Equilibrium and dominance in fuzzy games

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Abstract

In this paper, we study the generalization of (Nash) equilibrium and dominance solvability to interval fuzzy games in strategic form. We show that the more straightforward generalizations of these concepts do not inherit their most relevant results, either in terms of existence or refinement. To efficiently handle the fuzziness of the payoffs, we use the Hurwicz criterion and introduce new equilibrium concepts and dominance solutions which greatly overcome these drawbacks.

Keywords: Dominance solvability, Fuzzy interval payoffs, Hurwicz criterion

1 Introduction

In order to account for uncertainty in game-theoretical models, many papers in the literature focus on the randomness aspect of uncertainty, developing stochastic models where probability distributions describe uncertain parameters. However, the probability distribution may not be available in practice or difficult to estimate from limited data points. In this context, the fuzzy set theory is an appropriate modeling tool when probability distribution cannot describe uncertain parameters. Other papers consider uncertainty

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with different tools, for example interval uncertainty (there is an uncertain variable and its lower and upper bounds are known) or similar (Palancı et al., 2014).

In this paper, we study interval fuzzy games in strategic form, i.e., games (in strategic form) in which the payoff of an agent is generally a fuzzy set. Such fuzzy sets arise when there is uncertainty on the worth that an agent can get under certain strategy profiles. This approach to considering uncertainty fits with several concrete situations: for example in Xu et al. (2013) a class of supply chain, where the uncertainties of demands are described as fuzzy sets, is studied.

We present two generalizations of the Nash equilibrium (Nash, 1951) and of dominant solvable schemes (Moulin, 1979). To set a dominant solvable scheme, we eliminate equilibria in which some player's strategy is iteratively dominated.

First introduced by Moulin (1979) in the context of voting, dominance solvability relies on a straightforward procedure: if a player has a strictly dominated strategy, i.e., a strategy that generates worse payoffs than another regardless of what other players select, she should never use it. If strictly dominated strategies are eliminated, the reduced game may have further strictly dominated strategies that can then be eliminated, and so on. When this iterative procedure converges to a unique strategy profile, that profile turns out to be a Nash equilibrium, and the game is said dominance solvable. The dominated strategies elimination procedure allows to rule out some multiple Nash equilibria and thus it becomes an equilibrium refinement tool. See for example Carlsson and van Damme (1993).

Previous generalization of Nash equilibrium to fuzzy games are given by Maeda (2003); Cunlin and Qiang (2011), which focus on triangular two-person zero-sum games, and Chakeri and Sheikholeslam (2013), which defines a graded representation of Nash equilibria in crisp and fuzzy games. In Yu and Zhang (2010) a fuzzy Nash equilibrium definition based on a binary fuzzy ordering relation is presented and applied to a traffic flow problem.

The most direct generalization of weakly dominated strategies in the class of interval fuzzy games (*loose dominance*) requires one strategy to dominate another for some agent if it results in a weakly higher interval payoff when facing any set of opponent's profiles and in a strictly higher interval payoff against some of them. In the context of crisp games, a loosely dominant solution is a Nash equilibrium (Moulin, 1979). This result has been generalized to the class of multicriteria games (Gerasimou, 2019). However, it is no more true in the class of fuzzy interval games (see Example 4.1 in Section 4).

In order to overcome this difficulty, we consider the Hurwicz criterion (in line with

Mallozzi and Vidal-Puga (2021)) in fuzzy interval games in strategic form. We introduce three new concepts: the Hurwicz Nash equilibrium, the loose Hurwicz Nash equilibrium, and the interior Hurwicz Nash equilibrium. These concepts are equivalent to classical Nash equilibria in the context of crisp games. In particular, the Hurwicz Nash equilibrium coincides the the tight Nash equilibrium. Moreover, the equivalent dominance solution of the interior Hurwicz Nash equilibrium works as a refinement of both the loose Nash equilibrium and the interior Hurwicz Nash equilibrium.

Another backward of the two most direct generalizations of the Nash equilibrium deals with their existence for mixed strategies. A strategy profile is an equilibrium if any individual deviation provides a (weakly) worse payoff (*tight Nash equilibrium*) or if no deviation provides a (weakly) better payoff (*loose Nash equilibrium*). They both generalize the concept of Nash equilibrium in the context of fuzzy intervals. In the context of crisp games with a finite number of strategies, existence of Nash equilibria is guaranteed for mixed strategies Nash (1951). However, in the class of fuzzy interval games it only holds for loose Nash equilibria (see Section 6). On the other hand, existence also holds for interior Hurwicz Nash equilibria when fuzzy set are symmetric. The study of symmetric fuzzy sets is well motivated in literature from a theoretical point of view and a computational one (see for example Buckley and Feuring (2000); Ganesan and Veeramani (2006); Allahviranloo et al. (2007); Nasseri (2008); Veeramani et al. (2013)).

This paper is organized as follows. In Section 2, we present basic notions from the fuzzy interval theory. In Section 3, we present fuzzy interval games in strategic form. In Section 4, we describe the dominance concept. In Section 5, we consider the Hurwicz criterion and present the results. In Section 6, we study mixed strategies. In Section 7, we conclude.

2 Fuzzy intervals

Traditionally, a payoff $x_i \in \mathbb{R}$ of an agent i in a game is either possible or not possible to achieve. In the fuzzy logic, this possibility is not binary but uses a degree of membership that can vary in the closed real-valued interval $[0, 1]$. If the fuzzy number is 0, this payoff is not possible to achieve. If the fuzzy number is 1, this payoff is possible to achieve. However, all the intermediate situations are also feasible.

Formally, a *fuzzy set* in $\mathbb{R}$ is a real-valued function $F\colon \mathbb{R} \to [0, 1]$ which associates with each $x_i \in \mathbb{R}$ the grade of achievement $F(x_i)$ of x_i. This $F(x_i)$ is then called a *fuzzy number*. Following Zadeh (1965), who interprets a fuzzy number as the grade of membership, we say that $F(x_i)$ represents the grade of membership of x_i in the set of possible payoffs for player i. Another interpretation, due to Zadeh (1978), is that $F(x_i)$ represents the possibility of x_i to be achieved. Dubois and Prade (1997) acknowledges the existence of several interpretations of fuzzy numbers. Far away from being a weakness, this multiple existence provides a high potential in order to enrich its pervasiveness among many different fields. Dubois and Prade (1997) mentions three different types of interpretation, based on the degree of similarity, the degree of preference and the degree of uncertainty. In this paper, we consider the degree of uncertainty, where the grades of membership allow to rank the payoffs in terms of plausibility.

