Segmented assimilation: a minority's dilemma

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Segmented Assimilation: A Minority's Dilemma*

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Abstract

What factors determine a minority group's extent/pattern of assimilation with the mainstream population in a country? We study this question in a dynamic multi-generation model, and formalize the sociological theory of *segmented assimilation* propounded by Portés and Zhou (1993). Our key assumptions are: there exists cultural heterogeneity *within* a minority group, minority members can *shift* their inherited culture traits to an extent, shifting culture traits *closer* to the mainstream culture increases economic opportunities, benefits from being close to the dominant local culture generate *social interaction effects* in minority and mainstream locations, and minority members are motivated by *short-term goals*. We show that specific features of the socio-economic environment – regarding the extent of initial culture heterogeneity among the minority, and the influence of local social interaction effects on their payoffs – lead to *segmented assimilation* in the long run: In a sequence of generations, some minority members – those born with culture traits 'close enough' to the mainstream culture – move towards assimilating with the mainstream, while other members dissociate from the mainstream and become more entrenched in the traditional minority sub-culture. Such intertemporal segmentation, that arises in the *absence* of a minority preference for *oppositional identities*, can impose significant costs on the entire minority group in the long-run: poverty, inequality, and polarization. There can be *hysteresis* in the evolution of minority lineages. The efficacy of a policy intervention will depend on how it impacts the minority assimilation trajectory: an ill-timed affirmative-action policy can *lower* payoffs of all minority members in the long-run.

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1 Introduction

In many countries, various minority groups – minorities with respect to ethnicity, language, and religion – reside alongside a mainstream population. Some minority members might be recent immigrants, others might be the progeny of immigrants who came a long time ago, and still others might be the descendants of indigenous populations. Obvious examples include African-Americans, Latin-Americans, and Native Americans in the US, North Africans in France, West Indians and South Asians in the UK, Palestinians in Israel, Muslims in India, Hindus in Bangladesh and Pakistan, Tamils in Sri Lanka, and Aborigines in Australia. Any inquiry into the 'evenness' of economic progress of a nation must look into the extent to which the country's disparate minority communities find themselves included in (or excluded from) the development process.

There is substantial evidence that an important way for a minority community to share the fruits of economic progress is to *assimilate* with the mainstream culture. Chiswick (1978) and McManus, Gould and Welch (1983) estimate that learning the mainstream language generates better labour market outcomes in the US; Dustmann and Fabbri (2003) and Bisin, Patacchini, Verdier, and Zenou (2011a) find similar evidence for immigrants to European countries. Do such economic incentives induce minorities across the world to 'uniformly assimilate' with the majority culture? This question is very relevant today in light of persistent high poverty and inequality rates among many minority populations, and the current and possible future flows of refugees to developed countries.

In the sociology literature on immigrant group dynamics in the US, the *classical assimilation theory* of the 1920s predicted that over time, minority populations exhibit monotonic and complete assimilation with the mainstream; see Park (1928). While empirical evidence on early European immigrants to the US is consistent with that theory, the history of more recent (non-white) migrants to the US belies such a prediction of complete minority assimilation. In the late 1980s and early 1990s, sociological studies of immigrant experiences in the US revealed that while some sections of specific immigrant groups had indeed come closer to the mainstream, other sections of the same groups remained significantly apart; see, for instance, Hirschman and Falcon (1985) and Gans (1992).

The sociological theory of *segmented assimilation* of immigrant minorities was propounded by Portés and Zhou (1993). This theory recognizes that interaction with the mainstream can cause an immigrant community to fracture into multiple segments over successive generations. Some segments (especially ones with human capital advantages) succeed in 'assimilating upwards' with the mainstream, while other segments (that are socio-economically more disadvantaged) find it too costly to do so and remain dissociated from the mainstream. In surveying studies of assimilation of the descendants of 20^{th} century immigrants to the US, Zhou (1997) concludes: "[We] have observed the following possible patterns of adaptation among contemporary immigrants and their offspring: one [pattern] replicates the time-honored portrayal of growing acculturation and parallel integration into the white middle-class, while another leads straight into the opposite direction to permanent poverty and assimilation into the underclass."

It is important to recognize that the phenomenon of segmented assimilation need not be specific to immigrant experiences. It can be reasonably claimed that many minority communities in different countries have experienced such segmented assimilation. Note that such segmentation is likely to generate an intertemporal increase in socio-economic inequality *within* a minority group. Recent empirical evidence suggests that this has indeed been the case with respect to income inequality in many countries. Regarding African-Americans in the US, Nembhard, Pitts, and Mason (2005) write: "In 1966, the Gini coefficient for African-American families [in the US] was 0.35; in 1993, the figure had increased to 0.49. By way of comparison, the corresponding figures for white families were 0.36 and 0.41 respectively." In the UK Report of the National Equality Panel, Hills *et. al.* (2010) note that "the rise in inequality over the last forty years [in the UK] is mostly attributable to growing inequality within social groups, however those groups are defined; differences in outcomes between the more and less advantaged within each social group are much greater than differences between groups." In India, for a set of states with significant Muslim populations – Andhra Pradesh, Assam, Bihar, Kerala, Uttar Pradesh, and West Bengal – our own calculations (based on National Sample Survey data) show that the Gini coefficient for Muslim families in 2004-05 was 0.33 while in 2011-12 it had increased to 0.39; the corresponding figures for Hindu families were 0.34 and 0.36.

