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Abstract

We consider the licensing of a cost-reducing innovation in a Cournot oligopoly where an outside innovator uses three part tariffs that are combinations of upfront fees, per unit royalties and ad valorem royalties. The key insight of our analysis is per unit royalties have a location effect and ad valorem royalties have a scale effect on marginal costs. Using these two effects, we show that the same market outcome (price, quantities, operating profits) can be sustained by multiple combinations of per unit and ad valorem royalties. In the monopoly case, under three part tariffs it is optimal to set a pure upfront fee while the unique optimal two part royalty is a pure ad valorem royalty. In the case of a general oligopoly with linear demand, for relatively insignificant innovations, it is optimal to set a pure upfront fee; otherwise there is a continuum of optimal policies and there always exists an optimal policy consisting of a positive per unit royalty and upfront fee but no ad valorem royalty. For intermediate innovations, provided the demand intercept is relatively large, there exists an optimal policy that has both kinds of royalties but no fees. Finally in a Cournot duopoly it is illustrated that when the innovator is one of the incumbent firms rather than an outsider, market outcomes separately depend on two kinds of royalties and a pure ad valorem royalty is optimal among all three part tariffs.

Keywords: patent licensing; per unit royalties; ad valorem royalties; three part tariffs; acceptability and feasibility constraints

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1 Introduction

Ad valorem royalties, more commonly known as revenue sharing, are widely used in licensing of patented technologies. Under an ad valorem royalty a firm pays the patentee a percentage of its sales revenue for the use of a patented technology. This can sometimes lead to dispute, because even though the patented technology is essential to make its product, the firm may have improved many product features that are not related to that technology. If the firm earns an enhanced sales revenue due to these product improvements, it may consider it unfair to share a fraction of those higher earnings with the patentee. This was the subject of a legal dispute between Apple and Qualcomm, which was recently settled out of court (see Kraus, 2019).

Since the inception of the original i-phone, Qualcomm has supplied the required modem chips for i-phones. In addition, Qualcomm has a patented technology that enables the i-phones to connect to the internet through the modem chips. Apple has to pay a percentage of its revenue to use this patented technology. According to Apple, using this arrangement Qualcomm unfairly collects "exorbitant royalties" since the payment is based on the entire sales value of the i-phone, even though Qualcomm supplies only a single component of the product (see Rossignol, 2017):

> "As Apple innovates, Qualcomm demands more. Qualcomm had nothing to do with creating the revolutionary Touch ID, the world's most popular camera, or the Retina display Apple's customers love, yet Qualcomm wants to be paid as if these (and future) breakthroughs belong to it."

This paper provides an analysis of ad valorem royalties or revenue sharing in the context of patent licensing in oligopolies. Regarding the case of Apple vs. Qualcomm, it is worthwhile to note that apart from sharing a percentage of Apple's sales revenue, Qualcomm also charged a price on a per unit basis for the modem chips it supplied to Apple. This closely approximates a *two part royalty* contract, in which there is a per unit royalty (unit price of modem chips) together with an ad valorem royalty (a fraction of sales revenue). If there is also a lump-sum transfer between the two parties, the resulting contract will be a *three part tariff* consisting of three elements: a fixed fee, a unit royalty and an ad valorem royalty. Since contracting parties in practice can use such multiple payment schemes, the optimality and the implications of a specific scheme such as revenue sharing will depend on how much strategic freedom firms have to combine other schemes.

Given this background, in this paper we consider the problem of patent licensing in a Cournot oligopoly with an external patentee (an outside innovator) who has a patent on a technological innovation that lowers the initially identical, constant marginal cost of production. The innovator can license its patent to some or all firms in the oligopoly. We consider two general forms of licensing policies for the innovator: (i) a two part royalty policy, in which a licensee pays a per unit royalty for every unit it produces and also pays an ad valorem royalty which is a fraction of its revenue and (ii) a general three part tariff where in addition to the two kinds of royalties, a licensee pays an upfront fee. Under a three part tariff, the innovator can use the upfront fee to collect the surplus of a license while in a two part royalty policy, licensees are generally left with some surplus.

This paper is the very first work to study three part tariffs for licensing in a general oligopoly. The literature on three part tariffs is sparse. Bousquet et al. (1998) consider these policies for a new product innovation with one outside innovator and one risk averse firm. Savva and Taneri (2015) look at three part tariffs consisting of an equity (which is an *ad valorem profit* royalty where a licensee shares a fraction of its profit with the licensor), a per unit royalty and a fixed fee in a model of product innovation with uncertain demand conditions. Recently, Banerjee et al. (2021) study three part tariffs in a Hotelling duopoly where the innovator is one of the two competing firms.

In all of these papers on three part tariffs, the reservation payoff of a licensee does not depend on licensing policies, because there is only one potential licensee. This is not the case in an oligopoly where the profit a firm obtains without a license depends on the licensing policy as well as the number of other firms having licenses. This strategic aspect, which affects the willingness to pay for a license, is absent in the earlier works. Our analysis of three part tariffs fully explores the strategic interaction among licensees.

As a starting point of our analysis, a key question is: to what extent are the two kinds of royalties substitutable? In other words, can the same market outcome be sustained by slightly lowering one kind of royalty and raising the other? Attempting to answer this question, our analysis offers a simple but novel insight: a per unit royalty has a "location effect" on the marginal cost while an ad valorem royalty has a "scale effect" (see (7) for a precise expression of these two effects). As a result, the same *effective magnitude* of the innovation (denoted by δ) which determines the market outcome (price, quantities) can generally be supported by multiple combinations of per unit and ad valorem royalties.

Offering a licensing policy with any δ, the innovator has to consider two constraints: (i) an acceptability constraint that ensures the net profit from having a license is no less than the profit without a license and (ii) a feasibility constraint to ensure that the royalties are within their permissible bounds. We show that any positive δ can be always supported by a two part tariff consisting of a per unit royalty and an upfront fee but no ad valorem royalty. Moreover, relatively small values of δ can be also supported by two part royalties, while relatively large values of δ can be also supported by two part tariffs that are combinations of upfront fees and ad valorem royalties (Proposition 2).

The licensing revenue of the innovator under a three part tariff is completely determined by the effective magnitude,[1] so the same licensing revenue can be attained through multiple combinations of the two kinds of royalties. In the case of a monopoly, we show that among all three part tariffs, it is optimal to set a pure upfront fee, while

[1] It is important to note that this conclusion does not hold when the innovator is one of the competing firms rather than an outsider. In that case, the quantity the innovator chooses affects the market price which in turn affects the licensing revenue from ad valorem royalty. For this reason the market outcomes separately depend on ad valorem and per unit royalties. This is illustrated in the concluding section where the innovator is one of the incumbent firms in a duopoly. The unique optimal three part tariff in that case is a pure ad valorem royalty policy (see Proposition 5).

the unique optimal two part royalty is a pure ad valorem royalty policy (Proposition 1). In particular, a pure ad valorem royalty is superior to a pure unit royalty.

For a general oligopoly, we obtain a fairly complete characterization of optimal three part tariffs in the case of linear demand, where it is shown that: (i) for relatively insignificant innovations, it is optimal to set a pure upfront fee, (ii) otherwise, there is a continuum of optimal policies and there always exists an optimal policy that is a two part tariff consisting of a per unit royalty and an upfront fee, (iii) for relatively significant innovations, there exists an optimal policy that is a two part tariff consisting of an ad valorem royalty and an upfront fee and (iv) for intermediate innovations, provided the demand intercept is relatively large, there exists an optimal policy that is a two part royalty that has both kinds of royalties but no fees (Propositions 3,4).

Our results can shed light on the issue of reasonable royalties. There is generally an expectation of setting "fair, reasonable and nondiscriminatory" (FRAND) royalties. If a licensee alleges the royalties to be too high, the legal authorities often have to make a judgment on whether the asked royalties are reasonable. This can sometimes lead to a downward correction of the percentage of ad valorem royalties (see, e.g., the judgment TCL vs. Ericsson, discussed in Long, 2019).[2] Our analysis shows that whenever there is a continuum of optimal three part tariffs, there always exists an optimal policy that has a per unit royalty and an upfront fee but no ad valorem royalty. This shows that if the patentee has the strategic freedom to combine these different payment schemes, an imposed bound on ad valorem royalty does not affect the market outcomes.

Like the licensing agreement between Apple and Qualcomm, three part tariffs can be also applicable to franchising contracts where a franchisor allows a franchisee to use its trademark and duplicate its business in different locations (see, e.g., Section II of Rey, 1991; Chapter 3 of Blair and Lafontaine, 2005). Typically the franchisees pay an upfront fee to the franchisor at the time of granting a license, together with a fraction of sales revenues. In addition it can be the case that the franchisees are obliged to buy certain branded inputs produced by the franchisor on the basis of per unit prices. Such contracts are in the spirit of three part tariffs.

The paper is organized as follows. We present the related literature in Section 2. The model is presented in Section 3. Acceptability and feasibility constraints are determined in Section 4. Properties of optimal licensing policies are discussed in Section 5. Specific results with linear demand are obtained in Section 6. Most proofs are presented in the Appendix.

2 Related literature

The literature on patent licensing can be traced back to Arrow (1962) who concluded that a perfectly competitive industry provides a higher incentive to innovate than a monopoly. Licensing in oligopolies was studied in Katz and Shapiro (1986), Kamien and Tauman (1986) and Kamien et al. (1992) who considered licensing contracts based

[2]Rate of royalties can be also revised by the contracting parties as part of renegotiation of licensing contracts. See Xiao and Xu (2012) for the implications of such revision in a model of new product development with marketing.

on pure upfront fees and pure unit royalties for innovators who are outsiders to the industry. Although the early literature concluded that upfront fees are superior (e.g., Kamien and Tauman, 1986), later works showed that per unit royalties can be optimal for innovators who are incumbent firms in the industry (Shapiro, 1985; Wang, 1998). Royalties can be also optimal due to integer constraints on the number of licenses (Sen, 2005) or under differnt kinds of informational asymmetry such as regarding the value of the innovation (Gallini and Wright, 1990) or product quality (Hong et al., 2021). Two part tariffs that are combinations of upfront fees and per unit royalties are studied in Sen and Tauman (2007), but they do not consider ad valorem royalties.

Llobet and Padilla (2016) compare ad valorem and per unit royalties for an outside innovator who interacts with a monopolist in a downstream market. In the same setting, we show that among more general policies that combine both of these royalties, it is best to set only an ad valorem and no per unit royalty (Proposition 1). For an incumbent innovator in a Cournot duopoly, San Martín and Saracho (2010) find ad valorem royalties to be superior to per unit royalties. This conclusion is strengthened in Proposition 5, where we show that a pure ad valorem royalty is optimal among all three part tariffs.

In a model of asymmetric information about demand condition and two-sided moral hazard on costly investments by both the principal (the innovator) and the agent (a monopolist firm), Hagiu and Wright (2019) show that under two part tariffs with upfront fees and royalties, ad valorem royalties are superior to per unit royalties. For a new product innovation with demand uncertainty, Ma and Tauman (2020) show that a combination of upfront fee and ad valorem royalty is optimal when potential licensees are risk averse. For a differentiated Cournot duopoly with two part tariffs consisting of upfront fees and only one kind of royalties, whether ad valorem or per unit royalty is superior depends on the extent of product differentiation (San Martín and Saracho, 2015). With similar two part tariffs, a pure ad valorem royalty can be optimal in the presence of asymmetric information about the value of the patent (Heywood et al., 2014). The superiority of one kind of royalty over the other can also depend on factors such as relative efficiency of the innovator over the licensee (Fan et al., 2018) or scale economies of the production technology (Colombo and Filippini, 2015).

As in Savva and Taneri (2015), other works have also looked at ad valorem profit royalties, where a licensee shares a fraction of its profit with the innovator. In a model of asymmetric information on the quality of the innovation with ad valorem profit royalties, Jeon (2019) shows that innovators of superior innovations have incentive to voluntarily disclose information. Recently Colombo et al. (2021) find that for an outside innovator, ad valorem profit royalties are generally equivalent to upfront fees for an outside innovator.

Other related works include Hernandez-Murillo and Llobet (2006), who consider an outside innovator of a process innovation in the Dixit-Stiglitz model of monopolistic competition (Dixit and Stiglitz, 1977) with a continuum of firms. As in Bousquet et al. (1998), the reservation price of a licensee does not depend on the proportion of other licensees, so there is no strategic interaction. Under a regime of probabilistic patents, Jeon and Nishihara (2018) study licensing between an outside patent holder and a single downstream firm where the two parties bargain over the rate of ad valorem profit royalties prior to investment for the innovation. The key distinction of our paper in relation to these works is the presence of strategic interaction among potential licensees in the downstream market.

3 The model

Consider a Cournot oligopoly with $n \geq 2$ firms where the set of competing firms is $N = \{1, \ldots, n\}$. Initially any firm $j \in N$ produces under constant marginal cost $c > 0$. An outside innovator I has a patent for a new process innovation that reduces the per unit cost from c to $c - \varepsilon$ $(0 < \varepsilon < c)$, so ε is the magnitude of the innovation. For $j \in N$, let q_j be the quantity produced by firm j and $Q = \sum_{j \in N} q_j$. We maintain the following assumptions.

A1 The price function or the inverse demand function $p(Q)$: $\mathrm{R}_{++} \to \mathrm{R}_{+}$ is non-increasing and $\exists\, \overline{Q} > 0$ such that $p(Q)$ is decreasing and twice continuously differentiable for $Q \in (0, \overline{Q})$.

A2 $\overline{p} \equiv \lim_{Q \uparrow 0} p(Q) > c$ and $\exists\, 0 < Q^c < Q^{c-\varepsilon} < \overline{Q}$ such that $p(Q^c) = c > p(Q^{c-\varepsilon}) = c - \varepsilon > p(\overline{Q})$.

A3 $p(Q)$ is log-concave for $Q \in (0, \overline{Q})$.

Assumptions A1-A3 imply A4.

A4 For $p \in (0, \overline{p})$, the price elasticity $\eta(p) := -pQ'(p)/Q(p)$ is non-decreasing.

We also assume A5, which ensures a certain comparative-statics result.

A5 The revenue function $\gamma(Q) := p(Q)Q$ is strictly concave for $Q \in (0, \overline{Q})$.

The existence and uniqueness of Cournot equilibrium is ensured by Assumptions A1-A3 (Badia et al., 2014), or alternatively by A1-A2, A4-A5 (Kamien et al., 1992).[3]

In addition to linear demand, some examples of demand functions covered by our analysis include the constant elasticity inverse demand function $p(Q) = s/Q^t$ (where $s > 0$ and $0 < t < 1$) and $p(Q) = \max\{(a - Q)^t, 0\}$ (where $a, t > 0$ and $c < a^t$), both of which satisfy A1-A5.

Drastic and nondrastic innovations The notion of *drastic* innovations (Arrow, 1962) is useful for the analysis of patent licensing. A cost-reducing innovation is drastic if the monopoly price under the new technology does not exceed the old marginal cost c; otherwise it is nondrastic. If only one firm in an oligopoly has a drastic innovation, it becomes a monopolist with the new technology and drives all other firms out of the market. To classify drastic and nondrastic innovations, define

$$\theta \equiv c/\eta(c) = -Q(c)/Q'(c) \tag{1}$$

[3] The assumption A5 is needed to show that the equilibrium profit of a licensee is increasing in the effective magnitude of the innovation. For other sufficient conditions on the existence of Cournot equilibrium, see, e.g., Novshek (1985), Gaudet and Salant (1991).

For $k = 1, \ldots, n$, define the function $H^k : (0, \overline{p}) \to R$ as

$$H^k(p) := p[1 - 1/k\eta(p)] \tag{2}$$

Note that $H^k(p)$ is the marginal revenue of a firm in a k-firm oligopoly when each firm produces $Q(p)/k$. The following property is immediate from Assumption A4 that $\eta(p)$ is non-decreasing.

Observation 1 *Let $p, \widetilde{p} \in (0, \overline{p})$ and suppose $H^k(\widetilde{p}) > 0$. Then $H^k(p) > H^k(\widetilde{p})$ for $p > \widetilde{p}$ and $H^k(p) < H^k(\widetilde{p})$ for $p < \widetilde{p}$.*

To characterize drastic innovations, consider a monopolist who has unit cost $c - \varepsilon$. The profit of this monopolist at price p is

$$G(p) := (p - c + \varepsilon)Q(p) \tag{3}$$

Under assumptions A1-A5, there exists a unique $p_M(\varepsilon)$ (the monopoly price) that maximizes $G(p)$ and it satisfies $H^1(p_M(\varepsilon)) = c - \varepsilon$, where H^k is given in (2). The monopoly profit under cost $c - \varepsilon$ is denoted by $\phi_M(\varepsilon)$, that is, $\phi_M(\varepsilon) = G(p_M(\varepsilon))$.

An innovation of magnitude ε is drastic if and only if $p_M(\varepsilon) \leq c$. Since $c - \varepsilon > 0$, by Observation 1, $p_M(\varepsilon) \leq c \Leftrightarrow H^1(p_M(\varepsilon)) \leq H^1(c) \Leftrightarrow \varepsilon \geq c/\eta(c) \equiv \theta$. Thus, an innovation of magnitude ε is drastic if $\varepsilon \geq \theta$ and nondrastic if $\varepsilon < \theta$. If I has a drastic innovation, it can earn the monopoly profit by selling a sole license to any one firm using an upfront fee. The sole licensee earns the monopoly profit and I can collect this profit through the fee. For the rest of the paper, we consider nondrastic innovations.

3.1 Licensing policies

The innovator I chooses a licensing policy to license its innovation to some or all firms in N. A firm that becomes a licensee uses the innovation and pays I according to the chosen policy. We consider general policies of the form (k, f, r, v) where $f, r \geq 0$, $k = 1, \ldots, n$, and $0 \leq v \leq 1$. Under this policy I offers k licenses using a *three part tariff* that consists of: (i) an upfront fee f, (ii) a unit royalty r and (iii) an ad valorem royalty v. Any licensee pays the fee f upfront, pays r for every unit it produces and pays fraction v of its revenue to I. Note that several standard licensing policies can be obtained from these general policies by taking one or more of their components to be zero (for instance, taking $v = 0$ gives a licensing policy consisting of an upfront fee and a unit royalty).

Apart from the general three part tariffs, it is of interest to separately consider *two part royalty* policies that have two kinds of royalties but no upfront fee. Specifically, we consider the following policies:

(i) *Two part royalty policy*: Under a two part royalty policy $(k, r, v)_{RV}$, I offers k licenses (and commits to sell no more than k) at a unit royalty $r \geq 0$ and an ad valorem royalty $v \in [0, 1]$. Under this policy any licensee has to pay (i) r for every unit it produces and (ii) fraction v of its revenue to I. If k or less firms are willing to purchase a license, all of them are granted licenses. If more than k firms are willing purchase a license, k of them are chosen at random to be licensees. Note that a two part royalty policy $(k,r,v)_{RV}$ with $v=0$ gives a *pure unit royalty* policy, while $r=0$ gives a *pure ad valorem royalty* policy.

(ii) *General three part tariff policy*: Under a general three part tariff $(k,r,v)_{FRV}$, I offers k licenses (committing to sell no more than k) at a unit royalty $r \geq 0$, an ad valorem royalty $v \in [0,1]$ and in addition charges a non-negative upfront fee from each licensee. Under this policy any licensee has to pay (i) an upfront fee, (ii) r for every unit it produces and (iii) fraction v of its revenue to I.

The upfront fee is collected through an auction (possibly with a minimum bid).[4] Firms are asked to simultaneously place non-negative bids. If $m \leq k$ firms place bids, each of the bidding firms wins a license. If $m > k$ firms place bids, bids are arranged in ascending order as $f_1 \geq \ldots \geq f_k \geq \ldots \geq f_m$. If $f_k > f_{k+1}$, firms with k highest bids win licenses. If $f_k = f_{k+1}$, then (a) firms with bids strictly higher than f_k win licenses and (b) a random tie breaking process is run among the firms who place bid f_k to determine who get the remaining licenses.[5] Any firm that wins a license, pays its bid as upfront fee to I.

Remark 1 If I offers n licenses (that is, $k=n$), each firm is guaranteed a license, so no one will place a positive bid. To ensure positive upfront fees, I announces to auction n licenses together with a minimum bid $\hat{f}$. It can be shown that under this modified process, for suitably chosen $\hat{f}$, there is an equilibrium in which each firm will have a license with the minimum bid, so each will pay upfront fee $\hat{f}$ (See Lemma 2(II)).

The Licensing game Γ The strategic interaction between I and the firms in N is modeled as the *licensing game* Γ that has the following stages.

Stage 1: I offers to license the innovation by announcing a licensing policy.

Stage 2: Firms simultaneously decide whether to purchase a license or not. Any firm willing to purchase a license: (i) notifies I in the case of a two-part royalty policy and (ii) places its bid in the case of a three-part tariff. Following the rules described before, the set of licensees is determined and this set is commonly known.

Stage 3: Firms in N compete in quantities. Any licensee firm produces under the reduced marginal cost and pays I according to the licensing policy. Any non-licensee firm produces under the initial marginal cost.

Payoffs of firms Consider a licensing policy that has unit royalty r and ad valorem royalty v. Let $L \subseteq N$ be the set of licensees, so $\bar{L} = N \setminus L$ is the set of non-licensees. Any licensee $j \in L$ has marginal cost $c - \varepsilon$ and its payment for the license has the following components: (i) it pays r to I for every unit, (ii) it pays fraction v of its revenue to I, so it keeps the remaining fraction $1-v$ and (iii) it pays fee f_j (possibly

[4]The innovator can alternatively set the fee as a posted price. However, as recognized in earlier works (see, e.g., Kamien and Tauman, 1986; Katz and Shapiro, 1986), compared to a posted price, an auction generates more competition among firms that raises the willingness to pay for a license.