Given a fuzzy set F, for any $\alpha \in (0, 1]$, we denote

$$[F]^\alpha := \{x_i \in \mathbb{R} : \alpha \leq F(x)\}$$

and $[F]^0 := Cl\,\{x_i \in \mathbb{R} : F(x_i) > 0\}$, where $Cl(X)$ is the closure of X. A *fuzzy interval* is a fuzzy set satisfying

- compactness, i.e., $[F]^\alpha$ is compact for any $\alpha \in [0, 1]$
- convexity, i.e., $[F]^\alpha$ is convex for any $\alpha \in [0, 1]$
- normality, i.e., there exits $x \in \mathbb{R}$ such that $F(x) = 1$.

Let $\mathcal{F}$ be the set of fuzzy intervals. For any $F \in \mathcal{F}$, there exist $a, b, c, d \in \mathbb{R}$ with $a \leq b \leq c \leq d$, $L\colon [a, b] \to [0, 1]$ non-decreasing, and $R\colon [c, d] \to [0, 1]$ non-increasing such that $F(x_i) = L(x_i)$ if $x_i \in [a, b)$, $F(x_i) = 1$ if $x_i \in [b, c]$, $F(x_i) = R(x_i)$ if $x_i \in (c, d]$, and $F(x_i) = 0$ otherwise.

Some particular cases of fuzzy sets are the following:

- *Symmetric fuzzy sets*, where $b - a = d - c$ and $R(x) = L(a + d - x)$ for all $x \in [c, d]$.
- *Triangular fuzzy sets*, where L and R are linear functions and $b = c$. We denote a triangular fuzzy set as $F = (b, l, r)$, where $l = b - a$ and $r = d - b$.
- Any real number $y_i \in \mathbb{R}$ is identified with the fuzzy set with $a = b = c = d = y_i$, so that $F(y_i) = 1$ and $F(x_i) = 0$ otherwise. With some abuse of notation, we write $y_i \in \mathcal{F}$.
- Any closed interval $[\underline{e}, \overline{e}]$ with $\underline{e}, \overline{e} \in \mathbb{R}$ and $\underline{e} \leq \overline{e}$ is identified with the fuzzy set with $a = b = \underline{e}$ and $c = d = \overline{e}$, so that $F(x_i) = 1$ for all $x_i \in [\underline{e}, \overline{e}]$ and $F(x_i) = 0$ otherwise. With some abuse of notation, we write $[\underline{e}, \overline{e}] \in \mathcal{F}$.

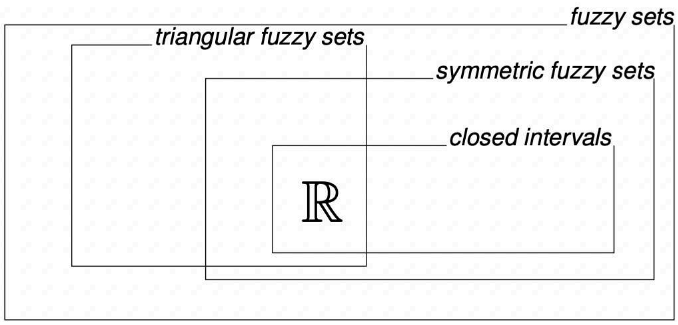

Figure 1: Relationships among some particular cases of fuzzy sets.

Relationships among these cases are depicted in Figure 1.

For any $F \in \mathcal{F}$ and $\alpha \in [0,1]$, let $\overline{F^\alpha}$ be the supremum of $[F]^\alpha$ and let $\underline{F^\alpha}$ be the infimum of $[F]^\alpha$. In particular, for each real number x, $\overline{x^\alpha} = \underline{x^\alpha} = x$; and for each closed interval $[x,z]$, $\overline{[x,z]^\alpha} = z$ and $\underline{[x,z]^\alpha} = x$.

Given two fuzzy intervals F, G, we say that $F \preceq G$ if $\underline{F^\alpha} \leq \underline{G^\alpha}$ and $\overline{F^\alpha} \leq \overline{G^\alpha}$ for all $\alpha \in [0,1]$ as in (Dubois and Prade, 1997; Mallozzi et al., 2011). Moreover, $F \prec G$ means $F \preceq G$ and $F \neq G$.

3 Fuzzy interval games in strategic form

A *fuzzy interval game in strategic form* is a tuple (N, A, u) where

- $N = \{1, \ldots, n\}$ is a set of players,
- $A = \prod_{i \in N} A_i$ with A_i a set of strategies, or actions, for each player $i \in N$, and
- $u = (u_i)_{i \in N}$ with $u_i \colon A \to \mathcal{F}$ a payoff function for each player $i \in N$.

A *(pure) strategy profile* $a \in A$ is a collection of strategies, one for each player.

Given $i \in N$, we use the notation $A_{-i} = \prod_{j \in N \setminus \{i\}} A_j$ and, given $a \in A$, we denote $a_{-i} = (a_j)_{j \in N \setminus \{i\}} \in A_{-i}$.

A Nash equilibrium is a strategy profile such that each strategy is a best response to all the other strategies. In the fuzzy logic, this idea can be stated in two ways:

Definition 3.1 *A* tight Nash equilibrium *(tNe) is a strategy profile $a^* \in A$ in which each strategy weakly improves any other, i.e., $u_i\left(a_i, a^*_{-i}\right) \preceq u_i\left(a^*_i, a^*_{-i}\right)$ for all $i \in N$ and all $a_i \in A_i$.*

	left	right
up	$3,3$	$[1,3],4$
down	$4,1$	$2,2$

Figure 2: Payoffs in Example 3.1.

A loose Nash equilibrium *(lNe) is a strategy profile* $a^* \in A$ *in which no other strategy makes any player weakly improved, i.e., there exist no* $i \in N$ *and* $a_i \in A_i$ *such that* $u_i\left(a_i, a^*_{-i}\right) \succ u_i\left(a^*_i, a^*_{-i}\right)$.

In both a loose and a tight Nash equilibrium, any unilateral deviation by some player that results in a gain for some $\alpha \in [0,1]$, also results in a loss for some other $\alpha' \in [0,1]$. Obviously, any tNe is also a lNe. However, not every lNe is a tNe.