1.1 Overview of Current Research

We study a dynamic model of minority assimilation over infinitely many generations. We identify conditions under which the long-run trajectory of a minority community generates segmented assimilation, and uncover socio-economic impacts of such segmentation on long-run minority welfare.

Our model is based on five key premises: (1) There is cultural heterogeneity *within* a minority community, with different community members exhibiting distinct *overt culture traits* (dialects, adherence to religious restrictions, social etiquette, dress codes, etc.), while no member has a preference for 'oppositional identity' against the mainstream. (2) In their youth, members get stochastic opportunities to *alter* their inherited overt culture traits to a limited extent. (3) Shifting overt culture traits *closer* to the mainstream culture improves the chances of securing lucrative mainstream employment. (4) There are distinct *location-specific social interaction effects* in the 'minority village $\mathbb{V}$' and the 'mainstream city $\mathbb{C}$' that include benefits from being close to the dominant local culture. (5) Minority decision-making is predominantly guided by *short-term goals* that do not take into account the impact of intra-generation decisions on descendants' well-being.

Different kinds of long-run assimilation patterns can arise in our model: eventual 'complete assimilation' of the entire minority population either in the minority village $\mathbb{V}$ or in the mainstream city $\mathbb{C}$, or eventual 'segmented assimilation' in which the minority community fractures into two subgroups, with one residing in the village and the other in the city generation after generation. We identify distinct sets of (minimal) sufficient conditions that guarantee the emergence of long-run segmented-assimilation outcomes of different forms. The sufficient conditions involve specifying rankings of a set of minority payoffs in $\mathbb{V}$ and in $\mathbb{C}$. The conditions are quite intricate since we construct them to be 'tight' – segmentation outcomes cannot be guaranteed to emerge in equilibrium if any condition is relaxed. At the same time, the conditions are simple in a specific way: they have to hold for an *initial* generation – once they do so, a 'recursion property' guarantees that they will hold for all subsequent generations. As we clarify below, there exist particular parameter configurations in our model that correspond to specific structural features of the socio-economic environment – regarding the nature of initial culture heterogeneity among the minority, and the influence of local social interaction effects on minority payoffs – that make it likely that our stated payoff-ranking conditions will hold and thus lead to equilibrium segmentation of the minority community.

In our model, when the minority group completely assimilates in $\mathbb{V}$ or in $\mathbb{C}$ in equilibrium, all members have 'very similar' culture traits and payoffs in the long-run due to the presence of local social-interaction effects in $\mathbb{V}$ and in $\mathbb{C}$. In contrast, when there is equilibrium segmented assimilation, the minority population gets *partitioned* into two homogenized subgroups, one in $\mathbb{V}$ and the other in $\mathbb{C}$, and that can impose large socio-economic costs on the community in the long run. There can be significant payoff poverty and inequality, and there can arise substantial polarization (*à la* Esteban and Ray (1994)) in culture traits and in payoffs across the entire minority population.

Along a segmented-assimilation trajectory, there can be *hysteresis* in the evolution of minority lineages: if two members born in the same period with identical inherited culture traits in $\mathbb{V}$ face different opportunity realizations, then their progeny can have very different life experiences; further, a minority parent and her offspring, born with identical inherited culture traits in $\mathbb{V}$, can have different life outcomes – with the offspring of a mainstream-oriented minority parent becoming more minority-focused – even when they experience identical opportunities.[1] Our analysis also clarifies that the efficacy of a policy intervention depends on the prevailing minority culture distribution in $\mathbb{V}$ at the time of implementation, and the short-term and long-term impacts of a policy can be very different as an intervention can alter the minority assimilation trajectory. Specifically, we uncover the following *policy dilemma*: a well-intentioned but ill-timed affirmative action policy intervention can lower payoffs of *all* minority members in the long-run, irrespective of their assimilation status.

In our study, within-minority-group culture trait heterogeneity plays a crucial role in generating long-run segmentation of the community, and the associated adverse effects on minority welfare. Within the minority community, we view the members' overt culture traits – e.g., their communication/dress/religious codes – as being located at different positions in a *minority culture interval.* Accordingly, we model a minority population that is distributed, in its first generation, over a set of culture positions '$\mathbb{P} = \{1, \cdots, P\}$'. In $\mathbb{P}$, we view the positions located more to the left to be ones steeped in traditional culture traits farthest from the mainstream, and ones located more to the right to be closer to the mainstream. We assume that while a minority member's birth culture position is determined by heredity, local social interactions in her youth (stochastically) provide her an opportunity to culture-shift to an *adjacent* culture position.[2]

[1] We note that such path dependence in minority lineages accords well with the principal theme of the Hanif Kureishi (1994) short story *My Son the Fanatic*.

[2] Our assumed structure of culture transmission follows that in the 'social economics' literature: one's culture traits arise partly from heredity and partly from local social interactions; see Bisin and Verdier (2011). Regarding the altering of overt culture trait by, say, learning the mainstream language, our assumption that a member can shift only to an *adjacent* culture position is intended to capture the feature that 'acquired mainstream language fluency' of an offspring can be significantly constrained by her parent's fluency.