[5]The only constraint we impose on the random tie breaking process is that every firm with bid f_k has a positive probability of winning a license. Similarly, for a two part royalty policy, if $m > k$ firms are willing to purchase a license, the only constraint on the random process is that each of these m firms has a positive probability of winning a license.

zero) to I. A non-licensee has marginal cost c and makes no payment. So the payoffs of firms are

$$\pi_j = \begin{cases} (1-v)p(Q)q_j - (c-\varepsilon)q_j - rq_j - f_j & \text{if } j \in L \\ [p(Q)-c]q_j & \text{if } j \in \bar{L} \end{cases} \tag{4}$$

The payoff of I has three components: (i) revenue from unit royalty $\sum_{j\in L} rq_j$, (ii) revenue from ad valorem royalty $\sum_{j\in L} vp(Q)q_j$ and (iii) revenue $\sum_{j\in L} f_j$ received from fees. This payoff is

$$\pi_I = \sum_{j\in L} rq_j + \sum_{j\in L} vp(Q)q_j + \sum_{j\in L} f_j \tag{5}$$

We confine to Subgame Perfect Nash Equilibrium (SPNE) outcomes of Γ.

Remark 2 Observe from (4) that if $v = 1$, a licensee obtains negative or zero payoff. Clearly a negative payoff will not be acceptable to a firm. With $v = 1$, the payoff of a licensee can be zero only if both $q_j = 0$ and $f_j = 0$, but in that case by (5), I has zero licensing revenue, so such a licensing policy is redundant. Henceforth we rule out unacceptable or redundant policies, so we only consider policies with $v < 1$.

Effective magnitude of the innovation To determine SPNE of Γ, consider stage 3 (Cournot stage) of this game where firms in N choose quantities. Since fee f_j is paid upfront, it does not affect the choice of quantities. So by (4), in stage 3 any licensee firm $j \in L$ has payoff

$$\hat{\pi}_j = (1-v)p(Q)q_j - (c-\varepsilon)q_j - rq_j = (1-v)[p(Q) - (c-\varepsilon+r)/(1-v)]q_j \tag{6}$$

For $r \geq 0$ and $0 \leq v < 1$, denote

$$\delta(r,v) := [\varepsilon - (r+cv)]/(1-v) \tag{7}$$

Then

$$\hat{\pi}_j = (1-v)[p(Q) - (c-\delta(r,v))]q_j \tag{8}$$

Since $1-v > 0$, by (8) at the Cournot stage any licensee solves the same problem as a firm that has marginal cost $c - \delta(r,v)$. This means for a licensing policy that has unit royalty r and ad valorem royalty v, the *effective magnitude* of the innovation is $\delta(r,v)$. Note from (7) that $\delta(r,v)$ is decreasing in each of r, v, so $\delta(r,v) \leq \delta(0,0) = \varepsilon$.

Any firm has marginal cost c without a license. Assuming no firm will accept a policy that raises its marginal cost, we only consider policies for which the effective magnitude of the innovation is non-negative, so we restrict r, v such that $\delta(r,v) \geq 0$. Thus $0 \leq \delta(r,v) \leq \varepsilon$.

Note that $\delta(r,v) \geq 0$ if and only if $r + cv \leq \varepsilon$. The line AB is Figure 1 has equation $r + cv = \varepsilon$, so $\delta(r,v) \geq 0$ if and only if (r,v) is in ΔOAB. In general, for any $\delta \in [0,\varepsilon]$, $\delta(r,v) = \delta$ if and only if $r + (c-\delta)v = \varepsilon - \delta$. This is demonstrated by line CD in Figure 1, which has equation $r + (c-\delta)v = \varepsilon - \delta$ for some $0 < \delta < \varepsilon$.

Remark 3 For $r \geq 0$ and $0 \leq v < 1$, the unique (r,v) that gives $\delta(r,v) = \varepsilon$ is $(r = 0, v = 0)$. However, as demonstrated by lines AB and CD in Figure 1, for any

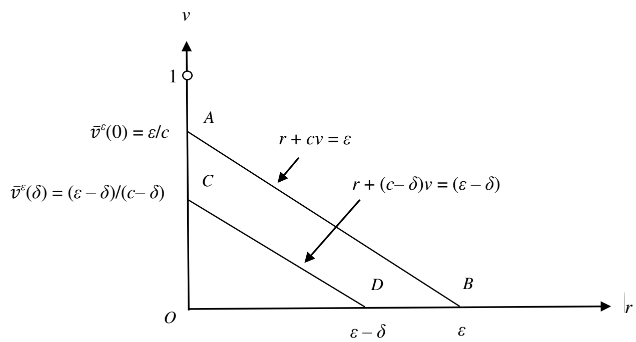


Figure 1: Effective magnitude of innovation $\delta(r, v)$

$\delta \in [0, \varepsilon)$, there is a continuum of (r, v) that can support $\delta(r, v) = \delta$. For any δ, the maximum v that can support δ, denoted by $\bar{v}^{\varepsilon}(\delta)$, is found by taking $r = 0$ in the equation $r + (c - \delta)v = \varepsilon - \delta$, so $\bar{v}^{\varepsilon}(\delta) = (\varepsilon - \delta)/(c - \delta)$, as shown in Figure 1.

Cournot oligopoly subgame $\mathcal{C}^n(k, \delta)$: For $k = 0, 1, \ldots, n$ and $\delta \in [0, \varepsilon]$, denote by $\mathcal{C}^n(k, \delta)$ the Cournot oligopoly game with n firms in which k firms (licensees) have marginal cost $c - \delta$ and the remaining $n - k$ firms (non-licensees) have marginal cost c. To determine (Cournot-Nash) equilibrium of $\mathcal{C}^n(k, \delta)$, the threshold θ/k will be useful (recall $\theta \equiv c/\eta(c)$). It is shown in Lemma 1 that if $\delta < \theta/k$, all firms are active in the market and if $\delta \geq \theta/k$, all non-licensees drop out of the market and a k-firm natural oligopoly is created with the k licensees.

Note that when $\delta = 0$, all firms (licensees as well as non-licenses) have marginal cost c. If there is no licensee (that is, $k = 0$), then again all firms have marginal cost c. Thus in terms of Cournot outcomes, for any k, δ, the game $\mathcal{C}^n(k, 0)$ is equivalent to the game $\mathcal{C}^n(0, \delta)$.

Lemma 1 *For any $k \in \{1, \ldots, n\}$ and $\delta \in [0, \varepsilon]$, the subgame $\mathcal{C}^n(k, \delta)$ has a unique equilibrium. Let $\overline{q}^n(k, \delta), \underline{q}^n(k, \delta)$ be the respective Cournot quantities of a licensee, non-licensee. Let $\overline{\phi}^n(k, \delta)$, $\underline{\phi}^n(k, \delta)$ be the corresponding Cournot profits and $p^n(k, \delta)$ be the Cournot price. The licensees always obtain positive Cournot profit. Specifically:*

(i) *If $\delta < \theta/k$, then $c < p^n(k, \delta) < \overline{p}$ and $p^n(k, \delta)$ is the unique solution of $H^n(p) = c - k\delta/n$ over $p \in (0, \overline{p})$. All firms obtain positive Cournot profit. The Cournot quantity and profit of any non-licensee, as well as the Cournot price, depend only on the product $k\delta$.*

(ii) *If $\delta \geq \theta/k$, then $c - \delta < p^n(k, \delta) \leq c$ [equality iff $\delta = \theta/k$] and $p^n(k, \delta)$ is the unique solution of $H^k(p) = c - \delta$ over $p \in (0, \overline{p})$. A k-firm natural oligopoly is created, k licensees obtain positive Cournot profit and the $n-k$ non-licensees drop out of the market.*

(iii) *For any $k \geq 1$, the Cournot price $p^n(k, \delta)$ is decreasing and the Cournot profit of a licensee $\overline{\phi}^n(k, \delta)$ is increasing in δ. The Cournot profit of a non-licensee $\underline{\phi}^n(k, \delta)$ is decreasing for $\delta \leq \theta/k$ and equals zero for $\delta \geq \theta/k$.*

(iv) *For $\delta = 0$, a licensee obtains the same Cournot profit as a non-licensee and the Cournot price and profits are same for all k.*

(v) *For any $\delta > 0$, the Cournot price and the Cournot profit of a licensee are decreasing in k and the Cournot profit of a non-licensee is non-increasing in k. Specifically:*

   (a) *$p^n(k-1, \delta) > p^n(k, \delta)$ for all k.*

   (b) *$\overline{\phi}^n(1, \delta) > \underline{\phi}^n(0, \delta) > \underline{\phi}^n(1, \delta)$.*

   (c) *For $k = 2, \ldots, n-1$: $\overline{\phi}^n(k-1, \delta) > \overline{\phi}^n(k, \delta) > \underline{\phi}^n(k-1, \delta) \geq \underline{\phi}^n(k, \delta)$, where*

   $\underline{\phi}^n(k-1, \delta) > \underline{\phi}^n(k, \delta)$ if $\delta < \theta/(k-1)$ and
   $\underline{\phi}^n(k-1, \delta) = \underline{\phi}^n(k, \delta) = 0$ if $\delta \geq \theta/(k-1)$.

   (d) *$\overline{\phi}^n(n-1, \delta) > \overline{\phi}^n(n, \delta) > \underline{\phi}^n(n-1, \delta)$.*

Proof Parts (i)-(ii) follow by Assumptions A1-A5 by applying the first order conditions. Part (iv) is immediate from (i). Part (v)(b) follows from (iii) by noting that $\underline{\phi}^n(0, \delta) = \overline{\phi}^n(1, 0) = \underline{\phi}^n(1, 0)$. See the Appendix for the proof of the remaining parts. ■

4 Acceptability and feasibility constraints

When there are k licensees under a licensing policy with unit royalty r and ad valorem royalty v, the Cournot oligopoly game $\mathcal{C}^n(k,\delta)$ is played in stage 3 of Γ where $\delta = \delta(r,v)$ given in (7). By Lemma 1, any non-licensee, having marginal cost c, obtains its Cournot profit $\underline{\phi}^n(k,\delta)$. Any licensee, having marginal cost $c-\delta$ has Cournot profit $\overline{\phi}^n(k,\delta)$. Note that the unit royalty is already included in δ. By (8), taking into account its ad valorem royalty payment, a licensee obtains net profit $(1-v)\overline{\phi}^n(k,\delta)$.

4.1 Acceptability constraint: relative gain in profit from a license

It will be useful to consider two kinds of acceptability constraints for a license that correspond to two different scenarios that may arise. If the number of firms willing to buy licenses does not exceed the number of licenses that I offers, any firm that has a license can lower the number of licensees by one if it chooses to not have a license. In this case when there are k licensees, a firm having a license would obtain $\underline{\phi}^n(k-1,\delta)$ by choosing to not have a license, so a license is acceptable if and only if

$$(1-v)\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(k-1,\delta) \tag{9}$$

On the other hand, if the number of firms willing to buy licenses is more than the number of licenses offered, the number of licensees does not alter when one firm refuses to have a license. In this case, when there are k licensees, a firm having a license would obtain $\underline{\phi}^n(k,\delta)$ without a license, so a license is acceptable if and only if

$$(1-v)\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(k,\delta) \tag{10}$$

Since $\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(m,\delta)$ for $m = k-1, k$ (Lemma 1(v)), both (9),(10) hold when $v = 0$, so the acceptability constraints hold when there is no ad valorem royalty.

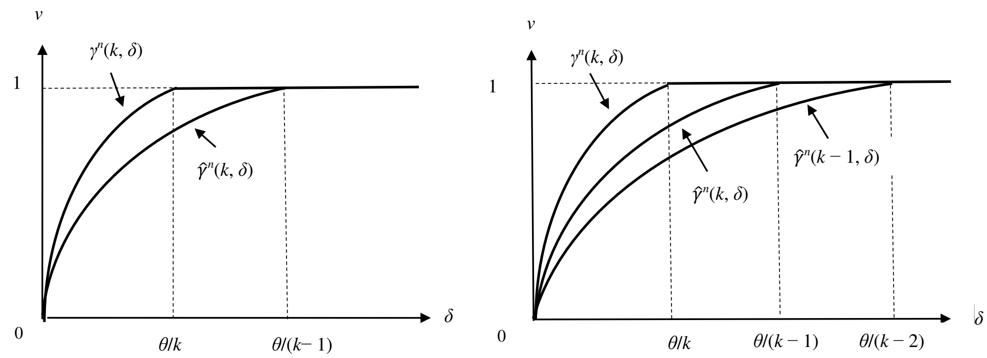
Remark 4 When there are k licensees where $1 \leq k \leq n-1$, if a licensee refuses to have a license, it obtains either $\underline{\phi}^n(k-1,\delta)$ or $\underline{\phi}^n(k,\delta)$. As $\underline{\phi}^n(k-1,\delta) \geq \underline{\phi}^n(k,\delta)$ (Lemma 1), by (9)-(10), the maximum upfront fee that can be obtained from a licensee is

$$f^n(k,v,\delta) = (1-v)\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta) \tag{11}$$

When all n firms are licensees, by refusing to have a license a firm obtains $\underline{\phi}^n(n-1,\delta)$. So the maximum upfront fee that can be obtained in this case is

$$\hat{f}^n(n,v,\delta) = (1-v)\overline{\phi}^n(n,\delta) - \underline{\phi}^n(n-1,\delta) \tag{12}$$

It follows from (12) that in the case of a three part tariff with $k = n$, for the modified auction process described in Remark 1, it is optimal for I to set the minimum bid $\hat{f} = \hat{f}^n(n,v,\delta)$ (by (9), this fee is non-negative if and only if the policy is acceptable).



Acceptable v for (k, δ), where $2 \leq k \leq n-1$

Figure 2(a): $\widehat{\gamma}^n(k,\delta) \leq \gamma^n(k,\delta)$

Figure 2(b): Assumption A6: $\widehat{\gamma}^n(k,\delta) \leq \widehat{\gamma}^n(k-1,\delta)$

Relative gains in profits Denote

$$\widehat{\gamma}^n(k,\delta) := [\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k-1,\delta)]/\overline{\phi}^n(k,\delta) \text{ for } k = 1,\dots,n \text{ and}$$

$$\gamma^n(k,\delta) := [\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta)]/\overline{\phi}^n(k,\delta) \text{ for } k = 1,\dots,n-1 \tag{13}$$

Note that $\widehat{\gamma}^n(k,\delta)$ and $\gamma^n(k,\delta)$ present the *relative gains in profits from a license.* From the acceptabilty constraints (9) and (10),

$$(1-v)\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(k-1,\delta) \Leftrightarrow v \leq \widehat{\gamma}^n(k,\delta) \text{ and}$$

$$(1-v)\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(k,\delta) \Leftrightarrow v \leq \gamma^n(k,\delta) \tag{14}$$

This means a license is acceptable if and only if the ad valorem royalty fraction v does not exceed the relative gain in profit from a license. The next observation summarizes the basic properties of these functions.

Observation 2 (i) $\gamma^n(k,0)=0$ *and* $\widehat{\gamma}^n(k,0)=0$ *for all* k.

(ii) *For* $1 \leq k \leq n-1$: $\gamma^n(k,\delta)$ *is increasing for* $0 \leq \delta < \theta/k$ *and* $\gamma^n(k,\delta)=1$ *for* $\delta \geq \theta/k$.

(iii) $\widehat{\gamma}^n(1,\delta)$ *is increasing for* $\delta \geq 0$ *and for* $2 \leq k \leq n$: $\widehat{\gamma}^n(k,\delta)$ *is increasing for* $0 \leq \delta < \theta/(k-1)$ *and* $\widehat{\gamma}^n(k,\delta)=1$ *for* $\delta \geq \theta/(k-1)$.

(iv) $\gamma^n(1,\delta) > \widehat{\gamma}^n(1,\delta)$ *for all* $\delta > 0$ *and for* $2 \leq k \leq n-1$: $\gamma^n(k,\delta) > \widehat{\gamma}^n(k,\delta)$ *for* $0 < \delta < \theta/(k-1)$ *and* $\gamma^n(k,\delta) = \widehat{\gamma}^n(k,\delta) = 1$ *for* $\delta \geq \theta(k-1)$.

(v) *For* $2 \leq k \leq n$: $\widehat{\gamma}^n(k-1,\delta) \leq 1 = \widehat{\gamma}^n(k,\delta)$ *for* $\delta \geq \theta/(k-1)$.

Proof Since the Cournot profit is increasing in δ for a licensee and non-increasing in δ for a non-licensee (Lemma 1), the ratio $\underline{\phi}^n(m,\delta)/\overline{\phi}^n(k,\delta)$ is non-increasing in δ for $m=k-1,k$, so both $\widehat{\gamma}^n(k,\delta)$ and $\gamma^n(k,\delta)$ are non-decreasing in δ. The properties are then immediate from Lemma 1(i)-(ii). ■

To demonstrate (i)-(iii), Figure 2(a) depicts $\gamma^n(k,\delta)$, $\widehat{\gamma}^n(k,\delta)$ as functions of δ for a fixed k. It can be shown that when the demand is linear, both these functions are concave (as drawn in Figure 2(a)). Since Cournot profits of both a licensee and a non-licensee are decreasing in k (Lemma 1), the effect of change in k on the ratios of profits is ambiguous. Beyond what is stated in Observation 2(iv), in general we cannot conclude more on the monotonicity of $\widehat{\gamma}^n(k,\delta)$ with respect to k. We make the following monotonicity assumption.

A6 For all $k=2,\ldots,n$: $\widehat{\gamma}^n(k-1,\delta) < \widehat{\gamma}^n(k,\delta)$ for $0<\delta<\theta/(k-1)$.

Figure 2(b) depicts $\widehat{\gamma}^n(k,\delta)$ that satisfies Assumption A6. It can be shown that this assumption holds for linear demand.

4.2 Acceptability versus feasibility constraints

To determine optimal licensing policies, together with acceptability constraints we need to also look at *feasibility* constraints of ad valorem and unit royalties. The feasibility constraints are determined by the bounds of r and v: both are non-negative, r is bounded above by ε and v by 1. Note from Remark 3 that for any δ, the maximum ad valorem royalty v that can support δ is

$$\bar{v}^\varepsilon(\delta) = (\varepsilon-\delta)/(c-\delta) \tag{15}$$

As shown in Figure 1, for any $\delta = \delta(r, v)$ (given by (7)), $v = \bar{v}^{\varepsilon}(\delta)$ if and only if $r = 0$. Note that $\bar{v}^{\varepsilon}(\delta)$ is decreasing and concave in δ.

For $1 \leq k \leq n-1$ and $\delta \in [0, \varepsilon]$, the acceptability constraint is $v \leq \gamma^n(k, \delta)$ and the feasibility constraint is $v \leq \bar{v}^{\varepsilon}(\delta)$. Therefore for any k, δ, for ad valorem royalty v to be both acceptable and feasible, we need

$$v \leq \min\{\gamma^n(k, \delta), \bar{v}^{\varepsilon}(\delta)\}$$

Since $\bar{v}^{\varepsilon}(0) = \varepsilon/c > \gamma^n(k, 0) = 0$ and $\bar{v}^{\varepsilon}(\varepsilon) = 0 < \gamma^n(k, \varepsilon)$, there exists a unique $\hat{\delta}^{n,\varepsilon}(k) \in (0, \varepsilon)$ such that $\gamma^n(k, \delta) \leq \bar{v}^{\varepsilon}(\delta)$ if and only if $\delta \leq \hat{\delta}^{n,\varepsilon}(k)$.

In Figures 3(a)-3(b), for any δ the set of all feasible and acceptable v corresponding to δ are those that are (a) on or below OA if $\delta < \hat{\delta}^{n,\varepsilon}(k, \delta)$ and (b) on or below AB if $\delta > \hat{\delta}^{n,\varepsilon}(k, \delta)$. Any (δ, v) on AB has (i) $v = \bar{v}^{\varepsilon}(\delta)$ (the maximum feasible v for that δ), so the per unit royalty r is zero and (ii) $v < \gamma^n(k, \delta)$, so a licensee is left with some positive surplus for a two part royalty, which can be collected through an upfront fee under three part tariff. On the other hand, any (δ, v) on OA has (i) $v < \bar{v}^{\varepsilon}(\delta)$, so the per unit royalty r is positive and (ii) $v = \gamma^n(k, \delta)$, so a licensee is left with no surplus.

Similarly for $k = n$ and $\delta \in [0, \varepsilon]$, the acceptability constraint is $v \leq \widehat{\gamma}^n(n, \delta)$ and the feasibility constraint is $v \leq \bar{v}^{\varepsilon}(\delta)$. In this case, for ad valorem royalty v to be both acceptable and feasible, we need

$$v \leq \min\{\widehat{\gamma}^n(n, \delta), \bar{v}^{\varepsilon}(\delta)\}$$

By similar reasoning as before, there is a unique $\hat{\delta}^{n,\varepsilon}(n) \in (0, \varepsilon)$ such that $\widehat{\gamma}^n(n, \delta) \leq \bar{v}^{\varepsilon}(\delta)$ if and only if $\delta \leq \hat{\delta}^{n,\varepsilon}(n)$. As before the region OAB in Figures 3(c)-(d) present the set of acceptable and feasible v for any δ.

