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Abstract

In a New Keynesian model, it is believed that combining active monetary policy using a Taylor rule with a passive fiscal rule can achieve local equilibrium determinacy. However, even with such policies, indeterminacy can occur from the emergence of a Shilnikov chaotic attractor in the region of the feasible parameter space. That result, shown by Barnett et al. (2021), implies that the presence of the Shilnikov chaotic attractor can cause the economy to drift towards and finally become stuck in the vicinity of lower-than-targeted inflation and nominal interest rates. The result can become the source of a liquidity trap phenomenon. We propose policy options for eliminating or controlling Shilnikov chaotic dynamics to help the economy escape from the liquidity trap or avoid drifting into it in the first place. We consider ways to eliminate or control the chaos by replacing the usual Taylor rule by an alternative policy design without interest rate feedback, such as a Taylor rule with monetary quantity feedback, an active fiscal policy rule with passive monetary rule, or an open loop policy without feedback. We also consider approaches that retain the Taylor rule with interest rate feedback and the associated Shilnikov chaos, while controlling the chaos through a well-known engineering algorithm using a second policy instrument. We find that a second instrument is needed to incorporate a long-run terminal condition missing from the usual myopic Taylor rule.

***Keywords*:** Shilnikov chaos criterion, global indeterminacy, long-term un-predictability, liquidity trap, long run anchor.

JEL classification: C61, C62, E12, E52, E63.

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1. Introduction

A long literature on chaos exists in macroeconomics, including empirical tests for chaos and findings of chaos in theoretical models. For example, one of the earliest papers, Barnett and Chen (1988a,b), found chaos in monetary aggregate data. In terms of theoretical models, Grandmont (1985) found evidence of chaos in classical models, while Benhabib, Schmitt-Grohè, and Uribe (2002) found it in New Keynesian (NK) models. Azariadis and Kaas (2016), Beaudry, Galizia, and Portier (2020), Gu et al. (2013), Bella, Mattana, and Venturi (2017), and Stockman (2010) are recent examples related to the topic of chaos in economics. However, despite its formidable potential in theory, policy relevance of chaos has remained limited, mainly owing to the fact that a finding of general chaos does not itself reveal a system's intrinsic stochastic properties implied by the model's fractal attractor set. As a result, there are limited discussions about solutions to policy problems caused by findings of chaos, while analyzing the implications of proposed solutions can be even more difficult.

Policy relevance of chaos in economics depends not only upon the existence of chaos at plausible settings of parameter values, but also the nature of the chaos, as reflected in the geometry of the resulting fractal attractor set. Many well-known types of chaos exist, such as Li-Yorke chaos (for dynamical systems generated by interval maps), Lorentz attractor (a type of chaos from atmospherical dynamical model), Smale horse-shoe chaos (with origin from celestial mechanics), and Shilnikov chaos for a particular scenario/criterion related to the Shilnikov homoclinic orbit. With a parameterization produced from recent data, Barnett et al. (2021) found Shilnikov chaos in the Benhabib et al. (2001a,b) model. Barnett et al. (2021) chose to investigate Shilnikov chaos, because it can be detected directly from the Shilnikov criterion and because of its broad relevance to phenomena in nature.

As explained by Alan Champneys (2010), "Over the years, Shilnikov's mechanism of chaos has proven to be one of the most robust and frequently occurring mechanisms chosen by nature." As pointed out by Afraimovich *et al*. (2014, p. 19):

> "Only starting from mid 70s–80s, when researchers became interested in computer studies of chaotic behavior in nonlinear models, it became clear that the Shilnikov saddle-focus is a pivotal element of chaotic dynamics in a broad range of real-world applications. In general, the number of various models from hydrodynamics, optics, chemical kinetics, biology etc., which demonstrated the numerically or experimentally strange attractors with the characteristic spiral structure suggesting the occurrence of a saddle focus homoclinic loop, was overwhelming. Indeed, this scenario has turned out to be typical for a variety of systems and models of very diverse origins."

While it is true that most findings of chaos in the physical sciences are adequately approximated by one of the few well-established classes of chaos, such as Li-York chaos, Lorentz attractor chaos, Smale horse-shoe chaos, or Shilnikov chaos, there is no result in mathematics proving that all chaotic systems are necessarily within a small number of such classes of chaotic systems or even within a finite number of such classes. This observation is similar to the well-known result in econometrics that no model having a finite number of parameters can span the entire space of increasing concave functions, but some such specifications are viewed as providing adequate approximations for most purposes.

Confirmation of Shilnikov chaos from the Shilnikov criterion with an economic model can be viewed as supporting the use of the Shilnikov fractal attractor set, but further investigations of robustness of inferences from that attractor set remain worthwhile. A local deviation from the geometry of the exact Shilnikov attractor set could result from a defect in the approximation properties of the economic model or in the settings of the model's parameters. Such deviations could contribute to a lack of robustness of some inferences based on the Shilnikov fractal attractor set's geometry, while some fundamental inferences can remain dominant and highly robust to modifications of the model or its parameter settings.

In this paper, we establish Shilnikov chaos in the Benhabib et al. (2001a,b) model using the Benhabib et al. (2001a,b) parameterization of the US economy, thereby confirming the robustness of the Barnett et al. (2021) fundamental findings, which were based on recent data, rather than on the Benhabib et al. (2001a,b) parameter settings. In particular, we confirm the Barnett et al. (2021) finding of Shilnikov chaos in a conventional NK model, when policy is based on active interest rate feedback with the consequent long run downward drift of interest rates within the Shilnikov fractal attractor set.[6]

The Barnett et al. (2021) finding, with a cashless economy and parameters calibrated using recent data, is viewed as relevant to the long run downward drift in US interest rates, beginning in 1981 and ending in a liquidity trap attained in 2009 after 28 years of downward drift (a remarkable match to their model's 27.5 year phenomenon).[7] Because of this discovery, now confirmed with Benhabib et al. (2002) parameter calibration and an NK model with money included, it is now possible to comprehend, from the Shilnikov fractal attractor set's geometry, the NK system's intrinsic stochastic properties. Previous findings of general chaos in NK models could not do so (see Benhabib et al. (2002)). This new finding has significant policy implications by suggesting that the downward drift in interest rates in recent years may have been an unintended consequence of Shilnikov chaos rather than of a deliberate policy choice.

The system becomes highly sensitive to initial conditions in the presence of a chaotic attractor, and irregular transitional dynamics may jeopardize an economy's ability to converge to a long-run stable equilibrium. There could be a continuum of initial values for the jump variables, given the initial value of the predetermined variable, resulting in admissible equilibria. Moreover, if the initial conditions of the jump variables are chosen far enough from the targeted steady state, the emerging aperiodic dynamics continue to

[6] Barnett et al. (2021) showed that the results on downward drift of interest rates within the geometry of the fractal attractor set are robust to the model's various assumptions, such as a plausible range of parameter settings, money in the utility function, money in the production function, or no money in the model at all, and we confirm that robustness further to a substantial difference in parameter settings. The downward drift result is therefore believed to be applicable to a wide range of NK models. But it should be observed that not all implications of the Shilnikov attractor set have been found to be similarly robust. For example, Barnett et al. (2021) found that the short run, high frequency phenomena within the attractor set do not necessarily share the same degree of robustness as displayed by the longer run downward drift phenomenon. As a result, Barnett et al. (2021) conclude that policy simulations based on the Shilnikov attractor set may not be dependable indicators for short run, high frequency "fine-tuning" policy.