In this environment, each minority member in every generation faces a trade-off. If she 'culture-shifts right' her chance of getting higher mainstream returns will rise; but, if she still fails to secure a city job (which is possible), she will be left behind in the village to interact with minority peers who, on average, will be more immersed in the traditional minority sub-culture. Then, given that 'closeness to the dominant local culture' matters for a $\mathbb{V}$-resident, her optimal culture-shift strategy might be *fractured*: "If I am born at a culture position *close enough* to the mainstream culture, I will culture-shift right to improve my chance of getting higher mainstream returns. But if I am not born *that close* to the mainstream, I will culture-shift left (rather than right) to stay culturally close to those minority peers who are more likely to remain behind in the village."

We show that such a fractured culture-shift strategy within a minority generation in $\mathbb{V}$ can lead to a partitioning of the culture position set $\mathbb{P}$ into an *entrenched culture subset* – with no one culture-shifting out of the subset and no migration to the city, and an *unentrenched culture subset* – from which there will be city-migration. We then establish that along an equilibrium trajectory of minority generations, if every intra-generation equilibrium has an entrenched culture subset while some (or all) of these equilibria also contain unentrenched culture subsets, then the trajectory can generate the kind of segmented assimilation outcomes that sociologists have identified. Our analysis clarifies that when the model parameters exhibit the following features, then equilibrium segmented assimilation is more likely to occur: (*a*) the 'initial' minority culture distribution is sufficiently right-skewed; and (*b*) for a class of right-skewed culture distributions, 'city payoffs' of minority members born *close enough* to the mainstream culture are significantly higher than their 'village payoffs' while the opposite is true for members born *sufficiently distant* from the mainstream culture.

1.2 Relation to the Literature

There is a burgeoning 'social economics' literature that studies the phenomenon of minority assimilation with the mainstream. Our paper builds upon that literature, and aims to contribute to it.

Starting from Lazear (1999), the 'minority assimilation' literature has assumed that minority members can improve their mainstream opportunities by acquiring mainstream-oriented socio-cultural attributes (e.g., by learning the mainstream language). We maintain that assumption, and additionally assume the following heterogeneity: within a generation, minority members decide on limited culture-shifting starting from different inherited culture positions. This specification creates a map from an inherited minority culture distribution to a distinct non-degenerate acquired-culture distribution. That creates differential opportunities to different minority members regarding mainstream returns, and thus opens up the possibility of segmented assimilation. Our specification is to be contrasted with that in Battu, Mwale, and Zenou (2007) and related papers, in which all minority members decide on culture-shifting from a common culture position, and so shift (if they do) to another common position closer to the mainstream, thus securing identical mainstream opportunities.

The study by Battu *et. al.* (2007) belongs in another important strand of the social economics literature that incorporates the feature that some minority group members in different societies have a preference for an *oppositional identity* vis-à-vis the mainstream, and that limits minority assimilation with the mainstream.[3] We recognize the logic of this argument and the empirical evidence in the US on the existence of identity preferences. But it is not clear to us how widespread/significant minority preference for oppositional identity is in other parts of the world.[4] Our aim is to complement the extant research by clarifying that the existence of 'oppositional identity preference' among the minority is *not necessary* for the emergence of segmented assimilation.

Further, we assume that once a minority member migrates to the mainstream, her welfare depends on her closeness to the mainstream culture in $\mathbb{C}$, and is unaffected by her 'culture distance' from her minority peers who have stayed back in $\mathbb{V}$. While minority members in $\mathbb{C}$ might indeed have residual social connections with their 'kin' in $\mathbb{V}$, we focus on the case where such ties are insignificant.[5] This modeling choice distinguishes our study from those in which minority members care about the (cultural) location of *all* their peers, either because minority interaction benefits are *global* (as in Akerlof (1997)), or because members moving to the mainstream are affected by the resentment of their left-behind peers (as in Austen-Smith and Fryer (2005)).

In a recent paper, Sato and Zenou (2020) study minority assimilation in a static model, where there is minority preference heterogeneity over the relative benefit of income *vis-a-vis* the cost of shifting from one's 'culture of origin'. Given that, the authors establish the possibility of a 'mixed social identity equilibrium' in which some minority members assimilate with the mainstream while others do not.[6] Our paper complements this research by proving the existence of dynamic equilibrium assimilation paths that generate minority segmentation over generations, when all minority members have *identical* preferences. Specifically, we show how minority assimilation trajectories leading to long-run segmented assimilation – along with their adverse consequences for the well-being of the entire minority community – can arise not due to heterogeneity in minority preferences, but due to heterogeneity in *initial conditions and subsequent opportunities* of different minority members.

The rest of our paper is organized as follows. Section 2 presents the model. Existence of an intra-generation *social interactions* equilibrium – when agents care about neighbors' culture traits and take discrete sequential decisions under uncertainty – is established in Section 3. In Section 4, we

[3] In the literature, papers that study minority assimilation given preference for oppositional identity include Battu and Zenou (2010), Bisin, Patacchini, Verdier, and Zenou (2011a, 2016), and Panebianco (2014); papers that explain emergence of minority oppositional identity preference as an equilibrium outcome include Akerlof (1997), Austen-Smith and Fryer (2005), and Bisin, Patacchini, Verdier, and Zenou (2011b).

[4] In the US, a large empirical literature finds evidence for the phenomenon of 'aversion to acting White' – the resentment of African-American youth toward those African-Americans moving toward the White culture; see, for instance, Fordham and Ogbu (1986) and Fryer and Torelli (2010). But the extent of resentment against 'acting White' possibly differs across different ethnic groups even in the US; a *New York Times* article on 22 May 2014 reported that more Hispanics are declaring themselves as Whites in the US.