5 Optimal licensing policies

We first look at the properties of optimal licensing policies in the benchmark case of the monopoly and then look at the case of an oligopoly.

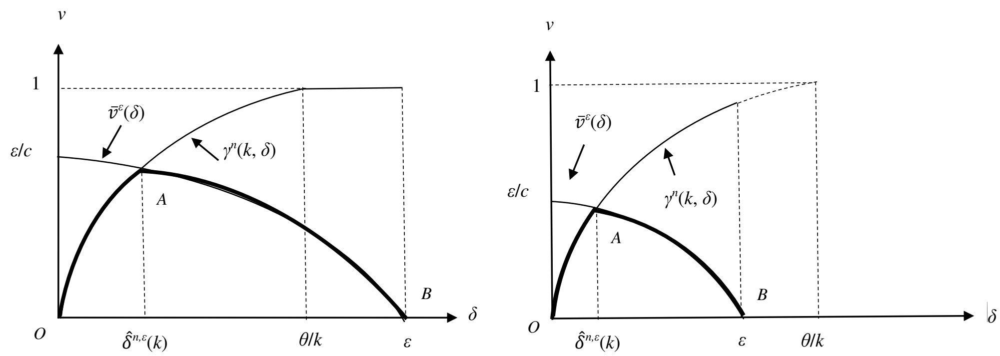
5.1 The monopoly case

The monopoly case corresponds to $n = 1$ in Lemma 1. Denote by $\phi_M(\delta)$ the monopoly profit under marginal cost $c - \delta$. If the monopolist has the innovation under a licensing policy with unit royalty r and ad valorem royalty v, the effective magnitude of the innovation is $\delta = \delta(r, v)$ (given in (7)). So the net profit of the monopolist with a license is $(1-v)\phi_M(\delta)$. The monopolist obtains $\phi_M(0)$ without a license, so its acceptability constraint is $(1-v)\phi_M(\delta) \geq \phi_M(0)$. For the monopolist, the relative gain in profit from a license is

$$\widehat{\gamma}_M(\delta) := [\phi_M(\delta) - \phi_M(0)]/\phi_M(\delta)$$

Noting that

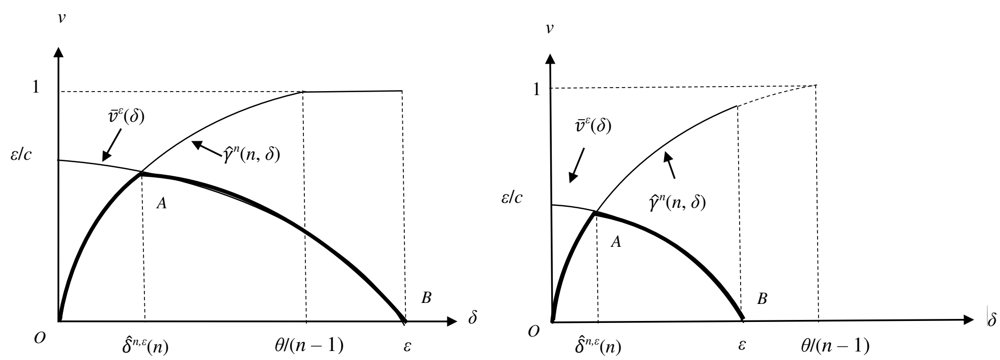
$$(1-v)\phi_M(\delta) \geq \phi_M(0) \Leftrightarrow v \leq \widehat{\gamma}_M(\delta) \tag{16}$$



Acceptable versus feasible v for (k, δ), $1 \leq k \leq n-1$

Figure 3(a): $\varepsilon \geq \theta/k$

Figure 3(b): $\varepsilon < \theta/k$



Acceptable versus feasible v for (n, δ)

Figure 3(c): $\varepsilon \geq \theta/(n-1)$

Figure 3(d): $\varepsilon < \theta/(n-1)$

the acceptability constraint in this case is $v \leq \widehat{\gamma}_M(\delta)$.

For any δ, the maximum v that I can set is $v = \bar{v}^{\varepsilon}(\delta)$ given by (15). Thus for any δ, ad valorem royalty v that I can set for the monopolist has to satisfy

$$v \leq \min\{\widehat{\gamma}_M(\delta), \bar{v}^{\varepsilon}(\delta)\}$$

Note that $\widehat{\gamma}_M(0) = 0$ and $0 < \widehat{\gamma}_M(\delta) < 1$ for $\delta \in (0, \varepsilon]$. Since $\phi_M(\delta)$ is increasing in δ, so is $\widehat{\gamma}_M(\delta)$. As $\bar{v}^{\varepsilon}(\delta)$ is decreasing in δ with $\bar{v}^{\varepsilon}(0) = \varepsilon/c > \widehat{\gamma}_M(0) = 0$ and $\bar{v}^{\varepsilon}(\varepsilon) = 0 < \widehat{\gamma}_M(\varepsilon)$, there is a unique $\hat{\delta}^{\varepsilon}_M \in (0, \varepsilon)$ such that $\widehat{\gamma}_M(\delta) \leq \bar{v}^{\varepsilon}(\delta)$ if and only if $\delta \leq \hat{\delta}_M^{\varepsilon}$. As shown in Figure 4, for any δ, the maximum v that is both feasible and acceptable lies on OA if $\delta \leq \hat{\delta}_M^{\varepsilon}$ and on AB if $\delta > \hat{\delta}_M^{\varepsilon}$.

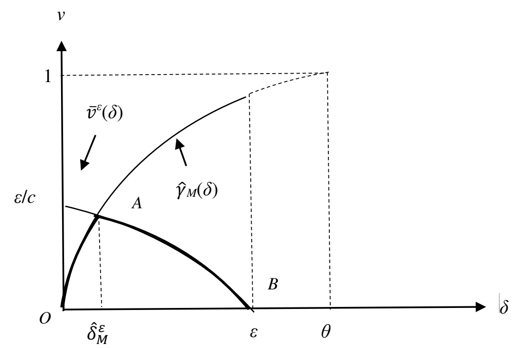


Figure 4: Monopoly: acceptable versus feasible v for δ

Proposition 1 *When an outside innovator interacts with a monopolist, the optimal licensing policies have the following properties.*

(i) *Any optimal two part royalty policy is a pure ad valorem royalty policy, that is, it has no per unit royalty. Consequently for the innovator, licensing by means of a pure ad valorem royalty is superior to pure per unit royalty licensing.*

(ii) *The unique optimal three part tariff policy is a pure upfront fee policy (that is, it has no ad valorem or per unit royalty) with fee* $\phi_M(\varepsilon) - \phi_M(0)$.

Proof See the Appendix. ∎

5.2 The case of oligopoly

Two part royalty: First consider an acceptable two part royalty policy with per unit royalty r and ad valorem royalty v. Let $\delta = \delta(r, v)$ given in (7). If there are k licensees under this policy, the Cournot quantity of any licensee is $\overline{q}^n(k,\delta)$. By (5), the payoff of I is

$$\Pi^n(k, r, v, \delta)_{RV} = kr\overline{q}^n(k,\delta) + kvp^n(k,\delta)\overline{q}^n(k,\delta) \tag{17}$$

Noting that $c - \delta = (c - \varepsilon + r)/(1 - v)$ (by (7)) and the Cournot profit of any licensee is $\overline{\phi}^n(k,\delta) = [p^n(k,\delta) - c + \delta]\overline{q}^n(k,\delta)$, from (17) we have

$$\Pi^n(k, r, v, \delta)_{RV} = \Pi^n(k, v, \delta)_{RV} = [p^n(k,\delta) - c + \varepsilon]k\overline{q}^n(k,\delta) - k(1 - v)\overline{\phi}^n(k,\delta)$$

As the Cournot profit of a non-licensee is $\underline{\phi}^n(k,\delta) = [p^n(k,\delta) - c]\underline{q}^n(k,\delta)$, using the industry quantity $Q^n(k,\delta) = k\overline{q}^n(k,\delta) + (n-k)\underline{q}^n(k,\delta)$ and the function $G(p)$ from (3), we have

$$\Pi^n(k, v, \delta)_{RV} = G(p^n(k,\delta)) - k(1-v)\overline{\phi}^n(k,\delta) - (n-k)\underline{\phi}^n(k,\delta) - \varepsilon(n-k)\underline{q}^n(k,\delta) \tag{18}$$

For any (k,δ), the payoff of (18) is increasing in v. So under two part royalty, for any (k,δ), it is best for I to choose the maximum v that is both feasible and acceptable, which is

$$v^*(k,\delta) := \min\{\gamma^{n,\varepsilon}(k,\delta), \bar{v}^{\varepsilon}(\delta)\} \text{ if } k \le n-1, v^*(n,\delta) := \min\{\widehat{\gamma}^{n,\varepsilon}(n,\delta), \bar{v}^{\varepsilon}(\delta)\} \tag{19}$$

Denote $\Pi^n(k,\delta)_{RV} \equiv \Pi^n(k, v^*(k,\delta), \delta)_{RV}$ (the payoff of I under two part royalty at (k,δ) when maximum possible v is chosen). For $1 \le k \le n-1$, noting that $\gamma^{n,\varepsilon}(k,\delta) \le \bar{v}^{\varepsilon}(\delta)$ iff $\delta \le \hat{\delta}^{n,\varepsilon}(k)$ (see Figures 3(a)-(b)), by (19) we have

$$\Pi^n(k,\delta)_{RV} = \begin{cases} \Pi^n(k, \gamma^{n,\varepsilon}(k,\delta), \delta)_{RV} & \text{if } \delta \le \hat{\delta}^{n,\varepsilon}(k) \\ \Pi^n(k, \bar{v}^{\varepsilon}(\delta), \delta)_{RV} < \Pi^n(k, \gamma^{n,\varepsilon}(k,\delta), \delta)_{RV} & \text{if } \delta > \hat{\delta}^{n,\varepsilon}(k) \end{cases} \tag{20}$$

Similarly for $k = n$, noting that $\widehat{\gamma}^{n,\varepsilon}(n,\delta) \le \bar{v}^{\varepsilon}(\delta)$ iff $\delta \le \hat{\delta}^{n,\varepsilon}(n)$ (see Figures 3(c)-(d)), by (19) we have

$$\Pi^n(n,\delta)_{RV} = \begin{cases} \Pi^n(k, \widehat{\gamma}^{n,\varepsilon}(n,\delta), \delta)_{RV} & \text{if } \delta \le \hat{\delta}^{n,\varepsilon}(n) \\ \Pi^n(n, \bar{v}^{\varepsilon}(\delta), \delta)_{RV} < \Pi^n(n, \widehat{\gamma}^{n,\varepsilon}(n,\delta), \delta)_{RV} & \text{if } \delta > \hat{\delta}^{n,\varepsilon}(n) \end{cases} \tag{21}$$

To determine these payoffs, note from (13) that if $v = \gamma^{n,\varepsilon}(k,\delta)$, then $(1-v)\overline{\phi}^n(k,\delta) = \underline{\phi}^n(k,\delta)$ and if $v = \widehat{\gamma}^{n,\varepsilon}(n,\delta)$, then $(1-v)\overline{\phi}^n(n,\delta) = \underline{\phi}^n(n-1,\delta)$. Using these in (18), we have

$$\Pi^n(k, \gamma^{n,\varepsilon}(k,\delta), \delta)_{RV} = G(p^n(k,\delta)) - n\underline{\phi}^n(k,\delta) - \varepsilon(n-k)\underline{q}^n(k,\delta) \text{ for } 1 \le k \le n-1,$$

$$\Pi^n(n, \widehat{\gamma}^{n,\varepsilon}(n, \delta), \delta)_{RV} = G(p^n(k, \delta)) - n\underline{\phi}^n(n-1, \delta) \tag{22}$$

Note that if $v = \bar{v}^\varepsilon(\delta)$, then $r = 0$. Using this in (17) we have

$$\Pi^n(k, \bar{v}^\varepsilon(\delta), \delta)_{RV} = \bar{v}^\varepsilon(\delta) p^n(k, \delta)[Q^n(k, \delta) - (n-k)\underline{q}^n(k, \delta)] \tag{23}$$

The payoffs under two part royalty are given by (20)-(23).

Three part tariffs: Consider an acceptable three part tariff with per unit royalty r and ad valorem royalty v, where as before $\delta = \delta(r, v)$ given in (7). If there are k licensees under this policy with $1 \leq k \leq n-1$, we know from (11) that the maximum upfront fee that I can set is $f^n(k, v, \delta) = (1-v)\phi^n(k, \delta) - \underline{\phi}^n(k, \delta)$. If I is able to collect this fee from each licensee, its payoff at the three part tariff will be simply the payoff at the corresponding two part royalty in (18) together with the sum of fees, which is $\Pi^n(k, v, \delta)_{RV} + kf^n(k, v, \delta)$. The only term in (18) that directly depends on v involves the net profit of a licensee $(1-v)\overline{\phi}^n(k, \delta)$. Since the fee $f^n(k, v, \delta)$ collects this net profit by leaving a licensee with its opportunity cost $\underline{\phi}^n(k, \delta)$ (the Cournot profit of a non-licensee), the payoff under three part tariff depends only on δ (and not separately on r, v), so we can denote it by $\Pi^n(k, \delta)_{FRV}$. By (18), we have

$$\begin{aligned}\Pi^n(k, \delta)_{FRV} &= \Pi^n(k, v, \delta)_{RV} + kf^n(k, v, \delta) \\ &= G(p^n(k, \delta)) - n\underline{\phi}^n(k, \delta) - \varepsilon(n-k)\underline{q}^n(k, \delta)\end{aligned} \tag{24}$$

If all n firms are licensees under a three part tariff, we know from (12) that the maximum upfront fee that I can set is $\hat{f}^n(n, v, \delta) = (1-v)\phi^n(n, \delta) - \underline{\phi}^n(n-1, \delta)$. If I is able to collect this fee from each licensee, as before its payoff at the three part tariff will be the payoff at the corresponding two part royalty in (18), together with the sum of fees, which is $\Pi^n(n, v, \delta)_{RV} + n\hat{f}^n(n, v, \delta)$. This payoff also depends only on δ, so we can denote it by $\Pi^n(n, \delta)_{FRV}$. Using (18) with $k = n$, we have

$$\Pi^n(n, \delta)_{FRV} = \Pi^n(n, v, \delta)_{RV} + n\hat{f}^n(n, v, \delta) = G(p^n(n, \delta)) - n\underline{\phi}^n(n-1, \delta) \tag{25}$$

Remark 5 Note from (20)-(22) and (24)-(25) that if $\delta \leq \hat{\delta}^{n,\varepsilon}(k)$, two part royalty policy gives the same payoff as a three part tariff. This is because in this case I can set the maximum acceptable ad valorem royalty ($v = \gamma^{n,\varepsilon}(k)$ for $k \leq n-1$ and $v = \widehat{\gamma}^{n,\varepsilon}(n)$ for $k = n$) to collect the entire surplus of each licensee, leaving each licensee with its opportunity cost (the Cournot profit of a non-licensee). On the other hand, if $\delta > \hat{\delta}^{n,\varepsilon}(k)$, two part royalty policy gives a lower payoff than a three part tariff. This is because in this case I cannot set the maximum acceptable ad valorem royalty, as it exceeds the maximum feasible level $\bar{v}^\varepsilon(\delta)$ (see Figures 3(a)-(d)). In this case under two part royalty, any licensee is left with a positive surplus ($(1-v)\overline{\phi}^n(k, \delta) - \underline{\phi}^n(k, \delta)$ for $k \leq n-1$ and $(1-v)\overline{\phi}^n(n, \delta) - \underline{\phi}^n(n-1, \delta)$ for $k = n$). Under three part tariffs, this surplus can be collected from each licensee using an upfront fee.

Proposition 2 (Relation between three part and two part tariffs)

(I) *The only acceptable three part tariff that supports $\delta = 0$ is the pure per unit royalty $r = \varepsilon$ and in this case it is best for I to offer licenses to all firms.*

(II) *The only combination of* r, v *that supports* $\delta = \varepsilon$ *is* $(r = 0, v = 0)$. *Consequently the only three part tariff that supports* $\delta = \varepsilon$ *is a pure upfront fee policy.*

(III) *For every* $\delta \in (0, \varepsilon)$, *there is a continuum of acceptable and feasible* v, *so there is a continuum of combinations of* r, v *that can support such* δ.

(IV) *For every* $\delta \in (0, \varepsilon)$, *there always exists a two part tariff consisting of a positive per unit royalty and upfront fee but no ad valorem royalty that can support such* δ. *Consequently under three part tariffs, it is either* (a) *optimal to set a pure upfront fee* $(\delta = \varepsilon)$ *or* (b) *optimal to set a pure per unit royalty* $(\delta = 0)$ *or* (c) *there exists an optimal policy that is a two part tariff consisting of a positive per unit royalty and upfront fee but no ad valorem royalty.*

(V) *A two part tariff consisting of a positive ad valorem royalty and upfront fee but no per unit royalty does not necessarily support every* $\delta \in (0, \varepsilon)$. *Specifically, when* I *offers* k *licenses*:

(i) *If* $\delta < \hat{\delta}^{n,\varepsilon}(k, \delta)$, *there does not exist any two part tariff consisting of a positive ad valorem royalty and upfront fee but no per unit royalty that can support such* δ. *Any three part tariff that supports such* δ *must include a positive per unit royalty.*

(ii) *If* $\delta > \hat{\delta}^{n,\varepsilon}(k, \delta)$, *there exists a two part tariff consisting of a positive ad valorem royalty* $(v = \bar{v}^{\varepsilon}(\delta))$ *and upfront fee, but no per unit royalty that can support such* δ.

(VI) *A two part royalty policy consisting of positive ad valorem and per unit royalties but no upfront fee does not necessarily support every* $\delta \in (0, \varepsilon)$. *Specifically, when* I *offers* k *licenses*:

(i) *If* $\delta < \hat{\delta}^{n,\varepsilon}(k, \delta)$, *there exists a two part royalty policy consisting of a positive ad valorem royalty* ($v = \gamma^n(k, \delta)$ *for* $1 \leq k \leq n - 1$ *and* $v = \widehat{\gamma}^n(n, \delta)$ *for* $k = n$) *and per unit royalty that can support such* δ.

(ii) *If* $\delta > \hat{\delta}^{n,\varepsilon}(k, \delta)$, *there does not exist any two part royalty policy that can support such* δ. *Any three part tariff that supports such* δ *must include a positive upfront fee.*

Proof (I) By Figures 3(a)-(d), the only feasible and acceptable v for $\delta = 0$ is $v = 0$ (no ad valorem royalty), so by (7), $r = \varepsilon$. Thus $\delta = 0$ corresponds to the pure unit royalty policy $r = \varepsilon$. Since a licensee obtains the same Cournot profit as a non-licensee, no licensee will pay any positive upfront fee to have a license. So with $\delta = 0$, any acceptable three part tariff policy is the pure unit royalty policy $r = \varepsilon$. Since Cournot quantities are same with or without a license, for $\delta = 0$, it is best for I to offer licenses to all firms (i.e., choose $k = n$) that gives licensing revenue $\varepsilon Q^n(n, 0)$.

(II) For $\delta = \varepsilon$, it is immediate from Figures 3(a)-(d) and Figure 1 that the only combination of r, v that supports $\delta = \varepsilon$ is $(r = 0, v = 0)$. Since for $\delta = \varepsilon$ a licensee obtains a higher Cournot profit than a licensee, I can use an upfront fee to collect the surplus of a licensee. Thus under three part tariff, $\delta = \varepsilon$ corresponds to a pure upfront fee policy.

(III) It is immediate from Figures 3(a)-(d) that any $\delta \in (0, \varepsilon)$ can be supported by a continuum of combinations of r, v.

(IV) Note from figures 3(a)-(d) that $v = 0$ (the line OB) is always acceptable and feasible for any δ. For any δ, taking $v = 0$ in (7) gives $r = \varepsilon - \delta$. This means for any $\delta \in (0, \varepsilon)$, there always exists a two part tariff consisting of a positive per unit royalty and upfront fee but no ad valorem royalty that can support that δ, which proves the first statement. The second statement follows by using the conclusions of (I)-(II).

(V) Follows by noting that a three part tariff has zero per unit royalty if and only if $v = \bar{v}^{\varepsilon}(\delta)$ and by Figures 3(a)-(d), this can happen if and only if $v \geq \hat{\delta}^{n,\varepsilon}(k)$.

(VI) Follows by noting that a three part tariff has zero upfront fee if and only if $v = \gamma^{n,\varepsilon}(k, \delta)$ for $1 \leq k \leq n-1$ and $v = \widehat{\gamma}^{n,\varepsilon}(n, \delta)$ for $k = n$ and by Figures 3(a)-(d), this can happen if and only if $v \leq \hat{\delta}^{n,\varepsilon}(k)$. ■

Using (24)-(25), we approach the licensing policies in terms of δ, determine optimal δ and then find r, v that can support an optimal δ. Applying Proposition 2 (in particular, parts (III)-(IV)), we can use the results on optimal two part tariffs (combinations of per unit royalties and upfront fees) from the existing literature to determine optimal δ and then find all combinations of r, v that can support that optimal δ as a three part tariff.

5.2.1 Properties of optimal policies

To determine optimal policies, we recall from Lemma 1 that if $\delta \geq \theta/k$ for $1 \leq k \leq n-1$ or $\delta \geq \theta/(n-1)$ for $k = n$, any non-licensee drops out of the market, so the opportunity cost of a license is zero. We first observe that at any optimal policy, δ is bounded above by θ/k for $k < n$ and $\theta/(n-1)$ for $k = n$.

