multiple discount rates. The *present bias* acception of this article must hence incorporate contain two separate dimensions, the first one relating to the upper bound of discount rates and the second one to the lower bound of discount rates.

Finally, Chakraborty [3] has just completed a generalized appraisal of present bias within the Fishburn & Rubinstein [7] approach where preferences are defined on the realization of a single outcome at a given date. Even though it builds from a from an approach that differs from the current *utility streams* appraisal, his *weak present bias* axiom **A4** shares some ressemblance with the current *decreasing temporal distance* axiom **B2**.

This study is organised as follows. Section two will illustrate how the introduction of time-dependencies in the preference order will provide a recursive representation that provides a new picture of multiple temporal biases in Section three. These are enriched and extended in Section four. The proofs are given in the Appendix.

2. Some basic Axioms and a Recursive Min-Max Representation

2.1 Fundamentals, Basic Axioms & the Construction of an Index Function

This study contemplates an axiomatization approach to the evaluation of infinite utility streams, the whole argument being cast for discrete time sequences. In order to avoid any confusion, letters like x, y, z will be used for sequences (of utils) with values in $\mathbb{R}$ while a notation $c\mathbf{1}$, $c'\mathbf{1}$, $c''\mathbf{1}$ will be used for constant sequences, the notation $\mathbf{1}$ being retained for the constant unitary sequence $(1, 1, \ldots,)$. In parallel to this, greek letters λ, η, μ will be preferred for constant scalars.

Recall first that the space of ℓ_∞ is defined as the set of real sequences $\{x_s\}_{s=0}^{\infty}$ such that $\sup_{s\geq 0} |x_s| < +\infty$. For every $x \in \ell_\infty$ and $T \geq 0$, let $x_{[0,T]} = (x_0, x_1, \ldots, x_T)$ denote its $T+1$ first components, $x_{[T+1,\infty[} = (x_{T+1}, x_{T+2}, \ldots)$ its *tail* starting from date $T+1$ and, finally, $(x_{[0,T]}, y_{[T+1,+\infty[}) = (x_0, x_1, \ldots, x_T, y_{T+1}, y_{T+2}, \ldots)$ that considers the $T+1$ first elements of the sequence x and the $T+1$-tail of the sequence y. Finally, $(z_{[0,T]}, x) = (z_0, z_1, \ldots, z_T, x_0, x_1, x_2, \ldots)$.

The following axiom introduces some fundamental properties for the order $\succeq$ on ℓ_∞.

Axiom **F.** The order $\succeq$ satisfies the following properties:

(i) *Completeness* For every $x, y \in \ell_\infty$, either $x \succeq y$ or $y \succeq x$.

(ii) *Transitivity* For every $x, y, z \in \ell_\infty$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. Denote as $x \sim y$ the case where $x \succeq y$ and $y \succeq x$. Denote as $x \succ y$ the case where $x \succeq y$ and $y \not\succeq x$.

(iii) *Monotonicity* If $x, y \in \ell_\infty$ and $x_s \geq y_s$ for every $s \in \mathbb{N}$, then $x \succeq y$.

(iv) *Non-triviality* There exist $x, y \in \ell_\infty$ such that $x \succ y$.

(v) *Archimedeanity* For $x \in \ell_\infty$ and $b\mathbf{1} \succ x \succ b'\mathbf{1}$, there are $\lambda, \mu \in [0, 1]$ such that

$$(1-\lambda)b\mathbf{1} + \lambda b'\mathbf{1} \succ x \text{ and } x \succ (1-\mu)b\mathbf{1} + \mu b'\mathbf{1}.$$

(vi) *Weak convexity* For every $x, y, b\mathbf{1} \in \ell_\infty$, and $\lambda \in]0, 1]$,

$$x \succeq y \Leftrightarrow (1-\lambda)x + \lambda b\mathbf{1} \succeq (1-\lambda)y + \lambda b1.$$

(vii) *Tail-insensitivity* For any $x, y, z \in \ell_\infty$, $\epsilon > 0$, there exists $T_0(\epsilon)$ such that, for any $T \geq T_0(\epsilon)$,

$$\big(x_{[0,T]}, y_{[T+1,\infty[}\big) \succeq \big(x_{[0,T]}, z_{[T+1,\infty[}\big) - \epsilon\mathbf{1}.$$

Now, and under conditions $(i) - (vi)$, the order $\succeq$ can be represented by an index function I which is homogeneous of degree 1 and constantly additive. More formally, and for $x \in \ell_\infty$, $\lambda > 0$, $I(\lambda x) = \lambda I(x)$ and, for $x \in \ell_\infty$ and a constant $b \in \mathbb{R}$, $I(x + b\mathbf{1}) = I(x) + b$.[2]

The *tail-insensitivity* condition implies that for any $x, y \in \ell_\infty$,

$$\lim_{T\to\infty} I\big(x_{[0,T]}, y_{[T+1,\infty[}\big) = I(x).$$

Usual conditions in the literature typically assume that the effect of the tail utilities converges to zero—*e.g.*, the *Continuity at infinity* of Chambers & Echenique [4], or

[2]For detailed comments about the signification of the axioms and the index function, see Drugeon & Ha-Huy [5].

the axioms ensuring insensitivity to the tail of a given sequence, or some sort of *negligible tail* for the distribution[3] .

2.2 Time-Dependency & the Emergence of a Max-Min Recursive Representation

The following axiom assumes that the decision maker can always evaluate the intertemporal stream starting from a certain date and that such an evaluation will take place independently from the previous values of the stream.