[5] In fact, minority members in the city might have incentives to break 'village kinship ties' so as to be free of kin-transfer obligations and thus be able to secure higher mainstream payoffs; see Hoff and Sen (2006).

[6] Another important aim of the Sato-Zenou analysis is to study the impact of urban city structures on minority assimilation choices. That is an issue that we do not address. We also do not study the impact of cultural leaders or of social networks on minority assimilation decisions; see Verdier and Zenou (2017, 2018).

run simulations on a set of parametric examples, and define alternative assimilation trajectories. Minimal sufficient conditions for the existence of equilibrium segmented-assimilation trajectories are identified in Section 5. In Section 6, we study the adverse consequences of segmented assimilation on minority welfare. We conclude in Section 7 by discussing the sensitivity of our results to some specific features of our model. All formal proofs are presented in Appendices A and B.

2 Minority and the Mainstream: A Model

There are two locations: a minority village $\mathbb{V}$, and a mainstream city $\mathbb{C}$. We study the intergenerational evolution of 'overt culture traits' and 'livelihood locations' of members of a minority community (whose first generation is born in $\mathbb{V}$), when, in every generation, the $\mathbb{V}$-born minority members receive two kinds of stochastic opportunities: an opportunity to alter inherited cultural traits to a limited extent, and a subsequent opportunity to migrate to a higher-paying city-job.

2.1 Minority Culture-shift and Migration Decisions over Generations

The first minority generation, consisting of a continuum of members of measure one, is born in $\mathbb{V}$ in period 1. The members differ in the degree of their adherence to minority norms and traditions, and this is reflected in distinguishable 'overt culture traits': dialects, social etiquette, dress codes, observance of religious edicts, etc.. We take $\mathbb{P} = \{1, \cdots, P\}$ (for some $P \geq 2$) to be the set of minority culture positions, and $\underline{\mu}_0 = \{\mu_0(1), \cdots, \mu_0(P)\}$ to be the *initial minority culture distribution* in $\mathbb{V}$ in period 1. We assume that initially a strictly positive measure of members $\mu_0(p) > 0$ is born at every $p \in \mathbb{P}$, and consider every position p to be 'more steeped in minority culture' than $p+1$.

In the first generation, and in every subsequent generation, each minority member born at every culture position p in $\mathbb{V}$ gets a chance – with probability $\sigma \in (0,1)$ – to culture-shift to any 'adjacent feasible culture position' (i.e., either to $\max\{1, p-1\}$ or to $\min\{p+1, P\}$). When she gets this chance, a member might want to culture-shift to improve her mainstream opportunities and/or to be closer to the dominant local culture in her livelihood-location (these conflicting objectives are clarified below). Culture-shift decisions made by the $\mathbb{V}$-residents, before they start working, generate the *acquired culture distribution* $\underline{\mu}_1^a = \{\mu_1^a(1), \cdots, \mu_1^a(P)\}$ in $\mathbb{V}$ in generation/period 1.[7]

We now describe the relation between the minority and the mainstream. At the beginning of period 1, $\mathbb{C}$ is populated entirely by a large mainstream population, which we assume (only for simplicity) to be culturally homogeneous at the common mainstream culture position $P^{main} > P$. In period 1, after the $\mathbb{V}$-born minority have made their culture-shift choices, each member located at every acquired position $p \in \mathbb{P}$ in $\mathbb{V}$ receives a city job offer (that pays higher wages than a village job) with probability $g(p) \in (0,1)$. The *mainstream access vector* $\mathbb{G} := (g(1), \cdots, g(P))$, with

[7] The terms 'generation t' and 'period t', for $t \geq 1$, will be used interchangeably in what follows. Further, the *acquired culture position* of a minority member will refer to her position after she has executed her culture-shift decision given her opportunity; for a person who does not get the opportunity or chooses not to culture-shift, her inherited culture position *will be* her acquired position.

$0 < g(1) \leq g(2) \leq \cdots \leq g(P) < 1$, states the probabilities with which minority members situated at different acquired culture positions in $\mathbb{V}$ get city job offers; $\mathbb{G}$ assumes that city access is more likely for those who are 'culturally closer' to the mainstream. When she gets a city offer, a member decides whether to migrate to $\mathbb{C}$ 'carrying her acquired culture position', or to stay back in $\mathbb{V}$ (where a village job is always available). The trade-off arises from the fact that in addition to wages, a member cares about her *net social-interaction benefits* in her final 'livelihood location'; in Section 2.2, we specify these benefits in $\mathbb{V}$ and in $\mathbb{C}$. In period 1, minority migration generates the *final culture distribution* of the minority population that stays back in $\mathbb{V}$: $\underline{\mu}_1 = \{\mu_1(1), \cdots, \mu_1(P)\}$, and the minority culture distribution in $\mathbb{C}$: $\underline{\kappa}_1 = \{\kappa_1(1), \cdots, \kappa_1(P)\}$, $\kappa_1(p)$ being the measure of members at acquired position p who migrate to $\mathbb{C}$.[8] After period 1 migration, minority residents in $\mathbb{V}$ and $\mathbb{C}$ get employed and receive wages; the wages and the social interaction benefits in their 'livelihood locations' (where they live and work) determine the members' payoffs $V(\cdot)$ in $\mathbb{V}$ and $C(\cdot)$ in $\mathbb{C}$; see Section 2.2.

