Evolutionary Stability of Behavioural Rules

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Abstract

I develop the notion of evolutionary stability of behavioural rules in a game-theoretic setting. Each individual chooses a strategy, possibly taking into account the game's history, and the manner in which he chooses his strategy is encapsulated by a behavioural rule. The payoffs obtained by individuals following a particular behavioural rule determine that rule's fitness. A population is stable if whenever some individuals from an incumbent behavioural rule mutate and follow another behavioural rule, the fitness of each incumbent behavioural rule exceeds that of the mutant behavioural rule. I show that any population comprised of more than one behavioural rule is not stable, and present necessary and sufficient conditions for stability of a population comprised of a single behavioural rule.

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1 Introduction

A basic tenet of evolution is the selection of the fitter at the expense of the less fit. In the context of decision-making in game-theoretic strategic situations, this principle has traditionally been expressed in terms of evolutionary stable strategies (abbreviated as ESS). A game is a representation of a strategic situation that is defined by three elements: *(i)* the set of players, *(ii)* each player's strategy set, which contains the feasible strategies that may chosen by the player, and *(iii)* a payoff function that describes the payoff received by each player corresponding to each combination of strategies chosen by the players. A particular strategy (or a particular mix of strategies) is said to be evolutionarily stable if it is able to withstand any mutant strategy in the sense of being fitter (i.e. obtaining a higher payoff) than the mutant strategy. The interpretation is that if an ESS is adopted by a population of players, then it is not possible for any other mutant strategy to invade the population.

In contrast to the ESS framework where each individual is associated with a particular strategy, I forward a notion of evolutionary stability where each individual is associated with a behavioural rule, and stability is based on the fitness of behavioural rules. An individual's behavioural rule determines his strategy choice, possibly taking into consideration the manner in which the game has unfolded in the past. Examples of behavioural rules include playing a best-response to some empirical distribution of strategies played in the past, imitation of most successful/popular strategies, always playing a fixed strategy (as in the ESS framework), or choosing strategies from the strategy set according to some probability distribution. I define a particular behavioural rule/combination of behavioural rules to be evolutionarily stable if the fitness exceeds that of any mutant behavioural rule – the interpretation is that a stable population cannot be invaded by any other mutant behavioural rule.

I show that if there is more than one behavioural rule in the population, then the population is unstable. Thus, a population may be stable only if all individuals choose according to the same behavioural rule, and I show that in this event, the population is stable at a particular strategy profile only if it leads to all individuals choosing an identical strategy. Furthermore, the characteristic of this identical strategy is that it must be an pure evolutionary stable strategy, i.e. it must be best-response to itself (or, alternatively stated, it must support a symmetric pure strategy Nash equilibrium), and in case there are alternative best responses to this strategy, then this strategy must be a better-response to the alternative best response than the latter is to itself. I also present a sufficient condition for stability that is 'reasonably' close to the necessary condition described above.

Evidently, this paper generalises the ESS approach pioneered by Smith and Price (1973); particularly useful expositions that expound on the properties and applicability of the ESS framework include Samuelson (1997), Sandholm (2010), and Weibull (1995). While Khan (2021) uses the evolutionary stability of behavioural rules approach in the specific context of bargaining game, in this paper, I develop this framework for general games, and present general results about the evolutionary stability of behavioural rules. Existing papers that examine the interaction between individuals who use different behavioural rules include Kaniovski, Kryazhimskii and Young (2000), Juang (2002), Josephson (2009) and Khan (2021); however, the aim of these papers is not to study the stability properties of the behavioural rules but rather, to examine the long-run outcomes that obtain when individuals display heterogeneity in their decision-making process.

2 The Framework

The strategic situation under consideration is a two player game G where both players have the same finite strategy set of pure strategies $S = \{s^1, \dots, s^P\}$. The power set of S (i.e. the set of all the *non-empty* subsets of S) is denoted by $\mathcal{P}(S)$. The payoff function $\pi : S \times S \rightarrow \mathbb{R}$ maps from the set of pure strategy combinations that can be used by two players to the real line. The payoff received by playing a pure strategy $s^i \in S$ against a pure strategy $s^j \in S$ is denoted by $\pi(s^i, s^j)$.

