Saving and dissaving under Ramsey - Rawls criterion

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Abstract

This article studies an inter-temporal optimization problem by using a criterion that is a combination of Ramsey's and Rawls's criteria. A detailed description of the saving behavior through time is provided. The optimization problem under $\alpha-$*maximin* criterion is also considered, and the optimal solution is characterized.

Keywords: Rawls criterion, Ramsey criterion, $\epsilon-$contamination, maximin principle, $\alpha-$MaxMin.

JEL classification numbers: C61, D11, D90.

1 Introduction

1.1 Motivation and concerns

Consensus has been reached that the main source of today's high living quality compared with other centuries is from the non-stop world economic growth that began 300 years ago. Nevertheless, the trade-off between efficiency and equality always causes debates among economists, politicians, and even historians. An extremist privilege could cause massive complications for human welfare. Especially

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in the context of climate change today, inter-generational inequality becomes one of the most important issues in economists' attention. So, the question is: how to *justly* reconcile these features?

This urges the need for ethnically accepted criteria evaluating inter-generational utilities. Chichilnisky, in seminar articles [11, 12], proposes to study Social Welfare Functions satisfying the *no-dictatorship of the present* and the *no-dictatorship of the future*, balancing the welfare of the present and of the future.[1] These criteria, weakening the *anonymity* in keeping *Paretian* property, rapidly become the inspiration source of a large range of contributions.

This article follows an alternative approche, considering a criterion balancing efficiency and equality, with the axiomatic foundation can be established by adding a *time invariance* property to the $\epsilon-$contamination configuration in Kopylov [25]. Precisely, given a discount rate β and a instataneous utility function u, the evaluation of a consumption stream $(c_0, c_1, c_2 \dots)$ is:

$$U(c_0, c_1, c_2, \dots) = \sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t),$$

where the parameter $a \geq 0$ captures the importance of equity in the inter-temporal generational evaluation in the choice of the economic agent. While the first term is the usual discounted utilitarian criterion, the second one, well-known as Rawls' one, measures the utility value of consumption streams in respect of equality.

We consider also the possibility balancing pessimism and optimism, taking into account the worst generation and the best generation.[2] This a a convex combination of the worst and the best:

$$U(c_0, c_1, \dots) = \alpha \sup_{t \geq 0} u(c_t) + (1 - \alpha) \inf_{t \geq 0} u(c_t), \tag{1.1}$$

for some $0 \leq \alpha \leq 1$ that can be considered as the *optimism* degree of the economic agent.

1.2 Approach and results

We begin the analysis by considering the following *modified* optimization problem: if we lower the value of the Rawls part to ϵ, what is the best we can make for the Ramsey part? By lowering the former, we have more room to improve the later.

[1] A generalization of these properties, with the convex parameter between the close future and the distant future may change in function of their values, can be found in Drugeon and Ha-Huy [13].

[2] For a general review of formulations in ambiguity, where $\alpha-$MaxMin is a special case, see Etner, Jeleva and Tallon [16].

The optimal ϵ can be considered as the efficiency-equality trade-off cost. This modification allows us to transform the initial problem into a classical optimization one with an additional constraint.

Beginning from a low level of capital accumulation, the utility of the early dates (or generations) has a tendency to be reduced as much as possible, for the sake of a rapid accumulation of capital. Nevertheless, because of the constraints imposed by the equality criteria of Rawls, the difference in utility between the early and later dates is not too high. In long term, the economy behaves as under the Ramsey criterion.

In the case of high capital accumulation, the economy converges to a higher steady-state than that of the Ramsey problem. In the long term, the economy behaves like a solution to the Ramsey problem with a higher discount factor, which is increasing with respect to the importance of the equality parameter a.

The $\alpha-$MaxMin problem is studied using a similar idea. By lowering the infimum part, we describe the optimal path for the supremum part. We determine the existence of an optimal value trade-off and a detailed properties of the economy.

Beginning from an initial value that is smaller than the *golden rule*, there exists a threshold for the parameter α. If α is small, corresponding to the situation where Rawls part dominates in the criterion, the solution coincides with Rawls part's one. Otherwise, with a sufficiently large value of α, the economy has an infinite number of solutions. Every optimal path fluctuates between two different values determined by the fundamental parameters of the problem.

A similar property is observed if the economy begins from an initial capital that is higher than the *golden rule* but not too much. In the case the initial capital level is sufficiently high, there is a unique solution and this path takes a constant value (smaller than the *golden rule*) from the date $t = 1$.

1.3 Related literature

In recent decades, a vast literature has expanded the results in decision theory to study inter-temporal axiomatization. The introduction of Gilboa and Schmeidler [19] of the *multiple priors* approach gives the inception of numerous works not only in ambiguity literature but also in multiple discount rates of inter-temporal analysis. To name some contributions, Wakai [33] provides an account of smoothing behaviours where the optimal discount assumes an MaxMin recursive representation. Chambers and Echenique [10] set an axiomatic approach to multiple exponential discount rates. Recently, in a similar axiomatic system as Chambers and Echanique [10], Bich, Dong-Xuan and Wigniolle [7] establish a multiple disount rates configuration combining temporal bias phenomena.

This article follows an alternative line of thinking, the idea of $\epsilon-$contamination, presented in Kopylov [25]. This configuration can be obtained by adding an $\epsilon-$contamination structure to the set of priors in Gilboa and Schmeidler [19]. The evaluation of an act is measured by a convex combination between a mean expected utility and a *maximin* expected utility with a given set of priors. The case of complete ignorance, *i.e.* the set of priors is the whole set of probabilities, is studied in Eichberger and Kelsey [15] and Nishimura and Ozaki [27]. Kopylov [25] obtains the general case by substituting the *certainty independence* of Gilboa and Schmeidler [19] by the $\Delta-$independence, with Δ is an information set of probabilities, given as a fundamental of the model. In our article, we add a supplementary condition, a weaker version of Koopmans's *time invariance* to obtain the Ramsey-Rawls presentation.

The contribution of Alvarez-Cuadrado and Van Long [1] studies a similar criterion (called the *mixed Bentham - Rawls* criterion) under a continuous time configuration. Their analysis is based on a maximization problem where the infimum of utility streams is supposed to be greater than a certain value $\underline{u}$, which is considered as a control parameter. The observation of the solution and the choice of the optimal parameter $\underline{u}$ provide the properties of the optimal path.

Many attempts have been made to study the solution of Rawls *maximin* critetion in [31]. The seminal contributions of Arrow [3], Calvo [8], Phelps and Riley [29], have been made to study the evolution of the economy by using this criterion to evaluate inter-temporal welfare. Arrow [3] assumes constant productivity. Calvo [8] studies under uncertain technology. Phelps and Riley [29] study a dynamic programming structure. The result is pessimistic: if the initial accumulation of capital is low, the economy remains in this low capital accumulation situation forever.

In a surprising and fascinating article, Zuber and Asheim [34] establish an axiomatic foundation for what, in our opinion, could be called the "*second Rawls criterion*". They assume the *anonymity*, a condition such that the value of a utility stream does not change after any permutation of the generations' utilities. Following the line of Koopmans [22, 23], and restricting *strong Pareto*, *separability* between the present and the future and *stationarity* on the set of streams which are increasing or can be rearranged as increasing, the criterion becomes:

$$U(c_0, c_1, \dots) = \inf_{p \in \Pi} \left[(1-\beta) \sum_{t=0}^{\infty} \beta^t u\left(c_{p(t)}\right) \right],$$

where Π represents the set of every permutation possible of the set of natural numbers $\{0, 1, 2, \dots\}$. Though the *strong Pareto* does not hold,[3] Zuber and Asheim [34] prove that this criterion satisfies Chichilnisky's no-dictatorship properties. They also apply it to a similar growth context as this article, with an additional condition that there is no depreciation of capital, and prove that the solution coincides with Ramsey's one.

As Chichilnisky's *no-dictatorship* criteria, the Ramsey-Rawls and $\alpha-$MaxMin criteria are time-inconsistent. However, an optimization problem under a time-incoherent criterion may have a coherent markovian solution. Drugeon *et al* [14] study the *maximin* criterion with multiple discount factors. The authors present a dynamic programming structure that has the same value function as the problem at stake and proves that the solutions in the two cases are coincide.

To overcome the difficulties caused by the generic non-existence, and the time inconsistency in *no-dictatorship* criteria, following the idea of Phelps and Pollack [28], Asheim and Ekeland [4] study the Markovian equilibrium of optimization problem under the Chichilnisky criterion, and come to interesting results about the influence of *no-dictatorship of the future* on long term behaviour of the economy.

1.4 Contents

This article is organized as follows. Section 2 analyses the *Ramsey-Rawls* problem. Section 3 solves the $\alpha-$MaxMin problem. Section 4 discusses the axiomatic foundation of Ramsey-Rawls and Chichilnisky's *no-dictatorship* criteria. Section 5 concludes. As an example, Appendix 6.1 studies the *Ramsey-Rawls* problem with linear production function and logarithmic utility function. The proofs and some intermediary analysis are given in the Appendix.

2 Dynamics under *Ramsey-Rawls* criterion

2.1 Fundamentals

Let $\beta \in (0, 1)$ be the discount factor of the discounted utilities part, and parameter $a \geq 0$ representing the importance of *equality* in the criterion. Let f and u be correspondingly the production and the utility functions. For a given capital stock

[3] See Basu and Mitra [6], for the impossibility theorem stating that there is no function that holds simultaneously *strong Pareto* and *anonymity* properties.

$k_0 \geq 0$, the economic agent solves the following optimisation problem (P):

$$V(k_0) = \max \left[\sum_{t=0}^{\infty} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t) \right],$$
$$\text{s.c } c_t + k_{t+1} \leq f(k_t),$$
$$k_t, c_t \geq 0, \forall\, t.$$

Assumption A1. *Utility function u is strictly concave, strictly increasing, satisfies Inada condition and bounded from below. The production function f is concave, strictly increasing, satisfying $f(0) = 0$ and $f'(0) > 1$.*

Let $\Pi(k_0)$ be the set of feasible paths $\{k_t\}_{t=0}^{\infty}$: $0 \leq k_{t+1} \leq f(k_t)$ for any t. Denote by $\overline{k}$ the solution to $f'(k) = 1$, which maximizes $f(k) - k$. In the case $f'(k) > 1$ for any $k \geq 0$, let $\overline{k} = \infty$. The value $\overline{k}$ is usually called the *golden rule*, the capital stock that maximizes the constant consumption.[4] Let k^s be the minimum of value(s) k such that $f'(k) = \frac{1}{\beta}$. If $f'(k) \geq \frac{1}{\beta}$ for every $k \geq 0$ then $k^s = \infty$. If $f'(k) < \frac{1}{\beta}$ for every $k \geq 0$ then $k^s = 0$.

It is well known in the literature that under suitable conditions, with respect to the product topology, the Ramsey part and the Rawls part are upper semi-continuous and the set of feasible paths $\Pi(k_0)$ is compact. To simplify the presentation, we assume directly this upper continuity property. Curious readers can refer to the classical book of Stokey, Lucas (with Prescott) [30], or the article by Le Van and Morhaim [21] for the details of conditions that ensure this property.

Assumption A2. *The following functions, being defined on the set of feasible paths $\Pi(k_0)$,*

$$\sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right) \ \textit{and} \ \inf_{t \geq 0} u\left(f(k_t) - k_{t+1}\right),$$

are well defined and upper semi-continuous with respect to the product topology.

Under this assumption, the problem (P) has an optimal solution, which is unique, due to the strict concavity of utility function u and the concavity of f. It is worth recall that in the case $a = 0$, there exists a strict increasing optimal policy function σ such that the optimal path $\{k_t^*\}_{t=0}^{\infty}$ satisfies $k_{t+1}^* = \sigma(k_t^*)$ for every t and this path converges to k^s when t tends to infinity.

In another extreme, when the evaluation criterion contains merely the Rawls part: $U(c_0, c_1, \ldots) = \inf_{t \geq 0} u(c_t)$, the long term behavior of solution depends strongly on the initial capital k_0. Precisely, let ν be the value function of the problem under

[4] See Gale [18].