Example 3.1 *Let* $N = \{1,2\}$, $A_1 = \{up, down\}$, $A_2 = \{left, right\}$, (u_1, u_2) *given in Figure 2. In this example,* $(down, right)$ *is a* lNe*, but not a* tNe *because* $u_1(down, right) = 2$ *and* $u_1(up, right) = [1,3]$ *are not comparable for player* 1.

4 Dominance

In this section, we generalize the concept of dominance (Moulin, 1979) to the class of fuzzy games.

Definition 4.1 *Given a fuzzy interval game in strategic form* (N, A, u)*, a strategy* $a^*_i \in A_i$ *of player* i loosely dominates *another strategy* $a_i \in A_i$ *in* A *if there exists no* $a_{-i} \in A_{-i}$ *such that* $u_i(a^*_i, a_{-i}) \prec u_i(a_i, a_{-i})$ *and there exists some* $a_{-i} \in A_{-i}$ *such that* $u_i(a^*_i, a_{-i}) \succ u_i(a_i, a_{-i})$.

This definition coincides with the standard definition of weak dominance (Moulin, 1979) when $u_i(a) \in \mathbb{R}$ for all $i \in N$ and $a \in A$. For $A' \subseteq A$, let $U(A')$ be the set of strategies profiles $a \in A'$ such that a_i is not loosely dominated for any player i.

Definition 4.2 *A fuzzy game* (N, A, u) *is* loosely dominance solvable *if there exists a finite sequence* $A = A^0 \supsetneq A^1 \supsetneq A^2 \supsetneq \cdots \supsetneq A^k$ *such that* $A^l = U(A^{l-1})$ *for all* $l = 1, \ldots, k$, $A^k = U(A^k)$*, and, for each* $i \in N$*, there exist no* $a_{-i} \in A^k_{-i}, a_i, a'_i \in A^k_i$ *such that* $u_i(a_i, a_{-i}) \prec u_i(a'_i, a_{-i})$*. Each* $a \in A^k$ *is then called a* loosely dominant solution *(lds).*

	left	center	right
up	$1, [3, 4]$	$[3, 4], 0$	$0, [2, 4]$
middle	$[0, 3], [0, 1]$	$2, [3, 4]$	$3, 0$
down	$2, [0, 1]$	$1, [1, 2]$	$3, [2, 4]$

Figure 3: Payoffs in Example 4.1.

The given definition generalizes the notion of dominance solvability as defined by Moulin (1979) when $u_i(a) \in \mathbb{R}$ for all $i \in N$ and $a \in A$. Notice that we discard the dominated strategies of all players in each step. As opposed to Moulin (1979), we do not impose equality among all the strategies in the last step. Instead, we impose either equality or that they are not comparable. When payoffs are real-valued, indifference between two strategies, given fix strategies by the other players, is equivalent to either payoff equality or no-dominance. Both concepts are equivalent. When payoffs are fuzzy sets, equality and no-dominance are not equivalent. We use no-dominance in the generalization of dominance solvability.

In the context of crisp cooperative games, a lNe is the same as a Nash equilibrium and a lds is the same as a dominant solution. As shown by (Moulin, 1979), each dominant solution is a Nash equilibrium. This result has been generalized to the class of multicriteria games (Gerasimou, 2019).

However, in fuzzy interval games, such result does not hold for these generalizations, as next example shows.

Example 4.1 *Let* $N = \{1, 2\}$, $A_1 = \{up, middle, down\}$, $A_2 = \{left, center, right\}$, $u = (u_1, u_2)$ *given in Figure 3.*

In this example, middle dominates down because $u_1(middle, center) = 2 > 1 = u_1(down, center)$, $u_1(middle, right) = u_1(down, right) = 3$ *and* $u_1(middle, left) = [0, 3]$ *is not comparable with* $u_1(down, left) = 2$*. Once down is excluded,* $left$ *dominates* $right$ *because* $u_2(up, left) = [3, 4] \succ [2, 4] = u_2(up, right)$ *and* $u_2(middle, left) = [0, 1] \succ 0 = u_2(middle, right)$*. Once right is excluded, up dominates middle since* $u_1(up, left) = 1$ *is not comparable with* $u_1(middle, left) = [0, 3]$ *and* $u_1(up, center) = [3, 4] \succ 2 = u_1(middle, center)$*. Finally, once middle and down are excluded,* $left$ *dominates center because* $u_2(up, left) = [3, 4] \succ 0 = u_2(up, center)$.

Hence, $(up, left)$ *is the only* lds*. However, it is not a* lNe *because* $u_1(up, left) = 1 \prec 2 = u_1(down, left)$*. The only (both loose and tight) Nash equilibrium in this example is* $(down, right)$.

5 Hurwicz criterion in fuzzy games

Hurwicz Hurwicz (1951) first stated what became the most well-known criterion to deal with uncertainly. Assume each player $i \in N$ has a coefficient $\eta_i \in [0,1]$ which determines its degree of optimism.[1] This means that, when the level of uncertainty is given by $\alpha \in [0,1]$, player i evaluates a fuzzy interval F as

$$\eta_i \circ F^\alpha := (1-\eta_i)\cdot \underline{F^\alpha} + \eta_i \cdot \overline{F^\alpha}.$$

The Hurwicz criterion allows us to redefine the partial ordering of fuzzy numbers, tight and loose Nash equilibria, and dominance.

5.1 Partial ordering

Given $F, G \in \mathcal{F}$, we say that $F \preceq^* G$ if $\eta_i \circ F^\alpha \leq \eta_i \circ G^\alpha$ for all $\alpha, \eta_i \in [0,1]$. Analogously, $F \prec^* G$ if $F \preceq^* G$ and $F \neq G$.

This definition is equivalent to the one presented in Section 2, as shown in the following result.

Proposition 5.1 *Order relations $\preceq$ and $\preceq^*$ are equivalent, i.e., $F \preceq G$ if and only if $F \preceq^* G$, and $F \prec G$ if and only if $F \prec^* G$.*

Proof. Let $F, G \in \mathcal{F}$. Assume first $F \preceq G$. Then, $\underline{F^\alpha} \leq \underline{G^\alpha}$ and $\overline{F^\alpha} \leq \overline{G^\alpha}$ for all $\alpha \in [0,1]$. Take $\alpha, \eta_i \in [0,1]$. Then, since both η_i and $1-\eta_i$ are nonnegative,

$$\eta_i \circ F^\alpha = (1-\eta_i)\cdot \underline{F^\alpha} + \eta_i \cdot \overline{F^\alpha} \leq (1-\eta_i)\cdot \underline{G^\alpha} + \eta_i \cdot \overline{G^\alpha} = \eta_i \circ G^\alpha$$

and hence $F \preceq^* G$. Assume now $F \prec G$, i.e., $F \preceq G$ and $F \neq G$. Hence, $F \preceq^* G$ and $F \neq G$, and thus $F \prec^* G$. Assume now $F \preceq^* G$. Given $\alpha \in [0,1]$, by taking $\eta = 0$ and $\eta = 1$, we deduce that $\underline{F^\alpha} \leq \underline{G^\alpha}$ and $\overline{F^\alpha} \leq \overline{G^\alpha}$, respectively. Hence, $F \preceq G$. Finally, assume $F \prec^* G$, i.e., $F \preceq^* G$ and $F \neq G$. Hence, $F \preceq G$ and $F \neq G$, and thus $F \prec G$.