The evolution of the minority group over generations occurs as follows. Each member lives for one period, and before her death, gives birth to one offspring who is born in her parent's livelihood-location at her parent's acquired culture position. Thus, in period 2, the second generation of minority members born in $\mathbb{V}$ has $\underline{\mu}_1$ as its inherited culture distribution. Then these members get similar opportunities as their parents; their culture-shift choices lead to the acquired culture distribution $\underline{\mu}_2^a$, and their responses to city offers determine the final culture distribution $\underline{\mu}_2$ in $\mathbb{V}$ of the members who stay behind in $\mathbb{V}$ and the culture distribution of the period 2 migrants to $\mathbb{C}$.

In $\mathbb{C}$ in period 2, the minority have $\underline{\kappa}_1$ as their inherited culture distribution. We posit that a minority member born in $\mathbb{C}$ at some $p < P$ in any period *always* gets a chance to 'culture-shift right' to $p + 1$.[9] In $\mathbb{C}$, culture-shifting right is induced by the gains from being closer to the mainstream culture; see Section 2.2. This culture-shifting is followed by migration of new members from $\mathbb{V}$; taken together, these movements determine the final culture distribution $\underline{\kappa}_2$ in $\mathbb{C}$ in period 2.

In periods $t = 3, 4, 5, \cdots$, successive minority generations experience the same chain of events in their lifetimes. In every period $t \geq 2$: $\underline{\mu}_{t-1}$ and $\underline{\kappa}_{t-1}$ are the initial culture distributions of members born in $\mathbb{V}$ and $\mathbb{C}$ respectively. Culture-shift decisions then generate acquired culture distributions $\underline{\mu}_t^a$ in $\mathbb{V}$ and $\underline{\kappa}_t^a$ in $\mathbb{C}$, and migration decisions generate final culture distributions $\underline{\mu}_t$ in $\mathbb{V}$ and $\underline{\kappa}_t$ in $\mathbb{C}$. Recognize that along a feasible outcome trajectory $\{\underline{\mu}_t, \underline{\kappa}_t\}$, population measures in $\mathbb{V}$ and in $\mathbb{C}$ (denoted subsequently by $m[\underline{\mu}_t]$ and $m[\underline{\kappa}_t]$) will add up to 1 in every period t.

Our model assumes that no minority member born in $\mathbb{C}$ can *reverse migrate* to $\mathbb{V}$. In Section 5.2, we identify conditions under which no reverse-migration incentives will exist along equilibrium paths. In Section 7, we briefly discuss the robustness of our results with regard to two other model features:

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[8] For the continuum of members born at any culture position p in $\mathbb{V}$, the law of large numbers imply that a fraction σ of them will get the chance to culture-shift. Analogously, for every $p \in \mathbb{P}$, a fraction $g(p)$ of the minority population measure situated at acquired culture position p in $\mathbb{V}$ will get a city job offer.

[9] It is plausible to assume that the chance to culture-shift will be much higher than σ in $\mathbb{C}$, as local interactions will provide minority children in $\mathbb{C}$ greater opportunities to shift toward the mainstream. Our central results will be unaffected even if the $\mathbb{C}$ shift-probability is less than 1; but see footnote 34.

(a) inability of minority members to *culture-shift globally* and jump over multiple culture positions, and (b) inability of members to be *multi-cultural* and occupy multiple culture positions concurrently.

2.2 Minority Payoffs in the Village and in the City

For any culture distribution $\underline{\mu}$ on $\mathbb{P}$, we define the following measures: $m[p \,|\, \underline{\mu}] := \sum_{p'=1}^{p} \mu(p')$ and $m[\underline{\mu}] := m[P \,|\, \underline{\mu}]$. We then define the culture distribution sets: $\Delta_+ := \{\underline{\mu} \in \mathbb{R}^P_{++} : m[\underline{\mu}] \leq 1\}$, and $\Delta_+[\underline{\mu}] := \{\underline{\mu}' \in \Delta_+ : m[\underline{\mu}'] \leq m[\underline{\mu}]\}$. Given our specification that $\underline{\mu}_0 \in \Delta_+$ with $m[\underline{\mu}_0] = 1$, recognize that if $\underline{\mu}_{t-1}$ is the inherited culture distribution in some generation t, then the inherited culture distribution $\underline{\mu}_t$ in the next generation $t+1$ must belong in $\Delta_+[\underline{\mu}_{t-1}]$.[10]

We now define village payoffs. In any generation, if $\underline{\mu}$ is the final culture distribution in $\mathbb{V}$, we posit that post migration, the payoff to a $\mathbb{V}$-resident situated at culture position p will be:

$$V(p \,|\, \underline{\mu}) := w + b\big(m[\underline{\mu}], d(p, \underline{\mu})\big),$$

where $w > 0$ is the village wage rate, and $b\big(m[\underline{\mu}], d(p, \underline{\mu})\big)$ is the social-interaction benefit at p. We posit that $b(\cdot,\cdot)$ at p depends positively on the measure $m[\underline{\mu}]$ of $\mathbb{V}$-residents, and negatively on the p-resident's aggregate culture distance from all $\mathbb{V}$ peers given by the *culture distance measure* $d(p, \underline{\mu})$:

$$d(p, \underline{\mu}) := \left(\frac{1}{P-1}\right) \cdot \sum_{p'=1}^{P} [|p - p'| \cdot \mu(p')].^{11}$$

We assume that for any $\underline{\mu}$ and $\underline{\mu}'$ of equal measure $m \in (0,1]$, and for any $p \in \mathbb{P}$, $b\big(m, d(p, \underline{\mu})\big) > b\big(m, d(p, \underline{\mu}')\big)$ if and only if $d(p, \underline{\mu}) < d(p, \underline{\mu}')$. Regarding the impact of a change in $\mathbb{V}$ population, in size and/or distribution, on $b(\cdot,\cdot)$, we assume that the following *dominance condition* **[D]** holds.