Rawls criterion. If $k_0 \leq \overline{k}$, then the unique optimal solution is $(k_0, k_0, k_0, \ldots)$ and $\nu(k_0) = u\left(f(k_0) - k_0\right)$. Otherwise, if $k_0 > \overline{k}$, there exists an infinite number of solution, all converge to $\overline{k}$ and $\nu(k_0) = u\left(f(\overline{k}) - \overline{k}\right)$. The details are presented in Appendix 6.2.

2.2 The dynamics

The intuition for studying this problem runs as follows. The maximum value possible for the Rawls part is $\nu(k_0)$. Naturally, the following question arises: if we accept a lower value of the Rawls part up to ϵ, what is the best improvement we can obtain for the Ramsey part? And which is the optimal acceptable sacrifice level ϵ? This optimal value may represent the cost of the trade-off between efficiency and equality.

Consider the following intermediary problem (P^ϵ), for a given k_0:

$$W(\epsilon) = \max \left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right],$$
$$\text{s.c } c_t + k_{t+1} \leq f(k_t), \forall\, t \geq 0,$$
$$u(c_t) \geq \nu(k_0) - \epsilon, \forall\, t \geq 0.$$

The optimal trade off value ϵ^* is defined as:

$$\epsilon^* = \underset{\epsilon \geq 0}{\operatorname{argmax}} \left[W(\epsilon) + a\left(\nu(k_0) - \epsilon\right)\right].$$

Let $\{k_t^*\}_{t=0}^{\infty}$ be the optimal path of problems (P) and (P^{ϵ^*}). A careful analysis of the solution of P^ϵ gives us the characterization of the economy under Ramsey-Ralws criterion. The details are given in Appendix 6.3.

Let $\{k_t^*\}_{t=0}^{\infty}$ the optimal path and $c_t^* = f(k_t^*) - k_{t+1}^*$.

Proposition 2.1. *Consider the case $0 < k_0 \leq k^s$. For any $a > 0$, there exists T such that*

i) *For $0 \leq t \leq T$, $u(c_t^*) = u\big(f(k_0) - k_0\big) - \epsilon^*$.*

ii) *For $t \geq T + 1$, $u(c_t^*) > u\big(f(k_0) - k_0\big) - \epsilon^*$.*

iii) *The sequence $\{k_t^*\}_{t=T+1}^{\infty}$ is the solution to the Ramsey problem with initial state k_{T+1}^*.*

In the case where productivity is high, $\left(f'(k_0) > \frac{1}{\beta}\right)$, the utility of the early dates (or generations) are lowered as much as possible for the purpose of a rapid accumulation of capital. Sacrificing even a little bit the value of the equality part is worth it, to have a better accumulation level of capital.

Once the capital accumulation level is sufficiently high, the economy follows a Ramsey path that does not violate the equality constraints and converges to the steady-state k^s. Because of the constraints imposed by the equality criterion of Rawls, the difference in utility between early dates and the later dates in the distant future is not too high. This difference depends negatively on the equality parameter a, which imposes a trade-off between equality and the speed of convergence to the steady-state.

The case of low productivity $\left(f'(k_0) < \frac{1}{\beta}\right)$, requires some considerations about threshold of parameter a. With a sufficiently high value of a, the equality part (if sufficiently high) causes the economy to converge to a higher steady-state than that of Ramsey problem. The difference between the lowest dates (in the distant future) and the highest dates (in present) is diminished. The optimal choice in long term behaves as if at a steady state of some Ramsey problem with a value of discount rate $\tilde{\beta}$ higher than β, which is defined as follows. Let $\tilde{k}_0 \leq \overline{k}$ be the capital accumulation that is solution to $u\big(f(k) - k\big) = \nu(k_0) - \epsilon^*$. The new discount factor $\tilde{\beta}$ satisfies:

$$f'(\tilde{k}_0) = \frac{1}{\tilde{\beta}}.$$

Moreover, there exists a threshold for equality parameter a. Beyond this threshold, the optimal sequence remains the same and every date (or generation) has the same utility level. If the equality parameter a is too low, there is no change in the behavior of the economy, compared with the one under Ramsey criterion.

Let $\tilde{\epsilon} = u\big(f(\overline{k}) - \overline{k}\big) - u\big(f(k^s) - k^s\big)$. By the strict concavity of u, the function W is strictly concave on $[0, \tilde{\epsilon}]$. In the Appendix, section 6.3, we prove that $W'(0)$ is finite. This value serves is an important threshold for parameter a.

PROPOSITION 2.2. *Consider the case $k_0 \geq k^s$.*

i) *For $W'(\tilde{\epsilon}) < a < W'(0)$, the optimal ϵ^* satisfies $0 < \epsilon^* < \tilde{\epsilon}$ and there exists T such that:*

   a) *For $0 \leq t \leq T$, $u(c_t^*) > \nu(k_0) - \epsilon^*$.*

   b) *For $t \geq T + 1$, $u(c_t^*) = \nu(k_0) - \epsilon^*$.*

ii) *For $a \geq W'(0)$, the optimal $\epsilon^* = 0$ and*

   a) *If $k^s \leq k_0 \leq \overline{k}$, for every t, $k_t^* = k_0$.*

   b) *If $k_0 \geq \overline{k}$, for every t sufficiently big, $k_t^* = \overline{k}$.*

iii) *For $0 \leq a \leq W'(\tilde{\epsilon})$, $\epsilon^* = \tilde{\epsilon}$ and the solution of (P) coincides with the solution of Ramsey problem.*

We conclude this section by Corollary 2.1, which establishes the dependence of the new discount factor $\tilde{\beta}$ in function of a. Remark that, as W is strictly concave, and ϵ^* is the solution to $W'(\epsilon) = a$, the trade-off between efficiency and equality depends negatively on the value of parameter a: $\frac{d\epsilon^*}{da} < 0$. This implies that $\tilde{\beta}$ is increasing in respect to a. With a slight abuse of notation, if $k_t = \overline{k}$ for every t big enough, we say that $\tilde{\beta} = 1$.

Corollary 2.1. *Assume that $k_0 \geq k^s$. In long term, the optimal path behaves as the solution of an economy under Ramsey criterion with a new discount factor $\tilde{\beta} \geq \beta$, which is increasing in respect to a. Precisely,*

i) *If $0 \leq a \leq W'(\tilde{\epsilon})$, $\tilde{\beta} = \beta$.*

ii) *If a increases from $W'(\tilde{\epsilon})$ to $W'(0)$, $\tilde{\beta}$ increases from β to $\min\{1/f'(k_0), 1\}$.*

iii) *If $a \geq W'(0)$, $\tilde{\beta} = 1$.*

3 Dynamics under $\alpha-MaxMin$ criterion

Consider the following problem, balancing the worst and best generations: for a given $k_0 \geq 0$,

$$\mathcal{V}(k_0) = \sup\left[\alpha \sup_{t\geq 0} u(c_t) + (1-\alpha)\inf_{t\geq 0} u(c_t)\right],$$
$$\text{s.c. } c_t + k_{t+1} \leq f(k_t),$$
$$c_t, k_t \geq 0 \text{ for all } t \geq 0.$$

The idea to resolve this is similar to the one in the previous section. To determine the supremum value of the optimization problem, we consider the following *sup*-modified problem. For $\epsilon > 0$, define

$$\mathcal{W}(\epsilon) = \max\left[\sup_{t\geq 0} u(c_t)\right],$$
$$\text{s.c } c_t + k_{t+1} \leq f(k_t), \text{ for all } t \geq 0,$$
$$u(c_t) \geq \nu(k_0) - \epsilon, \text{ for all } t \geq 0.$$

We have

$$\mathcal{V}(k_0) = \max_{\epsilon\geq 0}\left[\alpha\mathcal{W}(\epsilon) + (1-\alpha)(\nu(k_0) - \epsilon)\right].$$

With a careful analysis of the modified problem, we can solve the $\alpha-$*maximin* one. Beginning from a low level of capital accumulation, for a small value of α, the infimum part dominates and the optimal solution coincides with the solution of Ralws problem. For a big value of α, the supremum part has effects and the solution depends strongly on the initial capital stock k_0. If the initial capital is high, the supremum part always influences the result with every value $\alpha > 0$. We assume that $\alpha > 0$.

PROPOSITION 3.1. i) *Consider the case $k_0 < \overline{k}$. There exists $\alpha^* \in (0, 1)$ such that*

a) *If $\alpha > \alpha^*$, then there exist an infinite number of solution and two values $\underline{x} < k_0 < \overline{x}$ such that every optimal path satisfies*

$$\liminf_{t\to\infty} k_t^* = \underline{x},$$
$$\limsup_{t\to\infty} k_t^* = \overline{x}.$$

b) *If $\alpha < \alpha^*$, then the unique optimal path is the solution to Rawls' problem, $k_t^* = k_0, \forall\, t$.*

ii) *Consider the case $k_0 \geq \overline{k}$. There exist two values $\underline{x} < \overline{k} < \overline{x}$ such that*

a) *If if $k_0 \leq \overline{x}$, every optimal path satisfies*

$$\liminf_{t\to\infty} k_t^* = \underline{x},$$
$$\limsup_{t\to\infty} k_t^* = \overline{x}.$$

b) *If $k_0 > \overline{x}$, then the optimal path is unique and satisfies $k_t^* = \underline{x}$ for every $t \geq 1$. This is also a solution to Rawls' problem.*

It is worth noting that when α tends to zero, the two values $\underline{x}$ and $\overline{x}$ converge to k_0 if $k_0 \leq \overline{k}$ and to $\overline{k}$ otherwise. Every optimal path becomes a solution to Rawls problem, which can be considered as a special case of $\alpha-$MaxMin.

4 AXIOMATIC FOUNDATION AND NO-DICTATORSHIP CRITERIA

4.1 $\epsilon-$CONTAMINATION CRITERIA

In this article, we follow the ider of $\epsilon-$contamination. The agent has an "opinion" that the good discount system should be a $\sigma-$additive probability in π.[5] The

[5] We have $\pi_t \geq 0$ for every t and $\sum_{t=0}^{\infty} \pi_t = 1$.

use of the word "opinion" is similar to that of Kopylov [24], to define a state of mind that is less rigid than a "belief". The economic agent thinks that π is a good choice, but there are reasons that suggest to her or him that this conclusion could be hasty. She or he should also consider all other time discounting systems. Precisely, the evaluation can be presented as

$$U(c_0, c_1, \dots) = (1-\epsilon)\sum_{t=0}^{\infty} \pi_t u(c_t) + \epsilon \inf_{t\geq 0} u(c_t).$$

The parameter $\epsilon \in [0,1]$ represents the lack of confidence in the choice π^* of the agent. If $\epsilon = 1$, the ignorance is complete. We obtain Rawls' criterion $\inf_{t\geq 0} u(c_t)$. By contrast, if $\epsilon = 0$, she or he believes without doubt that π is the good one, and we find the usual discounted utilitarian configuration. The general case for $\epsilon \in [0,1]$ and a more general set of probabilities is provided by Kopylov [25].[6]

We assume that the preference order being represented by function U satisfies the four axioms $A1$ to $A4$ in Kopylov [25],[7] with the fundamental set Δ constitutes of *all countably additive* probabilities defining on the set $\{0, 1, 2, \dots\}$. We add a *time invariance* condition to establish the exponential discount rate form for the probability π. This property states that the comparison of two sequences with the same *infimum* does not depend on the period of departure. It is a weaker version of Koopman's standard condition in the time discounting literature. This guarantees that marginal rates between any two consecutive time periods are the same, a characterization of the exponential discounting.

Remark that in the general case of Kopylov [25] with π and Δ belong to the set of probabilities (which contains also the *non-countably additive* ones) on the set of natural numbers $\mathbb{N}$, we can also obtain an exponential representation for π with our version of *time invariance* and a property of Monotone Continuity, initiated by Villegas [32] and proven by Arrow [2]. This later one has the purpose to ensure *the countable additivity* of the subjective probability π.