[7] See, e.g., https://fred.stlouisfed.org/graph/?id=TB3MS.

evolve over a long period of time around lower-than-targeted inflation and nominal interest rates – a phenomenon known as the liquidity trap. The available fiscal policy that policymakers usually prescribe to avoid such indeterminacies is no longer considered effective.

In this paper, we discuss an innovative solution to these unfamiliar problems, if the central bank chooses to retain the Taylor rule and its consequent Shilnikov chaos. Specifically, we show that the chaotic dynamics can be controlled, in the sense of Ott, Grebogi and Yorke (1990), henceforth OGY. Under specific conditions, the announcement of a higher nominal interest rate at the steady state could anchor expectations to the long-run target. More generally, the long run nominal interest rate can be treated as an intermediate target of policy, with the instrument used to attain the target possibly being one of the new policy instruments available. This well-known engineering algorithm would not end the chaos but would eliminate the downward drift in interest rates and decrease the other undesirable properties of the chaos by imposing a long run anchor on interest rates. The undesired irregular and cyclical behavior can be superseded, and the intended fixed point can be targeted and attained in a relatively short period of time.

An alternative would be a non-Taylor policy rule that would not cause chaos and thus would not necessitate the use of a second instrument to control the chaotic drift. A possibility would be a feedback rule based on a monetary aggregate, rather than on an interest rate, to avoid Shilnikov chaos. Even if it did cause chaos, the inherent stochasticity that resulted would be harmless, since there is no lower bound on a monetary aggregate instrument. A similar policy approach was recently proposed by Belongia and Ireland (2019), where they showed that a flexible monetary feedback rule on Divisia M2 would have resulted in a faster recovery from the Great Recession and a shorter period of interest rates at the lower bound. Finally, it is important to note that we do not advocate one solution over the others for the problems of Shilnikov chaos, as one solution may be more relevant to some countries than to others.

We now present the plan of the paper. In Section 2, we briefly present the model, the implied three-dimensional system of first-order differential equations characterizing the solution of the model. Using the Benhabib et al. (2001a,b) parameterization of the US economy, we also outline the presence of Shilnikov chaos and its properties. In Section 3, we consider two policy approaches to solving the problems produced by the dynamics of the economy within the Shilnikov attractor set. First, we consider using the OGY algorithm to control chaos, and then we replace the Taylor rule with an alternative policy design that does not use interest rate feedback.

2. The model

Consider the optimization problem faced by household-firm *i* in the money-in-the utility-function, NK model in continuous time, as in Benhabib et al. (2001a,b). The model is consistent with the one used in Barnett et al (2021), but extended to include money in the utility function, as also considered in the appendix of Barnett et al. (2021). Prices are sticky in the sense of Rotemberg (1982). We shall call this problem Decision P.

Decision P:

$$\underset{c_i, m_i, l_i}{Max} \int_0^\infty \left[u(c_i, m_i) - f(l_i) - \frac{\eta}{2}(\pi_i - \pi^*)^2 \right] e^{-\rho t} dt$$

subject to

$$\dot{a}_i = (R - \pi_i)a_i - Rm_i + \tfrac{p_i}{p} y(l_i) - c_i - \tau$$

$$\dot{p}_i = \pi_i p_i$$

$$a_i(0) = a_{i0}$$

$$p_i(0) = p_{i0} \, .$$

The objective of the household-firm optimizer is to maximize the discounted sum of a *net* utility stream, where $u(c_i, m_i)$ measures utility derived by household-firm *i* from consumption of the composite good, c_i, and from real money balances, m_i, under the time discount rate, ρ. It is assumed that $u(.,.)$ is twice continuously differentiable in all its arguments and that

$$u_c(c_i, m_i) > 0; \quad u_{cc}(c_i, m_i) < 0; \quad u_m(c_i, m_i) > 0; \quad u_{mm}(c_i, m_i) < 0, \tag{1}$$

where the function subscripts denote partial derivatives.

The function $f(l_i)$ measures the disutility of labor, where $f(l_i)$ is twice continuously differentiable, with $f_l > 0$ and $f_{ll} < 0$. The term $\frac{\eta}{2}(\pi_i - \pi^*)^2$ is standard to account for deviations of the price change, $\pi_i = \frac{\dot{p}_i}{p_i}$, with regard to the intended rate π^*, where p_i is the price charged by individual i on the good it produces, and where the parameter η measures the degree to which household-firms dislike to deviate in their price-setting behavior from the intended rate of inflation, π^*.

In the household-firm budget constraint, a_i denotes real financial wealth, consisting of interest-bearing government bonds, where R is the nominal interest rate and $y(l_i)$ is the amount of perishable goods, produced according to a production function using labor, l_i, as the only input. Real lump-sum taxes are denoted by τ.

Recall that in the NK model, sales of good i are demand determined,

$$y(l_i) = \left(\frac{p_i}{p}\right)^{-\phi} y^d , \tag{2}$$

where $\phi > 1$ is the elasticity of substitution across varieties, and p is the aggregate price level. Consider a symmetric equilibrium in which all household-firm units' behaviors are based on the same equations. Then, given $c = y(l)$, we can derive the following three-dimensional system of differential equations, which we shall call System *M*.[8]

System *M*:

$$\dot{\mu}_1 = (\rho - R + \pi)\mu_1$$

$$\eta\dot{\pi} = \rho(\pi - \pi^*)\eta - c(\mu_1, \pi)\left[(1-\phi)\mu_1 + \phi c(\mu_1, \pi)^{\psi}\right]$$

$$\dot{a} = (R - \pi)a - Rm(c(\mu_1, \pi), R) - \tau,$$

where the subscripts are dropped to simplify notation. See Benhabib et al. (2001a,b) for details on the derivation.

[8] See Barnett et al. (2021) for derivation of the first order conditions

The first equation denotes the time evolution of the Lagrange multiplier associated with the continuous time budget constraint (or shadow price of the real value of aggregate per capita government liabilities, real balances, and bonds) at instant of time *t*. The second equation is the well-known New Keynesian Phillips Curve. The third equation is the budget constraint at time *t*.

We now turn our attention to the behavior of the public authorities. Following Benhabib et al. (2001a,b), we assume that the monetary authority adopts an interest rate policy described by the feedback rule, $R = R(\pi)$. The function $R(\pi)$ is continuous, strictly convex, and satisfies the following properties.

Assumption 1. *Monetary authorities set the nominal interest rate as an increasing function of the inflation rate, implying that* $R = R(\pi) > 0$; $R'(\pi) > 0$; and $R''(\pi) > 0$.

It is further assumed that there exists an inflation rate, π^*, at which the steady-state Fisher equation, $R(\pi^*) = \bar{R}$, is satisfied. Consider, moreover, the following definition (see Benhabib et al. (2001a,b)).