A strategy $s^j \in S$ is a *best-response* to $s^i \in S$ if, for all $s^k \in S$, the inequality $\pi(s^j, s^i) \geq \pi(s^k, s^i)$ holds. The *set of best-responses* to s^i is denoted by $BR(s^i)$, i.e. $BR(s^i) = \{s^j \in S :$ for all $s^k \in S, \pi(s^j, s^i) \geq \pi(s^k, s^i)\}$.

A strategy combination (s^i, s^j) is a *pure strategy Nash equilibrium* of the game G if $s^i \in BR(s^j)$ and $s^j \in BR(s^i)$. That is, in a Nash equilibrium, none of the two players experience an improvement in the payoff from a unilateral deviation from the chosen strategy. The strategy combination (s^i, s^i) is a *symmetric pure strategy Nash equilibrium* if $s^i \in BR(s^i)$.

A strategy $s^i \in S$ is a *pure strategy evolutionary stable strategy* in the game G if *(i)* $s_i \in BR(s^i)$ holds, and *(ii)* for any $s^j \in BR(s^i) \setminus \{s^i\}$, the inequality $\pi(s^i, s^j) > \pi(s^j, s^j)$ holds. Thus, a pure strategy evolutionary stable strategy is always a symmetric pure Nash equilibrium.

The game is said to be a *generic* game if, for any pair of strategies s^i and s^j that are different from each other, at most one of the following two equalities hold: *(i)* $\pi(s^i, s^i) = \pi(s^j, s^i)$ and *(ii)* $\pi(s^i, s^j) = \pi(s^j, s^j)$. The implication is that in generic games, if one considers the set of mixed strategies such that $p^i + p^j = 1$, then either: *(i)* exactly one of s^i and s^j is always a best-response to each mixed strategy in this set, or *(ii)* there exists $x \in (0, 1)$ such that exactly one of s^i and s^j is always a best-response to each strategy in the subset of mixed strategies where $p^i \leq x$ and $p^i + p^j = 1$, while the other strategy is always a best-response to each strategy in this set of mixed strategies where $p^i \geq x$ and $p^i + p^j = 1$. That is, if the game is generic, either there exists $\hat{x} \in [0, 1]$ such that $x\,\pi(s^i, s^i) + (1 - x)\pi(s^i, s^j) > x\,\pi(s^j, s^i) + (1 - x)\pi(s^j, s^j)$ holds for all $x \in [\hat{x}, 1)$, or there exists $\tilde{x} \in [0, 1]$ such that $x\,\pi(s^i, s^i) + (1 - x)\pi(s^i, s^j) < x\,\pi(s^j, s^i) + (1 - x)\pi(s^j, s^j)$ holds either for all $x \in (0, \tilde{x}]$. The example below illustrates a game that is not generic.

Example 1. The strategy set of the game below contains three strategies, i.e. $S = \{s^1, s^2, s^3\}$.

	s^1	s^2	s^3
s^1	2, 2	1, 2	0, 0
s^2	2, 1	1, 1	0, 0
s^3	0, 0	0, 0	3, 3

Figure 1

This game is not generic because $\pi(s^1, s^1) = \pi(s^2, s^1) = 2$, and $\pi(s^1, s^2) = \pi(s^2, s^2) = 1$. Let p denote any mixed strategy such that $p^1+p^2 = 1$. Then, $\pi(s^1, p) = \pi(s^2, p) = 2\,p^1+p^2 = 1+p^1$; so, both s^1 and s^2 are best-responses to any such mixed strategy. Hence, none of the two conditions described above are met. ■

The generic game G is played by individuals in a population of unit mass, and I suppose that the individuals are uniformly distributed over the unit interval $[0, 1]$. Each individual is identified by his respective location on the unit interval. Time is discrete, and in each time period, each player plays the two-player game G in a pairwise manner with all other players. Each individual chooses the same *pure strategy* from S for all his interactions in a particular period. This strategic interaction occurs in every time period.

The relative frequency of the individuals in the population playing a pure strategy $s^i \in S$ in time period t is denoted by f_t^i. The vector of relative frequencies with which each strategy is played in the population, and in a set A of individuals in the population, in period t is denoted by f_t and $f_{A,t}$, respectively. I refer to f_t and $f_{A,t}$ as the period t population strategy profile and the period t strategy profile in the set A, respectively. In order to simplify notation, I will drop the time subscript in the notation whenever it is convenient to do, and no confusion arises from doing so.