Axiom **B1.** Consider $T \geq 1$, $x \in \ell_\infty$ and a constant $c \in \mathbb{R}$. Either, for any $z \in \ell_\infty$,

$$\big(z_{[0,T-1]}, x_{[T,\infty)[}\big) \succeq \big(z_{[0,T-1]}, c\mathbf{1}_{[T,\infty[}\big),$$

or, for any $z \in \ell_\infty$,

$$\big(z_{[0,T-1]}, c\mathbf{1}_{[T,\infty)}\big) \succeq \big(z_{[0,T-1]}, x_{[T,\infty[}\big).$$

This axiom contemplates a variation of the classical *limited independence* condition of Koopmans [11] where the evaluation of some date T does not depend on the head of the utilities sequence. Either the sequence $x_{[T,+\infty[}$ dominates the constant sequence independently from the head of the utility sequences, or it is dominated by the constant sequence independently from the head of the utility sequences.

The analysis of the time-dependent order $\succeq_T$ rests upon the one of the properties of $\succeq$.

Definition **2.1.** *For any $x, y \in \ell_\infty$, the temporal order $\succeq_T$ is defined as: $x \succeq_T y$ if and only if for any $z_0, z_1, \ldots, z_{T-1}$,*

$$\big(z_{[0,T-1]}, x\big) \succeq \big(z_{[0,T-1]}, y\big).$$

Under axiom **B1**, the head utilities evaluation after some date T can be represented by the order $\succeq_T$ and an index function I_T satisfying axiom **F**.

Proposition **2.1.** *Assume that the order $\succeq$ satisfies axioms* **F** *and* **B1***.*

[3] As a basic illustration, consider the order represented by the index function $I(x) = (1-\delta)\sum_{s=0}^{\infty}\delta^s x_s$, for some $0 < \delta < 1$. Such an order satisfies both **F** and $\chi_g = \chi_\ell = 0$.

(i) *For every $T \geq 1$, the order $\succeq_T$ is complete.*

(ii) *If at least one of the two values $I\big(0\mathbf{1}_{[0,T-1]}, \mathbf{1}_{[T,\infty[}\big)$ and $-I\big(0\mathbf{1}_{[0,T-1]}, -\mathbf{1}_{[T,\infty[}\big)$ differs from zero, then the order $\succeq_T$ satisfies the axiom* **F** *and can be represented by an index function I_T satisfying* positive homogeneity *and* constant additive *properties:*

   a) $I_T(\lambda x) = \lambda I_T(x)$, *for every* $\lambda \geq 0$.

   b) $I_T(x + c\mathbf{1}) = I_T(x) + c$, *for every constant* $c \in \mathbb{R}$.

Define χ_g^T and χ_ℓ^T as the parameters measuring the gain and the loss in the future of T.

$$\chi_g^T = I_T\big(0, \mathbf{1}\big),$$
$$\chi_\ell^T = -I_T(0, -\mathbf{1}) = 1 - I_T\big(\mathbf{1}, 0\mathbf{1}\big).$$

First remark that the properties of these parameters directly result from the ones of the head utilities index. Indeed, $\chi_g^T > 0$ if and only if $I\big(0\mathbf{1}_{[0,T-1]}, \mathbf{1}_{[T,\infty[}\big) > 0$ while $\chi_\ell^T > 0$ if and only if $-I\big(0\mathbf{1}_{[0,T-1]}, -\mathbf{1}_{[T,\infty[}\big) > 0$. For the sake of simplicity, and by convention, in the case $\chi_g^T = \chi_l^T = 0$,[4] the order $\succeq_{T+1}$ becoming trivial, the temporal index function I_{T+1} is defined as: $I_{T+1}(x) = 0$ for any $x \in \ell_\infty$.

Proposition **2.2.** *Assume that the order $\succeq$ satisfies axioms* **F** *and* **B1**. *Let $\underline{\delta}_T = \min\{\chi_g^T, \chi_\ell^T\}$ and $\overline{\delta}_T = \max\{\chi_g^T, \chi_\ell^T\}$.*

(i) *If $\chi_g^T \leq \chi_\ell^T$, then for any $x \in \ell_\infty$:*

$$I_T\big(x_{[T,\infty[}\big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \big[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\big].$$

(ii) *If $\chi_g^T \geq \chi_\ell^T$, then for any $x \in \ell_\infty$:*

$$I_T\big(x_{[T,\infty[}\big) = \max_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \big[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\big].$$

At any given date, the evaluation of a utility stream builds from a *recursive convex sum* between the utility level at that date, and the evaluation at the subsequent

[4] This implies that the order $\succeq_{T'}$ is also trivial, for any $T' \geq T+1$: $\chi_g^{T'} = \chi_\ell^{T'} = 0$.

date of the utility stream. Interestingly, a multitude of choices are shown to be admissible for the weight parameters of this convex sum. The minimum solution hereby represents a configuration where the value of the future beyond some date T is not large enough to compensate the lost that is incurred in present, the maximum solution representing the opposite occurrence. It is finally to be stressed that, relying upon a system of axioms based upon *time-variability aversion,* Wakai [15] has provided an insightful account of smoothing behaviours with *gain/loss asymmetry* which explicitly builds upon a related recursive representation.