Definition 1. *Monetary policy is said to be active, if* $R'(\pi) > 1$, *and passive otherwise.*

Let us now turn our attention to fiscal policy. We assume that taxes are tuned according to fluctuations in total real government liabilities, a, so that $\tau = \tau(a)$. Similarly, for monetary policy, it is further assumed that there exists a tax rate corresponding to the steady-state state level of real government liabilities $\tau(a^*) = \bar{\tau}$.
As in Leeper (1991), Woodford (2003), and Kumhof, Nunes, and Yakadina (2010), we provide a definition of the fiscal policy stance. Let us consider the responses of a to its own variations. We have

$$\frac{\partial \dot{a}}{\partial a} = R(\pi) - \pi - \tau'(a). \tag{3}$$

The dynamic path of total government liabilities is locally stable or unstable, according to the magnitude of the marginal tax rate, $\tau'(a)$. Therefore, we have the following useful definition.

Definition 2. *Fiscal policy is said to be passive, if* $\tau'(a^*) > R(\pi^*) - \pi^*$, *and active otherwise.*

2.1. Steady states and local stability properties

The long-run properties of system *M* are well understood. Benhabib, Schmitt-Grohé, and Uribe (2001a,b) showed that if Assumption 1 holds, then, in general, two steady states exist: one where inflation is at the intended rate $\pi = \pi^*$ and one where $\pi = \bar{\pi} \neq \pi^*$. The unintended steady-state is labeled as a liquidity trap, in which the interest rate is zero or near-zero, and inflation is below the target level and possibly negative. Moreover, at the steady state where inflation is at the intended rate, $\pi = \pi^*$, it follows that μ_1^* exists and is unique. The local stability properties around the intended steady state, $\mathbf{P}^* \equiv (\mu_1^*, \pi^*, a^*)$, are also well described in Barnett et al. (2021).

Assumption 2. *Money and consumption are Edgeworth substitutes in the utility function, i.e.* $u_{cm}^* < 0$.

2.2. An explicit variant of the model

To proceed, we must first provide specific forms for the implicit functions presented in system M. We assume that the utility function has constant relative risk aversion in a composite good, which in turn is produced with consumption goods and real balances via a CES aggregator as follows:

$$u(c,m) = \frac{\left[\kappa c^{1-\beta} + (1-\kappa) m^{1-\beta}\right]^{\frac{1-\Phi}{1-\beta}}}{1-\Phi}, \tag{4}$$

where $0 < \kappa < 1$ is a share parameter, β measures the intra-temporal elasticity of substitution between the two arguments, c and m, and $\Phi > 0$ is the inverse of the intertemporal elasticity of substitution. Since we have, for now, assumed that consumption and real money balances are Edgeworth substitutes, the following parametric restriction is implied.

Remark 1. $Sign(u_{cm}^{*}) = Sign(\beta - \Phi)$. Therefore, Assumption 2 requires $\beta < \Phi$.

The disutility of labor is captured by the following functional form

$$f(l) = \frac{l^{1+\psi}}{1+\psi}, \tag{5}$$

where $\psi > 0$ measures the preference weight of leisure in utility. Furthermore, following Carlstrom and Fuerst (2003), we also assume that production is linear in labor, $y(l) = Al$, with A being the productivity level in the composite goods production. Without loss of generality, we set $A = 1$.

Additionally, we use the specification of the Taylor principle in Benhabib, Schmitt-Grohé, and Uribe (2001a,b), and assume that the monetary authority observes the inflation rate and conducts market operations to ensure that

$$R(\pi) = \bar{R}e^{(C/\bar{R})(\pi-\pi^{*})}, \tag{6}$$

where C is a positive constant. Notice that, from (6), our chosen functional form implies that $R(\pi^{*}) = \bar{R}$, and $R'(\pi^{*}) = C$.

Finally, to avoid a violation of the Transversality Condition, we assume that the economy satisfies a Ricardian monetary-fiscal regime. More specifically, equation (6) is complemented by the fiscal rule

$$\tau(a) = \alpha a - Rm, \tag{7}$$

where the marginal tax rate $\alpha \equiv \tau'(a) \in (0,1)$.

2.3. Conditions for the existence of Shilnikov chaos

In this section, we provide the mathematical underpinnings that guarantee the existence of a chaotic regime in system *M*. Consider the Shilnikov (1965) theorem.

Theorem 1. *Consider the dynamic system*

$$\frac{dY}{dt} = f(Y,\alpha), \quad Y \in \mathbb{R}^3, \quad \alpha \in \mathbb{R}^1,$$

with f sufficiently smooth. Assume f has a hyperbolic saddle-focus equilibrium point, $Y_0 = 0$*, at* $\alpha = 0$*, implying that eigenvalues of the Jacobian,* $J = Df$*, are of the form* γ *and* $\chi \pm \xi i$*, where* γ*,* χ*, and* ξ *are real constants with* $\gamma\chi < 0$ *. Assume that the following conditions also hold:*

$(H.1)$ *The saddle quantity,* $\sigma \equiv |\gamma| - |\chi| > 0$;

$(H.2)$ *There exists a homoclinic orbit,* Γ_0*, based at* Y_0.

Then the following results hold:

(1) *The Shilnikov map, defined in the neighborhood of the homoclinic orbit of the system, possesses an infinite number of Smale horseshoes in its discrete dynamics;*

(2) *For any sufficiently small* C^1*-perturbation,* g*, of* f*, the perturbed system has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map, defined in the neighborhood of the homoclinic orbit;*

(3) *Both the original and the perturbed system exhibit horseshoes chaos.*

A convenient statement of Shilnikov's famous theorem is in Chen and Zhou (2011). Because obtaining a general result on the critical parametric bifurcation surface is extremely difficult, we propose a numerical strategy based on the parameterization of the US economy from 1960(Q1) to 1998(Q3) proposed by Benhabib, Schmitt-Grohé, and Uribe (2001a,b) and extensively used in the succeeding literature. We show that there are regions in the parameter space where system M may satisfy the conditions of Theorem 1.

Example 1. *Denote the set of the deep parameters as* $D \equiv \{(\beta, \eta, \kappa, \phi, \psi, \rho, \Phi)\}$*, and assume*

$$\bar{\mathbf{D}} \equiv (1.975, 350, 0.90899, 21, 1, 0.018, 2) \in D.$$

Set the pair $(\bar{R}, \pi^*) = (0.06, 0.042)$ *to match the (average) three-month Treasury Bill rate and (average) inflation rate observed over the period for the US economy. Therefore, since* τ *cancels out in the calculations, the characteristic equation (A.2 in Appendix 1) is a function of the remaining policy parameters,* C *and* τ'*. Solving the characteristic equation gives*

$\lambda_1 = 0.018 - \tau'$,

$\lambda_{2,3} = 0.009 - 0.00058C \pm 0.00058\sqrt{(C - 1.00046)(C - 4.6543 \times 10^5)}$.