The pure strategy used by player $i \in [0, 1]$ in time period t is represented by $s_{i,t}$. The payoff received by player i from playing the game with another player j is $\pi(s_{i,t}, s_{j,t})$. Since each player plays the game with all other players in turn in a pairwise manner, the total payoff of player i in time period t on choosing $s_{i,t}$ when the strategy profile in the population is f_t is given by $\pi_{i,t} = \Sigma_{j=1}^{N} f_t^j\, \pi(s_{i,t}, s^j)$. The period t payoff profile π_t is the vector of relative frequencies of payoffs received by the players in the population in time period t. The entire history of strategy profiles and payoff profiles of each and every period till period t is denoted by $f_{1\to t}$ and $\pi_{1\to t}$.

The strategy chosen by the individuals in the very first period is specified exogenously, and I describe the manner in which players choose their respective strategies in period $t+1$ for any integer valued $t > 1$. Player i is associated with a behavioural rule that maps from the history of strategy profiles and payoff profiles into a subset of the identical finite set of pure strategies S. This latter set is called the period $t+1$ response set of player i and denoted by $R_i(f_{1\to t}, \pi_{1\to t})$. The strategy chosen by player i in period $t+1$ belongs to $R_i(f_{1\to t}, \pi_{1\to t})$, and each strategy in this set has positive probability of being chosen, and hence, is called a *feasible* period $t+1$ strategy for player i. A strategy profile (i.e. the vector of relative frequency of strategies in S) f' that can be generated by taking one feasible period $t+1$ strategy for each individual in $[0,1]$ is termed as a feasible period $t+1$ strategy profile. The set of all feasible period $t+1$ strategy profiles is denoted by Δ_{t+1}.

The collection of behavioural rules in use by the players in the population in the game G is denoted by $R = \{R^1, \ldots, R^N\}$. I clarify that I use lower case subscripts (such as R_i) to refer to the behavioural rule of the player located at point i in the interval $[0,1]$, and upper case superscripts (such as R^I) to denote a particular incumbent behavioural rule in the population without any reference to the players using that rule. A population is said to be *uniform at time period t* if the set of feasible period $t+1$ strategies is the same for all individuals, and it is *diverse at time period t* otherwise. So, a population is uniform at time period t as long as the response sets of all individuals are identical even though all of them may not follow the same behavioural rule. Thus, a sufficient (necessary) condition for a uniform (diverse) population at is that all individuals (not all individuals) in the population follow the same rule. Some examples of behavioural rules are as follows:

(i) Best-response: In period $t+1$, an individual plays a best response to some strategy profile – this may be the strategy profile till date (i.e. $f_{1\to t}$), or the strategy profile of the previous period (i.e. f_t), or the strategy profile of some selected time periods.

(ii) Imitation: An individual plays the strategy of the individual who received the highest payoff in some past period, or the highest average payoff in some selected time periods.

(iii) Stochastic play: An individual chooses a strategy from the strategy set according to some probability distribution.

(iv) Fixed strategy choice: An individual always plays the same strategy in all time periods.

For each behavioural rule $R^I \in R$, SoR^I_t and $\mid SoR^I_t \mid$ are the set of players and the relative frequency of the players in the population who play as per the behavioural rule R^I in time period t, respectively. The period t strategy profile of the individuals in the set SoR^I_t, and their payoff profile, is given by f^I_t, and π^I_t, respectively. A strategy profile, i.e. the vector of relative frequency of strategies in S, f' that can be generated by taking one feasible period $t+1$ strategy for each individual in SoR^I_t is referred to as a feasible period $t+1$ strategy profile for R^I, and Δ^I_{t+1} denotes the set of all feasible period $t+1$ strategy profiles for R^I. The example below illustrates the notion of feasible period $t+1$ strategy profiles.