COROLLARY 2.1. *Assume that for any $x, y \in \ell_\infty$ such that if $x \sim y$, one has $\frac{1}{2}x + \frac{1}{2}y \succeq x$, then for any T,*

$$I_T\big(x_{[T,\infty[}\big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} [(1-\delta)x_T + \delta I_{T+1}\big(x_{[T+1,\infty[}\big)].$$

The scope for separability being however central to this study, it is of interest to emphasize its specificity with respect to its earlier acceptions in the literature. Indeed, both the classical approach of Koopmans [11] or the more recent axiomatization of quasi-hyperbolic discounting due to Montiel Olea & Strzalecki [13] assume that the first or the first and the second components of two utility streams can be compared independently from its future components. Together with *stationarity* or *quasi-stationarity* postulates on the preferences ordering, these imply the existence of unique discount rate for every day or every generation, such a discount rate being constant for any $T \geqslant 0$ with *stationarity*, or constant from $T = 1$ to infinity with a *quasi-stationarity.*

In contradistinction with this, the approach of this article postulates that the components of two utility streams starting from a given date can be compared independently from their earlier past components, that gives rise to the possibility of multiple discount rates. The following example further proves that, for multiple discount rates, neither the *independence* nor the *extended independence* of Koopmans [11] are satisfied, the two approaches hence fundamentally differing, be it in their formulation or in their predictions.

EXAMPLE **2.1.** Consider the configuration where for any $T \geq 0$, $\underline{\delta}_T = 0.5$, $\overline{\delta}_T = 0.8$

and the operator is *min*. For any T, the order $\succeq_T$ is represented by

$$I_T\big(x_{[T,\infty[}\big) = \min_{0.5\leq\delta\leq 0.8}\left[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\right].$$

Consider these two utility streams $x = (1, 0, 0, 0, \ldots)$ and $y = (0.5, 0.5, 0, 0, \ldots)$. Obviously, $I_1\big(x_{[1,\infty[}\big) = 0$, and hence, $I_0(x_{[0,\infty[}) = (1-0.8)\times 1+0.8\times 0 = 0.2$. Similarly, since $I_1\big(y_{[1,\infty[}\big) = (1-0.8)\times 0.5 + 0.8\times 0 = 0.1$, one has $I_0\big(y_{[0,\infty[}\big) = (1-0.8)\times 0.5 + 0.8\times 0.1 = 0.18$. This implies $x \succ y$. Now consider $x' = (1, 0, 0.5, 0.5, \ldots)$ and $y' = (0.5, 0.5, 0.5, 0.5, \ldots)$. The two sequences x and y are changed by keeping the first two components intact. Obviously, $I_0(y'_{[0,\infty[}) = 0.5$. Calculus give $I_1\big(x'_{[1,\infty[}\big) = (1-0.5)\times 0 + 0.5\times 0.5 = 0.25$ and $I_0(x'_{[0,\infty[}) = (1-0.8)\times 1 + 0.8\times 0.25 = 0.4$. This implies $y' \succ x'$. The *extended stationarity* property of Koopmans [11] is not satisfied.

3. A Multiple Discount Formulation for Present-Biased Preferences

3.1 An Alternative Understanding of Multiple Present Biases

Present bias is commonly understood in the literature in terms of a behaviour that is controlled by the discount rate and according to which, what happens today affects more the decision maker that it would were this to happen by some day in the future. A gain—equivalently, a loss—today causes more happiness—more unhappiness—than the same occurrence in the future. This is one of sources of *time inconsistency*: the decision maker may prefer some small amount today than a larger one tomorrow, but that same small amount tomorrow is less enjoyable than the same larger one after-tomorrow.

This section is organized in order to examine the scope for such a phenomenon within the current multiple discounting setup. The following axiom, building from two separate parts, is a move in that direction. The first part of the axiom says that the delay of a perpetual gain to the next day and at time T diminishes the happiness of the decision maker more than it would do at time $T+1$ or for other dates in the future of T. The second part introduces another behaviour: delaying a perpetual loss at date T makes the decision maker more happy than if this were to take place in the future of T.

Axiom **B2.** For any $T \geq 1$ and a constant $c \geq 0$,

(i) If $\big(0\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty[}\big) \succeq \big(0\mathbf{1}_{[0,T-1]}, c\mathbf{1}_{[T,\infty[}\big)$, then $\big(0\mathbf{1}_{[0,T+1]}, \mathbf{1}_{[T+2,\infty[}\big) \succeq \big(0\mathbf{1}_{[0,T]}, c\mathbf{1}_{[T+1,\infty[}\big)$.

(ii) If $\big(0\mathbf{1}_{[0,T-1]}, (-c)\mathbf{1}_{[T,\infty[}\big) \succeq \big(0\mathbf{1}_{[0,T]}, -\mathbf{1}_{[T+1,\infty[}\big)$, then $\big(0\mathbf{1}_{[0,T]}, (-c)\mathbf{1}_{[T+1,\infty[}\big) \succeq \big(0\mathbf{1}_{[0,T+1]}, -\mathbf{1}_{[T+2,\infty[}\big)$.

The *supremum*—the greatest of the minorants—of the values of the parameter c in part (i) and part (ii) can both be used to figure out the *perception of the temporal distance* between date T and date $T+1$. These *extremum* values are evaluated using the perception at date T of the two sequences $(0, \mathbf{1})$ and $(0, -\mathbf{1})$. Axiom **B2** means that this *temporal distance* is decreasing as a function of T.[5]

Lemma 3.1 provides a straightforward implication of axiom **B2**.