Therefore, since $|u^*_{cm}| \cong 0.0008 < |\hat{u}^*_{cm}| \cong 15.7812$*, an active monetary-fiscal regime implies three eigenvalues with positive real parts for any reasonable value of the coefficient C. A fiscal policy switch to a passive rule implies one negative eigenvalue and two eigenvalues with positive real parts. In this example, the saddle quantity equals* $\sigma \equiv \tau' - 0.027 + 0.00058C$*. Therefore, if we set* $\tau' > 0.027 - 0.00058C$*, the saddle quantity is positive.*

We are now ready to provide the following result.

Lemma 1. (Fulfillment of pre-condition H.1 in Theorem 1). *There are regions of the parameter space where the intended steady-state,* $\mathbf{P}^*$*, is a saddle-focus equilibrium with* $\sigma > 0$.

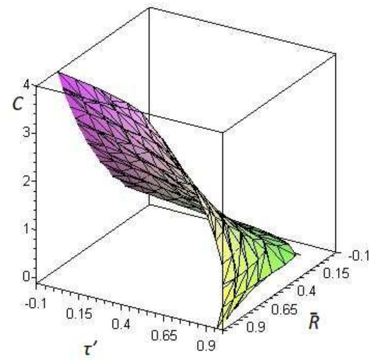
Proof. Set $C > 1.00046$ and $\tau' > 0.027 - 0.00058C$ as in Example 1. Then, the eigenvalues associated with system *M* are of the form required for $\mathbf{P}^*$ to be a saddle-focus equilibrium with $\sigma > 0$. ■

Furthermore, we can keep the deep parameters at $\bar{\mathbf{D}} \in D$, assume $\pi^* = 0.042$, and study the surface

$$\Omega \equiv \mathrm{B}(\mathbf{J}) + \mathrm{Tr}(\mathbf{J})^2 \tag{8}$$

in the remaining $(C, \bar{R}, \tau')$ parameter space. As shown in Bella, Mattana, and Venturi (2017), the vanishing of Ω corresponds to the critical parametric surface, at which a generic steady state is a saddle-focus equilibrium with null saddle quantity.

Figure 1. *Combinations of the* $(C, \bar{R}, \tau')$ *parameters at which* $\Omega = 0$.



$\Omega = 0$. Above the surface, the saddle quantity is positive. Below the surface, the saddle quantity is negative. Interestingly, the figure shows that a positive saddle quantity can be determined exactly, when the pair $(\bar{R}, \tau')$ is plausibly low and $C > 1$.

We end this section by noticing some further details regarding the form of the eigenvalues in Example 1. It is clear that for $C > 1.00046$, and irrespective of the stance of fiscal policy, one eigenvalue is real, while the remaining two eigenvalues are complex conjugate. This means that locally, when monetary policy is active, convergence towards $\mathbf{P}^*$ occurs typically through (damped) oscillating paths.[9] It is also useful to observe the following.

Remark 2. *In our simulations, the structure of the eigenvalues derived in Example* 1 *survives wide variations of the parameters. More specifically, when* $C > 1$ (*active monetary policy*), *there is always a small right neighborhood of* $C = 1$ *such that eigenvalues are all real. Things are different when* $C < 1$ (*passive monetary policy*). *In this case, eigenvalues are always real, and the convergence towards* $\mathbf{P}^*$*generally takes*

[9] It would be interesting here to confront the dynamics featured by these equilibria with the "volatile sequence of interest rates and inflation rates followed by sudden arrival at the low nominal interest rate steady state," pointed out by Bullard (2010, p. 344), regarding complicated or chaotic expectational dynamics. On that issue, see also Piazza (2016).

place along the monotonic perfect-foresight path.

Once it has been established that there are regions in the parameter space such that $\mathbf{P}^*$ is a saddle-focus equilibrium with $\sigma > 0$, we follow Bella, Mattana, and Venturi (2017) to show that system *M* admits homoclinic solutions (pre-condition *H*.2 in Theorem 1).

The procedure leads to the following split function[10]

$$\Sigma = \Xi + \frac{F_{3f}\Xi^2}{\gamma} + (2\chi - \gamma)\frac{F_{3a}\Psi\Omega + F_{3d}\Psi^2 + F_{3e}\Omega^2}{(2\chi-\gamma)^2 + 4\xi^2} = 0, \tag{9}$$

where $(\Xi, \Psi, \Omega) \in (0,1)^3$ are free constants, while $\gamma = \lambda_1$, $\chi = Re(\lambda_{2,3})$, and $\xi = Im(\lambda_{2,3})$. Then, with given parameters, conditions for the existence of the homoclinic loop, doubly asymptotic to the saddle-focus equilibrium point, rely on the existence of a triplet $(\Xi, \Psi, \Omega) \in (0,1)^3$ satisfying $\Sigma = 0$ (admissible solution).

To verify whether there are admissible solutions to (9) in the feasible parameter space, we specify further the calibration of the economy used in this section.

Example 2. *Let* $\bar{\mathbf{D}} \in D$ *and* $(\bar{R}, \pi^*) = (0.06, 0.042)$*, as in Example* 1. *Set* $C = 1.5$. *By* (8)*, the critical value of the marginal tax rate at which the saddle quantity is positive is*

$$\tilde{\tau}' = 0.027 - 0.00058C = 0.02613.$$

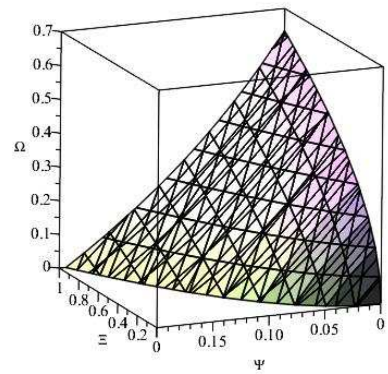
If $\tau' > \tilde{\tau}'$*, then* $\mathbf{P}^*$ *is a saddle-focus with positive saddle quantity. Let us now use the marginal tax rate,* τ'*, as the bifurcation parameter. More precisely, we iteratively increase* τ' *above* 0.02613 *with a grid of* 0.01 *until a solution for* $\Sigma = 0$ *with* $(\Xi, \Psi, \Omega) \in (0,1)^3$ *emerges. The procedure reveals that there exists an interval* $I_{\tau'} \cong (0.02613, 0.23543)$ *such that, for all* $\tau' \in I_{\tau'}$*, a family of homoclinic loops doubly asymptotic to the saddle-focus equilibrium point exists.*

Figure 2 depicts the combinations of (Ξ, Ψ, Ω) solving the split function (9) for

[10] Cf. Kuznetsov (1998, p. 198) for the geometrical interpretation of the split function in the context of homoclinic bifurcations.

the case of $\tau' = 0.15$.

Figure 2. *Coordinates in* $(\varXi, \Psi, \Omega)$ *that give rise to the homoclinic loop for* $\tau' = 0.15$.



Remark 3. *For this calibration of the economy, since* $\frac{\partial \Sigma}{\partial \tau'} > 0$ *for all values of* $(\varXi, \Psi, \Omega) \in (0,1)^3$, *there exists a unique critical value of* τ' *solving the split function* (9).