Example 2. Consider the game below. The strategy set comprises of three strategies s^1, s^2, and s^3, and the payoff function is depicted via the payoff matrix below.

	s^1	s^2	s^3
s^1	$1,1$	$5,0$	$4,0$
s^2	$0,5$	$2,2$	$4,0$
s^3	$0,4$	$0,4$	$3,3$

Figure 2

Suppose that all individuals play strategy s^3 in the very first time period so that the population level strategy profile is $(0, 0, 1)$. Also suppose that the population has the following composition in terms of the behavioural rule: each individual in the interval $[0, 0.5)$ plays a best-response to the population level strategy profile in the previous period, while each individual in the interval $[0.5, 1]$ imitates the strategy that has yielded the highest payoff in the previous time period. Then, in time period 2, the response set of each individual in the interval $[0, 0.5)$ is $\{s^1, s^2\}$, and the response set of each individual in the interval $[0.5, 1]$ is $\{s^3\}$. As a result, the feasible period 2 strategy profiles for the sub-population of best-responders and imitators are $(x, 1-x, 0)$ where x takes any value in $[0, 1]$, and $[0, 0, 1]$, respectively. Finally, the feasible period 2 strategy profile for the population is $[y, 0.5-y, 0.5]$ where y takes any value between $[0, 0.5]$. ■

The fitness of each behavioural rule $R^I \in R$ in time period t is determined by a fitness function F_t^I that maps from the period t payoff profile π_t to the set of real numbers. Thus, $F_t^I(\pi_t)$ is the fitness of the behavioural rule R^I in time period t, and this function may differ both across time periods and across behavioural rules. The only restriction I impose is that if, in time period t, the payoff distribution of one particular behavioural rule (strictly) first order stochastically dominates the payoff distribution of another behavioural rule, then the former rule is (fitter) at least as fit as the latter in time period t. Formally, let $\pi_t^I(x)$ and $\pi_t^J(x)$ denote the cumulative distribution function of the period t payoffs received by the individuals following the behavioural rules R^I and R^J, respectively; if $\pi_t^I(x) \leq \pi_t^J(x)$ holds for each real number x, then $F_t^I(\pi_t) \geq F_t^J(\pi_t)$; further, if $\pi_t^I(x) < \pi_t^J(x)$ holds for some real number x, then $F_t^I(\pi_t) > F_t^J(\pi_t)$

The criterion for evolutionary stability of behavioural rules that I define next is based on the fitness of behavioural rules.

3 Evolutionary Stability of Behavioural Rules

The stability criterion compares the fitness of each incumbent behavioural rule in the population to the fitness of a mutant behavioural rule. In this context, suppose that at the very beginning of time period $t+1$, a strict subset of individuals who follow one particular behavioural rule (say R^I) mutate, and these players adopt a mutant behavioural rule (say $R^{I'}$, with $R^{I'} \neq R^I$). The mutating behavioural rule is called the *source behavioural rule*, the mass of mutating individuals is denoted by ε where $\varepsilon <\mid SoR_t^I \mid$, and the set of mutating individuals is denoted by M_ε.

I will now define the concepts of an *effective mutation* and a *feasible ε effectively mutated strategy profile*. Take the set of individuals $M_\varepsilon \subset SoR_t^I$ who mutate at the beginning of time period $t+1$. Now, for each period $t+1$ feasible strategy profile of R^I, consider the corresponding feasible strategy profile of the individuals in this subset M_ε. That is, for each $g \in \Delta_{t+1}^I$, consider the strategy profile g_{M_ε}. This strategy profile g_{M_ε} represents the counterfactual strategy profile that would be chosen by the individuals in the set M_ε at the feasible period $t+1$ strategy profile g of the rule R^I in the absence of the mutation. The mutation in the behavioural rule R^I is *effective* at the strategy profile $g \in \Delta_{t+1}^I$ if the actual strategy profile $f_{M,t+1}$ played by the mutants is different from g_{M_ε}.

A strategy profile f is a *feasible ε effectively mutated period $t+1$ strategy profile* if there exists $g \in \Delta_{t+1}^I$ for some incumbent behavioural rule $R^I \in R$ and there exists some strict subset $M_\varepsilon \subset SoR_t^I$ with mass $\varepsilon \in (0, \mid SoR_t^I \mid)$ such that *(i)* $g_{M_\varepsilon} \neq f_{M_\varepsilon}$, *(ii)* $g_{SoR_t^I \setminus M_\varepsilon} = f_{SoR_t^I \setminus M_\varepsilon}$, and *(iii)* $f^J \in \Delta_{t+1}^J$ for all other incumbent behavioural rules $R^J \in R \setminus \{R^I\}$.