Q.E.D.

5.2 Nash equilibria

By using the Hurwicz criterion, we have two new equilibrium concepts. Given $\eta \in [0,1]^N$, $\alpha \in [0,1]$, and $F \in \mathcal{F}^N$, let $\eta \circ F^\alpha$ be the vector in $\mathbb{R}^N$ whose coordinates are given by $\eta_i \circ F_i^\alpha$ for each $i \in N$.

[1] We use the term η instead of the usual Hurwicz term α in order to avoid confusion with the coefficient in fuzzy numbers.

Definition 5.1 *We say that a strategy profile $a^* \in A$ is a* Hurwicz Nash equilibrium *(HNe) if each strategy is a best response to the other strategies under the Hurwicz criterion for all possible η and α, i.e.,*

$$\eta_i \circ u_i(a_i, a_{-i}^*)^\alpha \leq \eta_i \circ u_i(a^*)^\alpha$$

for all $i \in N$, $a_i \in A_i$, and $\eta_i, \alpha \in [0,1]$.

Analogously, we define the loose Hurwicz Nash equilibrium.

Definition 5.2 *We say that a strategy profile $a^* \in A$ is a* loose Hurwicz Nash equilibrium *(lHNe) if there exists $\eta \in [0,1]^N$ such that each strategy is a best response to the other strategies under the Hurwicz criterion for η, i.e.,*

$$\eta_i \circ u_i(a_i, a_{-i}^*)^\alpha \leq \eta_i \circ u_i(a^*)^\alpha$$

for all $i \in N$, $a_i \in A_i$, and $\alpha \in [0,1]$.

Obviously, any HNe is also a $lHNe$. However, not every $lHNe$ is a HNe. For example, the strategy profile described in Example 3.1 is a $lHNe$ but not a HNe.

We next study the relation between these new equilibrium concepts and the ones defined in Section 2.

Theorem 5.1 *Hurwicz Nash equilibria are equivalent to tight Nash equilibria, i.e., $a^* \in A$ is Hurwicz Nash equilibrium if and only if it is a tight Nash equilibrium.*

Proof. Let a^* be a HNe and take $i \in N$ and $a_i \in A_i$. We have to prove that $u_i\left(a_i, a_{-i}^*\right) \preceq u_i\left(a_i^*, a_{-i}^*\right)$, i.e., $\underline{u_i\left(a_i, a_{-i}^*\right)^\alpha} \leq \underline{u_i\left(a_i^*, a_{-i}^*\right)^\alpha}$ and $\overline{u_i\left(a_i, a_{-i}^*\right)^\alpha} \leq \overline{u_i\left(a_i^*, a_{-i}^*\right)^\alpha}$ for all $\alpha \in [0,1]$. Take any $\alpha \in [0,1]$. Then,

$$\underline{u_i\left(a_i^*, a_{-i}^*\right)^\alpha} = 0 \circ u_i\left(a_i, a_{-i}^*\right)^\alpha \leq 0 \circ u_i\left(a^*\right)^\alpha = \underline{u_i\left(a^*\right)^\alpha}$$

and

$$\overline{u_i\left(a_i^*, a_{-i}^*\right)^\alpha} = 1 \circ u_i\left(a_i, a_{-i}^*\right) \leq 1 \circ u_i\left(a^*\right)^\alpha = \overline{u_i\left(a^*\right)^\alpha}$$

where inequalities come from taking $\eta_i = 0$ and $\eta_i = 1$, respectively. Now, let a^* be a tNe and take $i \in N$, $a_i \in A_i$, and $\eta_i, \alpha \in [0,1]$. Then, since both $1 - \eta_i$ and η_i are nonnegative,

$$\begin{aligned}
\eta_i \circ u_i\left(a_i, a_{-i}^*\right)^\alpha &= (1-\eta_i) \cdot \underline{u_i\left(a_i, a_{-i}^*\right)^\alpha} + \eta_i \cdot \overline{u_i\left(a_i, a_{-i}^*\right)^\alpha} \\
&\leq (1-\eta_i) \cdot \underline{u_i\left(a_i^*, a_{-i}^*\right)^\alpha} + \eta_i \cdot \overline{u_i\left(a_i^*, a_{-i}^*\right)^\alpha} \\
&= \eta_i \circ u_i\left(a_i^*, a_{-i}^*\right)^\alpha .
\end{aligned}$$

	left	right
up	$[1,4],1$	$[1,4],1$
middle	$[0,5],1$	$[0,5],0$
down	$3,1$	$3,0$

Figure 4: Payoffs in Example 5.1.

	left	right
up	$0,1$	$F,0$
down	$[0,1],1$	$G,0$

Figure 5: Payoffs in Example 5.2.

Q.E.D.

However, $lHNe$ and lNe are not equivalent, as next examples show:

Example 5.1 *Let* $N = \{1,2\}$, $A_1 = \{up, middle, down\}$, $A_2 = \{left, right\}$, (u_1, u_2) *given in Figure 4.*

In this example, $(up, left)$, $(up, right)$,$(middle, left)$, *and* $(down, left)$ *are all* lNe *because* $[1,4]$, $[0,5]$, *and* 3 *are not comparable. However,* $(up, left)$ *and* $(up, right)$ *are not* $lHNe$ *because, for any* $\alpha \in [0,1]$, $\eta_i \circ [1,4]^\alpha \geq \eta_i \circ [0,5]^\alpha$ *implies* $\eta_i \leq \frac{1}{2}$, *whereas* $\eta_i \circ [1,4]^\alpha \geq \eta_i \circ 3^\alpha$ *implies* $\eta_i \geq \frac{2}{3}$, *which are incompatible conditions.*

Notice also that, in Example 5.1, $(up, left)$ is a lds, whereas $(up, right)$ is not. Hence, not all lNe that are not $lHNe$ are lds.