2.4. Existence and properties of the chaotic attractor

Let $\nu = \tau' - \bar{\tau}'$, where $\bar{\tau}' \in I_{\tau'}$ is the critical value of the marginal tax rate, such that an admissible solution of the split function exists for given coordinates, $(\varXi, \Psi, \Omega) \in (0,1)^3$. Let $V \subset \mathbb{R}$ be a small open neighborhood of **0**. We have the following result.

Example 3. *Set* $\bar{\mathbf{D}} \in D$, $(\bar{R}, \pi^*) = (0.06, 0.042)$, *and* $C = 1.5$, *as in Example* 2. *Then, we know that there exists* $I_{\tau'} \cong (0.02613, 0.23543)$ *such that for all* $\tau' \in I_{\tau'}$ *there exists a family of homoclinic loops doubly asymptotic to the saddle-focus equilibrium point. Consider the case of* $\tau' = 0.15$ *and initial conditions* $(\tilde{w}_1(0), \tilde{w}_2(0), \tilde{w}_3(0)) = (-0.01, -0.01, -0.01)$.

The attractor generated by this specific example is represented in Figure 3.

Figure 3. *The chaotic attractor in the* $(\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$ *space.*

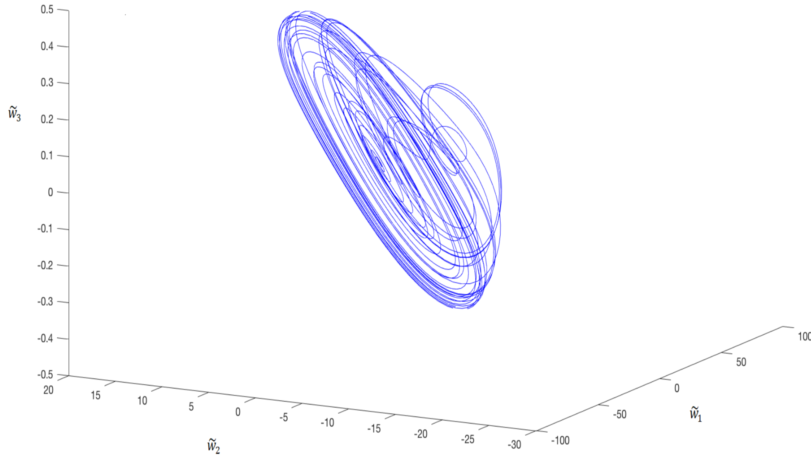
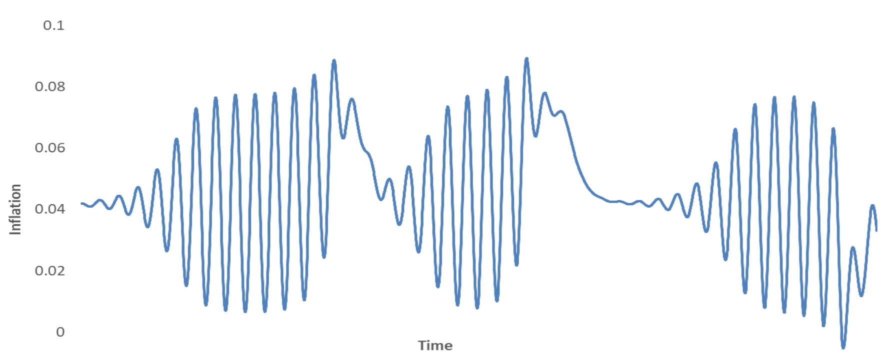


Figure 3 displays the distinct shape of the Shilnikov attractor. When the phase point approaches the saddle-focus point, the economy's dynamics along the spiral attractor experience periods of relative calm. When the phase point begins to spiral away from the saddle-focus point, irregular episodes of oscillatory activity begin. The presence of a chaotic attractor is very important to the dynamics implied by NK models and has important economic implications. Because of its presence, small changes in initial conditions can result in large changes in dynamics over time. Moreover, within a chaotic attractor, given the initial value of the predetermined variable, there exists a *continuum* of initial values of the jump variables giving rise to admissible equilibria. Therefore, the policy options required to recover the uniqueness suggested by the local analysis are exactly those which may cause global indeterminacy of the equilibrium.

The numerical simulations developed in the paper show that if the initial conditions of the jump variables are chosen far enough from the target steady-state, then the emerging aperiodic dynamics tend to evolve for a long time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon, which now depends on the existence of a chaotic attractor and not on the influence of an unintended steady state.

Consider the reconstructed time profile of the inflation rate (Figure 4). The time span is 10 years. First, observe the distinct shape of Shilnikov chaos. The wave train generated by the spiral attractor has long quiescence periods, when the phase point approaches the saddle-focus, followed by bursts of oscillatory activity. The average inflation rate can be persistently higher/lower than the steady-state value. This implies the possibility of long periods, during which inflation is stubbornly high, or long periods during which inflation is stubbornly low (akin to a deflationary equilibrium), and periods during which inflation is volatile. Also observe the downward drift, which was even more evident in Barnett et al. (2021), calibrated with parameter settings based on recent data.

Figure 4. *The time profile of the chaotic inflation rate.*



3. Policy solutions

3.1. Ending the chaos

Potential policy solutions to the problems produced by Shilnikov chaos can be divided into two groups. One group of approaches ends the Shilnikov chaos by removing the Taylor rule's closed-loop interest rate feedback dynamics, while introducing a fundamentally different monetary policy design. The other approach is to retain the Taylor principle and thereby the Shilnikov chaos, while imposing an algorithm to control the chaos. This latter approach requires introduction of a second policy instrument in addition to the interest rate that appears in the Taylor rule. The source of the drift problem, produced by augmenting the system's dynamics with a myopic Taylor rule, is the lack of a terminal condition. The additional policy instrument attaches a long run anchor to the feedback policy.

Alternatively, to end the chaos, the central bank could adopt any of the policy approaches that do not use interest rate feedback. There are many such policy designs in the literature that could be selected by the central bank in accordance with the central bank's mechanism design. It is not the purpose of this paper to advocate any one of those alternatives.

Examples could include using an active fiscal policy and a passive monetary policy. That approach produces its own dynamical problems in a NK Model, but not Shilnikov chaos. Another example could include monetary policy without interest rate feedback. An open loop fixed monetary quantity growth rate would be the simplest approach. More sophisticated modern closed-loop approaches could include those using Divisia monetary quantity aggregates, such as those approaches proposed by Belongia (1996), Serletis (2013), or Belongia and Ireland (2014, 2017, 2018, 2019), and as advocated by Peter Ireland in his role on the Shadow Open Market Committee. A long literature exists on alternatives to Taylor rule policies, such as Cochrane (2011).