Condition (i) in the definition above states that the mutation is effective, i.e. the strategy profile played by the mutating individuals is not the same as the counterfactual strategy profile that these individuals would play if they did not mutate. Condition (ii) and condition (iii) state that all other individuals in the population play a feasible period $t+1$ strategy as per their behavioural rule. This definition captures the feature that the mutation must be meaningful in the sense of causing a difference in the strategy profile of the set of mutating individuals, and so the strategy profile post-mutation is not an exact copy of the strategy profile had the mutation not taken place. The notion of stability of behavioural rules that I present next is based on a comparison of fitness of behavioural rules in feasible ε effectively mutated strategy profiles.

A population of individuals with behavioural rules in a set R is *stable* at period t if, for each behavioural rule $R^I \in R$, there exists $\bar{\varepsilon}^I \in (0, \mid SoR_t^I \mid)$ such that whenever any subset set of mass $\varepsilon \in (0, \bar{\varepsilon}^I]$ of individuals in SoR_t^I mutate to adopt another mutant behavioural rule $R^{I'}$, then the inequality $F_{t+1}^{I'}(\pi_{t+1}) < F_{t+1}^J(\pi_{t+1})$ holds for all incumbent behavioural rules $R^J \in R$ in each feasible ε effectively mutated period $t+1$ strategy profile.

Thus, stability requires that in each feasible ε effectively mutated period $t+1$ strategy profile that may be realised in period $t+1$, each incumbent behavioural rule should be fitter than any mutant behavioural rule. The notion of external stability is permissive in the sense that even though the framework of behavioural rules is dynamic, and involves the profile of strategies potentially changing over time, I only require each behavioural rule to be fitter than any mutant behavioural rule in the first time period in which the mutant behavioural rule appears in the population.

4 Results

I will first show that a stable diverse population does not exist by proving that in any such population, there exists a mutant behavioural rule that is at least as fit as an incumbent behavioural rule.

Proposition 1. *A stable diverse population of behavioural rules does not exist.*

Proof. If the population in period t is diverse, then there exist two different behavioural rules R^I and R^J in the population such that the period $t+1$ response set of these two behavioural rules is not identical. So, there exists a feasible period $t+1$ strategy profile where individuals who follow the behavioural rule R^I and R^J choose two different strategies, say s^i and s^j, respectively. I do not specify the period $t+1$ strategy choice of individuals who follow the other behavioural rules (if any). Now, let $\bar{\varepsilon}$ be any number in the interval $(0, \mid SoR_t^I \mid)$, and suppose that in period $t+1$, any measure $\varepsilon \in (0, \bar{\varepsilon}]$ of the individuals who follow the behavioural rule R^I mutate. Also suppose s^j is in the period $t+1$ response set of the mutant behavioural rule and each mutant individual chooses s^j. It follows that the resultant period $t+1$ is a feasible ε effectively mutated strategy profile. In this time period, each mutant individual obtains a payoff that is identical to the payoff obtained by each individual who follows the rule R^J. As a result, the cumulative distribution of the payoffs of the mutant behavioural rule is identical to that of R^J. Hence, the fitness of the mutant behavioural rule equals the fitness of R^J, and so, the population is not stable at time period t. This implies that there does not exist any stable diverse population. ■

The instability of diverse populations leads me to now analyse the stability of a uniform population. Proposition 3, and Proposition 4, below present a necessary condition, and a sufficient condition, respectively, for stability of the uniform population at time period t. I show that a necessary condition for stability in time period t is that there must be a unique feasible period $t+1$ strategy profile, and that the strategy played by each individual in this strategy profile is an ESS of G. A strengthening of this condition is also sufficient for stability of the population at time period t. After presenting the formal proposition and its proof, I discuss that the reason for the gap between the necessary and sufficient condition is due to the possibility of the response set of the mutant behavioural rule being non-singleton, which implies that the mutated strategy profile may involve multiple mutant strategies rather than just one mutant strategy.

Proposition 2. *Suppose that a uniform population is stable at time period t. Then, there exists only one feasible period $t+1$ strategy profile, and this strategy profile must represent a pure strategy ESS. That is, in period $t+1$, all individuals in the population play an identical strategy s^i such that:* (i) *$s^i \in BR(s^i)$, and* (ii) *for each $s^j \in BR(s^i) \setminus \{s^i\}$, the inequality $\pi(s^i, s^j) > \pi(s^j, s^j)$ holds.*

Proof. In Step 1, I will show that if the population is stable at time period t, then there must be only one feasible period $t+1$ strategy profile. In Step 2 and Step 3, I will demonstrate that in this strategy profile, the conditions (i) and (ii) in the above proposition must hold.