Observation 3 *For both two part royalties and three part tariffs*: (i) *if* $1 \leq k \leq n-1$, *it is not optimal for* I *to offer a policy with* $\delta > \theta/k$ and (ii) *if* $k = n$, *it is not optimal for* I *to offer a policy with* $\delta > \theta/(n-1)$.

Proof By Lemma 1(ii), when there are k licensees and $\delta \geq \theta/k$, Cournot quantity and profit of any non-licensee is zero. Using this in (24)-(25), the payoff of I under three part tariff is $G(p^n(k, \delta))$ if $\delta > \theta/k$ for $1 \leq k \leq n-1$ or $\delta > \theta/(n-1)$ for $k = n$. Since $c < p_M(\varepsilon)$ (the innovation is non-drastic), by (3), $G(p)$ is increasing for $p \leq c$. As $p^n(k, \delta) < c$ (by Lemma 1(ii)), we have $G(p^n(k, \delta)) < G(c) = G(p^n(k, \theta/k))$, showing that the policy with $\delta = \theta/k$ gives a higher payoff.

For two part royalties, note from Figures 3(a),3(c) that if $\delta > \theta/k$ for $1 \leq k \leq n-1$ or $\delta > \theta/(n-1)$ for $k = n$, the maximum possible v is $v = \bar{v}^{\varepsilon}(\delta)$. Then by (18), the payoff under two part royalty is $G(p^n(k, \delta)) - k(1 - \bar{v}^{\varepsilon}(\delta))\overline{\phi}^n(k, \delta)$. Noting that $(1 - \bar{v}^{\varepsilon}(\delta))$ and $\overline{\phi}^n(k, \delta)$ are both positive and increasing in δ, the result follows by the same reasoning as before by applying properties of $G(p)$. ■

In view of Observation 3, we consider $\delta \leq \theta/k$ for $1 \leq k \leq n-1$ and $\delta \leq \theta/(n-1)$ for $k = n$. The next lemma shows that I can sell k licenses (and collect the maximum upfront fee from each licensee in the case of three part tariff) as an SPNE outcome if and only if it offeres a policy that satisfies the corresponding acceptability constraint: $v \leq \gamma^n(k, \delta)$ for $1 \leq k \leq n-1$ and $v \leq \widehat{\gamma}^n(n, \delta)$ for $k = n$.

Lemma 2 *For the game Γ, the following hold for any licensing policy (which can be either a two part royalty or a three part tariff) with unit royalty r and ad valorem royalty v where $\delta = \delta(r, v)$ given by (7).*

(I) *Suppose I offers k licenses where $1 \leq k \leq n-1$ and $0 < \delta \leq \theta/k$.*

- (i) *If $v > \gamma^n(k, \delta)$, there is no SPNE in which there are k licensees.*
- (ii) *If $v \leq \gamma^n(k, \delta)$, there exists an SPNE with k licensees.*
  - (a) *For two part royalty policy, there exists an SPNE in which all n firms intend to buy licenses and k firms are chosen at random to be licensees.*
  - (b) *For three part tariff, there exists an SPNE in which at least $k+1$ firms place bids for licenses; the highest bid is $f^n(k, v, \delta) = (1-v)\phi^n(k, \delta) - \underline{\phi}^n(k, \delta)$, which is placed by at least $k+1$ firms and k firms are chosen at random to be licensees.*

(II) *Suppose I offers n licenses with $0 < \delta \leq \theta/(n-1)$.*

- (i) *If $v > \widehat{\gamma}^n(n, \delta)$, there is no SPNE in which there are n licensees.*
- (ii) *If $v \leq \widehat{\gamma}^n(n, \delta)$, for two part royalty policy, there exists an SPNE in which there are n licensees.*
- (iii) *If n licenses are offered with a three part tariff where $v \leq \widehat{\gamma}^n(n, \delta)$ and minimum bid $\hat{f}^n(n, v, \delta) = (1-v)\phi^n(n, \delta) - \underline{\phi}^n(n-1, \delta)$, then there exists an SPNE with n licensees.*

Proof See the Appendix. ■

Remark 6 Lemma 2 shows that when k licenses are offered, an acceptable policy always gives an equilibrium with k licensees. However, it should be noted that unless the ad valorem royalty is sufficiently smaller than the acceptable threshold (specifically, $v < \widehat{\gamma}^n(1, \delta)$, which is lower than both $\gamma^n(k, \delta), \widehat{\gamma}^n(n, \delta)$), there is also another equilibrium in which no firm buys a license. Henceforth we simply assume that when I offers a certain number of licenses using an acceptable policy, the equilibrium with the desired number of licensees is played.

Recall from Proposition 2 that for any $\delta \in (0, \varepsilon)$, there always exists a two part tariff consisting only of per unit royalty and upfront fee, but no ad valorem royalty (that is, $v = 0$) that can support such δ. When $v = 0$, no one buying a license is not an equilibrium. This shows using per unit royalty instead of ad valorem royalty can be useful to resolve the issue of no one buying a license.

Observation 4 *For both two part royalty and three part tariffs, there always exists an optimal policy where I offers $n-1$ or n licenses.*

Proof When $\delta = 0$, it is best to offer n licenses, so let $\delta > 0$. By Observation 3, for $1 \leq k \leq n-1$, it is not optimal for I to offer a policy with $\delta > \theta/k$. For any $1 \leq k \leq n-2$ and $0 < \delta \leq \theta/k$, let $\tilde{\delta} = k\delta/(n-1)$. The result is proved by showing that I obtains a higher payoff at $(n-1, \tilde{\delta})$ compared to (k, δ).

By Lemma 1(i), for any $\delta \leq \theta/k$, the Cournot quantity, profit of a non-licensee and the Cournot price are completely determined by the product $k\delta$. Since $(n-1)\tilde{\delta} = k\delta$, the first two terms of (24) are identical for policies (k, δ) and $(n-1, \tilde{\delta})$, but the last term that is subtracted is lower for the latter, which shows $\Pi^n(n-1, \tilde{\delta})_{FRV} \geq \Pi^n(k, \delta)_{FRV}$ (with equality iff $\delta = \theta/k$).

For two part royalty policies, if $\tilde{\delta} \leq \hat{\delta}^{n,\varepsilon}(n-1)$, then by (20), (22) and (24), $\Pi^n(n-1, \tilde{\delta})_{RV} = \Pi^n(n-1, \tilde{\delta})_{FRV}$ and $\Pi^n(k, \delta)_{RV} \leq \Pi^n(k, \delta)_{FRV}$ and by the same reasoning as in the case of three part tariffs, the payoff is higher at $(n-1, \tilde{\delta})$.

If $\tilde{\delta} > \hat{\delta}^{n,\varepsilon}(n-1)$, then $\gamma^{n,\varepsilon}(n-1, \tilde{\delta}) > \bar{v}^{\varepsilon}(\tilde{\delta})$. Since $\tilde{\delta} < \delta$, we have $\gamma^{n,\varepsilon}(k, \delta) \geq \gamma^{n,\varepsilon}(n-1, \tilde{\delta})$ (with equality iff $\delta = \theta/k$). Since $\bar{v}^{\varepsilon}(\delta)$ is decreasing, we have $\bar{v}^{\varepsilon}(\tilde{\delta}) > \bar{v}^{\varepsilon}(\delta)$. So $\gamma^{n,\varepsilon}(k, \delta) > \bar{v}^{\varepsilon}(\delta)$, implying $\delta > \hat{\delta}^{n,\varepsilon}(k)$. In this case the payoffs at both (k, δ) and $(n-1, \tilde{\delta})$ are determined from (23). Since $\bar{v}^{\varepsilon}(\tilde{\delta}) > \bar{v}^{\varepsilon}(\delta)$, using the same reasoning as three part tariffs, the payoff is higher at $(n-1, \tilde{\delta})$. ■

In view of Observation 4, we make the simplifying assumption that I offers either $n-1$ or n licenses.[6] When the demand curve is linear, optimal licensing policies can be more precisely characterized.

6 Linear demand

Suppose the demand curve is linear, given by

$$p(Q) = \max\{a - Q, 0\} \text{ where } a > c > 0 \tag{26}$$

Note that in this case $\theta = c/\eta(c) = a - c$. So a cost-reducing innovation of magnitude ε is drastic if $\varepsilon \geq a - c$ and nondrastic if $\varepsilon < a - c$.

In a Cournot oligopoly under linear demand (26), Sen and Tauman (2007) characterized optimal two part tariffs where licensing policies are combinations of upfront fees and per unit royalties. In terms of our framework, such a two part tariff is simply a three part tariff policy with $v = 0$. Recall that such a two part tariff corresponds to a point on the horizontal line OB in Figure 3(a)-(d).

Taking $v = 0$ in (7), any $r \in [0, \varepsilon]$ results in a unique δ, given $\delta(r, 0) = \varepsilon - r$. This shows that payoffs of I under combinations of upfront fees and per unit royalties are again functions of δ given by (24) and (25), but in this case there is a one-to-one relation between δ and r, so the multiplicity that we face for three part tariffs does not appear in the analysis of two part tariffs. Since as functions of δ, the payoffs that I seeks to maximize are same, we can apply the results of Sen and Tauman (2007)

[6] Offering $k \leq n-2$ licenses can be optimal only for a three part tariff with $\delta = \theta/k$. In that case all non-licensees drop out of the market, the Cournot price equals c and by (24), $\Pi^n(k, \theta/k)_{FRV} = G(c)$. The outcome of part (II) of Proposition 3 can be also obtained through k licensees and $\delta = \theta/k$ for any $2 \leq k \leq n-2$.

as follows: identify optimal δ from their results and then determine all feasible and acceptable v that can support that δ.

As shown in Proposition 2, if optimal $\delta = \varepsilon$, the corresponding policy is a pure up-front fee. However, if optimal $\delta \in (0, \varepsilon)$, there is a continuum of feasible and acceptable v that can support it. The next proposition presents optimal three part tariffs when the number of competing firms in the oligopoly is relatively large.

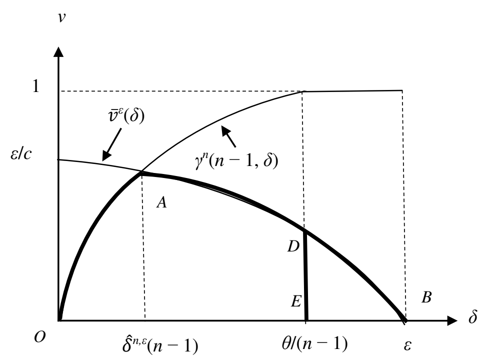


Figure 5(a): *DE* is the set of all optimal three part tariffs for part (II) of Proposition 3

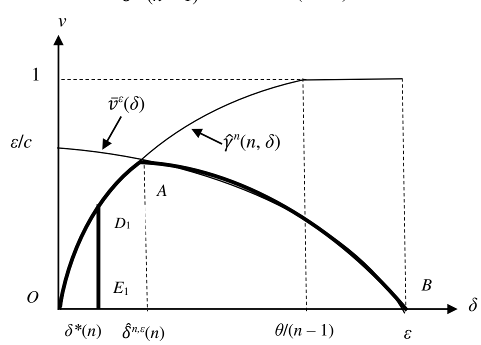


Figure 5(b): D_1E_1 is the set of all optimal three part tariffs for part (III)(i) of Proposition 3

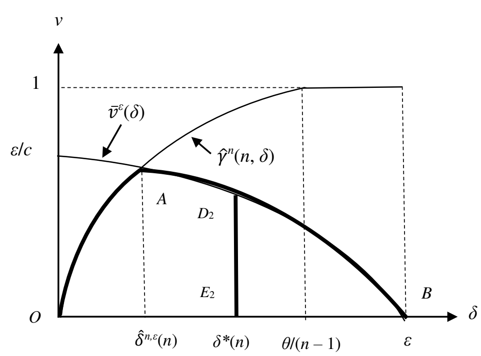


Figure 5(c): D_2E_2 is the set of all optimal three part tariffs for part (III)(ii) of Proposition 3

Proposition 3 *Consider a Cournot oligopoly under linear demand (26) with an outside innovator who has a nondrastic innovation of magnitude ε. For any $n \geq 4$, there is*

$1 < u(n) \leq 2$ *such that for three part tariffs that are combinations of upfront fees, unit royalties and ad valorem royalties, the following hold.*

(I) (**Relatively insignificant innovations**) *If* $\varepsilon < \theta/(2n-4)$, *the unique optimal policy is to sell* $n-1$ *or* n *licenses using a pure upfront fee.*

(II) (**Relatively significant innovations**) *If* $\varepsilon > \theta/u(n)$, *there is a continuum of optimal licensing policies. Under any optimal policy, the license is sold to* $n-1$ *firms, the effective magnitude of the innovation* δ *equals* $\theta/(n-1)$, *the Cournot price equals* c *and an* $(n-1)$*-firm natural oligopoly is created with the sole non-licensee dropping out of the market. The set of all optimal policies is*

$$\{(\delta, v) | \delta = \theta/(n-1), 0 \leq v \leq \bar{v}^{\varepsilon}(\theta/(n-1))\}$$

given by DE *in Figure* 5(a). *Specifically*

(i) *For any* v *that is part of an optimal policy, every licensee pays its net profit* $(1-v)\overline{\phi}^n(n-1, \theta/(n-1))$ *as upfront fee and obtains zero payoff. Any optimal policy has a positive upfront fee, so a two part royalty is not optimal.*

(ii) *There exists an optimal policy consisting of a positive per unit royalty and upfront fee but no ad valorem royalty (the policy with* $v = 0$, *given by the point* E *in Figure* 5(a)). *There also exists an optimal policy consisting of a positive ad valorem royalty and upfront fee but no unit royalty (the policy with* $v = \bar{v}^{\varepsilon}(\theta/(n-1))$, *given by the point* D *in Figure* 5(a)).

(III) (**Intermediate innovations**) *If* $\theta/5 < \varepsilon < \theta/2$ *and* $n \geq 7$, *there is a continuum of optimal licensing policies. Under any optimal policy, the license is sold to all* n *firms and the effective magnitude of the innovation is*

$$\delta^*(n) := [(n-1)\theta + (n+1)\varepsilon]/2(n^2 - n + 1) \tag{27}$$

There exists an optimal policy consisting of a positive per unit royalty and upfront fee but no valorem royalty (the policy with $v = 0$, *given by* E_1 *in Figure* 5(b), E_2 *in Figure* 5(c)). *Moreover, there are* $\underline{\kappa}(n) < \overline{\kappa}(n) < 1$ *such that*

(i) *If* $a > c/\underline{\kappa}(n)$ *(the demand intercept is relatively large in relation to the marginal cost), then* $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ *and the set of all optimal policies is*

$$\{(\delta, v) | \delta = \delta^*(n), 0 \leq v \leq \widehat{\gamma}^n(n, \delta^*(n))\}$$

given by D_1E_1 *in Figure* 5(b). *Consequently there exists an optimal policy that is a two part royalty having a positive ad valorem royalty and unit royalty but no upfront fee (the policy with* $v = \widehat{\gamma}^n(n, \delta^*(n))$, *given by the point* D_1 *in Figure* 5(b)).

(ii) *If* $a < c/\overline{\kappa}(n)$ (*the demand intercept is relatively small in relation to the marginal cost*), *then* $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ *and the set of all optimal policies is*

$$\{(\delta, v)|\delta = \delta^*(n), 0 \leq v \leq \bar{v}^{\varepsilon}(\delta^*(n))\}$$

given by D_2E_2 *in Figure* 5(c). *Consequently there exists an optimal policy consisting of a positive ad valorem royalty and upfront fee but no unit royalty* (*the policy with* $v = \bar{v}^{\varepsilon}(\delta^*(n))$, *given by the point* D_2 *in Figure* 5(c)).

(iii) *If* $c/\overline{\kappa}(n) < a < c/\underline{\kappa}(n)$ (*the demand intercept is of intermediate size in relation to the marginal cost*), *then* $\exists\ \underline{\varepsilon}(n) \in (\theta/5, \theta/2)$ *such that* (a) *if* $\varepsilon > \underline{\varepsilon}(n)$, *then* $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ *and the conclusion is the same as* (i) *and* (b) *if* $\varepsilon < \underline{\varepsilon}(n)$, *then* $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ *and the conclusion is the same as* (ii).

Proof Part (I) follows from Sen and Tauman (2007, Table A.5, p.183), where it is shown that for $n \geq 4$ and $\varepsilon < \theta/(2n-4)$, the optimal δ is $\delta = \varepsilon$. The result is immediate by noting that the unique r, v that can support $\delta = \varepsilon$ is $r = 0, v = 0$ (see Figures 3(a)-(d)).

For part (II), again we note from from Sen and Tauman (2007, Table A.5, p.183) that for $\varepsilon > \theta/u(n)$ (which in particular implies $\varepsilon > \theta/(n-1)$), it is optimal to sell $n-1$ licenses with $\delta = \theta/(n-1)$. Taking $k = n-1$ in Figure 3(a), we note that for $\delta = \theta/(n-1)$, the maximum feasible and acceptable v is $v = \bar{v}^{\varepsilon}(\theta/(n-1))$. The rest of the results are immediate by applying Lemma 2(I)(ii)(b) and part (I)(ii). See the Appendix for the proof of part (III). ■

Proposition 3 shows that for intermediate or significant innovations, there is a continuum of optimal three part tariffs and there always exists an optimal policy consisting of a positive unit royalty and upfront fee, but no ad valorem royalty. For relatively signficant innovations, there also exists an optimal policy consisting only of a positive ad valorem royalty and upfront fee, but a two part royalty is not optimal. For intermediate innovations, a two part royalty can be optimal provided the demand intercept a is relatively large. To see the intuition of this result, we note that for fixed n, c, ε, the relative gain in profit $\widehat{\gamma}^n(n, \delta)$ is decreasing in demand intercept a. Since $\bar{v}^{\varepsilon}(\delta)$ does not involve a, it follows that $\hat{\delta}^{n,\varepsilon}(n, \delta)$ is increasing in a. This is illustrated in Figure 6, where an increase in the demand intercept from a to $a_0 > a$ results in lowering of $\widehat{\gamma}^n(n, \delta)$ to $\widehat{\gamma}^n_0(n, \delta)$, which meets the curve $\bar{v}^{\varepsilon}(\delta)$ at $\hat{\delta}^{n,\varepsilon}_0(n, \delta) > \hat{\delta}^{n,\varepsilon}(n, \delta)$. As a result two part royalties can support a larger interval of δ. Note from Figure 6 that two part royalty policies are given by OA under demand intercept a and by OA_0 under demand intercept a_0.

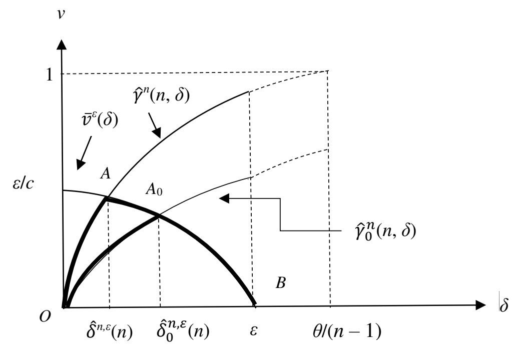


Figure 6: effects of increase in demand intercept a

The next proposition shows that for relatively small sizes of industry ($n = 2, 3$), the conclusions are qualitatively similar to Proposition 3.

Proposition 4 *Consider a Cournot oligopoly under linear demand* (26) *with an outside innovator who has a nondrastic innovation of magnitude* ε. *Suppose* n *is either* 2 *or* 3. *There exists* $n - 1 < s(n) < 2n - 1$ *such that*:

(I) *Suppose* $n = 3$ *and* $\varepsilon > \theta/2$. *Then the conclusion is the same as in Proposition* 3(II).

(II) *Suppose either* $[n = 3$ *and* $\varepsilon < \theta/2]$ *or* $n = 2$. *If* $\varepsilon < \theta/(2n-1)$, *the unique optimal three part tariff policy is to license the innovation to* $n-1$ *firms using a pure upfront fee and if* $\varepsilon > \theta/s(n)$, *the unique optimal three part tariff policy is to license the innovation to* n *firms firm using a pure upfront fee.*

(III) *Suppose either* $[n = 3$ *and* $\varepsilon < \theta/2]$ *or* $n = 2$. *If* $\theta/(2n-1) < \varepsilon < \theta/s(n)$, *there is a continuum of optimal three part tariffs. For any optimal policy, the innovation is licensed to all* n *firms and the effective magnitude of the innovation is* $\delta^*(n)$ *given in* (27). *There always exists an optimal three part tariff consisting of a positive per unit royalty and upfront fee but no ad valorem royalty. There is a constant* $\kappa(n)$ *with* $1/(2n-1) < \kappa(n) < 1/2$ *such that:*

  (i) *If* $a > c/\kappa(n)$ (*the demand intercept is relatively large in relation to the marginal cost*), *then* $\exists\ \underline{\varepsilon}(n) \in (\theta/(2n-1), \theta/s(n))$ *such that* (a) *if* $\varepsilon > \underline{\varepsilon}(n)$, *then* $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ *and the conclusion is the same as in Proposition* 3(III)(i) *and* (b) *if* $\varepsilon < \underline{\varepsilon}(n)$, *then* $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ *and the conclusion is the same as in Proposition* 3(III)(ii).

  (ii) *If* $a < c/\kappa(n)$ (*the demand intercept is relatively small in relation to the marginal cost*), *then* $\delta^*(n) > \hat{\delta}^{2,\varepsilon}(2)$ *and the conclusion is the same as in Proposition* 3(III)(ii).