3.2. Controlling the chaos

If the central bank were to decide to retain imposition of the Taylor Principle and thereby the resulting Shilnikov chaos, a second instrument of policy would need to be introduced to deal with the consequent eventual drift into a liquidity trap. The need for such a second instrument, in addition to the interest rate in the Taylor rule, is widely accepted and has been applied by most central banks in recent years. A survey of such new tools of monetary policy, such as forward guidance and intervention in long term bond markets, has been provided by Bernanke (2020) in his Presidential Address to the American Economic Association. But what is less well established is how to design a rule for use of such an instrument of policy. Under those circumstances, we would propose one of the available algorithms for controlling chaos. In addition, we find that when the economy is on a Shilnikov chaos attractor set associated with imposition of a Taylor rule, a second instrument of policy to control the undesirable properties of chaos should be adopted, even if the economy is not yet in a liquidity trap.

We now consider policy to control chaos. Assume the economy is enmeshed in a chaotic attractor. What should a policy maker do to alleviate the implied economic uncertainty, and bring agents' inflationary expectations back in line with those coherent with the intended steady state? In each such approach, one of the new tools of policy would be adopted as a second instrument of policy to target, as an intermediate target, a long run anchor consistent with an available algorithm for controlling chaos.

The methods of controlling chaotic dynamics in the engineering literature provide useful tools in this regard. Under certain conditions, undesired irregular or even cyclical behavior can be switched off.[11] An ingenious and well-known method for doing so is proposed by Ott, Grebogi, and Yorke (1990), also known as the OGY algorithm. It enables one to force a chaotic trajectory onto a desired target (a periodic orbit or a steady state of the system) by a correction mechanism. This mechanism has the form of a small, time-dependent perturbation of a certain control parameter. Suppose also that a neighborhood of the desired fixed point can be found, such that the system is guaranteed to be driven to the fixed point. If this neighborhood has points in common with a chaotic attractor, the neighborhood may be used as a controllable target to attain the fixed point.

To achieve control over a chaotic solution trajectory, the control parameter must be accessible to the central bank. This is a relevant point in our NK sticky-price model with Taylor rule interest rate feedback and a Ricardian fiscal policy. If we go back to the form of the eigenvalues in Example 1, it is clear that fiscal policy parameters can only govern the sign of the real eigenvalue.[12] As a consequence, fiscal policy is ineffective in

[11] There are examples of chaos control in the literature on optimal monetary models (Mendes and Mendes (2006)). However, they were developed in a discrete-time environment. Experiments with chaos control are common in other fields of economics. There are, for example, recent contributions in tâtonnement processes (Naimzada and Sordi (2017)) and disequilibrium macroeconomic models (Kaas (1998)).

[12] Recall that τ cancels out in the computation of the characteristic equation. See also (A.2), (A.3), and (A.4) in Appendix 1.

controlling chaos in the present setting.[13]

In this regard, we choose to operate through the manipulation of the long-run, steady-state nominal interest rate, $\bar{R}$, at which the steady-state Fisher equation ($R(\pi^*) = \bar{R}$) is satisfied. Since that interest rate is not under the direct control of the Central Bank, our algorithm treats manipulation of that interest rate as an intermediate target, rather than as an instrument of policy.

The choice of policy instrument or of market intervention operating procedure to be used in that intermediate targeting could depend upon the mechanism design of the central bank, which is not a topic of this research. An alternative OGY procedure could use the long run inflation rate, π^*, instead of the nominal interest rate, $\bar{R}$, as intermediate target. Although we believe that the two procedures would likely prove to be mathematically equivalent, we anticipate that an intermediate targeting OGY procedure using $\bar{R}$ would be more easily implemented by a Central Bank than an OGY procedure using the long run inflation rate, which is also a final target of policy.

From a more technical point of view, and to operate an informed choice between π^* and $\bar{R}$, we have also computed and evaluated at the intended steady-state, the partial derivatives of $G(\mathbf{J})$ in (A.5) with regard to π and R. This computation is helpful, since in correspondence of a saddle-focus, the Jacobian of system M presents negative $Tr(\mathbf{J})$ and $Det(\mathbf{J})$ at the intended steady-state. Therefore, chaos control according to the OGY mechanism translates into varying $\bar{R}$ or π^* in such a manner that $G(\mathbf{J})$ becomes negative. We found that, for parameters as in Example 3, $\left[\frac{\partial G(J)}{\partial \pi}\right]_{P^*} \ll \left[\frac{\partial G(J)}{\partial R}\right]_{P^*}$, implying that using $\bar{R}$ has proportionally much higher stabilization power than varying π^*.

Before proceeding with the implementation of the OGY algorithm, some

[13] Benhabib, Schmitt-Grohé, and Uribe (2002) discuss a similar issue. The authors describe the characteristics of fiscal policy schemes capable of eliminating the liquidity trap, while maintaining the assumed monetary policy stance. They discover that the government may be able to avert the unintended low-inflation equilibrium by using an inflation-sensitive revenue schedule of the generic form $\tau(a) = \alpha(\pi)a + Rm$. However, this exercise implies a modification of the standard model presented in Section 2. We therefore leave this type of analysis to future research.

preliminary steps need to be taken. First, we need to show that system *M* is controllable. Then, we will need to discuss the region of the parameter space supporting application of the OGY algorithm. Consider the following initial result.

Lemma 2. *System M satisfies the conditions for controllability.*
Proof. Cf. Appendix 2. ■

Once controllability of system *M* can be established, the OGY algorithm requires that the eigenvalues of the controlled system be chosen such that stability is implied. Stabilizing a system is thus translated into searching for values of the nominal, steady state interest rate, $\bar{R}$, such that all eigenvalues exhibit a negative real part (see Appendix 2).

We have anticipated that there exists a critical value, $|\hat{u}^*_{cm}(\bar{R})|$, such that if $|u^*_{cm}(\bar{R})| > |\hat{u}^*_{cm}(\bar{R})|$, then an active-passive monetary-fiscal regime implies stability of the intended steady state. For notational convenience, let us define

$$|\hat{u}^*_{cm}(\bar{R})| - |u^*_{cm}(\bar{R})| = \Theta(\bar{R})$$

and denote

$$\bar{R}^+_\Theta = \{\bar{R} \ : \ \Theta(\bar{R}) > 0\},$$
$$\bar{R}^-_\Theta = \{\bar{R} \ : \ \Theta(\bar{R}) < 0\}.$$

We are now ready to prove the following.