Lemma **3.1.** *Assume that the order $\succeq$ satisfies axioms* **F** *and* **B1**, **B2**. *Suppose that, for any T, at least one of the two values χ_g^T and χ_ℓ^T differs from 1 and at least one of the two values χ_g^T and χ_ℓ^T differs from zero. For any $T \geq 0$ and a constant $c \geq 0$,*

(i) *If $(0, \mathbf{1}) \sim_T c\mathbf{1}$, then $(0, \mathbf{1}) \succeq_{T+1} c\mathbf{1}$.*

(ii) *If $(-c)\mathbf{1} \sim_T (0, -\mathbf{1})$, then $(-c)\mathbf{1} \succeq_{T+1} (0, -\mathbf{1})$.*

Delaying gain and loss affects the decision maker more at time T than at time $T+1$. Indeed, at time T, delaying the gain for one day diminishes the welfare value from 1 to c. Delaying the same gain at time $T+1$ will diminish the welfare from 1 to some value $c' \geq c$. Similarly, delaying a loss at time T increases the welfare value from -1 to $-c$, which is higher than to delay the same loss at time $T+1$, which increases from -1 to some $(-c')$ smaller than $-c$.

[5]The axiom 10 in Montiel Olea & Strzalecki [13] correspond to the second part of axiom **B2**, or the second part of Lemma 3.1.

Otherwise stated, the *temporal distance* that is perceived between dates T and $T+1$ is larger that the one that is perceived between dates $T+1$ and $T+2$: at date T, the valuation of a constant sequence from tomorrow on is lower than its corresponding valuation at date $T+1$. This intuition is detailed in the following statement:

Proposition **3.1.** *Assume that the initial order* $\succeq$ *satisfies axioms* **F** *and* **B1**, **B2**.

(i) *The sequence* $\{\underline{\delta}_T\}_{T=0}^{\infty}$ *of the* min *occurrence is increasing according to* $\underline{\delta}_T \leq \underline{\delta}_{T+1}$.

(ii) *The sequence* $\{\overline{\delta}_T\}_{t=0}^{\infty}$ *of the* max *occurrence is increasing according to* $\overline{\delta}_T \leq \overline{\delta}_{T+1}$.

The order $\succeq$ is hence *present-biased*.

3.2 A Multiple Discount Acception for Generalized Quasi-Hyperbolic Preferences

The following *Quasi-stationarity* axiom, which is similar to the axiom 4 in Montiel Olea & Strzalecki [13], implies a generalization of the *quasi-hyperbolic discounting* in that the preferences satisfy a stationarity axiom for every $T \geq 1$.

Axiom **B3.** For any constants $c, c' \in \mathbb{R}$, utility streams $x, y \in \ell_\infty$,

$$(c, c', x) \succeq (c, c', y) \text{ if and only if } (c, x) \succeq (c, y).$$

Under axiom **B3**, one can establish a multiple acception of the *quasi-hyperbolic discounting* class of preferences.

Proposition **3.2.** *Assume that the order* $\succeq$ *satisfies axioms* **F** *and* **B1**, **B3**.

(i) *For any* $T \geq 1$, $\underline{\delta}_T = \underline{\delta}_1$ *and* $\overline{\delta}_T = \overline{\delta}_1$.

(ii) *As a result of the addition of axiom* **B2**, *this simplifies to* $\underline{\delta}_0 \leq \underline{\delta}_1$ *and* $\overline{\delta}_0 \leq \overline{\delta}_1$.

While, under Axiom **B1** and for each date T, there exists a set of possible discount rates, the quasi-stationarity axiom **B3** ensures that these sets are the same for any date $T \geq 1$. Moreover, and for any $T \geq 1$, the statement of Lemma 3.1 is strengthened to an equivalence condition. As this is clarified in Proposition 3.2(ii), combining with the axiom **B2**, the set of discount rates associated with date $T = 0$ assumes smaller lower and upper bounds that the subsequent sets of discount rates associated with $T \geq 1$.

REMARK **3.1.** This result does provide an interesting generalization of the *quasi-hyperbolic discounting* of Phelps & Pollack [14] and Laibson [12]. Consider, *e.g.*, the case where for any T, $\underline{\delta}_T = \overline{\delta}_T$ with $\delta_0 \leq \delta_1 = \delta$[6]. The comparison between two inter-temporal stream becomes: $x \succeq y$ if and only if

$$(1-\delta_0)x_0 + \delta_0 \left(\sum_{s=0}^{\infty} (1-\delta)\delta^s x_{1+s} \right) \geq (1-\delta_0)y_0 + \delta_0 \left(\sum_{s=0}^{\infty} (1-\delta)\delta^s y_{1+s} \right),$$

which is equivalent to

$$x_0 + \beta \left(\sum_{s=1}^{\infty} \delta^s x_s \right) \geq y_0 + \beta \left(\sum_{s=1}^{\infty} \delta^s y_s \right),$$

for $\beta = [(1-\delta_0)\delta]^{-1}\delta_0(1-\delta) \leq 1$.

Bich *et al* [1] also come to a multiple *quasi-hyperbolic* discounting and an MaxMin representation of the index function, with a similar set of possible discount rates $(\delta_0, \delta) \in [\underline{\delta}_0, \overline{\delta}_0] \times [\underline{\delta}, \overline{\delta}]$. The difference with the current work is that, while in Bich *et al* [1], the optimal discount rates are chosen in the beginning of the evaluation, in this article, they are chosen in *each* period, by comparing the utility values of the present and of the future. Moreover, in this article, a *present biais* property is present, with the under and upper bounds of possible sets increasing (or at least, not decreasing) over time.