PROPOSITION 4.1. *Assume that the evaluation has an $\epsilon-$contamination representation with Δ constitues of every countably additive probabilities. Assume also that for every consumption sequences $(c_0, c_1, c_2, \dots)$ and $(c'_0, c'_1, c'_2, \dots)$ such that $\inf_{t\geq 0} c_t = \inf_{t\geq 0} c'_t$, every consumption level c, we have*

$$U(c_0, c_1, c_2, \dots) \geq U(c'_0, c'_1, c'_2, \dots) \text{ if and only if } U(c, c_0, c_1, c_2, \dots) \geq U(c, c'_0, c'_1, c'_2, \dots).$$

Then there exists $\beta \in (0,1)$ such that $\pi_t = (1-\beta)\beta^t$ for every $t \geq 0$.

[6] The result of Kopylov, in the context of this article, can be presented as follow: $U(c_0, c_1, \dots) = (1-\epsilon)\sum_{t=0}^{\infty} \pi_t u(c_t) + \epsilon \inf_{p\in\Delta} \sum_{t=0}^{\infty} p_t u(c_t)$, where Δ is a set of probabilities being given as a fundamental of the model.

[7] The most two important axioms are $\Delta-$*monotonicity and* $\Delta-$*independence.*

The inter-temporal evaluation then becomes

$$U(c_0, c_1, c_2, \dots) = (1-\epsilon)\sum_{t=0}^{\infty}(1-\beta)\beta^t u(c_t) + \epsilon \inf_{t\geq 0} u(c_t).$$

If $0 < \epsilon < 1$, this is equivalent to a Ramsey-Rawls criterion with $a = \frac{\epsilon}{(1-\epsilon)(1-\beta)}$. The cases $\epsilon = 0$ and $\epsilon = 1$ correspond to the well-known Ramsey and Rawls criteria.

4.2 No-dictatorship criteria

4.2.1 The criteria and the non-existence of solution

To capture the idea of sustainable growth, and to maintain the equality between generations of the present and close future and generations of the distant future, Chichilnisky in [11, 12] considers criteria that are combinations of a sum of discounted utilities, exhibiting *dictatorship of the present* and a criteria that, disregarding the utilities of close future generations, exhibits *dictatorship of the future.*

The *no dictatorship of the present* requires the existence of two utility streams that the comparison can be reversed by careful changes in the distant future of these streams. By contrast, the *no dictatorship of the future* requires a similar one, with changes in the close future. As an example, the following mixed criterion precisely satisfies *no-dictatorship of the present* and *no-dictatorship of the future*:

$$\sum_{t=0}^{\infty}\beta^t u(c_t) + a \liminf_{t\to\infty} u(c_t),$$

with some parameter $a > 0$ that represents the importance of the distant future compared with the present and close future.

This approach is very appealing, and rapidly becomes an inspiration source for a large range of researches. However, under no-dictatorial criteria, an optimal solution may not exist, as proven in Heal [20], in an economy with renewable resources, or Ayong le Kama *et al* [5] in a growth context. The reason for this non-existence is that the optimal paths of the Ramsey part and the lim inf part converge to different values. While the former's converges to a steady-state depending on the value of the discount factor, the latter converges to the golden rule.

4.2.2 Towards a satisfying solution

Neither the no-dictatorship criteria nor the Ramsey-Rawls one is time-consistent. To overcome the difficulties caused by the generic non-existence, and the time inconsistency, following the idea of Phelps and Pollack [28], Asheim and Ekeland [4] study the Markovian equilibrium of optimization problem under the Chichilnisky criterion, and come to interesting results. If the economy begins from sufficiently high productivity of the initial stock, the lim inf part does not influence the determination of the solution. By contrast, from low productivity of stock, the distant future part leads the economy to larger stock conservation than the one which would have been under a standard discounted utilitarian configuration.

Our Propositions 2.1 and 2.2 go in line with the results of Asheim and Ekeland [4], especially in the long term behaviour of the economies. In Proposition 2.1, beginning from a high productivity capital stock, the equality has an effect on the first periods, avoiding a too low level of consumption. After that, the economy behaves as in the case there is no equality part. In Proposition 2.2, beginning from a high level of stock accumulation, in comparison with the Ramsey configuration, the economy converges to a higher accumulation for generations in the distant future, and even avoiding a possible collapse, as will be proven in Section 6.1.

In another contribution, Figuières and Tidball [17], in an economy where the economic agent enjoys consumption *and* natural resources, restrict themselves on the set of convex combinations between solutions of the Ramsey problem and the lim inf one. They prove the existence of an optimal combination and consider it as a "satisfying" response to the problem under Chichilnisky's criteria.

5 Conclusion

In this article, we establish the solution to the saving problems under Ramsey-Rawls and *maximin* criteria. The optimization of the *inf* part leads to a *status-quo* situation. It is important to note that the Ramsey-Rawls and $\alpha-$MaxMin criteria are time-inconsistent. Without commitment, future agents may want to revise past decisions.

A possibility to overcome this time incoherency is the approach that considers the *Markovian rules*, as presented in the seminal contribution of Phelps and Pollack [28], considering the *Markovian rules*. Phelps and Pollack [28] consider the existence and properties of linear *stationary Markov equilibria* in the context of *quasi-hyperbolic discounting*. In general, however, this question is difficult and complicated, even in the case of constant productivity, as pointed out in Krusell and Smith [26].[8]

According to our intuition, the Ramsey-Rawls criterion challenges us with similar

[8]For a review of this literature, and an excellent analysis of saving and dissaving under *quasi-hyperbolic discounting* criterion, see Cao and Werning [9].

difficulties. To address them, we can begin by following the ideas of Phelps and Pollack [28] and Cao and Werning [9] to study the linear Markovian rules and saving behavior.[9] Following this, we may come to similar results as Asheim and Ekeland [4], especially in the case where the initial level of resource is high. But this should be the subject of another study.

6 APPENDIX

6.1 Constant productivity function and logarithmic utility function

In this section, we provide computations for the case where productivity is constant ($f(k) = Ak$) and the utility function is logarithmic $u(c) = \ln c$. The optimal policy function is[10]

$$\sigma(k) = \beta Ak.$$

Assume that $A > 1$. Hence $\overline{k} = \infty$. By induction, we have

$$\begin{aligned} k_t^* &= (\beta A)^t k_0, \\ c_t^* &= A(1-\beta)(\beta A)^t k_0. \end{aligned}$$

The value function is defined as

$$\begin{aligned} v(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln c_t^* \\ &= \frac{\ln A + \ln(1-\beta) + \ln k_0}{1-\beta} + (\ln \beta + \ln A) \sum_{t=0}^{\infty} t\beta^t. \end{aligned}$$

1. Consider the case $A > \frac{1}{\beta}$. As $k_0 < k^s$ for every k_0, by Lemma 6.3, $W'(0) = \infty$ and $W'(\tilde{\epsilon}) = 0$. The optimal sacrifice level ϵ^* satisfies $W'(\epsilon^*) = a$. There is T such that for $0 \le t \le T$,

   $$u\left(f(k_t^*) - k_{t+1}^*\right) = u\left(f(k_0) - k_0\right) - \epsilon^*,$$

   which is equivalent to

   $$\ln\left(Ak_t^* - k_{t+1}^*\right) = \ln(A-1) + \ln k_0 - \epsilon^*.$$

[9] For another work following the same spirit, see Asheim and Ekeland [4].
[10] See Stokey and Lucas, with Prescott [30].

For $0 \leq t \leq T$,

$$k^*_{t+1} = Ak^*_t - \frac{(A-1)k_0}{e^{\epsilon^*}}.$$

The value T is the smallest such that

$$u\left(f(k^*_{T+1}) - \sigma(k^*_{T+1})\right) \geq u\left(f(k_0) - k_0\right) - \epsilon^*,$$

which is equivalent to

$$\ln\left(Ak^*_{T+1} - \beta Ak^*_{T+1}\right) \geq \ln\left(Ak_0 - k_0\right) - \epsilon^*.$$

This is equivalent to

$$\ln A + \ln(1-\beta) + \ln k^*_{T+1} \geq \ln(A-1) + \ln k_0 - \epsilon^*.$$

The value T is the first integer number satisfying

$$k^*_{T+1} \geq \frac{A-1}{A(1-\beta)} \times \frac{k_0}{e^{\epsilon^*}}.$$

The sequence $\{k^*_{T+t}\}_{t=0}^{\infty}$ is the solution to the Ramsey problem with initial state k^*_{T+1}.

2. Consider the case $A < \frac{1}{\beta}$. In this case, $k^s = 0$, every solution of the Ramsey problem converges to zero. The critical value $\tilde{\epsilon}$ is then

$$\begin{aligned}\tilde{\epsilon} &= u\left(f(k_0) - k_0\right) - u(0) \\ &= \infty.\end{aligned}$$

Next, we determine $W'(0)$. For ϵ close to zero, the critical time T from which $u(c^\epsilon_t) = u\left(f(k_0) - k_0\right) - \epsilon$ is $T = 1$.

Capital level k^ϵ_1 is solution to

$$u\left(f(k_1) - k_1\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

This implies

$$\ln\left(Ak^\epsilon_1 - k^\epsilon_1\right) = \ln(A-1) + \ln k_0 - \epsilon.$$

Hence

$$k^\epsilon_1 = \frac{k_0}{e^\epsilon}.$$

We have

$$\begin{aligned}W(\epsilon) &= u\left(f(k_0) - k_1^\epsilon\right) + \frac{\beta}{1-\beta}\left(u\left(f(k_0) - k_0\right) - \epsilon\right)\\ &= \ln\left(Ak_0 - \frac{k_0}{e^\epsilon}\right) + \frac{\beta}{1-\beta}\left(\ln\left(Ak_0 - k_0\right) - \epsilon\right)\\ &= \ln\left(A - \frac{1}{e^\epsilon}\right) + \frac{\beta}{1-\beta}\left(\ln(A-1) + \ln k_0 - \epsilon\right).\end{aligned}$$

Hence for ϵ close to zero,

$$W'(\epsilon) = \frac{e^{-\epsilon}}{A - e^{-\epsilon}} - \frac{\beta}{1-\beta}.$$

Let ϵ converge to zero, we obtain

$$W'(0) = \frac{1-\beta A}{(A-1)(1-\beta)}.$$

Then, we have Proposition 6.1. The equality parameter has a strong effect if it is sufficiently high. Otherwise, there is no difference between the behavior following Ramsey-Rawls criterion and the behavior following Rawls criterion.

Proposition 6.1. i) *For $a \leq \frac{1-\beta A}{(A-1)(1-\beta)}$, we have $\epsilon^* \geq 0$, and there exists T such that:*

a) *For $0 \leq t \leq T$, $u(c_t^*) > \ln(A-1) + \ln k_0 - \epsilon^*$.*

b) *For $t \geq T+1$, $u(c_t^*) = \ln(A-1) + \ln k_0 - \epsilon^*$.*

ii) *For $a \geq \frac{1-\beta A}{(A-1)(1-\beta)}$, $\epsilon^* = 0$. The optimal path is constant: $k_t^* = k_0$ for any $t \geq 0$.*

Thanks to the Rawls' part, even if the productivity is low, the economy does not collapse.

6.2 Optimal solution under *Rawlsian* criterion

We consider the problem under the Rawls criterion:

$$\max\left[\inf_{t\geq 0} u(c_t)\right],$$

under the constraint $c_t + k_{t+1} \leq f(k_t)$ for all t, with $k_0 > 0$ given.

Proposition 6.2. i) *Consider the case $0 \leq k_0 \leq \overline{k}$. The problem has a unique solution $\boldsymbol{k}^* = (k_0, k_0, \ldots)$ and*

$$\max_{\boldsymbol{k}\in\Pi(k_0)}\left[\inf_{t\geq 0} u(c_t)\right] = u\left(f(k_0) - k_0\right).$$

ii) *Consider the case $\overline{k}$ is finite and $k_0 \geq \overline{k}$. The problem has an infinite number of solutions and*

$$\max_{\boldsymbol{k}\in\Pi(k_0)} \left[\inf_{t\geq 0} u(c_t)\right] = u\left(f(\overline{k}) - \overline{k}\right).$$

Proof. Let $\mathbf{k}^*$ be a solution to the problem.