Step 1. Suppose that there exists more than one feasible period $t+1$ strategy profile. This can occur if and only if there exists an incumbent individual who has at least two period $t+1$ feasible strategies; since all incumbent individuals are identical, all individuals have the same non-singleton period $t+1$ response set containing two different strategies, say s^i and s^j. Then, there is a feasible period $t+1$ strategy profile given by $f^i = x$ and $f^j = 1-x$, where x takes any value in the interval $[0,1]$. In this strategy profile, the payoff obtained by individuals playing s^i and s^j is $x\pi(s^i,s^i) + (1-x)\pi(s^i,s^j)$ and $x\pi(s^j,s^i) + (1-x)\pi(s^j,s^j)$.

Since the game is generic, there exists $\hat{x} \in [0,1]$ such that $x\,\pi(s^i,s^i) + (1-x)\pi(s^i,s^j) > x\,\pi(s^j,s^i) + (1-x)\pi(s^j,s^j)$ holds either for all $x \in [\hat{x},1)$, or there exists $\tilde{x} \in [0,1]$ such that $x\,\pi(s^i,s^i) + (1-x)\pi(s^i,s^j) < x\,\pi(s^j,s^i) + (1-x)\pi(s^j,s^j)$ holds either for all $x \in (0,\tilde{x}]$.

First, suppose that such a $\hat{x}$ exists, and that in time period $t+1$, measure $f^i \in (\hat{x},1)$ of the population choose s^i while the complementary $1-f^i$ mass of individuals would have chosen s^j but for the event that some measure $(0, f^i - \hat{x})$ of these individuals mutate and play s^i. That is, in period $t+1$, I set $\bar{\varepsilon} = \hat{x} - f^i$, and a measure $\epsilon \in (0,\bar{\varepsilon}]$ of the individuals who would have chosen s^j mutate and choose s^i; a mass f^i of the incumbent individuals choose s^i and the remaining $1 - f^i - \varepsilon$ incumbent individuals choose s^j. Hence, this represents a feasible ε effectively mutated strategy profile. In this strategy profile, the measure of the individuals playing s^i exceeds $\hat{x}$. So, the payoff obtained from playing s^i is greater than that of playing s^j. Since each mutant individual plays s^i but some of the incumbent individuals play s^j, the cumulative distribution of payoffs of the mutant behavioural rule strictly first order stochastically dominates that of the incumbent behavioural rule. Consequently, the mutant behavioural rule is fitter than the incumbent behavioural rule, and the population is not stable at time period t.

An analogous reasoning establishes that the population is also not stable if $\tilde{x}$ exists but $\hat{x}$ does not exist. This establishes that there can only feasible period $t+1$ strategy profile. Let s^i be the strategy chosen by each individual in this time period. In Step 2 and Step 3, I argue for condition (i) and condition (ii) in the proposition, respectively.

Step 2. Suppose that $s^i \notin BR(s^i)$. Then, there exists $s^j \in S$, with $s^j \neq s^i$, such that $\pi(s^j,s^i) > \pi(s^i,s^i)$. As a result, there exists $\bar{\varepsilon} \in (0,1)$ such that, for all $\varepsilon \in (0,\bar{\varepsilon}]$, the inequality $(1-\varepsilon)\pi(s^j,s^i) + \varepsilon\pi(s^j,s^j) > (1-\varepsilon)\pi(s^i,s^i) + \varepsilon\pi(s^i,s^j)$ holds. Therefore, if any measure $\varepsilon \in [0,\bar{\varepsilon})$ of the incumbent individuals mutate in period $t+1$, and the mutant behavioural rule leads them to play s^j, then each mutant receives a payoff of $(1-\varepsilon)\pi(s^j,s^i) + \varepsilon\pi(s^j,s^j)$ while each incumbent receives a payoff of $(1-\varepsilon)\pi(s^i,s^i) + \varepsilon\pi(s^i,s^j)$. So, the cumulative distribution of payoffs of the mutant behavioural rule strictly first order stochastically dominates that of the incumbent behavioural rule. Hence, the mutant behavioural rule is fitter than the incumbent behavioural rule, and the population is not stable at time period t. Thus, if the population is stable at time period t, then it must be that $s^i \in BR(s^i)$.