4. THE ROBUST TEMPORAL PRE-ORDERS $\succeq^*_{\mathrm{T}}$

Define the robust time-dependent order $\succeq^*_{\mathrm{T}}$ as the satisfaction of $x \succeq^*_{\mathrm{T}} y$ if and only if, for any z, $x + z \succeq^*_{\mathrm{T}} y + z$. Lemma 4.1 then provides a characterization of the weights set Ω_{T} that represents the robustness order $\succeq^*_{\mathrm{T}}$.

[6]This property can be obtained by adding the following assertion: $x \succeq y$ if and only if $x + z \succeq y + z$ for any $z \in \ell_\infty$.

LEMMA **4.1.** *Assume that the order* $\succeq$ *satisfies axioms* **F** *and* **B1**. *For any* T, *there exists a convex set* $\Omega_T \subset \ell_1$ *of weights which can be considered as infinitely additive probabilistic measures* ω *on* $\mathbb{N}$ *such that*

(i) $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ *is a probability measure, i.e., a sequence of weights, belonging to* ℓ^1,

$$\sum_{s=0}^{\infty} \omega_s = 1.$$

(ii) *For every* $x, y \in \ell_\infty$, $x \succeq_T^* y$ *if and only if, for any* $\underset{\sim}{\omega} \in \Omega_T$,

$$\sum_{s=0}^{\infty} \omega_s x_s \geq \sum_{s=0}^{\infty} \omega_s y_s.$$

Proposition 4.1 then equips the analysis with a representation of the weights sets Ω_{T}.

PROPOSITION **4.1.** *Consider axioms* **F** *and* **B1**. *Assume that either for every* T, *the corresponding operateur is* min, *or for every* T, *the corresponding operator is* max. *Then the weights set* Ω_{T} *is the convex hull of the set*

$$\left\{\left(1-\delta_{\mathrm{T}}, \delta_{\mathrm{T}}(1-\delta_{\mathrm{T}+1}), \delta_{\mathrm{T}}\delta_{\mathrm{T}+1}(1-\delta_{\mathrm{T}+2}), \ldots, \delta_{\mathrm{T}}\delta_{\mathrm{T}+1}\ldots\delta_{\mathrm{T}+s}(1-\delta_{\mathrm{T}+s+1}), \ldots\right)\right\},$$

where $\delta_{\mathrm{T}+s} \in \left\{\underline{\delta}_{\mathrm{T}+s}, \overline{\delta}_{\mathrm{T}+s}\right\}$ *for any* s.

It is well known in the literature that, beside the initial order $\succeq_T$, there exists a robust or unanimous pre-order one, defining on a set of *linear* index functions, decribing a situation where a given utility stream being *robustly better* than an alternative, in the sense that such comparison is unanimous among a set of linear orders that can be understood as a set of possible evaluations. Proposition 4.1 provides a clear and precised description of this set.

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A. Proofs for Section 2

A.1 Proof of Proposition 2.1

(i) Define $C_T(x)$ the set of values c such that, for any $z_0, z_1, \ldots, z_{T-1}$,

$$\big(z_{[0,T-1]}, x_{[T,\infty[}\big) \succeq \big(z_{[0,T-1]}, c\mathbf{1}_{[T,\infty[}\big).$$

Define the order $\succeq_T$ as the holding of $x \succeq_T y$ if and only if $\sup C(x) \geq \sup C(y)$. Fix $x, y \in \ell_\infty$ and suppose that for any $z_0, z_1, \ldots, z_{T-1}$,

$$\big(z_{[0,T-1]}, x_{[T,\infty[}\big) \succeq \big(z_{[0,T-1]}, y_{[T,\infty[}\big).$$

This implies that $C_T(y) \subset C_T(x)$, or $x \succeq_T y$. First consider the case $\sup C_T(y) < +\infty$ and take $c_y^T = \sup C_T(y)$. It is readily checked that $C_T(y)$ is closed, whence the satisfaction of $c_y^T \in C_T(y) \subset C_T(x)$. Further and from the definition of c_y^T, which is finite, for any $z_0, z_1, \ldots, z_{T-1}$,

$$\big(z_{[0,T-1]}, x_{[T,\infty[}\big) \succeq \big(z_{[0,T-1]}, c_y^T\mathbf{1}_{[T,\infty[}\big) \succeq \big(z_{[0,T-1]}, y_{[T,\infty[}\big).$$

Secondly consider the case $\sup C_T(y) = +\infty$, that implies the holding of $\sup C_T(x) = +\infty$. Whence, for any $c \geq \sup_s y_s$ and for any $z_0, z_1, \ldots, z_{T-1}$:

$$\big(z_{[0,T-1]}, x_{[T,\infty[}\big) \succeq \big(z_{[0,T-1]}, c\mathbf{1}_{[T,\infty[}\big) \succeq \big(z_{[0,T-1]}, y_{[T,\infty[}\big).$$

(ii) For the *transitivity*, *monotonicity* and *weak convexity* properties, replicate the arguments used for the proof of Proposition **3.1** in Drugeon & Ha-Huy [5].

(iii) Suppose that at least one of two values $I\big(0\mathbf{1}_{[0,T-1]}, \mathbf{1}_{[T,\infty[}\big)$ and $-I\big(0\mathbf{1}_{[0,T-1]}, -\mathbf{1}_{[T,\infty[}\big)$ differs from zero. It is to be proved that the order $\succeq_T$ satisfies the technical *non-triviality* property. The *Archimedeanity* property would then follow as a direct corollary.