(i) Suppose that $k_t^* \leq k_0$ for some t. Observe that

$$\begin{aligned} k_0 - k_{t+1}^* &\geq f(k_0) - f(k_t^*) \\ &\geq f'(k_0)(k_0 - k_t^*) \\ &\geq k_0 - k_t^*, \end{aligned}$$

which implies $k_{t+1}^* \leq k_t^*$. By induction, we obtain $k_0 \geq k_t^*$ for all t. Furthermore, the sequence (k_t^*) is decreasing and then converges to $\hat{k} \leq k_0$. From the continuity of f, we have $f(\hat{k}) - \hat{k} \geq f(k_0) - k_0$. However, the function $f(x) - x$ is increasing in $[0, \overline{k}]$, thus, $f(\hat{k}) - \hat{k} \leq f(k_0) - k_0$. Then $\hat{k} = k_0$, then $k_t^* = k_0$ for all t.

(ii) Let $\mathbf{k}^*$ be an optimal path. Since the sequence $\mathbf{k} = (k_0, \overline{k}, \overline{k}, \ldots)$ is feasible, for every t, we have $f(k_t^*) - k_{t+1}^* \geq f(\overline{k}) - \overline{k}$. Hence,

$$\begin{aligned} \overline{k} - k_{t+1}^* &\geq f(\overline{k}) - f(k_t^*) \\ &\geq f'(\overline{k})(\overline{k} - k_t^*) \\ &= \overline{k} - k_t^*. \end{aligned}$$

This implies $k_{t+1}^* \leq k_t^*$ for any t. The sequence $\mathbf{k}^*$ is decreasing and converges to some $\hat{k}$. By the continuity of f, $f(\hat{k}) - \hat{k} \geq f(\overline{k}) - \overline{k}$. Since $\overline{k}$ maximizes $f(x) - x$, this implies $\hat{k} = \overline{k}$. By induction, we can construct an infinite number of sequence $\mathbf{k}$ that satisfy: for all t, $\overline{k} < k_{t+1} < f(k_t) - f(\overline{k}) + \overline{k}$. It is easy to verify that this sequence is decreasing and converges to $\overline{k}$, satisfying $\inf_{t\geq 0}(f(k_t) - k_{t+1}) = f(\overline{k}) - \overline{k}$. QED

6.3 The *Ramsey-modified* problem

Let ν be the value function of the problem under Ralws criterion:

$$\nu(k_0) = \max_{\mathbf{k}\in\Pi(k_0)} \left[\inf_{t\geq 0} u\left(f(k_t) - k_{t+1}\right)\right].$$

Given $k_0 \geq 0$, for each $\epsilon \geq 0$, we first consider the following intermediary problem

(P^ϵ) with a given k_0:

$$W(\epsilon) = \max\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right],$$
$$\text{s.c } c_t + k_{t+1} \le f(k_t), \forall\, t \ge 0,$$
$$u(c_t) \ge \nu(k_0) - \epsilon, \forall\, t \ge 0.$$

Proposition 6.1 states that the optimal solution of (P) is also the optimal solution of (P^ϵ), for some optimal value ϵ.

LEMMA 6.1. *For any $k_0 \ge 0$,*

$$V(k_0) = \max_{\epsilon \ge 0}\big[W(\epsilon) + a\left(\nu(k_0) - \epsilon\right)\big].$$

Using Lemma 6.1, to understand the behavior of the optimal solution of initial problem (P), we study the solution of problems (P^ϵ), with $\epsilon \ge 0$. For simplicity, henceforth, we use the term "equality constraint" to denote $u(c_t) \ge \nu(k_0) - \epsilon$. Let $\{c_t^\epsilon, k_{t+1}^\epsilon\}_{t=0}^\infty$ be the optimal solution of (P^ϵ). By the strict concavity of u, this sequence is unique.

Obviously, if ϵ is sufficiently large, the solution of the Ramsey problem also satisfies the equality constraint, and solving the problem (P^ϵ) becomes a trivial task. Let $\tilde{\epsilon}$ be the critical value for this property: if we lower the Rawls part to $\tilde{\epsilon}$, the solution of the Ramsey problem also satisfies the constraint of the Ramsey-modified problem and becomes the solution of the latter one.

Define

$$\tilde{\epsilon} = \begin{cases} u\big(f(k_0) - k_0\big) - u\big(f(k_0) - \sigma(k_0)\big) \text{ if } 0 \le k_0 \le k^s, \\ u\big(f(k_0) - k_0\big) - u\big(f(k^s) - k^s\big) \text{ if } k^s \le k_0 \le \overline{k} \\ u\big(f(\overline{k}) - \overline{k}\big) - u\big(f(k^s) - k^s\big) \text{ if } k_0 \ge \overline{k}. \end{cases}$$

As the equality constraint is satisfied, it is a trivial task to prove Lemma 6.2.

LEMMA 6.2. *Assume that $\epsilon \ge \tilde{\epsilon}$.*

i) *The optimal solution of the problem (P^ϵ) coincides with the solution of the Ramsey problem.*

ii) $W(\epsilon) = W(\tilde{\epsilon}) = v(k_0)$.

If $\epsilon = 0$, the optimal solution is $(k_0, k_0, \dots)$. We now consider the interesting case, where $0 < \epsilon \le \tilde{\epsilon}$.

Proposition 6.3 states as follows. If $0 \leq k_0 \leq k^s$, the equality constraints are binding in the early dates, the optimal solution behaves like a solution of the Ramsey problem when the accumulation of capital reaches a sufficiently high level. If $k_0 \geq k^s$, the equality constraints are binding from some date T that is sufficiently large, and in the long term, every date (or generation) has the same utility level, which is equal to exactly the lowest level acceptable.

PROPOSITION 6.3. i) *Consider the case* $0 < k_0 < k^s$. *If* $0 < \epsilon \leq \tilde{\epsilon}$, *there exists* T *such that:*

a) *For* $0 \leq t \leq T$, $u(c_t^\epsilon) = \nu(k_0) - \epsilon$.

b) *For* $t \geq T+1$, $u(c_t^\epsilon) > \nu(k_0) - \epsilon$.

c) *The sequence* $\{k_t^\epsilon\}_{t=T+1}^\infty$ *is the solution of the Ramsey problem with initial state* k_{T+1}^ϵ.

ii) *Consider the case* $k_0 > k^s$. *If* $0 < \epsilon \leq \tilde{\epsilon}$, *there exists* T *such that*

a) *For* $0 \leq t \leq T$, $u(c_t^\epsilon) > \nu(k_0) - \epsilon$.

b) *For* $t \geq T+1$, $u(c_t^\epsilon) = \nu(k_0) - \epsilon$.

For the case $k_0 \geq k^s$, define $\tilde{k}$ as the solution to

$$u\left(f(\tilde{k}) - \tilde{k}\right) = \nu(k_0) - \epsilon.$$

We easily verify that $k_t^\epsilon = \tilde{k}$ for a T that is sufficiently high. Let $\tilde{\beta}$ be the discount rate that satisfies

$$f'(\tilde{k}) = \frac{1}{\tilde{\beta}}.$$

By the choice of $\tilde{\epsilon}$, we have $k^s < \tilde{k} < \overline{k}$. Hence, $\tilde{\beta} > \beta$. In the long term, the optimal solution for the case $k_0 \geq k^s$ behaves as a solution of the Ramsey problem with a discount rate $\tilde{\beta}$, that is greater than β.

Lemma 6.3 is a direct consequence of Proposition 6.3. The function W is strictly concave with respect to ϵ belonging to $[0, \tilde{\epsilon}]$. This concavity implies the existence of the right derivative of W at 0 and the left derivative of W at $\tilde{\epsilon}$. In Section 2.2, these two values play the role of critical thresholds for equality parameter a. The behavior of the optimal solution depends strongly on the comparison between a and $W'(0)$, $W'_-(\tilde{\epsilon})$. Details are given in Section 2.2.

For instance, we provide some preparation results for $W'_+(0)$ and $W'_-(\tilde{\epsilon})$.

LEMMA 6.3. i) *For any* k_0, *the function* W *is strictly concave on* $[0, \tilde{\epsilon}]$.

ii) *If* $0 \leq k_0 < k^s$, *then* $W'(0) = +\infty$ *and* $W'(\tilde{\epsilon}) = 0$.

iii) *If* $k_0 > k^s$, *then* $W'(0) < +\infty$.

6.3.1 Proof of Lemma 6.1

By the very definition of ν, for every feasible sequence $\{k_t\}_{t=0}^{\infty}$,

$$\inf_{t\geq 0} u\big(f(k_t) - k_{t+1}\big) \leq \nu(k_0).$$

Let $\{k_t^*\}_{t=0}^{\infty}$ be the optimal solution of problem (P). Define

$$\epsilon^* = \nu(k_0) - \inf_{t\geq 0} u(c_t^*).$$

We have

$$\begin{aligned} V(k_0) &= \sum_{t=0}^{\infty} \beta^t u(c_t^*) + a \inf_{t\geq 0} u(c_t^*) \\ &= \sum_{t=0}^{\infty} \beta^t u(c_t^*) + a\left(\nu(k_0) - \epsilon^*\right) \\ &\leq W(\epsilon^*) + a\left(\nu(k_0) - \epsilon^*\right). \end{aligned}$$

Conversely, consider $\epsilon \geq 0$. Let $\{c_t^\epsilon\}_{t=0}^{\infty}$ be the consumption set corresponding to the solution of the modified problem:

$$\begin{aligned} W(\epsilon) + a\left(\nu(k_0) - \epsilon\right) &= \sum_{t=0}^{\infty} \beta^t u\left(c_t^\epsilon\right) + a\left(\nu(k_0) - \epsilon\right) \\ &\leq \sum_{t=0}^{\infty} \beta^t u\left(c_t^\epsilon\right) + a \inf_{t\geq 0} u\left(c_t^\epsilon\right) \\ &\leq V(k_0). \end{aligned}$$

The proof is completed.

6.3.2 Proof of Proposition 6.3

Obviously, W is increasing. Moreover, the concavity of W is from the concavity of utility function u and production function f. Let σ be the optimal policy function of the economy under Ramsey criterion.

First, we consider the case that $0 \leq k_0 \leq \overline{k}$. For each $\epsilon > 0$, let $x^*(\epsilon)$ such that $x \geq k_0$ and

$$u\big(f(x) - \sigma(x)\big) = u\big(f(k_0) - k_0\big) - \epsilon.$$

(i) We consider the case $k_0 < k^s$. Observe that $k_0 < x^*(\epsilon) < k^s$. Indeed, by the definition of $\tilde{\epsilon}$ and the choice of ϵ, $u\left(f(k_0) - \sigma(k_0)\right) < u\left(f(k_0) - k_0\right) - \epsilon$. Since

$\sigma(k^s) = k^s$, and $u\left(f(k^s) - k^s\right) > u\left(f(k_0) - k_0\right)$, the existence of $x^*(\epsilon) \in (k_0, k^s)$ is ensured.