Proposition 1. *Consider the case in which the policymaker runs an active-passive monetary-fiscal regime and assume that $\bar{R} \in \bar{R}^+_\Theta$. Then, one eigenvalue is negative, and two eigenvalues have positive real parts. Assume, furthermore, that the economy evolves within a chaotic attractor. Suppose the policy maker announces commitment to a higher steady state nominal interest rate, belonging to the set $\bar{R}^-_\Theta$. Then, the economy supersedes irregular and cyclical behavior and approaches the intended steady state.*
Proof. See Appendix 3. ■

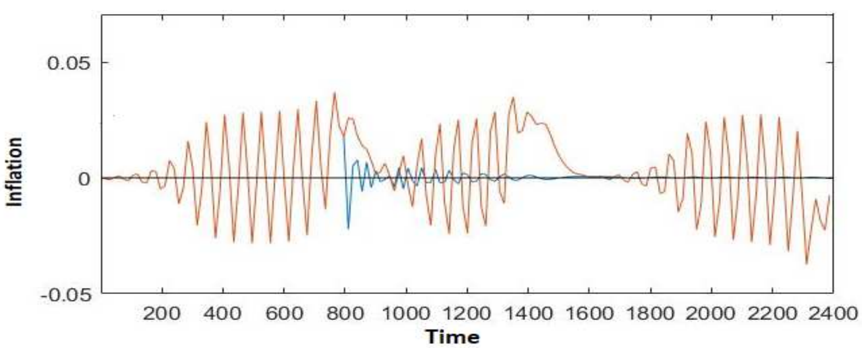
Example 4. *Let us denote by $\bar{\mathbf{D}}'$ the difference set $\bar{\mathbf{D}} - \{\beta\}$ and consider the following* *calibration*

$$\bar{\mathbf{D}}' = \widehat{\mathbf{D}}' \equiv (350, 0.78966, 21, 1, 0.018, 2).$$

Set the triplet $(\bar{R}, \pi^*, C)$ *at* $(0.06, 0.042, 1.5)$*, as in the preceding examples, and set* $\beta = 1.78$.[14]

Re-running the algorithm for the presence of a chaotic attractor, we find that system M has a saddle-focus equilibrium with positive saddle quantity and a family of homoclinic orbits for values of the bifurcation parameter τ' belonging to the (extended) interval, $I_{\tau'} \cong (0.02613, 0.27923)$. Set $\tau' = 0.15$. For this specific parameter configuration, J has three eigenvalues with negative real parts for any $\bar{R} \geq 0.06767$. Let us select $\bar{R} = 0.07$ and apply the OGY algorithm (as described in Appendix 3) to obtain the controlled system solution (blue curve), where the control has been initiated at iteration 800. In Figure 5, we superimpose the solution time profile with uncontrolled inflation (red curve).

Figure 5 *Un-controlled and controlled inflation rate* (*control activated at the* 800th *iteration*).



[14] Since the two elasticities, β and Φ, are very close, there is a very large divide between $|u^*_{cm}(\bar{R})|$ and $|\hat{u}^*_{cm}(\bar{R})|$, and $\bar{R}$ has to undergo a too large jump to make $|u^*_{cm}(\bar{R})| > |\hat{u}^*_{cm}(\bar{R})|$. In this example, we have therefore slightly decreased β.

closer look at the different comparative statics implied by our model provides a full explanation of this apparent contradiction.[15] Recall that Benhabib, Schmitt-Grohé, and Uribe (2002) maintain throughout their paper the complementarity condition in the utility function between real balances and consumption, $u_{cm} > 0$; this implies that a *drop* of the interest rate increases consumption and real economic activity in the model, via the implied increase in money holdings. In our case, the same stimulus to the real activity of the economy is obtained by an *increase* of the interest rate, provided that we have assumed $u_{cm} < 0$ (see Assumption 2 above).[16] In Benhabib, Schmitt-Grohé, and Uribe (2002), stability is achieved through the simultaneous modulation of nominal interest rates.[17] In our case, instead, it is the commitment to a long-run easing or tightening of the monetary policy stance, which is able to re-anchor expectations to the long-run target of inflation.

4. Conclusions

Indeterminacy can occur as a result of the emergence of a Shilnikov chaotic attractor in the region of a New Keynesian model's feasible parameter space. Due to the presence of the Shilnikov chaotic attractor, the system becomes highly sensitive to initial conditions, and irregular transitional dynamics may jeopardize an economy's ability to

[15] See Wang and Yip (1992), Table 1, p. 555, for a complete derivation of the comparative statics of models with productive and non-productive money.

[16] Other crucial differences deserve to be mentioned. Benhabib, Schmitt-Grohé, and Uribe (2002) modify the behavioral rules of public authorities, while the OGY algorithm merely uses a parameter bifurcation approach to induce a change in the topology of the dynamic system. Also, our case is aimed at stabilizing irregular chaotic dynamics as well as downward drift into a liquidity trap, whereas Benhabib, Schmitt-Grohé, and Uribe (2002) face solely the problem of eliminating a liquidity trap caused by an unintended steady state.

[17] Benhabib, Schmitt-Grohé, and Uribe (2002) present two examples involving a Taylor rule stipulating *low* (in fact zero) nominal interest rates at low rates of inflation. In one of the examples, the paths leading to the liquidity trap are characterized by nominal interest rates that converge to zero but never actually reach that floor. In the other example, the nominal interest rate hits the zero bound in finite time. Here the central bank must be committed to lower nominal interest rates to zero as inflation becomes sufficiently low.

converge to a long-run equilibrium. In fact, there could be a continuum of initial values for the jump variables, given the initial value of the predetermined variable, resulting in admissible equilibria. Moreover, if the initial conditions of the jump variables are chosen far enough away from the targeted steady state, the emerging aperiodic dynamics continue to evolve over a long period of time around lower-than-targeted inflation and nominal interest rates – a phenomenon known as the liquidity trap. Our results confirm those of Barnett et al. (2021), which strikingly resemble experience in developed economies in recent decades. While our results are based on the Benhabib et al. (2002) parameter calibration, the results in Barnett et al. (2021) use parameter settings based on recent data.

We propose two potential classes of solutions to the problem:

(a) The policy design could be altered from an active NK interest rate feedback policy to a fundamentally different design, such as active fiscal policy with passive monetary policy. That alternative is known to produce its own problems in economic dynamics with NK models, but not Shilnikov chaos. Among the many other alternatives in the literature is the targeting of a Divisia monetary aggregate, in accordance with such proposals as those in Belongia (1996), Serletis (2013), or Belongia and Ireland (2014, 2017, 2018, 2019) and as advocated by Lucas (2000, p. 270) in measuring welfare loss from inflation.[18] By removing the interest rate feedback rule from the NK model, such alternative approaches could prevent chaotic dynamics from occurring. A long literature exists on alternatives to Taylor Rule policies. See, e.g., Cochrane (2011).

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[18] Lucas (2000, p. 270) wrote: "I share the widely held opinion that M1 is too narrow an aggregate for this period [the 1990s], and I think that the Divisia approach offers much the best prospects for resolving this difficulty." Similarly, the International Monetary Fund (2008, pp. 183-184) has advocated Divisia monetary aggregation, available for the U.S. from the Center for Financial Stability (www.centerforfinancialstability.org/amfm.php) and for the UK from the Bank of England. Divisia monetary aggregates are also available at the European Central Bank and many other central banks, but only for internal use.

(b) A Taylor rule with interest rate feedback could continue to be used, but with the resulting Shilnikov chaos controlled through the use of a second policy instrument applied in accordance with the policy procedures advocated by engineers in the literature on controlling chaos. We find that the Ott, Grebogi, and Yorke (1990) algorithm could be particularly well suited to that objective and would serve to impose a long run anchor adding a terminal condition to the otherwise myopic Taylor feedback rule.

In subsequent research, we plan to explore robustness of our conclusions to different Taylor rules and to alternatives to the Ricardian fiscal policy laws.[19] But we do not expect fundamental changes in our conclusions, which are systems theory properties of sticky price New Keynesian macroeconomic dynamics, when augmented by closed loop interest rate feedback rules, as in Taylor rules. We find that our results do not depend upon whether money appears in the utility function, the production function, or not at all and are robust to different parameter settings. Our results do depend upon the existence of sticky prices. But without the existence of sticky prices, the use of an active interest rate feedback rule would be hard to justify.