Proof Parts (I)-(II) and the first three statements of part (III) follow from Table A.5 (p.183) of Sen and Tauman (2007). See the Appendix for the proof of part (III)(i)-(ii).
■

6.1 Concluding remarks: The case of an incumbent innovator

The key driving force for the analysis of three part tariffs with an outside innovator is for any r, v, the outcome at the Cournot stage is completely determined by $\delta(r, v)$ given in (7) and the Cournot price and quantities do not separately depend on r, v. This is not the case when the innovator is one of the incumbent firms in an oligopoly. This is because the quantity of an incumbent innovator affects the market price, which in turn affects the revenue from ad valorem royalty. To see this aspect in a simple setting, consider a Cournot duopoly with two firms $1, 2$ under linear demand curve (26). Initially both firms have the same constant marginal cost c. Firm 1 has a nondrastic cost reducing innovation that lowers the marginal cost from c to $c - \varepsilon$. Here firm 1 is an incumbent innovator.

Suppose firm 1 licenses the innovation to firm 2 using a licensing policy that has per unit royalty r and ad valorem royalty v, where $0 \le r \le \varepsilon$ and $0 \le v < 1$. Upfront fees, if any, do not affect the quantities at the Cournot stage. As in (6), the payoff of firm 2 at the Cournot stage is

$$\hat{\pi}_2 = (1-v)p(Q)q_2 - (c-\varepsilon)q_2 - rq_2 = (1-v)[p(Q) - (c-\delta)]q_2 \tag{28}$$

where $\delta = \delta(r, v)$ given in (7). Thus, as before, in the Cournot stage firm 2 solves the same problem as a firm that has marginal cost $c - \delta$. The payoff of firm 1 at the

Cournot stage is the sum of (i) the operating profit of firm 1, (ii) the revenue from ad valorem royalty and (iii) the revenue from unit royalty, which is

$$\hat{\pi}_1 = [p(Q)q_1 - (c - \varepsilon)q_1] + vp(Q)q_2 + rq_2 \tag{29}$$

Observe that the revenue from unit royalty rq_2 is not affected by q_1. However, q_1 affects the price $p(Q)$, which in turn affects the ad valorem royalty revenue $vp(Q)q_2$. For this reason, unlike the case of an outside innovator, in this case the (unique) Cournot price and quantities are functions of both δ and v.

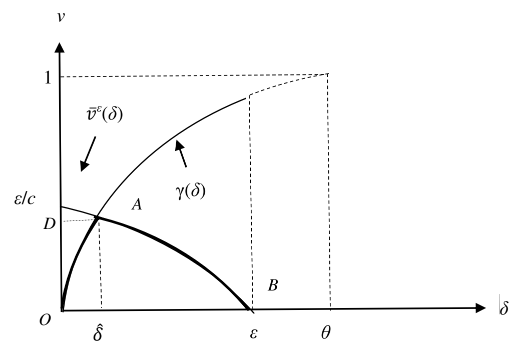


Figure 7: Incumbent innovator in a Cournot duopoly: acceptable versus feasible v

For any $\delta \in [0, \varepsilon]$ and $v \in (0, 1]$, let $p(v, \delta)$ be the Cournot price, $q_i(v, \delta), \phi_i(v, \delta)$ be the Cournot quantity, profit of firm i and $Q(v, \delta) = q_1(v, \delta) + q_2(v, \delta)$. The Cournot quantities and price are

$$q_1(v, \delta) = [(1 - v)(a - c) + 2\varepsilon - (1 + v)\delta]/(3 - v), q_2(v, \delta) = (a - c - \varepsilon + 2\delta)/(3 - v)$$

$$\text{and } p(v, \delta) = [a + (2 - v)c - \varepsilon - (1 - v)\delta]/(3 - v)$$

Observe that firm 2's net profit $\psi(v, \delta) = (1 - v)\phi_2(v, \delta)$ is decreasing in v and increasing in δ. Further, $\psi(0, 0) = \underline{\phi}$ and $\psi(0, \delta) > \underline{\phi}$ for $\delta > 0$. Also note that $\psi(1, \delta) = 0 < \underline{\phi}$ for all δ. Therefore for every $\delta \in [0, \varepsilon]$, $\exists$ a unique $\gamma(\delta)$ such that a policy is acceptable if and only if $v \leq \gamma(\delta)$. Moreover $\gamma(0) = 0$ and $\gamma(\delta)$ is increasing. Since $\bar{v}^\varepsilon(\delta)$ (the maximum feasible v for δ) is decreasing in δ, with $\bar{v}^\varepsilon(0) = \varepsilon/c > \gamma(0) = 0$ and $\bar{v}^\varepsilon(\varepsilon) = 0 < \gamma(\varepsilon)$, $\exists\ \hat{\delta} \in (0, \varepsilon)$ such that

$$\min\{\gamma(\delta), \bar{v}^\varepsilon(\delta)\} = \gamma(\delta) \text{ if } \delta \leq \hat{\delta} \text{ and } \min\{\gamma(\delta), \bar{v}^\varepsilon(\delta)\} = \bar{v}^\varepsilon(\delta) = \text{ if } \delta > \hat{\delta}$$

This is illustrated in Figure 7.

Proposition 5 *Consider a Cournot duopoly with two firms* $1, 2$ *under demand* (26) *where firm* 1 *has a nondrastic innovation of magnitude* ε. *The unique optimal three part tariff is a pure ad valorem royalty policy.*

Proof See the Appendix. ■

For an incumbent innovator in a Cournot duopoly, pure per unit royalties are superior to pure upfront fees (Wang, 1998) and they are also optimal among all combinations of unit royalties and upfront fees (Sen and Stamatopoulos, 2016). Moreover, pure ad valorem royalties are superior to pure per unit royalties (San Martín and Saracho, 2010). Proposition 5 strengthens these results by showing that a pure ad valorem policy is optimal among all three part tariffs. Comparing with the conclusions of Proposition 2(IV), this result also establishes the contrast between the cases of outside and incumbent innovators.

Appendix

Lemma A1 *Let*

$$\psi_\delta(Q) := [p(Q) - c + \delta]/[-p'(Q)] \tag{30}$$

If $p(Q) > c - \delta$, *then* $\psi_\delta(Q)$ *is decreasing in* Q.

Proof Observe that

$$\psi'_\delta(Q) = \left[p''(Q)(p(Q) - c + \delta) - (p'(Q))^2\right] / (p'(Q))^2$$

If $p(Q) > c - \delta$, then clearly the expression above is negative when $p''(Q) \leq 0$. So consider $p''(Q) > 0$. Then $p''(Q)(p(Q) - c + \delta) - (p'(Q))^2 < p''(Q)p(Q) - (p'(Q))^2$. Since $p(Q)$ is log concave (Assumption A3), the last expression is at most zero, which shows $\psi'_\delta(Q) < 0$, proving the assertion. ■

Proof of parts (iii), (v)(a), (v)(c) of Lemma 1 Part (iii): Let $\delta, \hat{\delta} \in (0, \varepsilon]$ and $\hat{\delta} > \delta$. If $\delta < \theta/k \le \hat{\delta}$, then by (i)-(ii), $p^n(k, \hat{\delta}) \le c < p^n(k, \delta)$. If $\delta < \hat{\delta} < \theta/k$, noting that $c - k\delta/n > c - k\hat{\delta}/n \ge c - \varepsilon > 0$, by part (i) and Observation 1 (page 6 of main text), $p^n(k, \hat{\delta}) < p^n(k, \delta)$. Finally if $\theta/k \le \delta < \hat{\delta}$, noting that $c - \delta > c - \hat{\delta} \ge c - \varepsilon > 0$, by part (ii) and Observation 1, $p^n(k, \hat{\delta}) < p^n(k, \delta)$.

For the proof of assertion that $\overline{\phi}^n(k, \delta)$ is increasing in δ, see the case of $\lambda = 0$ in Lemma A.2 of Sen and Tauman (2018, p.44), which follows along the same lines of the comparative static analysis of Dixit (1986).

By part (ii), $\underline{\phi}^n(k, \delta) = 0$ if $\delta \ge \theta/k$. To prove that $\underline{\phi}^n(k, \delta)$ is decreasing in δ for $\delta < \theta/k$, note that the profit function of a non-licensee firm j is $p(Q)q_j - cq_j$. So in this case from the first order condition: $q_j = [p(Q) - c]/[-p'(Q)] = \psi_0(Q)$ (where $\psi_\delta(Q)$ is given in (42)). Thus $\underline{q}^n(k, \delta) = \psi_0(Q^n(k, \delta))$.

Let $\delta < \hat{\delta} < \theta/k$. Then by (iii), $p^n(k, \hat{\delta}) < p^n(k, \delta)$. Thus $Q^n(k, \hat{\delta}) > Q^n(k, \delta)$ and by Lemma A1, $\psi_0(Q^n(k, \hat{\delta})) < \psi_0(Q^n(k, \delta))$, implying that $\underline{q}^n(k, \hat{\delta}) < \underline{q}^n(k, \delta)$. As $\hat{\delta}$ gives both lower Cournot price and lower Cournot quantity for a non-licensee, the Cournot profit of a non-licensee is also lower at $\hat{\delta}$ compared to δ.

Part (v)(a): First suppose $\delta \in (0, \varepsilon]$ such that $\delta < \theta/k$, so $\delta < \theta/(k-1)$. Since $c - (k-1)\delta/n > c - k\delta/n \ge c - \varepsilon > 0$, by part (i), $H^n(p^n(k-1, \delta)) > H^n(p^n(k, \delta)) > 0$ and the result follows by Observation 1.

Next suppose $\theta/k \le \delta < \theta/(k-1)$. In this case by (i)-(ii), $p^n(k-1, \delta) > c \ge p^{n-1}(k, \delta)$.

Finally suppose $\delta \ge \theta/(k-1)$. In this case by (ii), $H^{k-1}(p^n(k-1, \delta)) = H^k(p^n(k, \delta)) = c - \delta > 0$. Since $H^k(p) = p[1 - 1/k\eta(p)]$, if $H^{k-1}(p) > 0$ for a positive p, then $H^k(p) > H^{k-1}(p)$. Hence $H^k(p^n(k-1, \delta)) > H^{k-1}(p^n(k-1, \delta))$ and so $H^k(p^n(k-1, \delta)) > H^k(p^n(k, \delta)$. Then the result follows by again applying Observation 1.

Part (v)(b)-(c): The profit function of a licensee firm j is $p(Q)q_j - (c - \delta)q_j$. From the first order condition: $q_j = [p(Q) - c + \delta]/[-p'(Q)] = \psi_\delta(Q)$ (where $\psi_\delta(Q)$ is given in (42)). Thus $\overline{q}^n(k, \delta) = \psi_\delta(Q^n(k, \delta))$. Since $Q^n(k, \delta) > Q^n(k-1, \delta)$ (by (v)(a)), using Lemma A1: $\overline{q}^n(k-1, \delta) > \overline{q}^n(k, \delta)$. Since $p^n(k-1, \delta) > p^n(k, \delta) > c - \delta$, it follows that $\overline{\phi}^n(k-1, \delta) = [p^n(k-1, \delta) - c + \delta]\overline{q}^n(k-1, \delta) > \overline{\phi}^n(k, \delta) = [p^n(k, \delta) - c + \delta]\overline{q}^n(k, \delta)$. This proves the first inequality of both (v)(b), (v)(c).

Next we prove the last two statements of (v)(b). Let $k = 2, \ldots, n-1$. If $\delta \ge \theta/(k-1)$, then $\delta > \theta/k$ and by (ii), $\underline{\phi}^n(k-1, \delta) = \underline{\phi}^n(k, \delta) = 0$. If $\theta/k \le \delta < \theta/(k-1)$, then by (i)-(ii), $\underline{\phi}^n(k-1, \delta) > 0 = \underline{\phi}^n(k, \delta)$. Finally let $\delta < \theta/k$. Then $\delta < \theta/(k-1)$ and from the first order condition we have $\underline{q}^n(m, \delta) = \psi_0((Q^n(m, \delta))$ for $m = k-1, k$. Since $Q^n(k, \delta) > Q^n(k-1, \delta)$ (by (v)(a)), again using Lemma A1, we have $\underline{q}^n(k-1, \delta) > \underline{q}^n(k, \delta)$. Since $p^n(k-1, \delta) > p^n(k, \delta) > c$, it follows that $\underline{\phi}^n(k-1, \delta) > \underline{\phi}^n(k, \delta)$.

To prove the second inequality of (v)(b), first we show that $\overline{\phi}^n(k, \delta) > \underline{\phi}^n(k, \delta)$ for any $k = 2, \ldots, n-1$ and $\delta > 0$. For $\delta \ge \theta/k$, we have $\overline{\phi}^n(k, \delta) > 0 = \underline{\phi}^n(k, \delta)$, so let $\delta < \theta/k$. Then the quantities are determined by the first order conditions and we have

$$\overline{q}^n(k, \delta) = \psi_\delta(Q^n(k, \delta)), \underline{q}^n(k, \delta) = \psi_0(Q^n(k, \delta))$$

As $p' < 0$ and $\delta > 0$, by (42) $\overline{q}^n(k, \delta) > \underline{q}^n(k, \delta)$. So $\overline{\phi}^n(k, \delta) = [p^n(k, \delta) - c + \delta]\overline{q}^n(k, \delta) >$

$[p^n(k,\delta) - c]\underline{q}^n(k,\delta) = \underline{\phi}^n(k,\delta)$.

To complete the proof of the second inequality of (v)(b), note that if $\delta \geq \theta/(k-1)$, then $\underline{\phi}^n(k-1,\delta) = 0$ and the inequality clearly holds. So let $0 < \delta < \theta/(k-1)$. Let $\tilde{\delta}$ be such that $k\tilde{\delta} = (k-1)\delta$ (so $\tilde{\delta} < \theta/k$ and $\tilde{\delta} = (k-1)\delta/k < \delta$). By part (i), in this case $\underline{\phi}^n(k-1,\delta)$ depend only on the product $(k-1)\delta$, so we have $\underline{\phi}^n(k,\tilde{\delta}) = \underline{\phi}^n(k-1,\delta)$. By the conclusion of the last paragraph, $\overline{\phi}^n(k,\tilde{\delta}) > \underline{\phi}^n(k,\tilde{\delta})$. As $\tilde{\delta} < \delta$, by part (iii): $\overline{\phi}^n(k,\delta) > \overline{\phi}^n(k,\tilde{\delta})$. These inequalities together prove $\overline{\phi}^n(k,\delta) > \underline{\phi}^n(k-1,\delta)$.

Finally to prove the second inequality of (v)(c), note that if $\delta \geq \theta/(n-1)$, then $\underline{\phi}^n(n-1,\delta) = 0$ and the inequality clearly holds. So let $0 < \delta < \theta/(n-1)$. As before let $\tilde{\delta}$ be such that $n\tilde{\delta} = (n-1)\delta$ (so $\tilde{\delta} < \theta/n$ and $\tilde{\delta} < \delta$). By part (i), $p^n(n-1,\delta) = p^n(n,\tilde{\delta})$. So

$$\overline{\phi}^n(n,\tilde{\delta}) = [p^n(n,\tilde{\delta}) - c + \tilde{\delta}]Q^n(n,\tilde{\delta})/n = [p^n(n-1,\delta) - c + \tilde{\delta}]Q^n(n-1,\delta)/n$$

Since $\overline{q}^n(n-1,\delta) > \underline{q}^n(n-1,\delta)$ and $Q^n(n-1,\delta) = (n-1)\overline{q}^n(n-1,\delta) + \underline{q}^n(n-1,\delta)$, we have $Q^n(n-1,\delta)/n > \underline{q}^n(n-1,\delta)$. Hence

$$\overline{\phi}^n(n,\tilde{\delta}) > [p^n(n-1,\delta) - c]\underline{q}^n(n-1,\delta) = \underline{\phi}^n(n-1,\delta).$$

Since $\tilde{\delta} < \delta$, by (iii), $\overline{\phi}^n(n,\delta) > \overline{\phi}^n(n,\tilde{\delta})$, which proves $\overline{\phi}^n(n,\delta) > \underline{\phi}^n(n-1,\tilde{\delta})$. ∎

Proof of Proposition 1 (i) Consider a two part royalty policy with per unit royalty r and ad valorem royalty v and let $\delta = \delta(r,v)$, where

$$\delta(r,v) := [\varepsilon - (r + cv)]/(1-v) \tag{31}$$

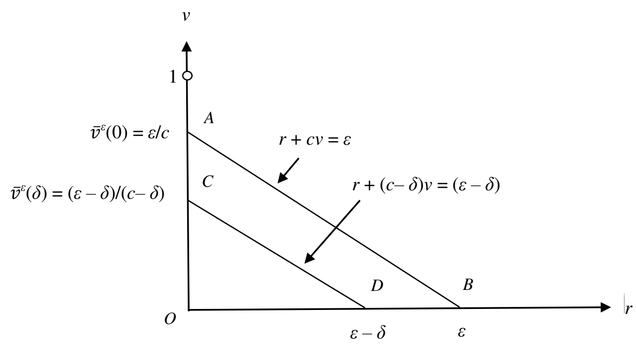


Figure 1: Effective magnitude of innovation $\delta(r, v)$

Let $p_M(\delta), Q_M(\delta)$ be the monopoly price and quantity under marginal cost $c - \delta$. The payoff of I at this policy is

$$\Pi_{RV}(r, v, \delta) = rQ_M(\delta) + vp_M(\delta)Q_M(\delta)$$

$$= [p_M(\delta) - c + \varepsilon]Q_M(\delta) - (1 - v)[p_M(\delta) - (c - \varepsilon + r)/(1 - v)]Q_M(\delta)$$

Noting that $(c - \varepsilon + r)/(1 - v) = c - \delta$ (by (43)) and using the function $G(p) =$

$(p - c + \varepsilon)Q(p)$ (the monopolist's profit at price p under marginal cost $c - \varepsilon$), we have

$$\Pi_{RV}(r, v, \delta) = \Pi_{RV}(v, \delta) = G(p_M(\delta)) - (1 - v)\phi_M(\delta) \tag{32}$$

For any δ, the payoff above is increasing in v, so for any δ, it is best for I to set the maximum possible v.

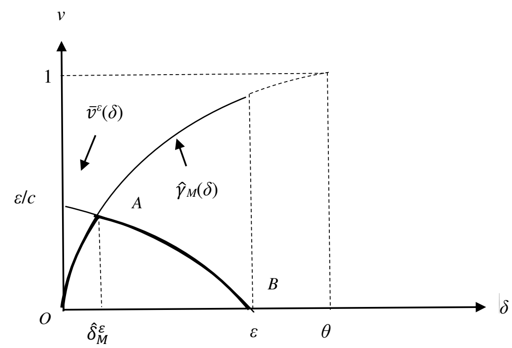


Figure 4: Monopoly: acceptable versus feasible v for δ

For any $\delta \leq \hat{\delta}_M^{\varepsilon}$, the maximum possible v that I can set is $v = \widehat{\gamma}_M(\delta)$ (see Figure 4), in which case $(1 - v)\phi_M(\delta) = \phi_M(0)$ (so the monopolist's net profit with a licensee equals its profit without a license) and by (44), I obtains $G(p_M(\delta)) - \phi_M(0)$.

As $p_M(\delta)$ is decreasing in δ and $\delta \leq \hat{\delta}_M^\varepsilon < \varepsilon$, we have $p_M(\delta) > p_M(\varepsilon)$. Since $G(p)$ is decreasing for $p > p_M(\varepsilon)$, it follows that $G(p_M(\delta))$ is increasing in δ. So the payoff of I is increasing for $\delta \leq \hat{\delta}_M^\varepsilon$, which shows that any optimal two part royalty policy of I must have $\delta \geq \hat{\delta}_M^\varepsilon$. In that case, the maximum possible v that I can set is $v = \bar{v}^\varepsilon(\delta)$ (see Figure 4), which implies $r = 0$. This shows that any optimal two part royalty policy has zero per unit royalty (any such policy is a pair (δ, v) that lies on curve AB in Figure 4). Since a pure ad valorem royalty is optimal among all two part royalties, clearly it is superior to a pure per unit royalty.

(ii) Under a policy with per unit royalty r and ad valorem royalty v, the monopolist has net profit $(1-v)\phi_M(\delta)$ with a license (where $\delta = \delta(r, v)$ given in (43)) and $\phi_M(0)$ without a license. So for a three part tariff policy with r, v, the maximum upfront fee I can set is $\hat{f}(v, \delta) = (1-v)\phi_M(\delta) - \phi_M(0)$. Adding this fee to its payoff at the corresponding two part royalty policy in (44), the payoff of I at this three part tariff policy is

$$\Pi_{FRV}(v, \delta) = \Pi_{RV}(v, \delta) + \hat{f}(v, \delta) = G(p_M(\delta)) - \phi_M(0) \tag{33}$$

Since $p_M(\delta)$ is decreasing for $\delta \in [0, \varepsilon]$ and $G(p)$ is decreasing for $p \geq p_M(\varepsilon)$, the unique maximum of the payoff in (45) is attained at $\delta = \varepsilon$, which implies both $r = 0$, $v = 0$ (see Figure 1). This proves that the unique optimal three part tariff is the pure upfront fee policy with fee $\hat{f}(0, \varepsilon) = \phi_M(\varepsilon) - \phi_M(0)$. ■

Proof of Lemma 2 Recall that

$$(1-v)\overline{\phi}^n(k, \delta) \geq \underline{\phi}^n(k-1, \delta) \Leftrightarrow v \leq \widehat{\gamma}^n(k, \delta) \text{ and}$$

$$(1-v)\overline{\phi}^n(k, \delta) \geq \underline{\phi}^n(k, \delta) \Leftrightarrow v \leq \gamma^n(k, \delta) \tag{34}$$

Part (I)(i) If there are k licensees, under either a two part royalty or a three part tariff policy, a firm with a license obtains at most $(1-v)\overline{\phi}^n(k, \delta)$. If such a firm unilaterally deviates to not have a license, it obtains either $\underline{\phi}^n(k-1, \delta)$ or $\underline{\phi}^n(k, \delta)$. Since $\underline{\phi}^n(k-1, \delta) \geq \underline{\phi}^n(k, \delta)$, it obtains at least $\underline{\phi}^n(k, \delta)$ following its deviation. Since $v > \gamma^n(k, \delta)$, by (46), the deviation is gainful, which proves the result.