We prove the following claim: $k_t^\epsilon < k^s$ for every t. First, observe that: if $k_t < x^*(\epsilon)$, then $k_{t+1} < k^s$. Indeed,

$$\begin{aligned} u\big(f(k_t^\epsilon) - k_{t+1}^\epsilon\big) &\geq u\big(f(k_0) - k_0\big) - \epsilon \\ &= u\big(x^*(\epsilon) - \sigma(x^*(\epsilon))\big). \end{aligned}$$

Since $k_t < x^*(\epsilon)$, this implies $k_{t+1} < \sigma(x^*(\epsilon)) < k^s$. Now, assume that for some T, $x^*(\epsilon) < k_{T+1}^\epsilon < k^s$. Let $\{\check{k}_t\}_{t=T+1}^\infty$ be the solution of the Ramsey problem with initial state k_{t+1}^ϵ. Since $k_{T+1}^\epsilon < k^s$, $\check{k}_t < k^s$ for any $t \geq T+1$ and

$$\begin{aligned} \inf_{t \geq T+1} u(\check{c}_t) &= u\big(f(k_{t+1}^\epsilon) - \sigma(k_{t+1}^\epsilon)\big) \\ &\geq u\big(f(x^*(\epsilon)) - \sigma(x^*(\epsilon))\big) \\ &= u\big(f(k_0) - k_0\big) - \epsilon. \end{aligned}$$

Hence the sequence $\{k_0, k_1^\epsilon, \ldots, k_T^\epsilon, k_{T+1}^\epsilon, \check{k}_{T+2}, \check{k}_{T+2}, \ldots\}$ is the optimal solution for the problem (P^ϵ), or $\check{k}_t = k_t^\epsilon$ for any $t \geq T+1$. The proof that $k_t^\epsilon < k^s$ for any t is completed.

It is impossible that $u\left(c_T^\epsilon\right) = u\big(f(k_0) - k_0\big) - \epsilon$ for every T. Indeed, assume the contrary. This implies $k_t^\epsilon < x^*(\epsilon)$ for every t and $k_1^\epsilon > k_0$. By induction, we have $k_{t+1}^\epsilon > k_t^\epsilon$ for every t and then the sequence $\{k_t^\epsilon\}_{t=0}^\infty$ converges to some k^* such that $k_0 < k^* \leq x^*(\epsilon)$. Taking the limit when T tends to infinity, we have $u\left(f(k^*) - k^*\right) < u\left(f(k_0) - k_0\right)$, a contradiction.

Consider the Lagrangian:

$$\begin{aligned} \mathcal{L} = &\sum_{t=0}^\infty \beta^t u(c_t) - \sum_{t=0}^\infty \beta^t \lambda_t \big[c_t + k_{t+1} - f(k_t)\big] \\ &- \sum_{t=0}^\infty \beta^t \mu_t \big[u\big(f(k_0) - k_0\big) - \epsilon - u(c_t)\big]. \end{aligned}$$

By the Inada condition of u, at optimal the consumption and capital level are strictly positive. Hence, the Lagrangian parameters for these constraints are zero.

For any t,

$$\begin{aligned} (1 + \mu_t) u'(c_t^\epsilon) &= \lambda_t, \\ \lambda_t &= \beta \lambda_{t+1} f'(k_{t+1}^\epsilon). \end{aligned}$$

This implies that for any t,

$$\begin{aligned} (1 + \mu_t) u'(c_t^\epsilon) &= \beta(1 + \mu_{t+1}) u'(c_{t+1}^\epsilon) f'(k_{t+1}^\epsilon) \\ &\geq \beta f'(k_{t+1}^\epsilon) u'(c_{t+1}^\epsilon). \end{aligned}$$

Let $T = T(\epsilon)$ be the smallest time such that $u\left(c_T^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$. The constraint does not bind, hence, $\mu_T = 0$. Since $f'(k_{T+1}^\epsilon) \geq \frac{1}{\beta}$, then $u'(c_T^\epsilon) \geq u'(c_{T+1}^\epsilon)$, hence, $c_{t+1}^\epsilon \geq c_T^\epsilon$. The $(T+1)^{th}$ constraint also does not bind: $u\left(c_{T+1}^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$. By induction, for any $t \geq T+1$, $u\left(c_t^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$ and $\mu_t = 0$. The sequence $\{(c_t^\epsilon, k_{t+1}^\epsilon)\}_{t=T}^\infty$ is increasing and satisfies Euler equations. Hence $\{k_t^\epsilon\}_{t=T}^\infty$ is the solution for Ramsey problem with initial state k_T^ϵ.

(ii) Consider the case $k^s < k_0 < \overline{k}$. In this case, k^s is finite. We will prove that $k_t^\epsilon > k^s$ for any $t \geq 0$. Assume that there exists T such that $k_T^\epsilon \leq k^s$. We have

$$\begin{aligned} u\big(f(k_T^\epsilon) - k_{T+1}^\epsilon\big) &\geq \nu(k_0) - \epsilon \\ &= u\big(f(k_0) - k_0\big) - \epsilon \\ &> u\big(f(k_0) - k_0\big) - \tilde{\epsilon} \\ &= u\big(f(k^s) - k^s\big), \end{aligned}$$

which implies $k_{T+1}^\epsilon < k_T^\epsilon < k^s$, since $f(x) - x$ is strictly increasing in $(0, k^s)$. By induction, the sequence $\{k_{T+t}^\epsilon\}_{t=0}^\infty$ is decreasing and converges to $\underline{k} < k^s$. Taking the limit, we obtain

$$\begin{aligned} u\big(f(k^s) - k^s\big) &> u\big(f(\underline{k}) - \sigma(\underline{k})\big) \\ &\geq \nu(k_0) - \epsilon \\ &\geq u\big(f(k_0) - k_0\big) - \epsilon \\ &> u\big(f(k^s) - k^s\big), \end{aligned}$$

a contradiction. The property that $k_t^\epsilon > k^s$ for any $t \geq 0$ is established, we re-consider the Lagrangian:

$$\begin{aligned} \mathcal{L} = &\sum_{t=0}^\infty \beta^t u(c_t) - \sum_{t=0}^\infty \beta^t \lambda_t \big[c_t + k_{t+1} - f(k_t)\big] \\ &- \sum_{t=0}^\infty \beta^t \mu_t \big[u\big(f(k_0) - k_0\big) - \epsilon - u(c_t)\big]. \end{aligned}$$

For any t,

$$\begin{aligned} (1 + \mu_t) u'(c_t^\epsilon) &= \lambda_t, \\ \lambda_t &= \beta \lambda_{t+1} f'(k_{t+1}^\epsilon). \end{aligned}$$

This implies for any t:

$$\begin{aligned} u'(c_t^\epsilon) &\leq (1 + \mu_t) u'(c_t^\epsilon) \\ &= \beta(1 + \mu_{t+1}) u'(c_{t+1}^\epsilon) f'(k_{t+1}^\epsilon). \end{aligned}$$

If $u\left(c_T^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$, the constraint does not bind, and $\mu_T = 0$. Since $f(k_T^\epsilon) < \frac{1}{\beta}$, we obtain $u'(c_{T-1}^\epsilon) < u'(c_T^\epsilon)$, which implies $c_{T-1}^\epsilon > c_T^\epsilon$, with the direct consequence

$$u\left(c_{T-1}^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon.$$

By induction, we obtain for any $0 \leq t \leq T$,

$$u\left(c_t^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon.$$

If this property is ensured for any $t \geq 0$, the the sequence $\{k_t^\epsilon\}_{t=0}^\infty$ satisfies Euler equations and the transversality condition, hence it is the optimal solution of the Ramsey problem and converges to k^s: a contradiction, since

$$u\big(f(k^s) - k^s\big) < u\big(f(k_0) - k_0\big) - \epsilon.$$

Hence there exists T such that for any $t \geq T$,

$$u\left(c_T^\epsilon\right) = u\big(f(k_0) - k_0\big) - \epsilon.$$

Obviously, for any $t \geq T$, we have

$$u\left(c_t^\epsilon\right) = u\big(f(k_0) - k_0\big) - \epsilon,$$

otherwise using the same arguments in the induction, we obtain $u\left(c_T^\epsilon\right) > u\big(f(k_0) - k_0\big) - \epsilon$, a contradiction.

For the last case $k_0 \geq \overline{k}$, we use the same arguments as those for the case $1 \leq f'(k_0) \leq \frac{1}{\beta}$, with the observation that the value of $\nu(k_0)$ is $u\left(f(\overline{k}) - \overline{k}\right)$ and $f(\overline{k}) - \overline{k} \geq f(k^s) - k^s$.

6.3.3 Proof of Lemma 6.3

From the concavity of the functions u and f, the function $W(\epsilon)$ is strictly concave in respect to ϵ on $[0, \tilde{\epsilon}]$.

(i) We prove that $W'(0) = +\infty$. Consider $T(\epsilon)$ in the proof of Proposition 6.3.

For any $0 \leq t \leq T(\epsilon)$:

$$\begin{aligned}
\epsilon &= u\big(f(k_0) - k_0\big) - u\big(f(k_t^\epsilon) - k_{t+1}^\epsilon\big) \\
&\geq u'\big(f(k_0) - k_0\big)\left(f(k_0) - k_0 - f(k_t^\epsilon) + k_{t+1}^\epsilon\right) \\
&\geq u'\big(f(k_0) - \sigma(k_0)\big)\left(f'(k_0)(k_0 - k_t^\epsilon) + k_{t+1}^\epsilon - k_0\right).
\end{aligned}$$

This implies

$$k_{t+1}^\epsilon - k_0 \leq \frac{\epsilon}{u'\big(f(k_0) - k_0\big)} + f'(k_0)(k_t^\epsilon - k_0).$$

By induction, we obtain for any $t \geq 0$,

$$k^{\epsilon}_{t+1} - k_0 \leq \frac{\big[f'(k_0)\big]^{t+1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'\big(f(x^*) - x^*\big)}.$$

Hence

$$\begin{aligned} x^*(\epsilon) - k_0 &\leq k_{T(\epsilon)+1} - k_0 \\ &\leq \frac{\big[f'(k_0)\big]^{T(\epsilon)+1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'\big(f(k_0) - k_0\big)}. \end{aligned}$$

$$\begin{aligned} W(\epsilon) &= \sum_{t=0}^{T(\epsilon)} \beta^t u\left(c^{\epsilon}_t\right) + \sum_{t=T(\epsilon)+1}^{\infty} \beta^t u\left(c^{\epsilon}_t\right) \\ &= \big(u\big(f(k_0 - k_0\big) - \epsilon\big) \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon)+1} v(k^{\epsilon}_{T(\epsilon)}). \end{aligned}$$

Hence

$$\begin{aligned} W(\epsilon) - W(0) &= -\epsilon \sum_{t=0}^{T(\epsilon)} \beta^t + \beta^{T(\epsilon)+1} \left(v\left(k^{\epsilon}_{T(\epsilon)}\right) - \frac{u\big(f(k_0) - k_0\big)}{1-\beta} \right) \\ &= -\epsilon \frac{1 - \beta^{T(\epsilon)+1}}{1-\beta} + \beta^{T(\epsilon)+1} \left(v\left(k^{\epsilon}_{T(\epsilon)}\right) - \frac{u\big(f(k_0) - k_0\big)}{1-\beta} \right). \end{aligned}$$

Now, we prove that

$$\lim_{\epsilon \to 0} \frac{\beta^{T(\epsilon)}}{\epsilon} = +\infty.$$

Indeed, recall that

$$\frac{\big[f'(k_0)\big]^{T(\epsilon)+1} - 1}{f'(k_0) - 1} \times \frac{\epsilon}{u'\big(f(k_0) - k_0\big)} \sim x^*(\epsilon) - k_0.$$

This implies

$$\big(f'(k_0)\big)^{T(\epsilon)} \epsilon \sim O(1).$$

Hence

$$T(\epsilon) \ln\big(f'(k_0)\big) \sim -\ln(\epsilon),$$

which is equivalent to

$$T(\epsilon) \sim -\frac{\ln(\epsilon)}{\ln\big(f'(k_0)\big)}.$$

We have

$$\begin{aligned}
\beta^{T(\epsilon)} &\sim \left(e^{\ln\beta}\right)^{-\frac{\ln(\epsilon)}{\ln\left(f'(k_0)\right)}} \\
&\sim \epsilon^{-\frac{\ln\beta}{\ln\left(f'(k_0)\right)}} \\
&\sim \epsilon^{\frac{\ln\left(\frac{1}{\beta}\right)}{\ln\left(f'(k_0)\right)}}.
\end{aligned}$$

Since $f'(k_0) > \frac{1}{\beta}$, we have

$$\lim_{\epsilon\to 0}\frac{\beta^{T(\epsilon)}}{\epsilon} = \lim_{\epsilon\to 0}\epsilon^{\frac{\ln\left(\frac{1}{\beta}\right)}{\ln\left(f'(k_0)\right)}-1} = \infty,$$

which implies $W'(0) = +\infty$.