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[19] Our conclusion about the relevancy of the OGY algorithm to control the chaos, or alternatively removal of the Taylor rule to end the chaos, remain relevant in models where we replace money in the utility function with money in the production function or in a cashless economy. But we find that the OGY long run anchor setting to control the chaos is affected. The results are available upon request. Future research could investigate the general sensitivity of the OGY policy to specification changes in the NK model.

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APPENDICES

Appendix 1: The invariants of system *M*.

Let **J** denote the Jacobian matrix of system *M*, evaluated at the long-run equilibrium, and let starred values denote steady-state state levels. Simple algebra leads to the following (3 × 3) matrix,

$$\mathbf{J} = \begin{bmatrix} 0 & (1 - R'(\pi^*))\mu_1^* & 0 \\ j_{21}^* & j_{22}^* & 0 \\ -\frac{\bar{R}}{u_{mm}^*}\left[\bar{R} - u_{cm}^* \frac{u_{mm}^* - u_{cm}^*\bar{R}}{u_{cc}^* u_{mm}^* - u_{cm}^{*2}}\right] & 0 & \bar{R} - \pi^* - \tau'(a^*) \end{bmatrix}, \quad \text{(A.1)}$$

where $j_{21}^* = -\frac{\psi\phi}{\eta} c^{*\psi} \frac{u_{mm}^* - u_{cm}^*\bar{R}}{u_{cc}^* u_{mm}^* - u_{cm}^{*2}} + \frac{(\phi-1)}{\eta} c^*$, and $j_{22}^* = \rho - \frac{\psi\phi c^{*\psi} u_{cm}^* \mu_1^*}{\eta(u_{cc}^* u_{mm}^* - u_{cm}^{*2})} R'(\pi^*)$. Following Tsuzuki (2016), we assume that $j_{21}^* > 0$, which implies that a lower bound exists for u_{cm}^*.

The eigenvalues of **J** are the solutions of the characteristic equation

$$det(\lambda\mathbf{I} - \mathbf{J}) = \lambda^3 - Tr(\mathbf{J})\lambda^2 + B(\mathbf{J})\lambda - Det(\mathbf{J}),$$

where **I** is the identity matrix and where

$$Tr(\mathbf{J}) = j_{22}^* + \bar{R} - \pi^* - \tau'(a^*) \quad \text{(A.2)}$$

$$Det(\mathbf{J}) = [\bar{R} - \pi^* - \tau'(a^*)][R'(\pi^*) - 1]\mu_1^* j_{21}^* \quad \text{(A.3)}$$

$$B(\mathbf{J}) = [R'(\pi^*) - 1]\mu_1^* j_{21}^* + [\bar{R} - \pi^* - \tau'(a^*)]j_{22}^* \quad \text{(A.4)}$$

are Trace, Determinant, and Sum of principal minors of **J**, respectively.

Also define

$$G(\mathbf{J}) = -B(\mathbf{J}) + \frac{Det(\mathbf{J})}{Tr(\mathbf{J})}. \quad \text{(A.5)}$$

Under the active monetary policy ($R'(\pi^*) > 1$), we can determine the sign of the real parts of the eigenvalues with active or passive fiscal policy. Consider first the case in which the fiscal policy is passive, implying $\bar{R} - \pi^* - \tau'(a^*) < 0$. In this case, $Det(\mathbf{J}) < 0$. Consider instead the case of active fiscal policy, implying $\bar{R} - \pi^* - \tau'(a^*) > 0$. Then, both $Tr(\mathbf{J})$ and $Det(\mathbf{J})$ are positive. In this case, irrespective of the sign of $G(\mathbf{J})$, we have one eigenvalue with negative real part and two eigenvalues with positive real parts; $\mathbf{P}^*$ is a saddle of index 2, and the equilibrium is locally unique. ■

Appendix 2: Proof of Lemma 2.

The algorithm for proving controllability of a given system requires that the nonlinear system be written in state-space notation. We first put the linear part of system *M* in the form

$$\dot{\mathbf{w}} = \mathbf{J}\mathbf{w} + \mathbf{M}\mathbf{K}\mathbf{w}, \qquad \text{(A.6)}$$

where $\mathbf{w} = (w_1, w_2, w_3)^T$, while $\mathbf{J}$ is as in A.1. Moreover, $\mathbf{M} = \left(\frac{\partial \dot{w}_1}{\partial \bar{R}}, \frac{\partial \dot{w}_2}{\partial \bar{R}}, \frac{\partial \dot{w}_3}{\partial \bar{R}}\right)^T$, while $\mathbf{K} = (k_1, k_2, k_3)$ is a (1×3) vector. System (A.6) now needs to be put into its first-companion form,

$$\dot{\boldsymbol{\omega}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\boldsymbol{\omega}. \qquad \text{(A.7)}$$

Here $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ from the following transformation $\mathbf{w} = \mathbf{T}\boldsymbol{\omega}$, where $\boldsymbol{A} = \mathbf{T}^{-1}\mathbf{J}\mathbf{T}$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}, \qquad \text{(A.8)}$$

and where $\mathbf{B} = \mathbf{T}^{-1}\mathbf{M}$. In detail, the transformation matrix $\mathbf{T}$ has to be chosen to satisfy the product $\mathbf{T} = \mathbf{N}\mathbf{W}$, with

$$\mathbf{N} = [\mathbf{B}, \mathbf{J}\mathbf{B}, \mathbf{J}^2\mathbf{B}] \qquad \text{(A.9)}$$

and

$$\mathbf{W} = \begin{bmatrix} \varepsilon_2 & \varepsilon_3 & 1 \\ \varepsilon_3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \qquad \text{(A.10)}$$

Controllability requires that matrix **N** have full rank. Since, in our case, matrix **A** is non-degenerate, the controllability of system *M* by means of changes in the nominal rate is feasible. ■

Appendix 3: Proof of Proposition 1.

The OGY algorithm requires that a desired form for the characteristic equation be obtained by varying the control parameter. In our case, the "desired" form implies three eigenvalues with negative real parts. We have shown that it can be done, if the policy-maker runs an active-passive monetary-fiscal regime, when $|\hat{u}_{cm}^{*}(\bar{R})| - |u_{cm}^{*}(\bar{R})| = \Theta(\bar{R}) < 0$. Assume now that the policy maker announces a commitment to a steady state nominal interest rate belonging to the set $\bar{R}_{\Theta}^{-}$. Then full stability of $\mathbf{P}^{*}$ is affirmed, and the statements in the proposition are implied. Since $\frac{\partial u_{cm}^{*}(\bar{R})}{\partial \bar{R}}$ is invariably positive, there is a commitment to a *higher* steady state nominal interest rate. Example 4 below completes the proof by showing that there are regions of the parameter space such that $\bar{R}_{\Theta}^{-} \neq \emptyset$. ■