Part (I)(ii) Since $0 < \delta \leq \theta/k$, by Assumption A6 $\widehat{\gamma}^n(1, \delta) < \gamma^n(1, \delta)$ and for $k \geq 2$, $\widehat{\gamma}^n(1, \delta) < \ldots < \widehat{\gamma}^n(k, \delta) < \gamma^n(k, \delta)$.

(I)(ii)(a) Let $v \leq \gamma^n(k, \delta)$. For two part royalty, note that if all n firms are willing to buy license, then there is some $0 < \lambda < 1$ such that any firm obtains $\lambda(1-v)\overline{\phi}^n(k, \delta) + (1-\lambda)\underline{\phi}^n(k, \delta)$, which is at least $\underline{\phi}^n(k, \delta)$ (since $v \leq \gamma^n(k, \delta)$). Since $k < n$, if a firm unilaterally deviates to not having a license, it obtains $\underline{\phi}^n(k, \delta)$, so the deviation is not gainful. This shows that if $v \leq \gamma^n(k, \delta)$, then all n firms intending to buy license (and k of them are chosen at random to be licensees) is an equilibrium for a two part royalty policy.

(I)(ii)(b) Next consider a three part tariff policy. Suppose at least $k+1$ firms place bids. Arrange the bids in ascending order as $f_1 \geq f_2 \ldots \geq f_m$. If $f_t > f_{t+1}$ for some $t = 1, \ldots, k$, then the firm that places bid f_t wins a license with certainty, so it is better off slightly reducing its bid. This means in equilibrium it must be the case that $f_t = f_{t+1}$ for all $t = 1, \ldots, k+1$ so that $f_1 = \ldots = f_{k+1}$. This shows the highest bid must be placed by at least $k+1$ firms. Let this highest bid be $\bar{f}$. Then a firm that places the highest bid obtains

$$\lambda[(1-v)\overline{\phi}^n(k,\delta)-\bar{f}]+(1-\lambda)\underline{\phi}^n(k,\delta)=\underline{\phi}^n(k,\delta)+\lambda[f^n(k,\delta,v)-\bar{f}]$$

If $\bar{f} > f^n(k,\delta,v)$, then any firm placing the highest bid can gain by not placing a bid, so $\bar{f} \leq f^n(k,\delta,v)$. If $\bar{f} < f^n(k,\delta,v)$, then any firm placing the bid $\bar{f}$ can unilaterally deviate to a bid slightly higher bid f where $\bar{f} < f < f^n(k,\delta,v)$. Then it will win a license with certainty to obtain $(1-v)\overline{\phi}^n(k,\delta)-f$, which is higher than the payoff before when f is close enough to $\bar{f}$. This shows that the following outcome is an equilibrium: at least $k+1$ firms place bids, the highest equals l $f^n(k,\delta,v)$, at least $k+1$ firms place the highest bid and k of them are chosen at random to be licensees. Since $f^n(k,v,\delta)=(1-v)\overline{\phi}^n(k,\delta)-\underline{\phi}^n(k,\delta)$, any firm that wins a license obtains $\underline{\phi}^n(k,\delta)$, which is the same profit as that of a non-licensee, so no one gains by a unilateral deviation.

Part (II)(i) If there are n licensees, under either a two part royalty or a three part tariff policy, a firm with a license obtains at most $(1-v)\overline{\phi}^n(n,\delta)$. If such a firm unilaterally deviates to not have a license, it obtains $\underline{\phi}^n(n-1,\delta)$. Since $v > \widehat{\gamma}^n(n,\delta)$, by (46), the deviation is gainful, which proves the result.

(II)(ii)-(iii) Since $0 < \delta \leq \theta/(n-1)$, by Assumption A6 $\widehat{\gamma}^n(1,\delta) < \ldots < \widehat{\gamma}^n(n,\delta)$.

(II)(ii) Let $v \leq \widehat{\gamma}^n(n,\delta)$. If all n firms intend to buy licenses under a two part royalty, each firm obtains $(1-v)\overline{\phi}^n(n,\delta)$. By unilaterally deviating to not having a license, any firm obtains $\underline{\phi}^n(n-1,\delta)$. Since $v \leq \widehat{\gamma}^n(n,\delta)$, by (46) this deviation is not gainful, so all firms buying licenses is an equilibrium.

Under three part tariff, when n licenses are offered, we show that all n firms placing the minimum bid is an equilibrium. When all n firms place the minimum bid, each one wins a license to obtain payoff $(1-v)\phi^n(n,\delta)-\hat{f}^n(n,\delta,v)=\underline{\phi}^n(n-1,\delta)$. By unilaterally deviating to not placing a bid, a firm gets $\underline{\phi}^n(n-1,\delta)$, showing there is no gain from the deviation. ■

Proof of part (III) of Proposition 3 By Table A.5 (p.185) of Sen and Tauman (2007), for $n \geq 7$ there are $u(n) < 2 < 5 < v(n)$ such that when $\varepsilon \in (\theta/v(n), \theta/u(n))$, any optimal (k,δ) for a three part tariff policy has $k=n$ and $\delta=\delta^*(n)$ where

$$\delta^*(n) := [(n-1)\theta+(n+1)\varepsilon]/2(n^2-n+1). \tag{35}$$

So this result holds for $\varepsilon \in (\theta/5, \theta/2)$. There is a continuum of (r,v) that can support $\delta^*(n)$.

We compare $\delta^*(n)$ with $\hat{\delta}^{n,\varepsilon}(n)$ (see Figure 3(d) in the main text). Note that

$$\delta^*(n) \leq \hat{\delta}^{n,\varepsilon}(n) \Leftrightarrow \bar{v}^{\varepsilon}(\delta^*) \geq \widehat{\gamma}^n(n,\delta^*(n)) \tag{36}$$

If $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$, then $\widehat{\gamma}^n(n,\delta^*(n)) < \bar{v}^{\varepsilon}(\delta^*(n))$ and the maximum v that can support $\delta^*(n)$ is $v=\widehat{\gamma}^n(n,\delta^*(n))$ (see Figure 5(b)). If $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$, then $\widehat{\gamma}^n(n,\delta^*(n)) < \bar{v}^{\varepsilon}(\delta^*(n))$ and the maximum v that can support $\delta^*(n)$ is $v=\bar{v}^{\varepsilon}(\delta^*(n))$ (see Figure 5(c)).

Using (48), we compare $\bar{v}^\varepsilon(\delta^*(n))$ with $\widehat{\gamma}^n(\delta^*(n))$. Note that in this case $\overline{\phi}^n(n,\delta^*(n)) = (\theta+\delta^*(n))^2/(n+1)^2$ and $\underline{\phi}^n(n-1,\delta) = (\theta-(n-1)\delta^*(n))^2/(n+1)^2$, where $\theta = a-c$. Noting $\widehat{\gamma}^n(n,\delta) = [\overline{\phi}^n(n,\delta) - \underline{\phi}^n(n-1,\delta)]/\overline{\phi}^n(n,\delta)$, using (47), we have

$$\widehat{\gamma}^n(n,\delta^*(n)) = n[(n-1)\theta+(n+1)\varepsilon][(3n^2-n+2)\theta-(n^2-n-2)\varepsilon]/[(2n^2-n+1)\theta+(n+1)\varepsilon]^2$$

$$\bar{v}^\varepsilon(\delta^*(n)) = (n-1)[(2n-1)\varepsilon-\theta]/[(n-1)\theta+(n+1)\varepsilon] \tag{37}$$

Using (49), note that $\bar{v}^\varepsilon(\delta^*(n)) \geq \widehat{\gamma}(\delta^*(n))$ if and only if $\tau_n(\varepsilon) \geq 0$ where

$$\tau_n(\varepsilon) := -(n^3-3n^2+n-1)(n+1)^2\varepsilon^3$$

$$+(n+1)[(9n^4-10n^3+16n^2-10n+3)a+(2n^5-13n^4+10n^3-14n^2+6n-3)c]\varepsilon^2$$

$$+\theta[(n-1)(8n^5-7n^4+16n^3-8n^2+6n-3)a-(12n^6-11n^5+23n^4-20n^3+18n^2-9n+3)c]\varepsilon$$

$$-(n-1)\theta^2[(n^2+1)^2a+(2n^2-n+1)(3n^3-3n^2+3n-1)c] \tag{38}$$

Note that $\tau_n(\varepsilon)$ is a cubic function of ε and its coefficient of ε^3 is negative. So the third order derivative of $\tau_n(\varepsilon)$ is negative, which implies $\tau_n''(\varepsilon)$ is decreasing in ε.

Since $\varepsilon > (a-c)/5$ and $c > \varepsilon$, we have $c > a/6$. Noting that $\tau_n''(\theta/2) > 0$ for $c > a/6$, it follows that $\tau_n''(\varepsilon) > 0$ for all $\varepsilon \in (\theta/5, \theta/2)$, thus $\tau_n'(\varepsilon)$ is increasing in ε in this interval.

Next observe that there exist $1/6 < \underline{t}(n) < \bar{t}(n) < 1$ given by

$$\underline{t}(n) := \frac{50n^6-72n^5+142n^4-132n^3+104n^2-73n+27}{70n^6-42n^5+152n^4-112n^3+134n^2-63n+27}$$

$$\bar{t}(n) := \frac{32n^6-27n^5+91n^4-60n^3+86n^2-61n+27}{40n^6-3n^5+107n^4-52n^3+110n^2-45n+27}$$

such that:
(i) If $a/6 < c < \underline{t}(n)a$, then $\tau_n'(\theta/5) > 0$ and hence $\tau_n'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. In this case $\tau_n(\varepsilon)$ is increasing in ε in this interval.
(ii) If $\underline{t}(n)a < c < \bar{t}(n)a$, then $\tau_n'(\theta/5) < 0$ and $\tau_n'(\theta/2) > 0$. In this case $\exists\ \varepsilon_0 \in (\theta/5, \theta/2)$ such that $\tau_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/5, \varepsilon_0)$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0, \theta/2)$.
(iii) If $\bar{t}(n)a < c < a$, then $\tau_n'(\theta/2) < 0$ and hence $\tau_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/5, \theta/2)$.

Next observe that there exist $1/6 < \underline{\kappa}(n) < \overline{\kappa}(n) < 1$ given by

$$\underline{\kappa}(n) := \frac{25n^6-57n^5+87n^4-102n^3+79n^2-48n+27}{(5n^2-2n+3)(26n^4-42n^3+48n^2-30n+9)}$$

$$\overline{\kappa}(n) := \frac{32n^6-51n^5+99n^4-96n^3+86n^2-57n+27}{(4n^2-n+3)(23n^4-30n^3+42n^2-24n+9)}$$

such that:
(a) If $a/6 < c < \underline{\kappa}(n)a$, then both $\tau_n(\theta/5)$, $\tau_n(\theta/2)$ are positive.
(b) If $\underline{\kappa}(n)a < c < \overline{\kappa}(n)a$, then $\tau_n(\theta/5) < 0$ and $\tau_n(\theta/2) > 0$.

(c) If $\overline{\kappa}(n)a < c < a$, then both $\tau_n(\theta/5)$, $\tau_n(\theta/2)$ are negative.

Noting that $\overline{\kappa}(n) < \underline{t}(n)$, from (i)-(iii) and (a)-(c), we conclude:

(1) If $a/6 < c < \underline{t}(n)a$, then $\tau_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/5, \theta/2)$.

- If $a/6 < c < \underline{\kappa}(n)a$, then $\tau_n(\theta/5) > 0$ and hence $\tau_n(\varepsilon) > 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

- If $\underline{\kappa}(n)a < c < \overline{\kappa}(n)a$, then $\tau_n(\theta/5) < 0 < \tau_n(\theta/2)$ and hence $\exists\ \underline{\varepsilon} \in (\theta/5, \theta/2)$ such that $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for $\varepsilon \in (\theta/5, \underline{\varepsilon})$ and $\tau_n(\varepsilon) > 0$ and hence $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ for $\varepsilon \in (\underline{\varepsilon}, \theta/2)$.

- If $\overline{\kappa}(n)a < c < \underline{t}(n)a$, then $\tau_n(\theta/2) < 0$ and hence $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

(2) If $\underline{t}(n)a < c < \overline{t}(n)a$, then $\exists\ \varepsilon_0 \in (\theta/5, \theta/2)$ such that $\tau_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/5, \varepsilon_0)$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0, \theta/2)$. Since both $\tau_n(\theta/5), \tau_n(\theta/2)$ are negative, in this case $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

(3) If $\overline{t}(n)a < c < a$, then $\tau_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/5, \theta/2)$. Since both $\tau_n(\theta/5) < 0$ are negative, in this case $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

Parts (III)(i)-(iii) of Proposition 3 follow from conclusions (1)-(3) above. ■

Proof of parts (III)(i)-(ii) of Proposition 4 Suppose either [$n = 3$ and $\varepsilon < \theta/2$] or $n = 2$. Note that $2n - 1 = 3$ for $n = 2$ and $2n - 1 = 5$ for $n = 3$. Taking $s(2) = t \equiv 3(\sqrt{2} - 1)$ and $s(3) = d_0 \equiv (8 + 2\sqrt{7})/3$, from Table A.5 (p.185) of Sen and Tauman (2007) it follows that for $n = 2, 3$, if $(a - c)/(2n - 1) < \varepsilon < (a - c)/s(n)$, then the optimal (k, δ) for a three part tariff policy has $k = n$ and $\delta = \delta^*(n) \in (0, \varepsilon)$ (given in (47)). There is a continuum of (r, v) that can support $\delta^*(n)$.

Taking $n = 2, 3$ in (49), note that $\delta^*(2) \le \hat{\delta}^{2,\varepsilon}(2)$ if and only if $\tau_n(\varepsilon) \ge 0$. Taking $n = 2$ in (50), we have

$$\tau_2(\varepsilon) = 27\varepsilon^3 + 333(a - c)\varepsilon^2 + 3(a - c)(83a - 227c)\varepsilon - (25a + 119c)(a - c)^2$$

and taking $n = 3$ in (50), we have

$$\tau_3(\varepsilon)/8 = -4\varepsilon^3 + 4(72a - 51c)\varepsilon^2 + 6(a - c)(73a - 157c)\varepsilon - (25a + 248c)(a - c)^2$$

Since $\varepsilon > (a - c)/(2n - 1)$ and $c > \varepsilon$, we have $c > a/2n$. Noting that for $n = 2, 3$, $\tau_n''(\varepsilon) > 0$ for all $0 < \varepsilon < c < a$, it follows that $\tau_n'(\varepsilon)$ is increasing in ε. Denoting $\rho(2) \equiv 10/19$, $\rho(3) \equiv 47/87$, note that if $a/2n < c < \rho(n)a$, then $\tau_n'(\theta/(2n - 1)) > 0$ and hence $\tau_n'(\varepsilon) > 0$ for all $\varepsilon > \theta/(2n-1)$. Thus $\tau_n(\varepsilon)$ is increasing for $a/2n < c < \rho(n)a$.

We note that $\tau(\theta/(2n - 1)) < 0$ and $\exists\ \kappa(n) \in (1/2n, \rho(n))$ (specifically $\kappa(2) \equiv 2689/(2185 + 2664\sqrt{2})$, $\kappa(3) \equiv 22513/(81313 + 14556\sqrt{7})$) such that $\tau(\theta/s(n)) \ge 0$ iff $c \le \kappa(n)a$. So we conclude that if $a/2n < c < \kappa(n)a$, then $\exists\ \underline{\varepsilon}(n) \in (\theta/2n, \theta/s(n))$ such that:

(1) If $\theta/(2n - 1) < \varepsilon < \underline{\varepsilon}(n)$, then $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$.

(2) If $\underline{\varepsilon}(n) < \varepsilon < (a - c)/s(n)$, then $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$.

Proposition 4(III)(i) follows by (1)-(2) above.

To prove (III)(ii), consider $c > \kappa(n)a$. Observe that when $\kappa(n)a < c < \rho(n)a$, then $\tau_n(\varepsilon)$ is increasing and $\tau_n(\theta/s(n)) < 0$. In this case for all $\varepsilon \in (\theta/(2n-1), \theta/s(n))$, we have $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$.

Next consider $c > \rho(n)$. There is $\tilde{\kappa}(n) > \rho(n)$ (specifically $\tilde{\kappa}(2) \equiv 3689/(9665 - 2880\sqrt{2})$, $\tilde{\kappa}(3) \equiv 12289/(19177 + 1212\sqrt{7})$ such that:
(1) If $\rho(n)a < c < \tilde{\kappa}(n)a$, then $\tau_n'(\theta/(2n-1)) < 0 < \tau_n'(\theta/s(n))$. Since $\tau_n'(\varepsilon)$ is increasing in ε, $\exists\ \varepsilon_0(n) \in (\theta/(2n-1), \theta/s(n))$ such that $\tau_n(\varepsilon)$ is decreasing for $\varepsilon \in (\theta/(2n-1), \varepsilon_0(n))$ and increasing for $\varepsilon \in (\varepsilon_0(n), \theta/s(n))$. Since $\tau_n(\varepsilon) < 0$ at both $\theta/(2n-1)$ and $\theta/s(n)$, it follows that $\tau_n(\varepsilon)$ is negative and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ throughout this interval.
(2) If $\tilde{\kappa}(n)a < c < a$, then $\tau_n'(\theta/s(n)) < 0$. Since $\tau_n'(\varepsilon)$ is increasing in ε, in this case $\tau_n'(\varepsilon) < 0$ and so $\tau_n(\varepsilon)$ is decreasing for all $\varepsilon \in ((\theta/(2n-1), \theta/s(n))$. Since $\tau_n(\theta/s(n)) < 0$, it follows that $\tau_n(\varepsilon) < 0$ throughout this interval. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ throughout this interval.

Proposition 4(III)(ii) follows by (1)-(2) above. ■

Proof of Proposition 5 Consider a three part tariff. Since the innovation is nondrastic, without a license firm 2 obtains a positive profit $\underline{\phi}$. For any feasible and acceptable r, v, the maximum upfront fee firm 1 can obtain from firm 2 is $\bar{f} = (1-v)\phi_2(v,\delta) - \underline{\phi}$. Noting that $c - \delta = (c - \varepsilon + r)/(1-v)$ and $\phi_2(\delta, v) = [p(v,\delta) - (c-\delta)]q_2(v,\delta)$, we have

$$\bar{f} = (1-v)p(v,\delta)q_2(v,\delta) - (c - \varepsilon + r)q_2(v,\delta) - \underline{\phi} \tag{39}$$

Recall the payoff of firm 1 at the Cournot stage is

$$\hat{\pi}_1 = [p(Q)q_1 - (c-\varepsilon)q_1] + vp(Q)q_2 + rq_2 \tag{40}$$

By (51) and (52), the payoff of firm 1 is

$$\pi_1(v,\delta) = \hat{\pi}_1 + \bar{f} = [p(v,\delta) - c + \varepsilon]Q(v,\delta) - \underline{\phi} = G(p(v,\delta)) - \underline{\phi} \tag{41}$$

where $G(p) = (p - c + \varepsilon)Q(p)$ is the monopolist's profit at price p under marginal cost $c - \varepsilon$.

Note that for $\delta \in [0, \varepsilon]$ and $v \in [0,1)$, $p(v,\delta)$ is increasing in v and decreasing in δ. Since $\lim_{v\uparrow 1} p(v, 0)$ equals the monopoly price $p_M(\varepsilon) = (a+c-\varepsilon)/2$, it follows that the Cournot price $p(v,\delta)$ is always lower than $p_M(\varepsilon)$. Since $G(p)$ is increasing for $p < p_M(\varepsilon)$, by (53), the payoff of firm 1 is increasing in the price $p(v,\delta)$ and it is maximum when $p(v,\delta)$ is maximum.