[D]: For $1 \geq \tilde{m} > m > 0$, $b(\tilde{m}, d) > b(m, d)$ for all $d \in (0,1]$; and for $\{\underline{\mu}, \underline{\tilde{\mu}}\} \in \Delta_+$, if $\tilde{\mu}(p'') > \mu(p')$ for some $p'' \in \mathbb{P}$ and $\tilde{\mu}(p') = \mu(p')$ for all other $p' \in \mathbb{P}$, then $b(m[\underline{\tilde{\mu}}], d(p, \underline{\tilde{\mu}})) \geq b(m[\underline{\mu}], d(p, \underline{\mu}))$ for all $p \in \mathbb{P}$.

[D] requires the impact of $\mathbb{V}$'s population-size on $b(\cdot,\cdot)$ to be positive and to dominate the negative impact of culture dispersion in $\mathbb{V}$, so that starting from any $\underline{\mu}$ in $\mathbb{V}$, if more minority members are 'added' to $\mathbb{V}$ given unchanged culture-positions of the original members, then interaction benefits to the original members go up; see a parametric representation of $b(\cdot,\cdot)$ in Section 4.1. Since $d(p, \underline{\mu}) = 0$ for all p when $m[\underline{\mu}] = 0$, **[D]** ensures that if $m[\underline{\mu}] > 0$ then $b\big(m[\underline{\mu}], d(p, \underline{\mu})\big) \geq b(0,0)$. Our assumptions thus imply that $b(0,0)$ is the smallest value and $b(1,0)$ is the largest value of $b(\cdot,\cdot)$.

Next, we define minority city-payoffs. Let $\underline{\mu}$ and $\underline{\kappa}$ be the final culture distributions of the minority sub-populations in $\mathbb{V}$ and in $\mathbb{C}$ in some period, with $m[\underline{\mu}] + m[\underline{\kappa}] = 1$. We posit that post migration,

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[10] As $\sigma < 1$, and $g(p) < 1$ and $\mu_0(p) > 0$ for all $p \in \mathbb{P}$, $\mu_t(p)$ will be strictly positive for all $p \in \mathbb{P}$ and $t \geq 1$; further, as there can be no reverse-migration, $m[\underline{\mu}_t]$ will be no larger than $m[\underline{\mu}_{t-1}]$ for all $t \geq 1$.

[11] The term $\left(\frac{1}{P-1}\right)$ in $d(.,.)$ normalizes the distance measure with respect to the number of culture position in $\mathbb{P}$, ensuring that $d(p, \underline{\mu}) \in [0,1]$. We study the case where $d(.,.)$ is *linear* in $|p - p'|$ because it is not obvious to us whether considering a concave or a convex distance measure would be more appropriate. We note, however, that our central results will remain valid for any $d(.,.)$ that is arbitrarily close to being linear.

payoff to a minority $\mathbb{C}$-resident at position p (defined in terms of $\underline{\mu}$, rather than in terms of $\underline{\kappa}$) will be:

$$C(p \,|\, \underline{\mu}) := [w + \Pi(p)] + [a(1 - m[\underline{\mu}]) - \Gamma(p)].$$

Here, $[w+\Pi(p)]$ is the city-wage that includes a 'city-premium' $\Pi(p) > 0$, and $[a(1-m[\underline{\mu}])-\Gamma(p)]$ is the *net* social-interaction benefit for a minority $\mathbb{C}$-resident situated at p. We posit that interaction effects in $\mathbb{C}$ arise from two sources: a minority $\mathbb{C}$-resident enjoys interaction benefit of $a(m[\underline{\kappa}]) \equiv a(1-m[\underline{\mu}]) > 0$ from minority co-residents in $\mathbb{C}$ (independent of their culture distribution in $\mathbb{C}$), but incurs a *discrimination cost* $\Gamma(p) > 0$ due to her distance from the mainstream culture $P^{main} > P$. We assume that the 'city wage premium net of discrimination costs' $[\Pi(\cdot)-\Gamma(\cdot)]$ is strictly increasing in p, and the 'minority complementarity effect' $a(\cdot)$ is strictly increasing in $m[\underline{\kappa}]$.[12]

Our posited city payoff structure implies that each $\mathbb{C}$-born minority member will culture-shift right from every birth-position $p < P$ to raise received wages and to reduce discrimination. Thus, when studying a generational equilibrium in any period, we will take as given this culture-shift behaviour of the $\mathbb{C}$-minority, and focus only on the culture-shift and migration decisions of the $\mathbb{V}$-minority.

2.3 Equilibria within and across Minority Generations

We assume that minority members have *short-term* (*within-generation*) *goals* – they ignore the impact of their decisions on their progeny. Below, we describe the implications of such 'short-termism' for generational minority decision-making; we then discuss our reasons for assuming short-termism.