(ii) First assume that $k^s < k_0 \leq \overline{k}$. Now we prove that $W'(0) < +\infty$. For ϵ small:

$$\begin{aligned}
W(\epsilon) - W(0) &= \sum_{t=0}^{\infty}\beta^t\left[u\big(f(k_t^\epsilon) - k_{t+1}^\epsilon\big) - u\big(f(k_0) - k_0\big)\right] \\
&\leq u'\big(f(k_0) - k_0\big)\sum_{t=0}^{\infty}\beta^t\left[f(k_t^\epsilon) - f(k_0) - k_{t+1}^\epsilon + k_0\right] \\
&\leq u'\big(f(k_0) - k_0\big)\sum_{t=0}^{\infty}\beta^t\left[f'(k_0)(k_t^\epsilon - k_0) - k_{t+1}^\epsilon + k_0\right] \\
&\leq u'\big(f(k_0) - k_0\big)\sum_{t=0}^{\infty}\beta^t\left[f'(k_0)(k_t^\epsilon - k_0)\right] \\
&\leq u'\big(f(k_0) - k_0\big)f'(k_0)\sum_{t=0}^{\infty}\beta^t\left[k_t^\epsilon - k_0\right] \\
&\leq u'\big(f(k_0) - k_0\big)f'(k_0)\sum_{t=0}^{\infty}\beta^t\frac{\left[f'(k_0)\right]^{t+1} - 1}{f'(k_0) - 1}\times\frac{\epsilon}{u'\big(f(k_0) - k_0\big)} \\
&= f'(k_0)\sum_{t=0}^{\infty}\beta^t\frac{\left[f'(k_0)\right]^{t+1} - 1}{f'(k_0) - 1}\times\epsilon \\
&= O(\epsilon),
\end{aligned}$$

since $\beta f'(k_0) < 1$.

This implies $W(\epsilon) - W(0) = O(\epsilon)$, or $W'(0) < +\infty$.

Now assume that $\overline{k}$ is finite and $k_0 \geq \overline{k}$. We use exactly the same arguments in the proof of part (ii), by changing the constrains $u(c_t) \geq u\left(f(k_0) - k_0\right) - \epsilon$ by $u(c_t) \geq u\left(f(\overline{k}) - \overline{k}\right)$.

Now we prove that $W'(\tilde{\epsilon}) = 0$. For ϵ close enough to $\tilde{\epsilon}$, the critical time $T(\epsilon)$ from which the optimal path behaves as a solution of Ramsey problem with initial state

$k^{\epsilon}_{T(\epsilon)}$ is $T(\epsilon) = 1$. We then have

$$u\left(f(k_0) - k^{\epsilon}_1\right) = u\left(f(k_0) - k_0\right) - \epsilon,$$

and the sequence $\{k^{\epsilon}_{1+t}\}_{t=0}^{\infty}$ is the solution of the Ramsey problem with initial state k^{ϵ}_1. Denote by v the value function of the Ramsey problem. Then

$$W(\epsilon) = u\left(f(k_0) - k_0\right) - \epsilon + \beta v(k^{\epsilon}_1),$$

and

$$W'(\epsilon) = -1 + \beta v'(k^{\epsilon}_1) \times \frac{dk'_1}{d\epsilon}.$$

By the implicit function theorem, we have

$$\frac{dk^{\epsilon}_1}{d\epsilon} = \frac{1}{u'\left(f(k_0) - k^{\epsilon}_1\right)}.$$

We observe that by letting ϵ converge to $\tilde{\epsilon}$, we have

$$\lim_{\epsilon \to \tilde{\epsilon}} k^{\epsilon}_1 = \sigma(k_0).$$

This implies

$$W'_{-}(\tilde{\epsilon}) = -1 + \beta v'\left(\sigma(k_0)\right) \times \frac{1}{u'\left(f(k_0) - \sigma(k_0)\right)}.$$

Recall that it is well-known in dynamic programming literature that

$$\begin{aligned} v(k_0) &= \max_{0 \le k_1 \le f(k_0)} \left[u\left(f(k_0) - k_1\right) + \beta v(k_1)\right] \\ &= u\left(f(k_0 - \sigma(k_0)\right) + \beta v\left(\sigma(k_0)\right). \end{aligned}$$

Combined with the Inada condition, this implies

$$-u'\left(f(k_0) - \sigma(k_0)\right) + \beta v'\left(\sigma(k_0)\right) = 0,$$

which is equivalent to

$$W'(\tilde{\epsilon}) = 0.$$

(iii) For any $0 \le \epsilon \le \tilde{\epsilon}$, there exists T such that the equality constraint corresponding to T bind. Hence, the solutions corresponding to different values of ϵ differ. We combine this with the strict concavity of u, and obtain that W is strictly concave in $[0, \tilde{\epsilon}]$.

6.4 Proof of Proposition 2.1

For any $0 \leq \epsilon \leq \tilde{\epsilon}$, the optimal solution satisfies the following property: there exists t such that $u\,(c_t^\epsilon) = u\big(f(k_0) - k_0\big)$. Hence, the solutions corresponding to difference values of ϵ also differ. Combined with the strict concavity of u, the function W is strictly concave in $[0, \tilde{\epsilon}]$. This implies the existence of an unique left derivative of W.

Since for any $a > 0$, we have $W'_-(\tilde{\epsilon}) = 0 < a < W'(0) = \infty$, there exists a unique $0 < \epsilon < \tilde{\epsilon}$ such that $W'(\epsilon^*) = a$. The statement of this Proposition is a consequence of Propositions 6.1 and 6.3.

6.5 Proof of Proposition 2.2

We use exactly the results in Proposition 6.3 and the same arguments as in the proof of Proposition 2.1 and Proposition 2.2.

6.6 Proof of Corollary 2.1

The result is a direct consequence of the definition of $\tilde{k}_0$ and $\tilde{\beta} \times f'(\tilde{k}_0) = 1$.

6.7 The $\alpha-$*MaxMin* criterion

6.7.1 The *sup*-modified problem

Consider the following problem: for a given $k_0 \geq 0$,

$$\mathcal{V}(k_0) = \sup \left[\alpha \sup_{t\geq 0} u(c_t) + (1-\alpha) \inf_{t\geq 0} u(c_t)\right],$$
$$\text{s.c. } c_t + k_{t+1} \leq f(k_t) \text{ for all } t \geq 0.$$

The idea to resolve this is similar to the one in the previous section. To determine the supremum value of the optimization problem, we consider the following *sup*-modified problem: For $\epsilon > 0$, define

$$\mathcal{W}(\epsilon) = \max \left[\sup_{t\geq 0} u(c_t)\right],$$
$$\text{s.c } c_t + k_{t+1} \leq f(k_t), \text{ for all } t \geq 0,$$
$$u(c_t) \geq \nu(k_0) - \epsilon, \text{ for all } t \geq 0.$$

Let $\Pi^\epsilon(k_0)$ be the set of feasible paths of this problem.

Lemma 6.4 states the existence of an optimal level of trade-off.

Lemma 6.4. *We have*

$$\mathcal{V}(k_0) = \max_{\epsilon \geq 0} \left[\alpha \mathcal{W}(\epsilon) + (1-\alpha)\left(\nu(k_0) - \epsilon\right) \right].$$

6.7.2 Solution of the *sup*-modified problem

With Lemma 6.4, we solve the modified problem, with some $\epsilon > 0$. Let $\underline{x}^\epsilon$ and $\overline{x}^\epsilon$ be correspondingly the solution in $[0, k_0]$ and (k_0, ∞) to the equation

$$u\left(f(x) - x\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

In the case $u\left(f(x) - x\right) \geq u\left(f(k_0) - k_0\right) - \epsilon$ for any $x \geq k_0$, let $\overline{x}^\epsilon = \infty$. Obviously, if $\lim_{x\to\infty} f'(x) < 1$, the value $\overline{x}^\epsilon$ is finite.

Proposition 6.4. *Consider the case* $0 \leq k_0 \leq \overline{k}$.

i) *For any* $\epsilon \geq 0$,

$$\mathcal{W}(\epsilon) = u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right).$$

ii) *For any optimal path* $\{k_t\}_{t=0}^\infty$*, we have* $\underline{x}^\epsilon \leq k_t \leq \overline{x}^\epsilon$*, and*

$$\liminf_{t\to\infty} k_t = \underline{x}^\epsilon,$$
$$\limsup_{t\to\infty} k_t = \overline{x}^\epsilon.$$

The case $\overline{k}$ is finite and $k_0 \geq \overline{k}$ deserves a slight change in the treatment. The optimal value ϵ^*does not depend on k_0. Let $\underset{\sim}{x}^\epsilon$ be the unique solution in $[0, \overline{k}]$ to the equation:

$$u\left(f(x) - x\right) = u\left(f(\overline{k}) - \overline{k}\right) - \epsilon.$$

Since $f(x) - x$ is decreasing on $[\overline{k}, \infty)$, there is a unique $\tilde{x}^\epsilon$ in $[\overline{k}, +\infty)$, that is solution to

$$u\left(f(x) - x\right) = u\left(f(\overline{k}) - \overline{k}\right) - \epsilon.$$

Notably, contrary to the case $k_0 \leq \overline{k}$, in this case the values $\underset{\sim}{x}^\epsilon$ and $\tilde{x}^\epsilon$ are independent from k_0. If $k_0 \leq \tilde{x}^\epsilon$, thus there exists an infinite number of solutions, and every optimal path fluctuates between $\underset{\sim}{x}^\epsilon$ and $\tilde{x}^\epsilon$. Only for k_0 sufficiently large, there exists a unique solution, and it is constant from date $t = 1$.

Proposition 6.5. *Consider the case* $k_0 \geq \overline{k}$.

i) *If* $\overline{k} \leq k_0 \leq \tilde{x}^\epsilon$, *then*

$$\mathcal{W}(\epsilon) = u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right).$$

Moreover, there is an infinite number of solutions. Every optimal paths $\{k_t\}_{t=0}^\infty$ *satisfies* $\underset{\sim}{x}^\epsilon \leq k_t \leq \tilde{x}^\epsilon$, *and*

$$\liminf_{t\to\infty} k_t = \underset{\sim}{x}^\epsilon,$$
$$\limsup_{t\to\infty} k_t = \tilde{x}^\epsilon.$$

ii) *If* $k_0 \geq \tilde{x}^\epsilon$, *then* $k_t = \underset{\sim}{x}^\epsilon$ *for any* $t \geq 1$ *and*

$$\mathcal{W}(\epsilon) = u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).$$

6.7.3 Proof of Lemma 6.4

Consider any feasible sequence $\{k_t\}_0^\infty \in \Pi(k_0)$, with $c_t = f(k_t) - k_{t+1}$, and define

$$\hat{\epsilon} = \nu(k_0) - \inf_{t\geq 0} u\left(c_t\right).$$

Obviously,

$$\begin{aligned}(1-\alpha)\sup_{t\geq 0} u(c_t) + \alpha \inf_{t\geq 0} u(c_t) &= \alpha \sup_{t\geq 0} u(c_t) + (1-\alpha)\left(\nu(k_0) - \hat{\epsilon}\right) \\ &\leq \alpha\mathcal{W}(\hat{\epsilon}) + (1-\alpha)\left(\nu(k_0) - \hat{\epsilon}\right) \\ &\leq \sup_{\epsilon\geq 0}\left[(1-\alpha)\mathcal{W}(\epsilon) + \alpha\left(\nu(k_0) - \epsilon\right)\right].\end{aligned}$$

Now, consider any feasible sequence $\{k_t\}_0^\infty \in \Pi(k_0)$ that satisfies the constraints of the modified problem.

$$\begin{aligned}\alpha \sup_{t\geq 0} u(c_t) + (1-\alpha)\left(\nu(k_0) - \epsilon\right) &\leq \alpha \sup_{t\geq 0} u(c_t) + (1-\alpha) \inf_{t\geq 0} u(c_t) \\ &\leq \mathcal{V}(k_0).\end{aligned}$$

Taking the supremum on the left side, the proof of the Lemma is completed.