Next observe that firm 2's net profit $\psi(v,\delta) = (1-v)\phi_2(v,\delta)$ is decreasing in v and increasing in δ. Further, $\psi(0,0) = \underline{\phi}$ and $\psi(0,\delta) > \underline{\phi}$ for $\delta > 0$. Also note that $\psi(1,\delta) = 0 < \underline{\phi}$ for all δ. Therefore for every $\delta \in [0,\varepsilon]$, $\exists$ a unique $\gamma(\delta)$ such that a policy is acceptable if and only if $v \leq \gamma(\delta)$. Moreover $\gamma(0) = 0$ and $\gamma(\delta)$ is increasing. Since $\bar{v}^\varepsilon(\delta)$ (the maximum feasible v for δ) is decreasing in δ, with $\bar{v}^\varepsilon(0) = \varepsilon/c > \gamma(0) = 0$ and $\bar{v}^\varepsilon(\varepsilon) = 0 < \gamma(\varepsilon)$, $\exists\ \hat{\delta} \in (0,\varepsilon)$ (see Figure 7) such that

$$\min\{\gamma(\delta), \bar{v}^\varepsilon(\delta)\} = \gamma(\delta) \text{ if } \delta \leq \hat{\delta} \text{ and } \min\{\gamma(\delta), \bar{v}^\varepsilon(\delta)\} = \bar{v}^\varepsilon(\delta) = \text{ if } \delta > \hat{\delta}$$

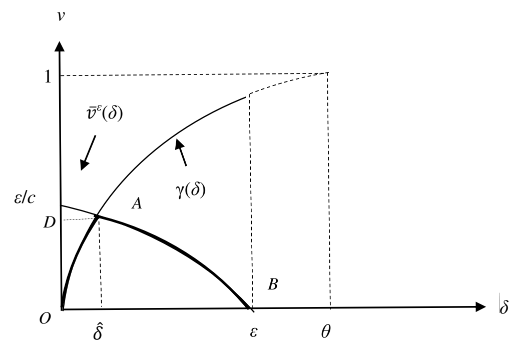


Figure 7: Incumbent innovator in a Cournot duopoly: acceptable versus feasible v

Since for any δ, the Cournot price $p(v, \delta)$ is increasing in v, by (53), for any δ, the payoff of firm 1 is maximum when v is maximum. Thus any optimal pair (δ, v) must be on the curve OA for $\delta \leq \hat{\delta}$ and on the curve AB for $\delta > \hat{\delta}$ (see Figure 7).

Next observe that choosing $\delta > \hat{\delta}$ cannot be optimal. This is because the curve AB (presenting $\bar{v}^{\varepsilon}(\delta)$) is decreasing, so $p(\bar{v}^{\varepsilon}(\hat{\delta}), \hat{\delta}) > p(\bar{v}^{\varepsilon}(\delta), \delta)$ for any $\delta > \hat{\delta}$ (recall $p(v, \delta)$ is increasing in v and decreasing in δ). Therefore any optimal δ must be $\delta \leq \hat{\delta}$ and any optimal (δ, v) pair must lie on curve OA. For any such (δ, v), we have $\psi(v, \delta) = \underline{\phi}$, so the net profit of firm 2 equals its profit without a license and the upfront fee is zero.

Noting that $\psi(v,\delta) = (1-v)\phi_2(v,\delta) = (1-v)(a-c-\varepsilon+2\delta)^2/(3-v)^2$ and $\underline{\phi} = \psi(0,0) = (a-c-\varepsilon)^2/9$, we have

$$\psi(v,\delta) = \underline{\phi} \Leftrightarrow \delta = h(v) \text{ where } h(v) := (a-c-\varepsilon)[\{(3-v)/6\sqrt{1-v}\} - 1/2]$$

Note from Figure 7 that for any $v \in [0, \gamma(\hat{\delta})]$ (any such v is on the line OD on the vertical axis), $h(v)$ is given by the curve OA. When $\delta = h(v)$, the Cournot price is $p(v, h(v)) = (a+c-\varepsilon)/2 - \{(a-c-\varepsilon)\sqrt{1-v}\}/6$, which is increasing in v. This shows that for all $(v, h(v))$ on the curve OA, the Cournot price is maximum when $v = \gamma(\hat{\delta})$. So the unique optimal three part tariff corresponds to the point A in Figure 7. Noting that $\gamma(\hat{\delta}) = \bar{v}^{\varepsilon}(\hat{\delta})$, at point A we have $v = \bar{v}^{\varepsilon}(\hat{\delta})$ (the maximum feasible ad valorem royalty), so the unit royalty r is zero and the unique optimal three part tariff is a pure ad valorem royalty policy. ∎

Appendix

Lemma A1 *Let*

$$\psi_\delta(Q) := [p(Q) - c + \delta]/[-p'(Q)] \tag{42}$$

If $p(Q) > c - \delta$, then $\psi_\delta(Q)$ is decreasing in Q.

Proof Observe that

$$\psi'_\delta(Q) = \left[p''(Q)(p(Q) - c + \delta) - (p'(Q))^2\right]/(p'(Q))^2$$

If $p(Q) > c - \delta$, then clearly the expression above is negative when $p''(Q) \leq 0$. So consider $p''(Q) > 0$. Then $p''(Q)(p(Q)-c+\delta) - (p'(Q))^2 < p''(Q)p(Q) - (p'(Q))^2$. Since $p(Q)$ is log concave (Assumption A3), the last expression is at most zero, which shows $\psi'_\delta(Q) < 0$, proving the assertion. ∎

Proof of parts (iii), (v)(a), (v)(c) of Lemma 1 Part (iii): Let $\delta, \hat{\delta} \in (0, \varepsilon]$ and $\hat{\delta} > \delta$. If $\delta < \theta/k \leq \hat{\delta}$, then by (i)-(ii), $p^n(k, \hat{\delta}) \leq c < p^n(k, \delta)$. If $\delta < \hat{\delta} < \theta/k$, noting that $c - k\delta/n > c - k\hat{\delta}/n \geq c - \varepsilon > 0$, by part (i) and Observation 1 (page 6 of main text), $p^n(k, \hat{\delta}) < p^n(k, \delta)$. Finally if $\theta/k \leq \delta < \hat{\delta}$, noting that $c - \delta > c - \hat{\delta} \geq c - \varepsilon > 0$, by part (ii) and Observation 1, $p^n(k, \hat{\delta}) < p^n(k, \delta)$.

For the proof of assertion that $\overline{\phi}^n(k,\delta)$ is increasing in δ, see the case of $\lambda = 0$ in Lemma A.2 of Sen and Tauman (2018, p.44), which follows along the same lines of the comparative static analysis of Dixit (1986).

By part (ii), $\underline{\phi}^n(k,\delta) = 0$ if $\delta \geq \theta/k$. To prove that $\underline{\phi}^n(k,\delta)$ is decreasing in δ for $\delta < \theta/k$, note that the profit function of a non-licensee firm j is $p(Q)q_j - cq_j$. So in this case from the first order condition: $q_j = [p(Q) - c]/[-p'(Q)] = \psi_0(Q)$ (where $\psi_\delta(Q)$ is given in (42)). Thus $\underline{q}^n(k,\delta) = \psi_0(Q^n(k,\delta))$.

Let $\delta < \hat{\delta} < \theta/k$. Then by (iii), $p^n(k,\hat{\delta}) < p^n(k,\delta)$. Thus $Q^n(k,\hat{\delta}) > Q^n(k,\delta)$ and by Lemma A1, $\psi_0(Q^n(k,\hat{\delta})) < \psi_0(Q^n(k,\delta))$, implying that $\underline{q}^n(k,\hat{\delta}) < \underline{q}^n(k,\delta)$. As $\hat{\delta}$ gives both lower Cournot price and lower Cournot quantity for a non-licensee, the Cournot profit of a non-licensee is also lower at $\hat{\delta}$ compared to δ.

Part (v)(a): First suppose $\delta \in (0, \varepsilon]$ such that $\delta < \theta/k$, so $\delta < \theta/(k-1)$. Since $c-(k-1)\delta/n > c-k\delta/n \geq c-\varepsilon > 0$, by part (i), $H^n(p^n(k-1,\delta)) > H^n(p^n(k,\delta)) > 0$ and the result follows by Observation 1.

Next suppose $\theta/k \leq \delta < \theta/(k-1)$. In this case by (i)-(ii), $p^n(k-1,\delta) > c \geq p^{n-1}(k,\delta)$.

Finally suppose $\delta \geq \theta/(k-1)$. In this case by (ii), $H^{k-1}(p^n(k-1,\delta)) = H^k(p^n(k,\delta)) = c-\delta > 0$. Since $H^k(p) = p[1-1/k\eta(p)]$, if $H^{k-1}(p) > 0$ for a positive p, then $H^k(p) > H^{k-1}(p)$. Hence $H^k(p^n(k-1,\delta)) > H^{k-1}(p^n(k-1,\delta))$ and so $H^k(p^n(k-1,\delta)) > H^k(p^n(k,\delta)$. Then the result follows by again applying Observation 1.

Part (v)(b)-(c): The profit function of a licensee firm j is $p(Q)q_j - (c-\delta)q_j$. From the first order condition: $q_j = [p(Q)-c+\delta]/[-p'(Q)] = \psi_\delta(Q)$ (where $\psi_\delta(Q)$ is given in (42)). Thus $\overline{q}^n(k,\delta) = \psi_\delta(Q^n(k,\delta))$. Since $Q^n(k,\delta) > Q^n(k-1,\delta)$ (by (v)(a)), using Lemma A1: $\overline{q}^n(k-1,\delta) > \overline{q}^n(k,\delta)$. Since $p^n(k-1,\delta) > p^n(k,\delta) > c-\delta$, it follows that $\overline{\phi}^n(k-1,\delta) = [p^n(k-1,\delta)-c+\delta]\overline{q}^n(k-1,\delta) > \overline{\phi}^n(k,\delta) = [p^n(k,\delta)-c+\delta]\overline{q}^n(k,\delta)$. This proves the first inequality of both (v)(b), (v)(c).

Next we prove the last two statements of (v)(b). Let $k = 2,\ldots,n-1$. If $\delta \geq \theta/(k-1)$, then $\delta > \theta/k$ and by (ii), $\underline{\phi}^n(k-1,\delta) = \underline{\phi}^n(k,\delta) = 0$. If $\theta/k \leq \delta < \theta/(k-1)$, then by (i)-(ii), $\underline{\phi}^n(k-1,\delta) > 0 = \underline{\phi}^n(k,\delta)$. Finally let $\delta < \theta/k$. Then $\delta < \theta/(k-1)$ and from the first order condition we have $\underline{q}^n(m,\delta) = \psi_0((Q^n(m,\delta))$ for $m = k-1, k$. Since $Q^n(k,\delta) > Q^n(k-1,\delta)$ (by (v)(a)), again using Lemma A1, we have $\underline{q}^n(k-1,\delta) > \underline{q}^n(k,\delta)$. Since $p^n(k-1,\delta) > p^n(k,\delta) > c$, it follows that $\underline{\phi}^n(k-1,\delta) > \underline{\phi}^n(k,\delta)$.

To prove the second inequality of (v)(b), first we show that $\overline{\phi}^n(k,\delta) > \underline{\phi}^n(k,\delta)$ for any $k = 2,\ldots,n-1$ and $\delta > 0$. For $\delta \geq \theta/k$, we have $\overline{\phi}^n(k,\delta) > 0 = \underline{\phi}^n(k,\delta)$, so let $\delta < \theta/k$. Then the quantities are determined by the first order conditions and we have

$$\overline{q}^n(k,\delta) = \psi_\delta(Q^n(k,\delta)), \underline{q}^n(k,\delta) = \psi_0(Q^n(k,\delta))$$

As $p' < 0$ and $\delta > 0$, by (42) $\overline{q}^n(k,\delta) > \underline{q}^n(k,\delta)$. So $\overline{\phi}^n(k,\delta) = [p^n(k,\delta)-c+\delta]\overline{q}^n(k,\delta) > [p^n(k,\delta)-c]\underline{q}^n(k,\delta) = \underline{\phi}^n(k,\delta)$.

To complete the proof of the second inequality of (v)(b), note that if $\delta \geq \theta/(k-1)$, then $\underline{\phi}^n(k-1,\delta) = 0$ and the inequality clearly holds. So let $0 < \delta < \theta/(k-1)$. Let $\tilde{\delta}$ be such that $k\tilde{\delta} = (k-1)\delta$ (so $\tilde{\delta} < \theta/k$ and $\tilde{\delta} = (k-1)\delta/k < \delta$). By part (i), in this case $\underline{\phi}^n(k-1,\delta)$ depend only on the product $(k-1)\delta$, so we have $\underline{\phi}^n(k,\tilde{\delta}) = \underline{\phi}^n(k-1,\delta)$. By the conclusion of the last paragraph, $\overline{\phi}^n(k,\tilde{\delta}) > \underline{\phi}^n(k,\tilde{\delta})$. As $\tilde{\delta} < \delta$, by part (iii): $\overline{\phi}^n(k,\delta) > \overline{\phi}^n(k,\tilde{\delta})$. These inequalities together prove $\overline{\phi}^n(k,\delta) > \underline{\phi}^n(k-1,\delta)$.

Finally to prove the second inequality of (v)(c), note that if $\delta \geq \theta/(n-1)$, then $\underline{\phi}^n(n-1,\delta) = 0$ and the inequality clearly holds. So let $0 < \delta < \theta/(n-1)$. As before let $\tilde{\delta}$ be such that $n\tilde{\delta} = (n-1)\delta$ (so $\tilde{\delta} < \theta/n$ and $\tilde{\delta} < \delta$). By part (i), $p^n(n-1,\delta) = p^n(n,\tilde{\delta})$. So

$$\overline{\phi}^n(n,\tilde{\delta}) = [p^n(n,\tilde{\delta}) - c + \tilde{\delta}]Q^n(n,\tilde{\delta})/n = [p^n(n-1,\delta) - c + \tilde{\delta}]Q^n(n-1,\delta)/n$$

Since $\overline{q}^n(n-1,\delta) > \underline{q}^n(n-1,\delta)$ and $Q^n(n-1,\delta) = (n-1)\overline{q}^n(n-1,\delta) + \underline{q}^n(n-1,\delta)$, we have $Q^n(n-1,\delta)/n > \underline{q}^n(n-1,\delta)$. Hence

$$\overline{\phi}^n(n,\tilde{\delta}) > [p^n(n-1,\delta)-c]\underline{q}^n(n-1,\delta) = \underline{\phi}^n(n-1,\delta).$$

Since $\tilde{\delta} < \delta$, by (iii), $\overline{\phi}^n(n,\delta) > \overline{\phi}^n(n,\tilde{\delta})$, which proves $\overline{\phi}^n(n,\delta) > \underline{\phi}^n(n-1,\tilde{\delta})$. ■

Proof of Proposition 1 (i) Consider a two part royalty policy with per unit royalty r and ad valorem royalty v and let $\delta = \delta(r,v)$, where

$$\delta(r,v) := [\varepsilon - (r+cv)]/(1-v) \tag{43}$$

Let $p_M(\delta), Q_M(\delta)$ be the monopoly price and quantity under marginal cost $c-\delta$. The payoff of I at this policy is

$$\Pi_{RV}(r,v,\delta) = rQ_M(\delta) + vp_M(\delta)Q_M(\delta)$$

$$= [p_M(\delta)-c+\varepsilon]Q_M(\delta) - (1-v)[p_M(\delta)-(c-\varepsilon+r)/(1-v)]Q_M(\delta)$$

Noting that $(c-\varepsilon+r)/(1-v) = c-\delta$ (by (43)) and using the function $G(p) = (p-c+\varepsilon)Q(p)$ (the monopolist's profit at price p under marginal cost $c-\varepsilon$), we have

$$\Pi_{RV}(r,v,\delta) = \Pi_{RV}(v,\delta) = G(p_M(\delta)) - (1-v)\phi_M(\delta) \tag{44}$$

For any δ, the payoff above is increasing in v, so for any δ, it is best for I to set the maximum possible v.

For any $\delta \le \hat{\delta}^\varepsilon_M$, the maximum possible v that I can set is $v = \widehat{\gamma}_M(\delta)$ (see Figure 4), in which case $(1-v)\phi_M(\delta) = \phi_M(0)$ (so the monopolist's net profit with a licensee equals its profit without a license) and by (44), I obtains $G(p_M(\delta)) - \phi_M(0)$.

As $p_M(\delta)$ is decreasing in δ and $\delta \le \hat{\delta}^\varepsilon_M < \varepsilon$, we have $p_M(\delta) > p_M(\varepsilon)$. Since $G(p)$ is decreasing for $p > p_M(\varepsilon)$, it follows that $G(p_M(\delta))$ is increasing in δ. So the payoff of I is increasing for $\delta \le \hat{\delta}^\varepsilon_M$, which shows that any optimal two part royalty policy of I must have $\delta \ge \hat{\delta}^\varepsilon_M$. In that case, the maximum possible v that I can set is $v = \bar{v}^\varepsilon(\delta)$ (see Figure 4), which implies $r = 0$. This shows that any optimal two part royalty policy has zero per unit royalty (any such policy is a pair (δ, v) that lies on curve AB in Figure 4). Since a pure ad valorem royalty is optimal among all two part royalties, clearly it is superior to a pure per unit royalty.

(ii) Under a policy with per unit royalty r and ad valorem royalty v, the monopolist has net profit $(1-v)\phi_M(\delta)$ with a license (where $\delta = \delta(r,v)$ given in (43)) and $\phi_M(0)$ without a license. So for a three part tariff policy with r, v, the maximum upfront fee I can set is $\hat{f}(v,\delta) = (1-v)\phi_M(\delta) - \phi_M(0)$. Adding this fee to its payoff at the corresponding two part royalty policy in (44), the payoff of I at this three part tariff policy is

$$\Pi_{FRV}(v,\delta) = \Pi_{RV}(v,\delta) + \hat{f}(v,\delta) = G(p_M(\delta)) - \phi_M(0) \tag{45}$$

Since $p_M(\delta)$ is decreasing for $\delta \in [0,\varepsilon]$ and $G(p)$ is decreasing for $p \ge p_M(\varepsilon)$, the unique maximum of the payoff in (45) is attained at $\delta = \varepsilon$, which implies both $r = 0$, $v = 0$ (see Figure 1). This proves that the unique optimal three part tariff is the pure upfront fee policy with fee $\hat{f}(0,\varepsilon) = \phi_M(\varepsilon) - \phi_M(0)$. ■

Proof of Lemma 2 Recall that

$$(1-v)\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(k-1,\delta) \Leftrightarrow v \leq \widehat{\gamma}^n(k,\delta) \text{ and}$$

$$(1-v)\overline{\phi}^n(k,\delta) \geq \underline{\phi}^n(k,\delta) \Leftrightarrow v \leq \gamma^n(k,\delta) \tag{46}$$

Part (I)(i) If there are k licensees, under either a two part royalty or a three part tariff policy, a firm with a license obtains at most $(1-v)\overline{\phi}^n(k,\delta)$. If such a firm unilaterally deviates to not have a license, it obtains either $\underline{\phi}^n(k-1,\delta)$ or $\underline{\phi}^n(k,\delta)$. Since $\underline{\phi}^n(k-1,\delta) \geq \underline{\phi}^n(k,\delta)$, it obtains at least $\underline{\phi}^n(k,\delta)$ following its deviation. Since $v > \gamma^n(k,\delta)$, by (46), the deviation is gainful, which proves the result.

Part (I)(ii) Since $0 < \delta \leq \theta/k$, by Assumption A6 $\widehat{\gamma}^n(1,\delta) < \gamma^n(1,\delta)$ and for $k \geq 2$, $\widehat{\gamma}^n(1,\delta) < \ldots < \widehat{\gamma}^n(k,\delta) < \gamma^n(k,\delta)$.

(I)(ii)(a) Let $v \leq \gamma^n(k,\delta)$. For two part royalty, note that if all n firms are willing to buy license, then there is some $0 < \lambda < 1$ such that any firm obtains $\lambda(1-v)\overline{\phi}^n(k,\delta) + (1-\lambda)\underline{\phi}^n(k,\delta)$, which is at least $\underline{\phi}^n(k,\delta)$ (since $v \leq \gamma^n(k,\delta)$). Since $k < n$, if a firm unilaterally deviates to not having a license, it obtains $\underline{\phi}^n(k,\delta)$, so the deviation is not gainful. This shows that if $v \leq \gamma^n(k,\delta)$, then all n firms intending to buy license (and k of them are chosen at random to be licensees) is an equilibrium for a two part royalty policy.

(I)(ii)(b) Next consider a three part tariff policy. Suppose at least $k+1$ firms place bids. Arrange the bids in ascending order as $f_1 \geq f_2 \ldots \geq f_m$. If $f_t > f_{t+1}$ for some $t = 1, \ldots, k$, then the firm that places bid f_t wins a license with certainty, so it is better off slightly reducing its bid. This means in equilibrium it must be the case that $f_t = f_{t+1}$ for all $t = 1, \ldots, k+1$ so that $f_1 = \ldots = f_{k+1}$. This shows the highest bid must be placed by at least $k+1$ firms. Let this highest bid be $\bar{f}$. Then a firm that places the highest bid obtains

$$\lambda[(1-v)\overline{\phi}^n(k,\delta) - \bar{f}] + (1-\lambda)\underline{\phi}^n(k,\delta) = \underline{\phi}^n(k,\delta) + \lambda[f^n(k,\delta,v) - \bar{f}]$$

If $\bar{f} > f^n(k,\delta,v)$, then any firm placing the highest bid can gain by not placing a bid, so $\bar{f} \leq f^n(k,\delta,v)$. If $\bar{f} < f^n(k,\delta,v)$, then any firm placing the bid $\bar{f}$ can unilaterally deviate to a bid slightly higher bid f where $\bar{f} < f < f^n(k,\delta,v)$. Then it will win a license with certainty to obtain $(1-v)\overline{\phi}^n(k,\delta) - f$, which is higher than the payoff before when f is close enough to $\bar{f}$. This shows that the following outcome is an equilibrium: at least $k+1$ firms place bids, the highest equals 1 $f^n(k,\delta,v)$, at least $k+1$ firms place the highest bid and k of them are chosen at random to be licensees. Since $f^n(k,v,\delta) = (1-v)\overline{\phi}^n(k,\delta) - \underline{\phi}^n(k,\delta)$, any firm that wins a license obtains $\underline{\phi}^n(k,\delta)$, which is the same profit as that of a non-licensee, so no one gains by a unilateral deviation.