A minority member born in $\mathbb{V}$ in period t will observe her generation's inherited $\underline{\mu}_{t-1}$, and form a *conjecture* $\underline{\mu}_t^e$ about the final culture distribution in $\mathbb{V}$.[13] Given $\underline{\mu}_t^e$, the expected-utility maximizing member will then determine her feasible *sequentially-optimal decision rule*: she will identify her optimal migration decision from all feasible acquired culture positions, and then backward-induct her optimal culture-shift decision. Under a *commonly-held conjecture* $\underline{\mu}_t^e$, the collection of sequentially-optimal decision rules of all $\mathbb{V}$-born minority in period t will constitute a *sequentially-optimal decision profile* for generation t. Every such decision profile will map the generation's inherited distribution $\underline{\mu}_{t-1}$ to a *unique* final culture distribution $\underline{\mu}_t$ under a common conjecture $\underline{\mu}_t^e$.

We define an *intra-generation equilibrium outcome* for a minority generation t in terms of the final culture distribution in $\mathbb{V}$. Given inherited $\underline{\mu}_{t-1}$ in period $t \geq 1$, $\underline{\mu}_t^*$ will be an intra-generation (rational-expectations) equilibrium outcome if: (a) the $\mathbb{V}$-born minority in period t, given short-term goals, pursue a sequentially-optimal decision profile X_t^* under the common conjecture $\underline{\mu}_t^*$; and (b) X_t^* generates $\underline{\mu}_t^*$ (from $\underline{\mu}_{t-1}$) as the final culture distribution in $\mathbb{V}$. Starting from $\underline{\mu}_0$, we will call an infinite sequence of such equilibrium outcomes $\{\underline{\mu}_t^* : t \geq 1 \,|\, \underline{\mu}_0\}$ an *equilibrium trajectory* if

[12] Our 'minority complementarity in $\mathbb{C}$' assumption (that $a'(\cdot) > 0$) is similar to that in Sato and Zenou (2020). We assume that 'within-minority culture dispersion' is ignored by the $\mathbb{C}$-minority as that is vastly dominated by culture heterogeneity between the mainstream and the minority in $\mathbb{C}$.

[13] When a member observes $\underline{\mu}_{t-1}$, she also learns the measure of the minority population $(1 - m[\underline{\mu}_{t-1}])$ born in $\mathbb{C}$, and $\underline{\mu}_t^e$ also includes her conjecture about the final minority population $(1 - m[\underline{\mu}_t^e])$ in $\mathbb{C}$.

$\underline{\mu}_t^*$ is an intra-generation equilibrium outcome given inherited $\underline{\mu}_{t-1}^*$ in every period $t \geq 1$.[14]

The assumption of minority short-termism greatly simplifies our analysis by ensuring that the only inter-generational link that exists in our model is the following: for all $t \geq 1$, minority decisions period in t determine the inherited culture distributions in $\mathbb{V}$ and $\mathbb{C}$ in period $t+1$. But we assume minority short-termism also for an additional reason. To explain that, we clarify that our analysis will be carried out under the assumption that the following *payoff-ranking condition* [**R**] holds.

[**R**]: $V_{max} \equiv w+b(1,0) > C_{min} \equiv w+\Pi(1)-\Gamma(1)+a(0)$; $C_{max} \equiv w+\Pi(P)-\Gamma(P)+a(1) > V_{min} \equiv w+b(0,0)$.

[**R**] requires the minimum $\mathbb{C}$-payoff C_{min} to be below the maximum $\mathbb{V}$-payoff V_{max} (since otherwise all $\mathbb{V}$-born minority will always prefer to migrate), and the minimum $\mathbb{V}$-payoff V_{min} to be below the maximum $\mathbb{C}$-payoff C_{max} (since otherwise all $\mathbb{V}$-born minority will always prefer to stay back). Given that, recognize that if each minority member in every period cares *predominantly and sufficiently* about the generational welfare of her distant progeny, then the following will be true: If $C_{max} > V_{max}$ (respectively, if $V_{max} > C_{max}$), there will exist an equilibrium trajectory in which *everyone* born in $\mathbb{V}$ in every period will prefer to migrate to $\mathbb{C}$ (respectively, to culture-shift towards some common culture position $\bar{p}$ in $\mathbb{V}$). This equilibrium trajectory – which will generate the long-run outcome in which the minority population will be completely assimilated at P in $\mathbb{C}$ (respectively, completely assimilated at $\bar{p}$ in $\mathbb{V}$) – will Pareto-dominate all other equilibrium trajectories, with individual minority payoffs attaining the *optimum optimorum* level in the long-run. Thus, if we focused on Pareto-dominant equilibrium trajectories, we would conclude that minority members' concerns about descendant welfare induce the community either to get 'fully assimilated' with the mainstream or to get 'fully homogenized' in the minority village in the long-run. Without denying real-world progeny concerns, we aim to explore the consequences of decision-making by successive minority generations when members have significantly shorter-term objectives.[15]

3 Existence of Intra-generation Equilibria

In this section, we prove that at least one *intra-generation equilibrium* exists in every period $t \geq 1$, given inherited distribution $\underline{\mu}_{t-1} \in \Delta_+$. To establish existence, we need to find a culture distribu-

[14] In any period t, an intra-generation equilibrium outcome for the minority population spread across $\mathbb{V}$ and $\mathbb{C}$ needs to be described by a vector $\{\underline{\mu}_t^*, \underline{\kappa}_t^*\}$. But given that apart from $\underline{\mu}_t^*$, only $m[\underline{\kappa}_t^*] = 1 - m[\underline{\mu}_t^*]$ matters for decision-making, each generation's equilibrium outcome can indeed be adequately represented by $\underline{\mu}_t^*$. We will thus use the sequence $\{\underline{\mu}_t^* : t \geq 1 \,|\, \underline{\mu}_0\}$ to represent an (intertemporal) equilibrium trajectory. Further, as is well understood, there can be multiple (rational-expectations) equilibrium outcomes in any given generation – disparate individual expectations of aggregate behaviors can be self-fulfilling prophecies; as a result, many equilibrium trajectories can arise from a particular initial culture distribution.