Example 5.2 *Let* $N = \{1,2\}$, $A_1 = \{up, down\}$, $A_2 = \{left, right\}$, (u_1, u_2) *given in Figure 5 (*F, G *are fuzzy intervals).*

In this example, $(up, left)$ *is a* $lHNe$ *with* $\eta_1 = 0$, *but not a* lNe *because* $0 \prec [0,1]$.

Notice that the strategy profile described in Example 5.2 is a somehow unstable equilibrium. Player 1 has incentives to deviate for any $\eta_1 > 0$, and for $\eta_1 = 0$ it is indifferent.

In order to rule out this kind of equilibria, we may request the feasible η_i not to be a extreme case (0 or 1).

Formally, we give the definition.

Definition 5.3 *We say that a strategy profile $a^* \in A$ is an* interior Hurwicz Nash equilibrium *(iHNe) if there exists $\eta \in (0,1)^N$ such that each strategy is a best response to the other strategies under the Hurwicz criterion for η, i.e.,*

$$\eta_i \circ u_i(a_i, a^*_{-i})^\alpha \leq \eta_i \circ u_i(a^*)^\alpha$$

for all $i \in N$, $a_i \in A_i$, and $\alpha \in [0,1]$.

Each tNe is an $iHNe$.

Theorem 5.2 *Each tight Nash equilibrium is an interior Hurwicz Nash equilibrium.*

Proof. Let us suppose that $a^* \in A$ is a tNe, i.e., $u_i\left(a_i, a^*_{-i}\right) \preceq u_i\left(a^*_i, a^*_{-i}\right)$ for all $i \in N$ and all $a_i \in A_i$ that by Proposition 5.1 is equivalent to

$$\eta_i \circ u_i(a_i, a^*_{-i})^\alpha \leq \eta_i \circ u_i(a^*)^\alpha$$

for all $i \in N$, $a_i \in A_i$, and $\alpha, \eta_i \in [0,1]$. This means that $a^* \in A$ an $iHNe$ for any $\eta \in (0,1)^N$.

Q.E.D.

The reciprocal is not true. In Example 3.1, the strategy profile $(down, right)$ is an $iHNe$ for any $\eta_1 \in (0, \frac{1}{2}]$ and not a tNe.

Moreover, each $iHNe$m is also a $lHNe$.

Theorem 5.3 *Each interior Hurwicz Nash equilibrium is a loose Hurwicz Nash equilibrium.*

Proof. Let a^* be an $iHNe$, i.e., there exists $\eta \in (0,1)^N$ such that $\eta_i \circ u_i(a_i, a^*_{-i})^\alpha \leq \eta_i \circ u_i(a^*)^\alpha$ for all $i \in N, a_i \in A_i$ and $\alpha \in [0,1]$. Hence, $\eta \in [0,1]^N$ and it satisfies the conditions of $lHNe$.

Q.E.D.

The reciprocal is not true as, for example, the $lHNe$ described in Example 5.2 is not an $iHNe$.

Finally, every $iHNe$ is also a lNe.

Theorem 5.4 *Each interior Hurwicz Nash equilibrium is a loose Nash equilibrium.*

Proof. Let a^* be an $iHNe$, i.e., there exists $\eta^* \in (0,1)^N$ such that

$$\eta_i^* \circ u_i(a_i, a_{-i}^*)^\alpha \leq \eta_i^* \circ u_i(a^*)^\alpha \text{ for all } i \in N, a_i \in A_i, \alpha \in [0,1]. \quad (1)$$

We proceed by contradiction. Assume a^* is not a lNe, i.e., there exist $i \in N$, $a_i \in A_i$ such that $u_i(a_i, a_{-i}^*) \succ u_i(a^*)$. Under Proposition 5.1, $u_i(a_i, a_{-i}^*) \succ^* u_i(a^*)$. Thus

$$\eta_i \circ u_i(a_i, a_{-i}^*)^\alpha \geq \eta_i \circ u_i(a^*)^\alpha \text{ for all } \eta_i, \alpha \in [0,1] \quad (2)$$

and there exist $\eta_i', \alpha' \in [0,1]$ such that

$$\eta_i' \circ u_i(a_i, a_{-i}^*)^{\alpha'} > \eta_i' \circ u_i(a^*)^{\alpha'}. \quad (3)$$

Combining (1) and (2), we get $\eta_i^* \circ u_i(a_i, a_{-i}^*)^{\alpha'} = \eta_i^* \circ u_i(a^*)^{\alpha'}$. By definition, such equality and inequality (3) become

$$(1-\eta_i^*) \cdot \underline{u_i(a_i, a_{-i}^*)^{\alpha'}} + \eta_i^* \cdot \overline{u_i(a_i, a_{-i}^*)^{\alpha'}} = (1-\eta_i^*) \cdot \underline{u_i(a^*)^{\alpha'}} + \eta_i^* \cdot \overline{u_i(a^*)^{\alpha'}} \quad (4)$$

and

$$(1-\eta_i') \cdot \underline{u_i(a_i, a_{-i}^*)^{\alpha'}} + \eta_i' \cdot \overline{u_i(a_i, a_{-i}^*)^{\alpha'}} > (1-\eta_i') \cdot \underline{u_i(a^*)^{\alpha'}} + \eta_i' \cdot \overline{u_i(a^*)^{\alpha'}} \quad (5)$$

respectively. Obviously, $\eta_i' \neq \eta_i^*$. Assume $\eta_i' > \eta_i^* > 0$ (the alternative case, $1 > \eta_i^* > \eta_i'$, is analogous). By multiplying each term in (4) by $\frac{-\eta_i'}{\eta_i^*}$ and adding (5), we get

$$\left[1 - \eta_i' - \frac{\eta_i'}{\eta_i^*}(1-\eta_i^*)\right] \cdot \underline{u_i(a_i, a_{-i}^*)^{\alpha'}} > \left[1 - \eta_i' - \frac{\eta_i'}{\eta_i^*}(1-\eta_i^*)\right] \cdot \underline{u_i(a^*)^{\alpha'}}.$$

Since $1-\eta_i'-\frac{\eta_i'}{\eta_i^*}(1-\eta_i^*) = \frac{\eta_i^*-\eta_i'}{\eta_i^*} < 0$, we deduce $\underline{u_i(a_i, a_{-i}^*)^{\alpha'}} < \underline{u_i(a^*)^{\alpha'}}$. Since $\underline{F^{\alpha'}} = 0 \circ F^{\alpha'}$ for all $F \in \mathcal{F}$, we conclude that $0 \circ u_i(a_i, a_{-i}^*)^{\alpha'} < 0 \circ u_i(a^*)^{\alpha'}$, which contradicts (2).