6.7.4 Proof of Proposition 6.4

i) We prove that for any feasible sequence $\{k_t\}_{t=0}^\infty$ of the modified problem, we have for any $t \geq 0$,

$$\underline{x}^\epsilon \leq k_t < \overline{x}^\epsilon.$$

Assume the contrary of the first inequality, that is for some T, $k_T < \underline{x}^\epsilon$. Since the function $f(x) - x$ is strictly increasing in $[0, k_0]$, we have

$$\begin{aligned} u\left(f(k_T) - k_T\right) &< u\left(f(\underline{x}^\epsilon) - \underline{x}^\epsilon\right) \\ &= u\left(f(k_0) - k_0\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right). \end{aligned}$$

This implies that $k_{T+1} \leq k_T < \underline{x}^\epsilon$. By induction, the sequence $\{k_{T+t}\}_{t=0}^{\infty}$ is decreasing and converges to some $0 \leq k^* < \underline{x}^\epsilon$. Hence,

$$\begin{aligned} u\left(f(k^*) - k^*\right)) &< u\left(f(\underline{x}^\epsilon) - \underline{x}^\epsilon\right) \\ &= u\left(f(k_0) - k_0\right) - \epsilon \\ &\leq u\left(f(k^*) - k^*\right), \end{aligned}$$

a contradiction.

Consider the sequence $\{\overline{k}_t\}_{t=0}^{\infty}$ determined as

$$\begin{aligned} \overline{k}_0 &= k_0, \\ u\left(f(\overline{k}_t) - \overline{k}_{t+1}\right) &= u\left(f(k_0) - k_0\right) - \epsilon. \end{aligned}$$

We easily verify that the sequence $\{\overline{k}_t\}_{t=0}^{\infty}$ is increasing and converges to $\overline{x}^\epsilon$, whether this value is finite or infinite.

Fix any feasible sequence $\{k_t\}_{t=0}^{\infty}$ of the modified problem. Assume that for some T, $k_T \leq \overline{k}_T$. As a consequence,

$$\begin{aligned} u\left(f(\overline{k}_t) - \overline{k}_{t+1}\right) &= u\left(f(k_0) - k_0\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right) \\ &\leq u\left(f(\overline{k}_T) - k_{T+1}\right), \end{aligned}$$

which implies $k_{T+1} \leq \overline{k}_T$. By induction, for any $t \geq 0$,

$$k_{T+t} \leq \overline{k}_{T+t} < \overline{x}^\epsilon.$$

As for every t, $\underline{x}^\epsilon < k_t < \overline{x}^\epsilon$, the following inequality holds:

$$\sup_{t \geq 0} u\left(f(k_t) - k_{t+1}\right) \leq u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right).$$

Now, we prove that the existence of feasible paths $\{k_t\}_{t=0}^{\infty}$ such that

$$\sup_{t \geq 0} u\left(f(k_t) - k_{t+1}\right) = u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right).$$

Fix any two sequences $\{\underline{x}_n\}_{n=0}^{\infty}$ and $\{\overline{x}_n\}_{n=0}^{\infty}$ such that the former is strictly decreasing and converges to $\underline{x}^{\epsilon}$ and the later is strictly increasing and converges to $\overline{x}^{\epsilon}$:

$$\underline{x}_0 > \underline{x}_1 > \ldots > \underline{x}_n > \ldots \to \underline{x}^{\epsilon},$$
$$\overline{x}_0 < \overline{x}_1 < \ldots < \overline{x}_n < \ldots \to \overline{x}^{\epsilon}.$$

We construct the sequence $T_0 < T_1 < \ldots < T_n$ and the sequence $\{k_t\}_{t=0}^{\infty}$ as follows. For $0 \leq t \leq T_0$,

$$u\left(f(k_t) - k_{t+1}\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

If we continue to use this equation to define k_{t+1} from k_t to infinity, the sequence converges to $\overline{x}^{\epsilon}$. Hence, there exists a T_0 that is the smallest one t satisfying $k_{T_0} \geq \overline{x}_0$. Let $k_{T_0+1} = \underline{x}_0$. We have

$$u\left(f(k_{T_0}) - k_{T_0+1}\right) \geq u\left(f(\overline{x}_0) - \underline{x}_0\right).$$

For $t \geq T_0 + 1$, define the sequence as

$$u\left(f(k_t) - k_{t+1}\right) = u\left(f(k_0) - k_0\right) - \epsilon.$$

Using the same argument for the definition of T_0, there exists a T_1 that is the smallest satisfying $k_t \geq \overline{x}_1$. Let $k_{T_1+1} = \overline{x}_1$. We have

$$u\left(f(k_{T_1}) - k_{T_1+1}\right) \geq u\left(f(\overline{x}_1) - \underline{x}_1\right).$$

Additionally, we define in the same manner, by induction T_{n+1} in function of T_n. For any $n \geq 0$ we have $k_{T_n} \geq \overline{x}_n$, $k_{T_n+1} = \underline{x}_n$ and:

$$u\left(f(k_{T_n}) - k_{T_n+1}\right) \geq u\left(f(\overline{x}_n) - \underline{x}_n\right).$$

Let n converge to infinity,

$$\lim_{n\to\infty} u\left(f(k_{T_n}) - k_{T_n+1}\right) \geq u\left(f(\overline{x}^{\epsilon}) - \underline{x}^{\epsilon}\right).$$

Hence,

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) = u\left(f(\overline{x}^{\epsilon}) - \underline{x}^{\epsilon}\right).$$

Since the two sequences $\{\underline{x}_n\}_{n=0}^{\infty}$ and $\{\overline{x}_n\}_{n=0}^{\infty}$ can be chosen arbitrarily, there exist an infinite number of optimal solution.

Consider a optimal path $\{k_t\}_{t=0}^{\infty}$. It is an easy task to verify that if $k_T = \underline{x}^{\epsilon}$, by induction, we obtain that $k_{T+t} = \underline{x}^{\epsilon}$ for any $t \geq 0$. As for $0 \leq t \leq T$,

$k_t < \overline{x}^\epsilon$, the strict inequality $\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) < u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right)$ holds, implying that this sequence is not optimal. Hence, for every optimal path, the inequality $\underline{x}^\epsilon < k_t < \overline{x}^\epsilon$ is satisfied for every t. Moreover, as there exits an infinite number $T_0 < T_1 < \ldots < T_n < \ldots$ such that

$$\lim_{n\to\infty} u\left(f(k_{T_n}) - k_{T_n+1}\right) = u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right),$$

the following limits are verified:

$$\lim_{n\to\infty} k_{T_n} = \overline{x}^\epsilon,$$
$$\lim_{n\to\infty} k_{T_n+1} = \underline{x}^\epsilon.$$

ii) This part is a direct consequence of the proof of the first part.

6.7.5 Proof of Proposition 6.5

i) First, we prove that for any $\overline{k} \leq k_0 \leq \tilde{x}^\epsilon$, and for every feasible path $\{k_t\}_{t=0}^{\infty} \in \Pi^\epsilon(k_0)$, we have

$$\utilde{x}^\epsilon \leq k_t \leq \tilde{x}^\epsilon.$$

Assume that there is some T such that $k_T < \utilde{x}^\epsilon$. Then

$$\begin{aligned} u\left(f(k_T) - k_T\right) &< u\left(f(\utilde{x}) - \utilde{x}\right) \\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right), \end{aligned}$$

which implies $k_{T+1} \leq k_T < \tilde{x}^\epsilon$. By induction, the sequence $\{k_{T+t}\}_{t=0}^{\infty}$ is decreasing and converges to some $k^* < \tilde{x}^\epsilon$, and

$$\begin{aligned} u\left(f(k^*) - k^*\right) &< u\left(f(\tilde{x}) - \tilde{x}\right) \\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon, \end{aligned}$$

a contradiction.

By induction, assume that for some T, we have $\utilde{x}^\epsilon \leq k_T \leq \tilde{x}^\epsilon$. Since

$$\begin{aligned} u\left(f(\tilde{x}^\epsilon) - \utilde{x}^\epsilon\right) &\leq u\left(f(k_T) - k_{T+1}\right) \\ &\leq u\left(f(\tilde{x}^\epsilon) - k_{T+1}\right), \end{aligned}$$

we have $k_{T+1} \leq \tilde{x}^\epsilon$. As this property is satisfied by k_0, by induction, we obtain $k_t \leq \tilde{x}^\epsilon$ for all $t \geq 0$.

Since for any t, $\underset{\sim}{x}^\epsilon \leq k_t \leq \tilde{x}^\epsilon$,

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) \leq u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right). \tag{6.1}$$

To prove that the left side is equal to the right side in the aforementioned inequality (6.1), and that there exists an infinite number of solutions for the modified problem, we prove that for any $\underset{\sim}{x}^\epsilon \leq k_0 \leq \tilde{x}^\epsilon$, the sequence $\{\tilde{k}_t\}_{t=0}^\infty$ defined as follows is increasing and converges to $\tilde{x}^\epsilon$:

$$\begin{aligned}\tilde{k}_0 &= k_0,\\ u\left(f(\tilde{k}_t) - \tilde{k}_{t+1}\right) &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon, \text{ for all } t \geq 0.\end{aligned}$$

Indeed, using the same aforementioned arguments, we have for any t, $\underset{\sim}{x}^\epsilon \leq \tilde{k}_t \leq \tilde{x}^\epsilon$. Then

$$\begin{aligned}u\left(f(\tilde{k}_t) - \tilde{k}_{t+1}\right) &= u\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon\right)\\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon\\ &\leq u\left(f(\tilde{k}_t) - \tilde{k}_t\right).\end{aligned}$$

This implies $\tilde{k}_t \leq \tilde{k}_{t+1}$, and the sequence $\{\tilde{k}_t\}_{t=0}^\infty$ is increasing and converges to the solution of $u\left(f(x) - x\right) = u\left(f(\overline{k}) - \overline{k}\right) - \epsilon$, or

$$\lim_{t\to\infty} \tilde{k}_t = \tilde{x}^\epsilon.$$

Now, we fix two sequences $\{\underset{\sim}{x}_n\}_{n=0}^\infty$ which is strictly decreasing and converges to $\underset{\sim}{x}^\epsilon$, and $\{\tilde{x}_n\}_{n=0}^\infty$, which is strictly increasing and converges to $\tilde{x}^\epsilon$.

Using the same arguments as in the Proof of Proposition 6.4, we can construct a feasible sequence $\{k_t\}_{t=0}^\infty \in \Pi^\epsilon(k_0)$ and a sequence of index $T_0 < T_1 < \ldots < T_n < \ldots$ such that for any n,

$$u\left(f(k_{T_n}) - k_{T_n+1}\right) \geq u\left(f(\tilde{x}_n) - \underset{\sim}{x}_n\right).$$

Additionally, we have

$$\begin{aligned}\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) &\geq \lim_{n\to\infty} u\left(f(\tilde{x}_n) - \underset{\sim}{x}_n\right)\\ &= u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right).\end{aligned}$$

Since the two sequences $\{\tilde{x}_n\}_{n=0}^\infty$ and $\{\tilde{x}_n\}_{n=0}^\infty$ are chosen arbitrarily, there exists an infinite number of optimal paths.