Assume that $I\big(0\mathbf{1}_{[0,T-1]}, \mathbf{1}_{[T,\infty[}\big) > 0$. Then $I\big(0\mathbf{1}_{[0,T-1]}, \mathbf{1}_{[T,\infty[}\big) > I\big(0\mathbf{1}_{[0,T-1]}, 0\mathbf{1}_{[T,\infty[}\big)$. This implies $\mathbf{1} \succ_T 0\mathbf{1}$. Use the same argument for the case $-I\big(0\mathbf{1}_{[0,T-1]}, -\mathbf{1}_{[T,\infty[}\big) > 0$, which implies $0\mathbf{1} \succ_T -\mathbf{1}$.

(iv) Suppose that at least one of two critical values is different from zero. From (i), (ii) and (iii), $\sup C_T(x) < +\infty$ for any x and the order $\succeq_T$ satisfies every property in axiom **F**. The index function $I_T(x) = \sup C(x)$ therefore satisfies every property listed for the index I.

The satisfaction of *tail-insensitivity* is obvious. QED

A.2 Proof of Proposition 2.2

Fix $x \in \ell_\infty$, let $c = I_{T+1}\big(x_{[T+1,\infty[}\big)$ and consider the case $x_T \leq c$. From Proposition 2.1 and as $c - x_T \geq 0$,

$$\begin{aligned} I_T\big(x_{[T,\infty[}\big) &= I_T\big(x_T, c\mathbf{1}\big) \\ &= x_T + I_T\big(0, (c - x_T)\mathbf{1}\big) \\ &= x_T + (c - x_T) I_T(0, \mathbf{1}) \\ &= \big(1 - I_T(0, \mathbf{1})\big) x_T + I_T(0, \mathbf{1}) c. \end{aligned}$$

Likewise and for $x_T \geq c$:

$$\begin{aligned} I_T\big(x_{[T,\infty[}\big) &= I_T\big(x_T, c\mathbf{1}\big) \\ &= I_T\big(x_T - c, 0\mathbf{1}\big) + c \\ &= \big(x_T - c\big) I_T(1, 0\mathbf{1}) + c \\ &= I_T(1, 0) x_T + \big(1 - I_T(1, 0\mathbf{1})\big) c. \end{aligned}$$

First suppose that $\chi_g^T \leq \chi_\ell^T$, or $I_T(0, \mathbf{1}) + I_T(1, 0\mathbf{1}) \leq 1$, and let $\underline{\delta}_T = \chi_g^T = I(0, \mathbf{1})$ and $\overline{\delta}_T = \chi_\ell^T = 1 - I(1, 0\mathbf{1})$. It follows that $0 < \underline{\delta}_T \leq \overline{\delta}_T < 1$ and

$$I_T\big(x_{[T,\infty[}\big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1 - \delta) x_T + \delta I_{T+1}\big(x_{[T+1,\infty[}\big)\right].$$

Consider the remaining case $\chi_g^T \geq \chi_\ell^T$, or $I_T(0, \mathbf{1}) + I_T(1, 0\mathbf{1}) \geq 1$ and let $\underline{\delta}_T = \chi_\ell^T = 1 - I_T(1, 0\mathbf{1})$, $\overline{\delta}_T = \chi_g^T = I_T(0, \mathbf{1})$. It follows that $0 < \underline{\delta}_T \leq \overline{\delta}_T < 1$ and

$$I_T\big(x_{[T,\infty[}\big) = \max_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1 - \delta) x_T + \delta I_{T+1}\big(x_{[T+1,\infty[}\big)\right],$$

which establishes the statement. QED

B. Proofs for Section 3

B.1 Proof of Lemma 3.1

(i) Consider the constant c such that $(0, \mathbf{1}) \sim_T c\mathbf{1}$. From Proposition 2.1, this implies that:

$$\big(0\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty[}\big) \sim \big(0\mathbf{1}_{[0,T-1]}, c\mathbf{1}_{[T,\infty[}\big).$$

The order $\succeq$ further satisfying every property in axiom **F**, for any $c' < c$, the following is to hold:

$$\big(0\mathbf{1}_{[0,T]}, \mathbf{1}_{[T+1,\infty[}\big) \succeq \big(0\mathbf{1}_{[0,T-1]}, c'\mathbf{1}_{[T,\infty[}\big).$$

From axiom **B2**, this can be strengthened to:

$$\big(0\mathbf{1}_{[0,T+1]}, \mathbf{1}_{[T+2,\infty[}\big) \succeq \big(0\mathbf{1}_{[0,T]}, c'\mathbf{1}_{[T+1,\infty[}\big).$$

It follows that:

$$(0, \mathbf{1}) \succeq_{T+1} c'\mathbf{1}.$$

As this is true for any $c' < c$, letting c' converge to c, it derives that:

$$(0, \mathbf{1}) \succeq_{T+1} c\mathbf{1}.$$

(ii) Follow the same line of arguments as for (i). QED

B.2 Proof of Proposition 3.1

First observe that, for any T,

$$\underline{\delta}_T = \min\big\{I_T(0, \mathbf{1}), 1 - I_T(1, 0\mathbf{1})\big\},$$
$$\overline{\delta}_T = \max\big\{I_T(0, \mathbf{1}), 1 - I_T(1, 0\mathbf{1})\big\}.$$

But and from Lemma 3.1, both of the two sequences $\{I_T(0, \mathbf{1})\}_{T=0}^{\infty}$ and $\{1 - I_T(1, 0\mathbf{1})\}_{T=0}^{\infty}$ are increasing. This in its turn implies that the two sequences $\{\underline{\delta}_T\}_{T=0}^{\infty}$ and $\{\overline{\delta}_T\}_{T=0}^{\infty}$ are also increasing.