Q.E.D.

There are strategy profiles that are both $lHNe$ and lNe but not $iHNe$, as next example shows:

Example 5.3 *Let $N = \{1,2\}$, $A_1 = \{up, down\}$, $A_2 = \{center\}$, (u_1, u_2) given in Figure 6, where G, H are arbitrary fuzzy intervals and F is a fuzzy interval defined by*

$$F(x) = \begin{cases} \frac{x}{4} & \text{if } x \in [0,2) \\ 3-x & \text{if } x \in [2,3] \\ 0 & \text{otherwise.} \end{cases}$$

for each $x \in \mathbb{R}$ (see Figure 7).

In this example, all the strategy profiles are lNe because F and $(2,1,1)$ are not comparable. Moreover, they are also $lHNe$ with $\eta_1 = 1$. However, $(down, center)$ is not an $iHNe$ because, for any $\eta_1 < 1$,

$$\eta_1 \circ F^{\frac{1}{2}} = 2 + \frac{\eta_1}{2} > \frac{3}{2} + \eta_1 = \eta_1 \circ (2,1,1)^{\frac{1}{2}}.$$

	center
up	F, G
down	$(2,1,1), H$

Figure 6: Payoffs in Example 5.3.

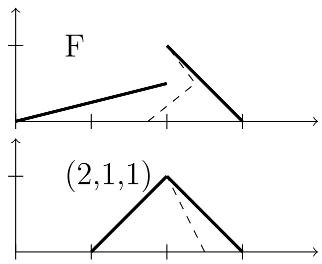

Figure 7: Fuzzy intervals in Example 5.3: F and $(2,1,1)$. For each $\eta_i \in (0,1)$, $\eta_i \circ F \leq \eta_i \circ (2,1,1)$ for some α's and $\eta_i \circ F \leq \eta_i \circ (2,1,1)$ for others.

5.3 Hurwicz dominance

In the following, we define the concept of Hurwicz dominance. Hurwicz dominance arises when there exists a common η in each step of the procedure. We have two such possible dominances, depending on whether the common η can be chosen interiorly or not.

Definition 5.4 *Given a fuzzy interval game in strategic form (N, A, u), a strategy $a_i^* \in A_i$ of player i* loosely Hurwicz dominates *another strategy $a_i \in A_i$ in A if there exists $\bar{\eta} \in [0,1]^N$ such that*

$$\bar{\eta}_i \circ u^\alpha(a_i^*, a_{-i}) \geq \bar{\eta}_i \circ u^\alpha(a_i, a_{-i}), \quad \text{for all } a_{-i} \in A_{-i}$$

and

$$\bar{\eta}_i \circ u^\alpha(a_i^*, a_{-i}) > \bar{\eta}_i \circ u^\alpha(a_i, a_{-i}), \quad \text{for some } a_{-i} \in A_{-i}$$

hold for any $\alpha \in [0,1]$.

For $A' \subseteq A$ and $\bar{\eta} \in [0,1]^N$, let $U^{\bar{\eta}}(A')$ be the set of strategies profiles $a \in A'$ such that a_i is not loosely Hurwicz dominated for any player i under $\bar{\eta}$.

Definition 5.5 *A fuzzy interval game in strategic form (N, A, u) is* loosely Hurwicz dominance solvable *(resp.,* interiorly Hurwicz dominance solvable*) if there exist $\bar{\eta} \in [0,1]^N$ (resp., $\bar{\eta} \in (0,1)^N$) and a finite sequence $A = A^0 \supsetneq A^1 \supsetneq A^2 \supsetneq \cdots \supsetneq A^k$ such* *that* $A^l = U^{\bar{\eta}}(A^{l-1})$ *for all* $l = 1, \ldots, k$*,* $A^k = U^{\bar{\eta}}(A^k)$*, and, for each* $i \in N$*, for any* $a_{-i} \in A^k_{-i}, a_i, a'_i \in A^k_i$

$$\bar{\eta}_i \circ u^\alpha(a'_i, a_{-i}) = \bar{\eta}_i \circ u^\alpha(a_i, a_{-i}), \forall \alpha \in [0,1].$$

Each $a \in A^k$ *is then called a* loosely Hurwicz dominant solution *(lHds) (resp.,* interiorly Hurwicz dominant solution *(iHds)).*

Obviously, each $iHds$ is also a $lHds$. However, not every $lHds$ is an $iHds$. For example, the strategy profile $(down, center)$ in example 5.3 is a $lHds$ but not an $iHds$.

In line with Moulin (1979), we have the following results.

Theorem 5.5 *Each loosely Hurwicz dominant solution of a fuzzy interval game is a loose Hurwicz Nash equilibrium.*

Proof. Let us suppose that $a^* \in A$ is a loosely Hurwicz dominant solutions for a given $\bar{\eta} \in [0,1]^N$ of the fuzzy interval game (N, A, u) and for each $\alpha \in [0,1]$ consider the crisp game $(N, A, \bar{\eta} \circ u^\alpha)$. By using the result in Moulin (1979) we obtain that $a^* \in A$ is a pure-strategy Nash equilibrium of $(N, A, \bar{\eta} \circ u^\alpha)$, for any α. Then $a^* \in A$ is a loose Hurwicz Nash equilibrium.

Q.E.D.

We point out that a loosely Hurwicz dominant solution is not necessary a loose Nash equilibrium. For example, $(up, left)$ in Example 5.2 is a loosely Hurwicz dominant solution and not a loose Nash equilibrium.

Theorem 5.6 *Each interiorly Hurwicz dominant solution of a fuzzy interval game is an interior Hurwicz Nash equilibrium.*

Proof. Let us suppose that $a^* \in A$ is an interiorly Hurwicz dominant solution for a given $\bar{\eta} \in (0,1)^N$ of the fuzzy interval game (N, A, u). Then, for each $\alpha \in [0,1]$, the dominated strategy elimination process in the crisp game $(N, A, \bar{\eta} \circ u^\alpha)$ ends to the profile $a^* \in A$ that is a Nash equilibrium of the crisp game, then an interior Hurwicz Nash equilibrium.

Q.E.D.