Step 3. In order to establish condition (ii) of the proposition, suppose that a measure ε of the population mutate, and each mutant plays s^j, where $s^j \neq s^i$. Let the resulting strategy profile be denoted by f. The payoff obtained by each incumbent individual and each mutant individual is $(1-\varepsilon)\pi(s^i, s^i)+\varepsilon\pi(s^i, s^j)$ and $(1-\varepsilon)\pi(s^j, s^i)+\varepsilon\pi(s^j, s^j)$, respectively. Since the population is stable at time period t, the fitness of incumbent behavioural rule exceeds that of the mutant behavioural rule. Hence, if $s^j \in BR(s^i)$, then it must be that $\pi(s^i, s^j) > \pi(s^j, s^j)$ holds. This proves the statement of the proposition. ■

Proposition 3. *A uniform population is stable at time period t if there is only one feasible period $t+1$ strategy profile where all individuals in the population play an identical strategy s^i such that:* (i) *$s^i \in BR(s^i)$, and* (ii) *for all $s^j, s^k \in BR(s^i) \setminus \{s^i\}$, the inequality $\pi(s^i, s^k) > \pi(s^j, s^k)$ holds.*

Proof. Consider a time period t where the conditions of the propositions hold. In time period $t+1$, suppose that a measure $\varepsilon > 0$ of the individuals mutate to another behavioural rule. Each incumbent individual chooses s^i in period $t+1$. Hence, in any ε effectively mutated strategy profile, a positive measure of the mutant individuals choose a strategy other than s^i. So, let $\nu \in [0, \varepsilon)$ denote the measure of the mutant sub-population who choose s^i, and let f denote any ε effectively mutated period $t+1$ strategy profile.

The payoff received by each incumbent individual, and a mutant individual choosing s^j (where $s^j \neq s^i$), in the strategy profile f equals $(1-\varepsilon+\nu)\pi(s^i, s^i) + \Sigma_{k\neq i} f^k \pi(s^i, s^k)$ and $(1-\varepsilon+\nu)\pi(s^j, s^i) + \Sigma_{k\neq i} f^k \pi(s^j, s^k)$, where $\Sigma_{k\neq i} f^k = \varepsilon - \nu$. Whenever ε small enough, the inequality $(1-\varepsilon+\nu)(\pi(s^i, s^i) - \pi(s^j, s^i)) + \Sigma_{k\neq i} f^k(\pi(s^i, s^k) - \pi(s^j, s^k)) > 0$ holds under the conditions of the proposition for all strategy profiles f with $f^i = 1-\varepsilon+\nu$ and $\Sigma_{k\neq i} f^k = \varepsilon-\nu$. That is, in the strategy profile f, the payoff received by playing each strategy s^j that is different from s^i is lower than the payoff received by playing the strategy s^i. So, the cumulative distribution of payoffs of the incumbent behavioural rule strictly first order dominates that of the mutant behavioural rule. Hence, the incumbent behavioural rule is fitter than the mutant behavioural rule, and the population is stable at time period t. ■

A comparison of the two propositions above shows that condition (ii) is stronger in case of sufficiency than in case of necessity in a very revealing manner. To convey the intuition behind this difference succinctly, consider the specific case where there exist two different strategies $s^j, s^k \in BR(s^i) \setminus \{s^i\}$. Necessity requires $\pi(s^i, s^j) > \pi(s^j, s^j)$ and $\pi(s^i, s^k) > \pi(s^k, s^k)$ while sufficiency requires $\pi(s^i, s^j) > \pi(s^k, s^j)$ and $\pi(s^i, s^k) > \pi(s^j, s^k)$ in addition. The reason is that both s^j and s^k may be feasible period $t+1$ strategies for the mutant behavioural rule. So, if a measure ε of the incumbent individuals mutate, and a measure ν and $\varepsilon - \nu$ of the mutant individuals choose s^j and s^k respectively, then the payoff received from playing s^i, s^j, and s^k equals $(1-\varepsilon)\pi(s^i, s^i)+\nu\pi(s^i, s^j)+(\varepsilon-\nu)\pi(s^i, s^k)$, $(1-\varepsilon)\pi(s^j, s^i)+\nu\pi(s^j, s^j)+(\varepsilon-\nu)\pi(s^j, s^k)$, and $(1-\varepsilon)\pi(s^k, s^i) + \nu\pi(s^k, s^j) + (\varepsilon - \nu)\pi(s^k, s^k)$, respectively. Since $s^j, s^k \in BR(s^i) \setminus \{s^i\}$, one obtains $(1-\varepsilon)\pi(s^i, s^i) = (1-\varepsilon)\pi(s^j, s^i) = (1-\varepsilon)\pi(s^j, s^i)$. So, if $\pi(s^i, s^j) > \pi(s^k, s^j)$ and $\pi(s^i, s^k) > \pi(s^j, s^k)$ do not hold, and in particular, if $\pi(s^i, s^j) < \pi(s^k, s^j)$ and $\pi(s^i, s^k) < \pi(s^j, s^k)$ hold, then it might be possible that the payoff obtained from playing s^i is lower than the payoff obtained from playing s^j or s^k. Then, depending on the fitness function – which has considerable flexibility – the mutant behavioural rule may be at least as fit as the incumbent behavioural rule thereby resulting in instability of the population.