B.3 Proof of Proposition 3.2

Fix any $T \geq 1$. Suppose that $(0, \mathbf{1}) \sim_{T+1} c\mathbf{1}$, which is equivalent to $(0\mathbf{1}_{[0,T+1]}, \mathbf{1}) \sim (0\mathbf{1}_{[0,T]}, c\mathbf{1})$. By Lemma 4.3, this implies $(0\mathbf{1}_{[0,T]}, \mathbf{1}) \sim (0\mathbf{1}_{[0,T-1]}, c\mathbf{1})$, which is equivalent to $(0, \mathbf{1}) \sim_T c\mathbf{1}$. Hence $\chi_g^T = \chi_g^{T+1}$. Use the same arguments, one gets $\chi_\ell^T = \chi_\ell^{T+1}$. Hence for any $T \geq 1$, $\underline{\delta}_T = \underline{\delta}_1$ and $\overline{\delta}_T = \overline{\delta}_1$.

The second part appears as a direct consequence of the *present bias* property. QED

B.4 Proof of Lemma 4.1

First, recall that the dual space of ℓ_∞ , *i.e.*, the set of real sequences such that $\sup_s |x_s| < +\infty$, can be decomposed into the direct sum of two subspaces, ℓ_1 and ℓ_1^d: $(\ell_\infty)^* = \ell_1 \oplus \ell_1^d$. The subspace ℓ_1 satisfies σ-additivity. The subspace ℓ_1^d, the *disjoint complement* of ℓ_1, is the one of finitely additive measures defined on $\mathbb{N}$. More precisely, for each measure $\phi \in \ell_1^d$, for any $x \in \ell_\infty$, the value of $\phi \cdot x$ depends only on the distant behaviour of x, and does not change if there only occurs a change in a finite number of values x_s, $s \in \mathbb{N}$.

Define $\mathcal{P}_T^*$ as the positive polar cone of $\mathcal{P}_T = \left\{x \in \ell_\infty \text{ such that } x \succeq_T^* 0\mathbf{1}\right\}$ in the dual space $\left(\ell_\infty\right)^*$:

$$\mathcal{P}^* = \left\{\mathrm{P} \in (\ell_\infty)^* \text{ such that } \mathrm{P} \cdot x \geq 0 \text{ for every } x \succeq_T^* 0\mathbf{1}\right\}.$$

Observe that by the very definition of the order $\succeq^*$, $\mathcal{P}$ is convex and separable by the vector $-\mathbf{1}$, the cone $\mathcal{P}^*$ does not degenerate to $\{0\mathbf{1}\}$.

For each $\mathrm{P} \in \mathcal{P}^*$, define

$$\pi(\mathrm{P}) = \frac{1}{\mathrm{P} \cdot \mathbf{1}}\mathrm{P}.$$

Since $x \succeq_T^* 0\mathbf{1}$ for every $x \in \ell_\infty$ satisfying $x_s \geq 0$ for all s, it follows that $\mathrm{P} \cdot x \geq 0$ for every x such that $x_s \geq 0$ for every s. Let then $\Omega_T = \pi(\mathcal{P}_T)$. As $\mathrm{P} \cdot x \geq 0$ if and only if $\pi(\mathrm{P}) \cdot x \geq 0$, $x \succeq_T^* 0\mathbf{1}$ is equivalent to $\pi(\mathrm{P}) \cdot x \geq 0$ for every $\mathrm{P} \in \mathcal{P}$. For every P, $\pi(\mathrm{P})$ can be decomposed as $\pi(\mathrm{P}) = (1 - \lambda)\underset{\sim}{\omega} + \lambda\phi$, where $0 \leq \lambda \leq 1$, $\underset{\sim}{\omega} = (\omega_0, \omega_1, \cdots, \omega_s, \cdots) \in \ell_1$ and $\phi \in \ell_1^d$. From the *tail-insensitivity* property, it is obvious that $\lambda = 0$ and $\phi = 0$. QED

B.5 Proof of Proposition 4.1

Let Ω_T is the convex hull of the set

$$\left\{\left(1-\delta_T, \delta_T(1-\delta_{T+1}), \delta_T\delta_{T+1}(1-\delta_{T+2}), \ldots, \delta_T\delta_{T+1}\ldots\delta_{T+s}(1-\delta_{T+s+1}), \ldots\right)\right\},$$

where $\delta_{T+s} \in \left\{\underline{\delta}_{T+s}, \overline{\delta}_{T+s}\right\}$ for any s.