The next example shows interior Hurwicz Nash equilibria that are not interiorly Hurwicz dominant solutions.

Example 5.4 *Let* $N = \{1,2\}$ *,* $A_1 = \{up, down\}$*,* $A_2 = \{left, right\}$*,* (u_1, u_2) *given in Figure 8. Then,* $(up, left)$ *and* $(down, right)$ *are interior Hurwicz Nash equilibria but not interiorly Hurwicz dominant solutions.*

	left	right
up	$1, 1$	$-3, 0$
down	$0, [-1, 0]$	$1, 0$

Figure 8: Payoffs in Example 5.4.

	left	right
up	$0, [0, 1]$	$[0, 1], 0$
down	$[0, 1], 0$	$0, [0, 1]$

Figure 9: Payoffs in Example 6.1.

6 Mixed strategies in fuzzy games

In this section, we study the role of mixed strategies. Assume players have von Neumann-Morgenstern utility functions, so that the utility given by the lottery in which $F \in \mathcal{F}$ happens with probability $p \in [0, 1]$ and $G \in \mathcal{F}$ with probability $q = 1 - p$ is $pF + qG \in \mathcal{F}$ defined as follows. For each $\alpha \in [0, 1]$,

$$\underline{[pF + qG]^\alpha} = p\underline{[F]^\alpha} + q\underline{[G]^\alpha}$$

and

$$\overline{[pF + qG]^\alpha} = p\overline{[F]^\alpha} + q\overline{[G]^\alpha}.$$

Let us consider the following example.

Example 6.1 *Let $N = \{1, 2\}$, $A_1 = \{up, down\}$, $A_2 = \{left, right\}$, (u_1, u_2) given in Figure 9. In this example, there are not equilibria, neither lNe nor $lHNe$ (and hence, neither $iHNe$ nor tNe).*

As it happens in crisp finite games, mixing can be of help. Let $(p, 1-p)$ and $(q, 1-q)$, for $p, q \in [0, 1]$, be probability distributions on the sets A_1, A_2 respectively, and consider the expected payoff functions

$$\hat{u}_1(p, q) = p(1-p)[0, 1] + q(1-p)[0, 1] = [0, p + q - 2pq]$$
$$\hat{u}_2(p, q) = pq[0, 1] + (1-p)(1-q)[0, 1] = [0, 2pq + 1 - p - q].$$

The game $\left(N, \hat{A}, \hat{u}\right)$, where $\hat{u} = (\hat{u}_1, \hat{u}_2)$, $\hat{A} = \hat{A}_1 \times \hat{A}_2$ and $\hat{A}_i$ is the set of probability distributions on A_i (for $i = 1, 2$), is a fuzzy interval game and $((p, 1-p), (q, 1-q))$ with $p = q = \frac{1}{2}$ is an interior Hurwicz Nash equilibrium. Remark that for any $\alpha \in [0, 1]$, any $\eta \in (0, 1)^2$, $\eta \circ u^\alpha = \big(\eta_1(p + q - 2pq), \eta_2(2pq + 1 - p - q)\big)$.

Then, in Example 6.1 we find a mixed interior Hurwicz Nash equilibrium for the game (N, A, u), as specified in the following definition.

We consider finite fuzzy games, namely tuples (N, A, u) where $N = \{1, \dots, n\}$ is a set of players, $A = \prod_{i \in N} A_i$ with A_i a finite set of strategies, for each player $i \in N$, and $u = (u_i)_{i \in N}$ with $u_i \colon A \to \mathcal{F}$ a payoff function for each player $i \in N$. We consider the mixed extension of the game, say $\left(N, \hat{A}, \hat{u}\right)$, being $\hat{A} = \prod_{i \in N} \hat{A}_i$ with $\hat{A}_i$ the set of the probability distributions on the finite set A_i and $\hat{u}_i$ the expected payoff (averages) of player i $(i = 1, ..., N)$.

Definition 6.1 *We say that a strategy profile $\hat{a}^* \in \hat{A}$ is a* mixed loose Nash equilibrium *if each mixed strategy $\hat{a}^*$ is not worse than any other mixed strategy, i.e., there exist no $i \in N$ and $\hat{a}_i \in \hat{A}_i$ such that*

$$\hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^\alpha \succeq \hat{u}_i(\hat{a}^*)^\alpha$$

for all $\alpha \in [0, 1]$ and $\hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^{\alpha^*} \succ \hat{u}_i(\hat{a}^*)^{\alpha^*}$ for some $\alpha^* \in [0, 1]$.

We say that a strategy profile $\hat{a}^* \in \hat{A}$ is a mixed interior Hurwicz Nash equilibrium *if there exists $\eta \in (0, 1)^N$ such that each mixed strategy $\hat{a}^*$ is a best response to any other mixed strategy under the Hurwicz criterion for η, i.e.,*

$$\eta_i \circ \hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^\alpha \leq \eta_i \circ \hat{u}_i(\hat{a}^*)^\alpha$$

for all $i \in N$, $\hat{a}_i \in \hat{A}_i$, and $\alpha \in [0, 1]$.

Theorem 6.1 *Each finite fuzzy game has a mixed loose Nash equilibrium.*

Proof. Given a fuzzy set F, we define $E(F)$ as the expected value of the middle point of F^α, i.e.,

$$E(F) = \int_0^1 0.5 \circ F^\alpha \, d\alpha.$$

It is straightforward to check that $\phi(\alpha) = 0.5 \circ F^\alpha$ is a left-continuous function. Hence, $0.5 \circ F^\alpha \leq 0.5 \circ G^\alpha$ for all $\alpha \in [0, 1]$ and $0.5 \circ F^{\alpha^*} < 0.5 \circ G^{\alpha^*}$ for some $\alpha^* \in [0, 1]$ imply $E(F) < E(G)$. Given (N, A, u), let $(N, A, E(u))$ be the crisp game where, for each $i \in N$, $E(u)_i$ is defined as