Consider an optimal path $\{k_t\}_{t=0}^\infty$. We verify easily that if $k_T = \underset{\sim}{x}^\epsilon$, by the constraint, $k_{T+t} = \underset{\sim}{x}^\epsilon$ for any $t \geq 0$, which implies $\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) <$

$u\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon\right)$, a contradiction. Hence, for any t, $\underset{\sim}{x}^\epsilon < k_t < \tilde{x}^\epsilon$. Moreover, there exists an infinite number $T_0 < T_1 < \ldots < T_n < \ldots$ such that

$$\lim_{n\to\infty} u\left(f(k_{T_n}) - k_{T_n+1}\right) = u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right).$$

Hence, we have

$$\lim_{n\to\infty} k_{T_n} = \tilde{x}^\epsilon,$$
$$\lim_{n\to\infty} k_{T_n+1} = \underset{\sim}{x}^\epsilon.$$

ii) Now, we consider the case $k_0 \geq \tilde{x}^\epsilon$. Take any feasible sequence $\{k_t\}_{t=0}^\infty \in \Pi^\epsilon(k_0)$. We claim that for any $t \geq 0$,

$$\underset{\sim}{x}^\epsilon \leq k_t \leq k_0.$$

Using the same arguments as in the proof of the part (i), we have $k_t \geq \underset{\sim}{x}^\epsilon$ for any $t \geq 0$. We prove by induction that $k_t \leq k_0$ for any t. Indeed, this is true for $t = 0$. Assume that $k_t \leq k_0$ for any $0 \leq t \leq T-1$. If $k_T > k_0$, which is bigger thant $\tilde{x}^\epsilon$, then

$$\begin{aligned} u\left(f(k_T) - k_T\right) &\leq u\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon\right) \\ &= u\left(f(\overline{k}) - \overline{k}\right) - \epsilon \\ &\leq u\left(f(k_T) - k_{T+1}\right), \end{aligned}$$

which implies $k_{T+1} \leq k_T \leq k_0$, a contradiction. The claim is proved. Thus, for any t,

$$\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) \leq u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).$$

We verify easily that the sequence $\{k_t^*\}_{t=0}^\infty = (k_0, \underset{\sim}{x}^\epsilon, \underset{\sim}{x}^\epsilon, \underset{\sim}{x}^\epsilon, \ldots)$ is feasible and

$$\sup_{t\geq 0} u\left(f(k_t^*) - k_{t+1}^*\right) = u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).$$

To prove that this sequence is unique solution, take any feasible sequence $\{k_t\}_{t=0}^\infty$. Assume that $k_1 > \underset{\sim}{x}^\epsilon$. Hence,

$$u\left(f(k_0) - k_1\right) < u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).$$

If $k_1 \geq \tilde{x}^\epsilon$, then

$$\begin{aligned} \sup_{t\geq 1} u\left(f(k_t) - k_{t+1}\right) &\leq u\left(f(k_1) - \underset{\sim}{x}^\epsilon\right) \\ &< u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right). \end{aligned}$$

If $k_1 \leq \tilde{x}^\epsilon$, then

$$\begin{aligned}\sup_{t\geq 1} u\left(f(k_t) - k_{t+1}\right) &\leq u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right) \\ &< u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).\end{aligned}$$

Combining these inequalities, we obtain

$$\begin{aligned}\sup_{t\geq 0} u\left(f(k_t) - k_{t+1}\right) &= \max\left\{u\left(f(k_0) - k_1\right), \sup_{t\geq 1} u\left(f(k_t) - k_{t+1}\right)\right\} \\ &< u\left(f(k_0) - \underset{\sim}{x}^\epsilon\right).\end{aligned}$$

For the case $k_1 = \underset{\sim}{x}^\epsilon$, to maintain the path being feasible, we must have $k_t = \underset{\sim}{x}^\epsilon$ for any $t \geq 1$. The uniqueness of the optimal solution is proven.

6.8 Proof of Proposition 3.1

The optimal trade-off value ϵ^* is defined as

$$\epsilon^* = \underset{\epsilon\geq 0}{\operatorname{argmax}}\left[\alpha\mathcal{W}(\epsilon) + (1-\alpha)(\nu(k_0) - \epsilon)\right].$$

We will prove that if $k_0 < \overline{k}$, $\mathcal{W}'(0) < \infty$. The value $\overline{x}^\epsilon$ is defined as solution to $u\left(f(x) - x\right) = u\left(f(k_0) - k_0\right)$. Hence,

$$\begin{aligned}\epsilon &= u\left(f(k_0) - k_0\right) - u\left(f(\overline{x}^\epsilon) - \overline{x}^\epsilon\right) \\ &\geq u'\left(f(k_0) - k_0\right)\left(f(k_0) - k_0 - f(\overline{x}^\epsilon) + \overline{x}^\epsilon\right) \\ &\geq u'\left(f(k_0) - k_0\right)\left(f'(k_0) - 1\right)\left(\overline{x}^\epsilon - k_0\right).\end{aligned}$$

Hence, $\overline{x}^\epsilon - k_0 = O(\epsilon)$. Using the same arguments, we have $k_0 - \underline{x}^\epsilon = O(\epsilon)$. We have

$$\begin{aligned}\mathcal{W}(\epsilon) - \mathcal{W}(0) &= u\left(f(\overline{x}^\epsilon) - \underline{x}^\epsilon\right) - u\left(f(k_0) - k_0\right) \\ &\leq u'\left(f(k_0) - k_0\right)\left(f'(k_0)(\overline{x}^\epsilon - k_0) - (\underline{x}^\epsilon - k_0)\right) \\ &= O(\epsilon).\end{aligned}$$

Hence, $\mathcal{W}'(0) < \infty$. Let $\alpha^* \in (0,1)$ such that

$$\mathcal{W}'(0) = \frac{1-\alpha^*}{\alpha^*}.$$

If $\alpha \leq \alpha^*$, the Rawls' part dominates, and $\epsilon^* = 0$. The optimal path is solution to Rawls' problem. Otherwise, $\epsilon^* > 0$. In two cases, the results are direct consequences of Proposition 6.4.

Consider the case $k_0 > \overline{k}$. If ϵ^* is big enough such that $k_0 < \tilde{x}^{\epsilon^*}$, apply part (i) of Proposition 6.5. Otherwise, ϵ^* is close to 0 and $k_0 > \tilde{x}^{\epsilon^*}$, apply part (ii) of Proposition 6.5.

Consider the remaining case, $k_0 = \overline{k}$. We prove that $\mathcal{W}'(0) = +\infty$. Indeed, from $u\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon\right) = u\left(f(\overline{k}) - \overline{k}\right) - \epsilon$ and the Mean Value Theorem, we have

$$\begin{aligned}\epsilon &= u\left(f(\overline{k}) - \overline{k}\right) - u\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon\right)\\ &= u'(\zeta)\left(f(\overline{k}) - \overline{k} - f(\tilde{x}^\epsilon) + \tilde{x}^\epsilon\right)\\ &= u'(\zeta)\left(f'(\xi) - 1\right)\left(\tilde{x}^\epsilon - \overline{k}\right),\end{aligned}$$

with some $f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon \leq \zeta \leq f(\overline{k}) - \overline{k}$ and $\underline{k} \leq \xi \leq \tilde{x}^\epsilon$. Since ζ is bounded from below and above,

$$f(\overline{k}) - \overline{k} - f(\tilde{x}^\epsilon) + \tilde{x}^\epsilon = O(\epsilon).$$

Moreover, remark that when ϵ converges to 0, $f'(\xi)$ converges to 1. This implies

$$\lim_{\epsilon\to 0} \frac{\tilde{x}^\epsilon - \overline{k}}{\epsilon} = +\infty.$$

We have

$$\begin{aligned}\mathcal{W}(\epsilon) - \mathcal{W}(0) &= u\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right) - u\left(f(\overline{k}) - \overline{k}\right)\\ &\geq u'\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right)\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon - f(\overline{k}) + \overline{k}\right)\\ &= u'\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right)\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon - f(\overline{k}) + \overline{k} + \tilde{x}^\epsilon - \underset{\sim}{x}^\epsilon\right)\\ &\geq u'\left(f(\tilde{x}^\epsilon) - \underset{\sim}{x}^\epsilon\right)\left(f(\tilde{x}^\epsilon) - \tilde{x}^\epsilon - f(\overline{k}) + \overline{k} + \tilde{x}^\epsilon - \overline{k}\right).\end{aligned}$$

Hence,

$$\lim_{\epsilon\to 0} \frac{\mathcal{W}(\epsilon) - \mathcal{W}(0)}{\epsilon} = +\infty.$$

This implies that $\epsilon^* > 0$ for every $\alpha > 0$. We then apply Proposition 6.5.

6.9 Proof of Proposition 4.1

For each $T \geq 0$, let π^T the probability being defined as

$$\pi^T_s = \frac{\pi_{T+s}}{\sum_{s'=0}^{\infty} \pi_{T+s'}}.$$

We will prove that $\pi = \pi^T$ for every $T \geq 0$. First, we prove the following Claim: for every consumption sequences $(c_0, c_1, \ldots)$, $(c'_0, c'_1, \ldots)$,

$$\sum_{t=0}^{\infty} \pi_t u(c_t) \geq \sum_{t=0}^{\infty} \pi_t u(c'_t) \text{ if and only if } \sum_{t=0}^{\infty} \pi^1_t u(c_t) \geq \sum_{t=0}^{\infty} \pi^1_t u(c'_t).$$

Consider the "*only if*" case. Fix any constant $b > 0$, a constant $c \geq 0$ such that $c \leq \min\{\inf_{t\geq 0} c_t, \inf_{t\geq 0} c'_t\}$. Fix $T_0 \geq 0$ big enough such that for every $T \geq T_0$,

$$\sum_{t=0}^{T} \pi_t u(c_t + b) + \pi_{T+1} u(c) + \sum_{t=T+2}^{\infty} \pi_t u(c_t + b) \geq \sum_{t=0}^{T} \pi_t u(c'_t) + \pi_{T+1} u(c) + \sum_{t=T+2}^{\infty} \pi_t u(c'_t).$$

Since the infimum of the consumption sequence on the left-hand side is equal to the infimum of the one on the right-hand side, we have

$$U(c_0 + b, c_1 + b, \ldots, c_T + b, c, c_{T+2} + b, \ldots) \geq U(c'_0, c'_1, \ldots, c'_T, c, c'_{T+2}, \ldots).$$

This implies

$$U(c, c_0 + b, c_1 + b, \ldots, c_T + b, c, c_{T+2} + b, \ldots) \geq U(c, c'_0, c'_1, \ldots, c'_T, c, c'_{T+2}, \ldots).$$

Hence,

$$\sum_{t=1}^{T+1} \pi_t u(c_{t-1} + b) + \pi_{T+2} u(c) + \sum_{t=T+3}^{\infty} \pi_t u(c_{t-1} + b) \geq \sum_{t=1}^{T+1} \pi_t u(c'_{t-1}) + \pi_{T+2} u(c) + \sum_{t=T+3}^{\infty} \pi_t u(c'_{t-1}).$$

Let T converges to infinity, we get

$$\sum_{t=1}^{\infty} \pi_t u(c_{t-1} + b) \geq \sum_{t=1}^{\infty} \pi_t u(c'_{t-1}).$$

Since $b > 0$ is chosen arbitrarily, we obtain

$$\sum_{t=1}^{\infty} \pi_t u(c_{t-1}) \geq \sum_{t=1}^{\infty} \pi_t u(c'_{t-1}),$$

which is obviously equivalent to

$$\sum_{t=0}^{\infty} \pi_t^1 u(c_t) \geq \sum_{t=0}^{\infty} \pi_t^1 u(c'_t).$$

In the "*if*" case, we apply the same argument, and the Claim is proven. The satisfaction of the Claim proves that $\pi = \pi^1$. By induction, we have $\pi = \pi^T$ for every $T \geq 0$. Hence,

$$\frac{\pi_{T+t+1}}{\pi_{T+t}} = \frac{\pi_{t+1}}{\pi_t}, \forall\, T, s \geq 0.$$

Let $\beta = \frac{\pi_1}{\pi_0}$, we have $\pi_{t+1} = \beta \pi_t$, for every t. Since $\sum_{t=0}^{\infty} \pi_t = 1$, it is easy to verify that $\pi_t = (1 - \beta)\beta^t$, for $t \geq 0$.

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