Part (II)(i) If there are n licensees, under either a two part royalty or a three part tariff policy, a firm with a license obtains at most $(1-v)\overline{\phi}^n(n,\delta)$. If such a firm unilaterally deviates to not have a license, it obtains $\underline{\phi}^n(n-1,\delta)$. Since $v > \widehat{\gamma}^n(n,\delta)$, by (46), the deviation is gainful, which proves the result.

(II)(ii)-(iii) Since $0 < \delta \leq \theta/(n-1)$, by Assumption A6 $\widehat{\gamma}^n(1,\delta) < \ldots < \widehat{\gamma}^n(n,\delta)$.

(II)(ii) Let $v \leq \widehat{\gamma}^n(n,\delta)$. If all n firms intend to buy licenses under a two part royalty, each firm obtains $(1-v)\overline{\phi}^n(n,\delta)$. By unilaterally deviating to not having a license, any firm obtains $\underline{\phi}^n(n-1,\delta)$. Since $v \leq \widehat{\gamma}^n(n,\delta)$, by (46) this deviation is not gainful, so all firms buying licenses is an equilibrium.

Under three part tariff, when n licenses are offered, we show that all n firms placing the minimum bid is an equilibrium. When all n firms place the minimum bid, each one wins a license to obtain payoff $(1-v)\phi^n(n,\delta) - \hat{f}^n(n,\delta,v) = \underline{\phi}^n(n-1,\delta)$. By unilaterally deviating to not placing a bid, a firm gets $\underline{\phi}^n(n-1,\delta)$, showing there is no gain from the deviation. ■

Proof of part (III) of Proposition 3 By Table A.5 (p.185) of Sen and Tauman (2007), for $n \geq 7$ there are $u(n) < 2 < 5 < v(n)$ such that when $\varepsilon \in (\theta/v(n), \theta/u(n))$, any optimal (k,δ) for a three part tariff policy has $k = n$ and $\delta = \delta^*(n)$ where

$$\delta^*(n) := [(n-1)\theta + (n+1)\varepsilon]/2(n^2-n+1). \tag{47}$$

So this result holds for $\varepsilon \in (\theta/5, \theta/2)$. There is a continuum of (r,v) that can support $\delta^*(n)$.

We compare $\delta^*(n)$ with $\hat{\delta}^{n,\varepsilon}(n)$ (see Figure 3(d) in the main text). Note that

$$\delta^*(n) \leq \hat{\delta}^{n,\varepsilon}(n) \Leftrightarrow \bar{v}^\varepsilon(\delta^*) \geq \widehat{\gamma}^n(n,\delta^*(n)) \tag{48}$$

If $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$, then $\widehat{\gamma}^n(n,\delta^*(n)) < \bar{v}^\varepsilon(\delta^*(n))$ and the maximum v that can support $\delta^*(n)$ is $v = \widehat{\gamma}^n(n,\delta^*(n))$ (see Figure 5(b)). If $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$, then $\widehat{\gamma}^n(n,\delta^*(n)) < \bar{v}^\varepsilon(\delta^*(n))$ and the maximum v that can support $\delta^*(n)$ is $v = \bar{v}^\varepsilon(\delta^*(n))$ (see Figure 5(c)).

Using (48), we compare $\bar{v}^\varepsilon(\delta^*(n))$ with $\widehat{\gamma}^n(\delta^*(n))$. Note that in this case $\overline{\phi}^n(n,\delta^*(n)) = (\theta+\delta^*(n))^2/(n+1)^2$ and $\underline{\phi}^n(n-1,\delta) = (\theta-(n-1)\delta^*(n))^2/(n+1)^2$, where $\theta = a-c$. Noting $\widehat{\gamma}^n(n,\delta) = [\overline{\phi}^n(n,\delta) - \underline{\phi}^n(n-1,\delta)]/\overline{\phi}^n(n,\delta)$, using (47), we have

$$\widehat{\gamma}^n(n,\delta^*(n)) = n[(n-1)\theta+(n+1)\varepsilon][(3n^2-n+2)\theta-(n^2-n-2)\varepsilon]/[(2n^2-n+1)\theta+(n+1)\varepsilon]^2$$

$$\bar{v}^\varepsilon(\delta^*(n)) = (n-1)[(2n-1)\varepsilon-\theta]/[(n-1)\theta+(n+1)\varepsilon] \tag{49}$$

Using (49), note that $\bar{v}^\varepsilon(\delta^*(n)) \geq \widehat{\gamma}(\delta^*(n))$ if and only if $\tau_n(\varepsilon) \geq 0$ where

$$\begin{aligned}\tau_n(\varepsilon) &:= -(n^3-3n^2+n-1)(n+1)^2\varepsilon^3\\ &+(n+1)[(9n^4-10n^3+16n^2-10n+3)a + (2n^5-13n^4+10n^3-14n^2+6n-3)c]\varepsilon^2\\ &+\theta[(n-1)(8n^5-7n^4+16n^3-8n^2+6n-3)a-(12n^6-11n^5+23n^4-20n^3+18n^2-9n+3)c]\varepsilon\\ &-(n-1)\theta^2[(n^2+1)^2a+(2n^2-n+1)(3n^3-3n^2+3n-1)c]\end{aligned} \tag{50}$$

Note that $\tau_n(\varepsilon)$ is a cubic function of ε and its coefficient of ε^3 is negative. So the third order derivative of $\tau_n(\varepsilon)$ is negative, which implies $\tau_n''(\varepsilon)$ is decreasing in ε.

Since $\varepsilon > (a-c)/5$ and $c > \varepsilon$, we have $c > a/6$. Noting that $\tau_n''(\theta/2) > 0$ for $c > a/6$, it follows that $\tau_n''(\varepsilon) > 0$ for all $\varepsilon \in (\theta/5, \theta/2)$, thus $\tau_n'(\varepsilon)$ is increasing in ε in this interval.

Next observe that there exist $1/6 < \underline{t}(n) < \bar{t}(n) < 1$ given by

$$\underline{t}(n) := \frac{50n^6 - 72n^5 + 142n^4 - 132n^3 + 104n^2 - 73n + 27}{70n^6 - 42n^5 + 152n^4 - 112n^3 + 134n^2 - 63n + 27}$$

$$\bar{t}(n) := \frac{32n^6 - 27n^5 + 91n^4 - 60n^3 + 86n^2 - 61n + 27}{40n^6 - 3n^5 + 107n^4 - 52n^3 + 110n^2 - 45n + 27}$$

such that:
(i) If $a/6 < c < \underline{t}(n)a$, then $\tau_n'(\theta/5) > 0$ and hence $\tau_n'(\varepsilon) > 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. In this case $\tau_n(\varepsilon)$ is increasing in ε in this interval.
(ii) If $\underline{t}(n)a < c < \bar{t}(n)a$, then $\tau_n'(\theta/5) < 0$ and $\tau_n'(\theta/2) > 0$. In this case $\exists\ \varepsilon_0 \in (\theta/5, \theta/2)$ such that $\tau_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/5, \varepsilon_0)$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0, \theta/2)$.
(iii) If $\bar{t}(n)a < c < a$, then $\tau_n'(\theta/2) < 0$ and hence $\tau_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/5, \theta/2)$.

Next observe that there exist $1/6 < \underline{\kappa}(n) < \overline{\kappa}(n) < 1$ given by

$$\underline{\kappa}(n) := \frac{25n^6 - 57n^5 + 87n^4 - 102n^3 + 79n^2 - 48n + 27}{(5n^2 - 2n + 3)(26n^4 - 42n^3 + 48n^2 - 30n + 9)}$$

$$\overline{\kappa}(n) := \frac{32n^6 - 51n^5 + 99n^4 - 96n^3 + 86n^2 - 57n + 27}{(4n^2 - n + 3)(23n^4 - 30n^3 + 42n^2 - 24n + 9)}$$

such that:
(a) If $a/6 < c < \underline{\kappa}(n)a$, then both $\tau_n(\theta/5)$, $\tau_n(\theta/2)$ are positive.
(b) If $\underline{\kappa}(n)a < c < \overline{\kappa}(n)a$, then $\tau_n(\theta/5) < 0$ and $\tau_n(\theta/2) > 0$.
(c) If $\overline{\kappa}(n)a < c < a$, then both $\tau_n(\theta/5)$, $\tau_n(\theta/2)$ are negative.

Noting that $\overline{\kappa}(n) < \underline{t}(n)$, from (i)-(iii) and (a)-(c), we conclude:
(1) If $a/6 < c < \underline{t}(n)a$, then $\tau_n(\varepsilon)$ is increasing in ε for all $\varepsilon \in (\theta/5, \theta/2)$.

- If $a/6 < c < \underline{\kappa}(n)a$, then $\tau_n(\theta/5) > 0$ and hence $\tau_n(\varepsilon) > 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

- If $\underline{\kappa}(n)a < c < \overline{\kappa}(n)a$, then $\tau_n(\theta/5) < 0 < \tau_n(\theta/2)$ and hence $\exists\ \underline{\varepsilon} \in (\theta/5, \theta/2)$ such that $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for $\varepsilon \in (\theta/5, \underline{\varepsilon})$ and $\tau_n(\varepsilon) > 0$ and hence $\delta^*(n) < \hat{\delta}^{n,\varepsilon}(n)$ for $\varepsilon \in (\underline{\varepsilon}, \theta/2)$.

- If $\overline{\kappa}(n)a < c < \underline{t}(n)a$, then $\tau_n(\theta/2) < 0$ and hence $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

(2) If $\underline{t}(n)a < c < \bar{t}(n)a$, then $\exists\ \varepsilon_0 \in (\theta/5, \theta/2)$ such that $\tau_n(\varepsilon)$ is decreasing in ε for $\varepsilon \in (\theta/5, \varepsilon_0)$ and it is increasing in ε for $\varepsilon \in (\varepsilon_0, \theta/2)$. Since both $\tau_n(\theta/5), \tau_n(\theta/2)$ are negative, in this case $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.
(3) If $\bar{t}(n)a < c < a$, then $\tau_n(\varepsilon)$ is decreasing in ε for all $\varepsilon \in (\theta/5, \theta/2)$. Since both $\tau_n(\theta/5) < 0$ are negative, in this case $\tau_n(\varepsilon) < 0$ for all $\varepsilon \in (\theta/5, \theta/2)$. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ for all ε in this interval.

Parts (III)(i)-(iii) of Proposition 3 follow from conclusions (1)-(3) above. ■

Proof of parts (III)(i)-(ii) of Proposition 4 Suppose either $[n = 3$ and $\varepsilon < \theta/2]$ or $n = 2$. Note that $2n - 1 = 3$ for $n = 2$ and $2n - 1 = 5$ for $n = 3$. Taking $s(2) = t \equiv 3(\sqrt{2} - 1)$ and $s(3) = d_0 \equiv (8 + 2\sqrt{7})/3$, from Table A.5 (p.185) of Sen and Tauman (2007) it follows that for $n = 2, 3$, if $(a - c)/(2n - 1) < \varepsilon < (a - c)/s(n)$, then the optimal (k, δ) for a three part tariff policy has $k = n$ and $\delta = \delta^*(n) \in (0, \varepsilon)$ (given in (47)). There is a continuum of (r, v) that can support $\delta^*(n)$.

Taking $n = 2, 3$ in (49), note that $\delta^*(2) \le \hat{\delta}^{2,\varepsilon}(2)$ if and only if $\tau_n(\varepsilon) \ge 0$. Taking $n = 2$ in (50), we have

$$\tau_2(\varepsilon) = 27\varepsilon^3 + 333(a - c)\varepsilon^2 + 3(a - c)(83a - 227c)\varepsilon - (25a + 119c)(a - c)^2$$

and taking $n = 3$ in (50), we have

$$\tau_3(\varepsilon)/8 = -4\varepsilon^3 + 4(72a - 51c)\varepsilon^2 + 6(a - c)(73a - 157c)\varepsilon - (25a + 248c)(a - c)^2$$

Since $\varepsilon > (a - c)/(2n - 1)$ and $c > \varepsilon$, we have $c > a/2n$. Noting that for $n = 2, 3$, $\tau_n''(\varepsilon) > 0$ for all $0 < \varepsilon < c < a$, it follows that $\tau_n'(\varepsilon)$ is increasing in ε. Denoting $\rho(2) \equiv 10/19$, $\rho(3) \equiv 47/87$, note that if $a/2n < c < \rho(n)a$, then $\tau_n'(\theta/(2n - 1)) > 0$ and hence $\tau_n'(\varepsilon) > 0$ for all $\varepsilon > \theta/(2n-1)$. Thus $\tau_n(\varepsilon)$ is increasing for $a/2n < c < \rho(n)a$.

We note that $\tau(\theta/(2n - 1)) < 0$ and $\exists\ \kappa(n) \in (1/2n, \rho(n))$ (specifically $\kappa(2) \equiv 2689/(2185 + 2664\sqrt{2})$, $\kappa(3) \equiv 22513/(81313 + 14556\sqrt{7})$) such that $\tau(\theta/s(n)) \ge 0$ iff $c \le \kappa(n)a$. So we conclude that if $a/2n < c < \kappa(n)a$, then $\exists\ \underline{\varepsilon}(n) \in (\theta/2n, \theta/s(n))$ such that:

(1) If $\theta/(2n - 1) < \varepsilon < \underline{\varepsilon}(n)$, then $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$.

(2) If $\underline{\varepsilon}(n) < \varepsilon < (a - c)/s(n)$, then $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$.

Proposition 4(III)(i) follows by (1)-(2) above.

To prove (III)(ii), consider $c > \kappa(n)a$. Observe that when $\kappa(n)a < c < \rho(n)a$, then $\tau_n(\varepsilon)$ is increasing and $\tau_n(\theta/s(n)) < 0$. In this case for all $\varepsilon \in (\theta/(2n - 1), \theta/s(n))$, we have $\tau_n(\varepsilon) < 0$ and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$.

Next consider $c > \rho(n)$. There is $\tilde{\kappa}(n) > \rho(n)$ (specifically $\tilde{\kappa}(2) \equiv 3689/(9665 - 2880\sqrt{2})$, $\tilde{\kappa}(3) \equiv 12289/(19177 + 1212\sqrt{7})$ such that:

(1) If $\rho(n)a < c < \tilde{\kappa}(n)a$, then $\tau_n'(\theta/(2n - 1)) < 0 < \tau_n'(\theta/s(n))$. Since $\tau_n'(\varepsilon)$ is increasing in ε, $\exists\ \varepsilon_0(n) \in (\theta/(2n - 1), \theta/s(n))$ such that $\tau_n(\varepsilon)$ is decreasing for $\varepsilon \in (\theta/(2n - 1), \varepsilon_0(n))$ and increasing for $\varepsilon \in (\varepsilon_0(n), \theta/s(n))$. Since $\tau_n(\varepsilon) < 0$ at both $\theta/(2n - 1)$ and $\theta/s(n)$, it follows that $\tau_n(\varepsilon)$ is negative and hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ throughout this interval.

(2) If $\tilde{\kappa}(n)a < c < a$, then $\tau_n'(\theta/s(n)) < 0$. Since $\tau_n'(\varepsilon)$ is increasing in ε, in this case $\tau_n'(\varepsilon) < 0$ and so $\tau_n(\varepsilon)$ is decreasing for all $\varepsilon \in ((\theta/(2n - 1), \theta/s(n))$. Since $\tau_n(\theta/s(n)) < 0$, it follows that $\tau_n(\varepsilon) < 0$ throughout this interval. Hence $\delta^*(n) > \hat{\delta}^{n,\varepsilon}(n)$ throughout this interval.

Proposition 4(III)(ii) follows by (1)-(2) above. ■

Proof of Proposition 5 Consider a three part tariff. Since the innovation is nondrastic, without a license firm 2 obtains a positive profit $\underline{\phi}$. For any feasible and acceptable r, v, the maximum upfront fee firm 1 can obtain from firm 2 is $\bar{f} = (1 - v)\phi_2(v, \delta) - \underline{\phi}$.

Noting that $c-\delta=(c-\varepsilon+r)/(1-v)$ and $\phi_2(\delta,v)=[p(v,\delta)-(c-\delta)]q_2(v,\delta)$, we have

$$\bar{f}=(1-v)p(v,\delta)q_2(v,\delta)-(c-\varepsilon+r)q_2(v,\delta)-\underline{\phi} \tag{51}$$

Recall the payoff of firm 1 at the Cournot stage is

$$\hat{\pi}_1=[p(Q)q_1-(c-\varepsilon)q_1]+vp(Q)q_2+rq_2 \tag{52}$$

By (51) and (52), the payoff of firm 1 is

$$\pi_1(v,\delta)=\hat{\pi}_1+\bar{f}=[p(v,\delta)-c+\varepsilon]Q(v,\delta)-\underline{\phi}=G(p(v,\delta))-\underline{\phi} \tag{53}$$

where $G(p)=(p-c+\varepsilon)Q(p)$ is the monopolist's profit at price p under marginal cost $c-\varepsilon$.

Note that for $\delta\in[0,\varepsilon]$ and $v\in[0,1)$, $p(v,\delta)$ is increasing in v and decreasing in δ. Since $\lim_{v\uparrow 1}p(v,0)$ equals the monopoly price $p_M(\varepsilon)=(a+c-\varepsilon)/2$, it follows that the Cournot price $p(v,\delta)$ is always lower than $p_M(\varepsilon)$. Since $G(p)$ is increasing for $p<p_M(\varepsilon)$, by (53), the payoff of firm 1 is increasing in the price $p(v,\delta)$ and it is maximum when $p(v,\delta)$ is maximum.

Next observe that firm 2's net profit $\psi(v,\delta)=(1-v)\phi_2(v,\delta)$ is decreasing in v and increasing in δ. Further, $\psi(0,0)=\underline{\phi}$ and $\psi(0,\delta)>\underline{\phi}$ for $\delta>0$. Also note that $\psi(1,\delta)=0<\underline{\phi}$ for all δ. Therefore for every $\delta\in[0,\varepsilon]$, $\exists$ a unique $\gamma(\delta)$ such that a policy is acceptable if and only if $v\le\gamma(\delta)$. Moreover $\gamma(0)=0$ and $\gamma(\delta)$ is increasing. Since $\bar{v}^\varepsilon(\delta)$ (the maximum feasible v for δ) is decreasing in δ, with $\bar{v}^\varepsilon(0)=\varepsilon/c>\gamma(0)=0$ and $\bar{v}^\varepsilon(\varepsilon)=0<\gamma(\varepsilon)$, $\exists\ \hat{\delta}\in(0,\varepsilon)$ (see Figure 7) such that

$$\min\{\gamma(\delta),\bar{v}^\varepsilon(\delta)\}=\gamma(\delta)\text{ if }\delta\le\hat{\delta}\text{ and }\min\{\gamma(\delta),\bar{v}^\varepsilon(\delta)\}=\bar{v}^\varepsilon(\delta)=\text{ if }\delta>\hat{\delta}$$

Since for any δ, the Cournot price $p(v,\delta)$ is increasing in v, by (53), for any δ, the payoff of firm 1 is maximum when v is maximum. Thus any optimal pair (δ,v) must be on the curve OA for $\delta\le\hat{\delta}$ and on the curve AB for $\delta>\hat{\delta}$ (see Figure 7).

Next observe that choosing $\delta>\hat{\delta}$ cannot be optimal. This is because the curve AB (presenting $\bar{v}^\varepsilon(\delta)$) is decreasing, so $p(\bar{v}^\varepsilon(\hat{\delta}),\hat{\delta})>p(\bar{v}^\varepsilon(\delta),\delta)$ for any $\delta>\hat{\delta}$ (recall $p(v,\delta)$ is increasing in v and decreasing in δ). Therefore any optimal δ must be $\delta\le\hat{\delta}$ and any optimal (δ,v) pair must lie on curve OA. For any such (δ,v), we have $\psi(v,\delta)=\underline{\phi}$, so the net profit of firm 2 equals its profit without a license and the upfront fee is zero.

Noting that $\psi(v,\delta)=(1-v)\phi_2(v,\delta)=(1-v)(a-c-\varepsilon+2\delta)^2/(3-v)^2$ and $\underline{\phi}=\psi(0,0)=(a-c-\varepsilon)^2/9$, we have

$$\psi(v,\delta)=\underline{\phi}\Leftrightarrow\delta=h(v)\text{ where }h(v):=(a-c-\varepsilon)[\{(3-v)/6\sqrt{1-v}\}-1/2]$$

Note from Figure 7 that for any $v\in[0,\gamma(\hat{\delta})]$ (any such v is on the line OD on the vertical axis), $h(v)$ is given by the curve OA. When $\delta=h(v)$, the Cournot price is $p(v,h(v))=(a+c-\varepsilon)/2-\{(a-c-\varepsilon)\sqrt{1-v}\}/6$, which is increasing in v. This shows that for all $(v,h(v))$ on the curve OA, the Cournot price is maximum when $v=\gamma(\hat{\delta})$. So the unique optimal three part tariff corresponds to the point A in Figure 7. Noting that $\gamma(\hat{\delta}) = \bar{v}^{\varepsilon}(\hat{\delta})$, at point A we have $v = \bar{v}^{\varepsilon}(\hat{\delta})$ (the maximum feasible ad valorem royalty), so the unit royalty r is zero and the unique optimal three part tariff is a pure ad valorem royalty policy. ■

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