$$E(u)_i(a) = E(u_i(a))$$

for all $a \in A$. It follows from the well-known Nash's theorem on the existence of mixed Nash equilibria for finite games applied to the crisp game $(N, A, E(u))$ that there exists at least one Nash equilibrium (maybe with mixed strategies) $\hat{a}^* \in \hat{A}$. We check that $\hat{a}^*$ is also a mixed loose Nash equilibrium in (N, A, u). Assume, on the contrary, that there exist $i \in N$ and $\hat{a}_i \in \hat{A}_i$ such that $\hat{u}_i(\hat{a}^*)^\alpha \preceq \hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^\alpha$ for all $\alpha \in [0,1]$ and $\hat{u}_i(\hat{a}^*)^{\alpha^*} \prec \hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^{\alpha^*}$ for some $\alpha^* \in [0,1]$. Thus, $0.5 \circ \hat{u}_i(\hat{a}^*)^\alpha \leq 0.5 \circ \hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^\alpha$ for all $\alpha \in [0,1]$ and $0.5 \circ \hat{u}_i(\hat{a}^*)^{\alpha^*} < 0.5 \circ \hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})^{\alpha^*}$. Hence,

$$E(\hat{u})_i(\hat{a}^*) = E(\hat{u}_i(\hat{a}^*)) < E(\hat{u}_i(\hat{a}_i, \hat{a}^*_{-i})) = E(\hat{u})_i(\hat{a}_i, \hat{a}^*_{-i}))$$

which is a contradiction because $\hat{a}^*$ is a Nash equilibrium in $(N, A, E(u))$.

Q.E.D.

Theorem 6.2 *Each finite fuzzy game with symmetric fuzzy sets has a mixed interior Hurwicz Nash equilibrium.*

Proof. Take $\eta_i = 0.5$ for all $i \in N$. Given $\alpha \in [0,1]$, it follows from the well-known Nash's theorem on the existence of mixed Nash equilibria for finite games to the crisp game $(N, A, \eta \circ u^\alpha)$ that there exists at least one Nash equilibrium (maybe with mixed strategies) $\hat{a}^* \in \hat{A}$. Given the symmetry of the fuzzy sets, $\eta \circ u^\alpha$ is independent of α, and hence $\hat{a}^*$ is also a mixed interior Hurwicz Nash equilibrium.

Q.E.D.

Theorem 6.2 does not apply for general fuzzy games. Take for example the case $N = \{1, 2\}$, $A_1 = \{up, down\}$, $A_2 = \{center\}$, $u_1(up, center) = (2, 0, 2)$ and $u_1(down, center) = (3, 2, 0)$. For any $\alpha \in [0,1]$, we have $(2,0,2)^\alpha = [2, 4 - 2\alpha]$ and $(3,2,0)^\alpha = [2\alpha + 1, 3]$. Hence, given $\eta_1 \in [0,1]$,

$$\eta_1 \circ (2,0,2)^\alpha = \eta_1 \circ [2, 4 - 2\alpha] = 2\eta_1 - 2\alpha\eta_1 + 2$$

and

$$\eta_1 \circ (3,2,0)^\alpha = \eta_1 \circ [1 + 2\alpha, 3] = 2\eta_1 + 2\alpha(1 - \eta_1) + 1$$

which implies that, independently of η_1, for $\alpha < 0.5$, $\eta_1 \circ (2,0,2)^\alpha > \eta_1 \circ (3,2,0)^\alpha$ and, for $\alpha > 0.5$, $\eta_1 \circ (2,0,2)^\alpha < \eta_1 \circ (3,2,0)^\alpha$. Hence, neither $(up, center)$ nor $(down, center)$ are $iHNe$. In the mixed strategy where *up* is played with probability $p \in [0,1]$ and *down* with probability $1 - p$, $p = 0$ is always optimal for $\alpha > 0.5$ and $p = 1$ for $\alpha < 0.5$.

7 Conclusion

In this paper, we define five reasonable extensions of the Nash equilibria for fuzzy games, as well as three extensions of the notion of a dominant solution. We show that the most natural extension of a dominant solution misses its most appealing properties in the context of fuzzy games, since it may fail to be the most weak extensions of a Nash equilibrium (namely, the so-called loose Nash equilibria and loose Hurwicz Nash equilibrium).

We find that loosely Hurwicz dominant solutions are loose Hurwicz Nash equilibria and that interiorly Hurwicz dominant solutions are interior Hurwicz Nash equilibria. Both these concepts, based on the classical Hurwicz criterion, provide a refinement tool. However, in a finite fuzzy game, existence of a mixed loose Nash equilibrium is guaranteed, while in a finite symmetric fuzzy game (i.e. payoffs are symmetric fuzzy sets) there exist mixed interior Hurwicz Nash equilibria. Then, the concept of interiorly Hurwicz dominance, introduced in this paper, turns out to be the most appropriate generalization of the classical dominance notion in crisp games. In the class of symmetric fuzzy games, the concept of interior Hurwicz Nash equilibrium turns out to be the most appropriate generalization of the Nash equilibrium one.

We summarize the relationship between these solution concepts in Figure 10.

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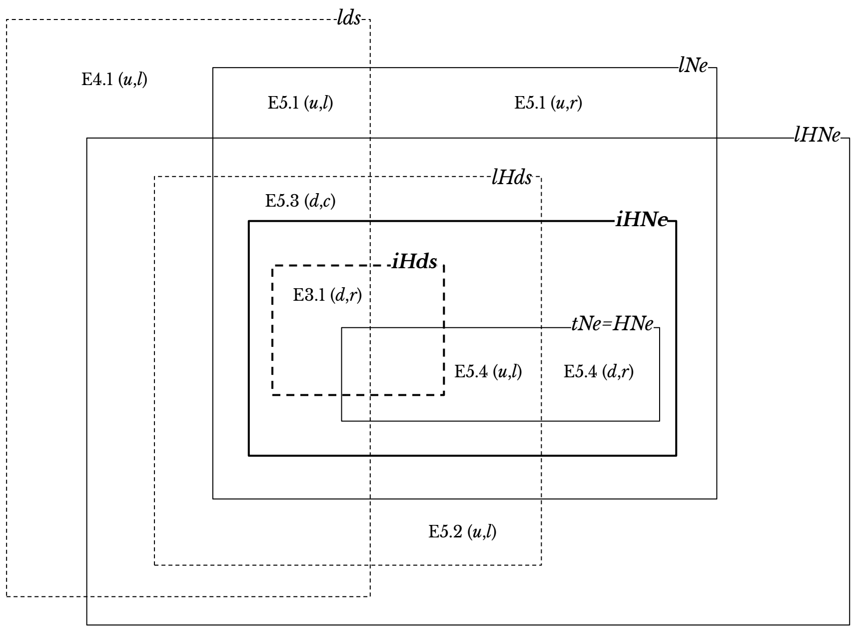

Figure 10: Summary of the results. "E4.1 (u, l)" stands for the strategy profile $(up, left)$ presented in Example 4.1, and so on.

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