Obviously, we have only to prove the Proposition for $T = 0$. Consider the case where every operator is *min*. For $T \geq 0$,

$$I_T\left(x_{[T,\infty[}\right) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \left[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\right].$$

We have

$$I(x) = \min_{\underline{\delta}_0 \leq \delta_0 \leq \overline{\delta}_0, \cdots, \underline{\delta}_T \leq \delta_T \leq \overline{\delta}_T} \left\{(1-\delta_0)x_0 + \delta_0(1-\delta_1)x_1 + \cdots + \delta_0\delta_1\cdots\delta_{T-1}(1-\delta_T)x_T \right.$$
$$\left. + \delta_0\delta_1\cdots\delta_T I_{T+1}(x_{[T+1,\infty[})\right\}. \tag{1}$$

First, observe that

$$\lim_{T\to\infty} \overline{\delta}_0\overline{\delta}_1\cdots\overline{\delta}_T = 0.$$

Indeed, consider some value $c > 0$. Using (1), it is easy to verify that

$$I\left(0\mathbf{1}_{[0,T]}, (-c)\mathbf{1}\right) = (\overline{\delta}_0\overline{\delta}_1\cdots\overline{\delta}_T) \times (-c).$$

By the *tail-insensitivity* property, one has

$$\lim_{T\to\infty} I\left(0\mathbf{1}_{[0,T]}, (-c)\mathbf{1}\right) = 0,$$

which implies that

$$\lim_{T\to\infty} \overline{\delta}_0\overline{\delta}_1\cdots\overline{\delta}_T = 0.$$

Now, consider some sequence $x \in \ell_\infty$. First, we prove that

$$I(x) = \inf_{\omega\in\Omega_0} (\omega \cdot x).$$

Denote by $\{\delta^*_T\}_{T=0}^{\infty}$ the sequence of discount rates such that for every $T \geq 0$,

$$I_T\left(x_{[T,\infty[}\right) = (1-\delta^*_T)x_T + \delta^*_T I_{T+1}(x_{[T+1,\infty[}).$$

Recall that

$$I(x) = (1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \cdots + \delta_0^*\delta_1^*\cdots\delta_{T-1}^*(1-\delta_T^*)x_T + \delta_0^*\delta_1^*\cdots\delta_T^* I_{T+1}(x_{[T+1,\infty[}).$$

Let T converges to infinity, since $\delta_0^*\delta_1^*\cdots\delta_T^*$ converges to zero, we have

$$I(x) = \lim_{T\to\infty}\big((1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \cdots + \delta_0^*\delta_1^*\cdots\delta_{T-1}^*(1-\delta_T^*)x_T\big).$$

Assume that $I(x) > \inf_{\omega\in\Omega_0}(\omega\cdot x)$. Then there exists a sequence $\{\delta_T\}_{T=0}^{\infty}$ such that for every T, $\underline{\delta}_T \le \delta_T \le \overline{\delta}_T$, and

$$\begin{aligned} I(x) &= \lim_{T\to\infty}\big((1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \cdots + \delta_0^*\delta_1^*\cdots\delta_{T-1}^*(1-\delta_T^*)x_T\big) \\ &> \lim_{T\to\infty}\big((1-\delta)x_0 + \delta_0(1-\delta_1)x_1 + \cdots + \delta_0\delta_1\cdots\delta_{T-1}(1-\delta_T)x_T\big). \end{aligned}$$

Hence, for T sufficiently large, one gets

$$\begin{aligned} &(1-\delta_0^*)x_0 + \delta_0^*(1-\delta_1^*)x_1 + \cdots + \delta_0^*\delta_1^*\cdots\delta_{T-1}^*(1-\delta_T^*)x_T + \delta_0^*\delta_1^*\cdots\delta_T^* I_{T+1}(x_{[T+1,\infty[}) \\ &> (1-\delta)x_0 + \delta_0(1-\delta_1)x_1 + \cdots + \delta_0\delta_1\cdots\delta_{T-1}(1-\delta_T)x_T + \delta_0\delta_1\cdots\delta_T I_{T+1}(x_{[T+1,\infty[}), \end{aligned}$$

a contradiction with (1).

Let $P_0 = \pi(\mathcal{P}_0)$, with $\mathcal{P}_0$ is defined in the proof of Lemma 4.1. The set P_0 represents the weights set corresponding to the robuste order $\succeq^*$. We have to prove that $P_0 = \Omega_0$.

If Ω_0 is not a subset of P_0, then there exists $x \succeq^* 0\mathbf{1}$ such that $\omega\cdot x < 0$, for some $\omega\in\Omega_0$. This implies $I(x) < 0$: a contradiction. Hence, $\Omega_0 \subset P_0$.

Now, assume that for $x, y \in \ell_\infty$, we have $\omega\cdot x \ge \omega\cdot y$ for every $\omega\in\Omega_0$. It is easy to verify that for any $z\in\ell_\infty$,

$$\begin{aligned} I(x+z) &= \inf_{\Omega_0}\omega\cdot(x+z) \\ &\ge \inf_{\Omega_0}\omega\cdot(y+z) \\ &= I(y+z), \end{aligned}$$

which implies $x \succeq^* y$, by the definition of the robuste order $\succeq^*$. Hence, $P_0 \subset \Omega_0$.

Consider the case where every operateur is max. For $T \ge 0$,

$$I_T\big(x_{[T,\infty[}\big) = \max_{\underline{\delta}_T\le\delta\le\overline{\delta}_T}\left[(1-\delta)x_T + \delta I_{T+1}(x_{[T+1,\infty[})\right].$$

Define the order $\hat{\succeq}$ as: $x\hat{\succeq}y$ if and only if $(-y) \succeq (-x)$. We can verify that $\hat{\succeq}$ satisfies Axiom **F** and can be represented by and index function $\hat{I}$.

Obviously $x\hat{\succeq}^*y$ if and only if $(-y) \succeq^* (-x)$. Moreover,

$$\hat{I}_T\big(x_{[T,\infty[}\big) = \min_{\underline{\delta}_T \leq \delta \leq \overline{\delta}_T} \Big[(1-\delta)x_T + \delta \hat{I}_{T+1}(x_{[T+1,\infty[})\Big].$$

Apply the same arguments as in the first part of the proof, the claim of this Proposition is proven.