The corollary below, which states the relationship between evolutionary stability of behavioural rules, ESS, population level Nash equilibrium, and symmetric Nash equilibrium, follows directly from the earlier propositions.

Corollary 1. *Suppose that a population is stable in terms of the behavioural rules in time period t. Then, the strategy profile in time period $t+1$ must be a symmetric pure strategy population level Nash equilibrium as well.*

The converse of Proposition 3, and the corollary above, is not true in general, and this is illustrated by the example below – this example presents a game where an ESS in pure strategies exists but there does not exists any population that is evolutionary stable in terms of behavioural rules.

Example 3. Consider the game below. The only pure strategy symmetric Nash equilibrium of this game is (s^1, s^1), and it is easily verified that this is also a pure strategy ESS.

	s^1	s^2	s^3
s^1	$3,3$	$2,3$	$3,3$
s^2	$3,2$	$1,1$	$5,5$
s^3	$3,3$	$5,5$	$1,1$

Figure 3

However, I will argue that there does not exist any behavioural rule that is stable at any time period t. By Proposition 3, a population is stable at time period t only if the population is uniform and the only feasible period $t+1$ strategy profile is $(p^1, p^2, p^3) = (1, 0, 0)$. So, suppose that a measure ε of the individuals mutate to another behavioural rule, and measure $\frac{\varepsilon}{2}$ of the mutant individuals choose s^2 while the remaining measure $\frac{\varepsilon}{2}$ of the mutant individuals choose s^3. Then, in period $t+1$, the payoff obtained by each incumbent individual, each mutant individual who chooses s^2, and each mutant individual who plays s^3 is $3(1-\varepsilon)+2\frac{\varepsilon}{2}+3\frac{\varepsilon}{2} = 3-\frac{\varepsilon}{2}$, $3(1-\varepsilon) + \frac{\varepsilon}{2} + 5\,\frac{\varepsilon}{2} = 3$, and $3(1-\varepsilon) + 5\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = 3$. Hence, for any positive ε, no matter how small, the payoff received by each mutant individual exceeds the payoff received by each incumbent individual. As a result, the cumulative distribution of period $t+1$ payoffs of the mutant behavioural rule strictly first order stochastically dominates that of the incumbent behavioural rule. Consequently, the mutant behavioural rule is fitter than the incumbent behavioural rule in period $t+1$, and so, the population is not stable in period t. ■

5 Conclusion

In this paper, I forward and develop a framework for evolutionary stability of a population, where the stability is conceptualised in terms of a behavioural rule. An individual's action in a strategic situation is a response to the manner in which the game has evolved in the past, and the manner in which an individual responds is captured by a behavioural rule. Therefore, the population can be thought of as being comprised of a set of behavioural rules, and this set of behavioural rules is stable if it is able to withstand the invasion by any other mutant behavioural rule in the sense of being fitter than the mutant rule. This represents a substantial generalisation over the traditional concept of evolutionary stability of strategies. I show that any population which comprises of more than one incumbent behavioural rule is unstable. Next, I present fairly close necessary and sufficient conditions for stability of a population comprising of a single behaviorial rule, and show the relationship between evolutionary stability of the behavioural rule on the one hand and evolutionary stability of strategies on the other hand.

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