[15] We note that our current results will continue to hold when minority parents care about their children's payoffs to *a limited extent*; see footnote 21 indicating the robustness of our simulation results in this regard. Also note that minority short-termism can generate reverse-migration incentives: '$(t+n)^{\text{th}}$ generation $\mathbb{C}$-born progeny' of a 'period t-born $\mathbb{C}$-migrant' may want to reverse-migrate to $\mathbb{V}$ depending on how the 'left behind culture distribution' in $\mathbb{V}$ evolved over the intervening n periods; we study this issue in Section 5.2.

tion $\underline{\mu}_t^*$ that maps onto itself when $\mathbb{V}$-residents make their optimal decisions under the conjecture $\underline{\mu}_t^*$ (starting from $\underline{\mu}_{t-1}$). This exercise is complicated by the fact that the agents' decisions are sequential and their choice spaces are discrete. Our analysis clarifies that equilibrium existence can be guaranteed for each generation by requiring specific fractions of the minority population to make distinct choices precisely when they are indifferent between those choices.

We begin by describing *feasible* and *sequentially optimal* decision rules of a minority member either born at or (after culture-shifting) situated at some $p \in \mathbb{P}$ in $\mathbb{V}$. Within a generation, a feasible decision rule χ_p at p will be specified as follows (where $\mathsf{S}^\mathsf{n} := \{\mathsf{x} \in \mathbb{R}_+^\mathsf{n} : \sum_{\mathsf{k}=1}^{\mathsf{n}} \mathsf{x}_\mathsf{k} \leq 1\}$ is the n-dimensional unit simplex): $\chi_p = \{\chi_p^S = (\chi_p^{SL}, \chi_p^{SR}) \in \mathsf{S}^2; \chi_p^M \in \mathsf{S}^1\}$, where χ_p^{SL} (respectively, χ_p^{SR}) is the probability with which a member born at p will culture-shift left (respectively, right), and χ_p^M is the probability with which a member situated at acquired position p will migrate to $\mathbb{C}$.[16]

To identify optimal decisions of an expected-utility maximizing member, we define $U^*(p \,|\, \underline{\mu}^e)$ to be the maximal expected payoff of a member situated at acquired p in $\mathbb{V}$ under conjecture $\underline{\mu}^e$:

$$U^*(p \,|\, \underline{\mu}^e) := [1 - g(p)].V(p \,|\, \underline{\mu}^e) + [g(p)].\max\{V(p \,|\, \underline{\mu}^e), C(p \,|\, \underline{\mu}^e)\}.$$

Under $\underline{\mu}^e$, it will be sequentially optimal for a member born at p' to act as follows: if given the chance to culture-shift then situate at any $p \in \{\max\{1, p'-1\}, p', \min\{P, p'+1\}\}$ for which $U^*(p \,|\, \underline{\mu}^e)$ is maximal, and then if given the chance, migrate if (resp., only if) $C(p \,|\, \underline{\mu}^e) >$ (resp., $\geq$) $V(p \,|\, \underline{\mu}^e)$.

In order to state a (symmetric) sequentially-optimal decision rule $\bar{\chi}_p(.) \equiv \{\bar{\chi}_p^S(.); \bar{\chi}_p^M(.)\}$, we need to specify a vector of fractions: $\underline{\varphi}_p \equiv (\varphi_p^{ls}, \varphi_p^{rs}, \varphi_p^{lr}, \varphi_p^l, \varphi_p^r) \in [0,1]^3 \times \mathsf{S}^2$, and a scalar $\psi_p \in [0,1]$, such that a member's culture-shift and/or migration will occur with a specific probability when she is indifferent. While the optimal decision rule $\bar{\chi}_p(.)$ for $p \in \mathbb{P}$ is fully stated in Appendix A, it is easy to see its structure for every $p \in \mathbb{P}$, given $\{V(p \,|\, \underline{\mu}^e), C(p \,|\, \underline{\mu}^e), U^*(p \,|\, \underline{\mu}^e)\}$. For instance, if $U^*(p-1 \,|\, \underline{\mu}^e) > \max\{U^*(p \,|\, \underline{\mu}^e), U^*(p+1 \,|\, \underline{\mu}^e)\}$ then $\bar{\chi}_p^{SL}(\underline{\mu}^e \,|\, \underline{\varphi}_p) = 1$ and $\bar{\chi}_p^{SR}(\underline{\mu}^e \,|\, \underline{\varphi}_p) = 0$, while if $U^*(p-1 \,|\, \underline{\mu}^e) = U^*(p \,|\, \underline{\mu}^e) = U^*(p+1 \,|\, \underline{\mu}^e)$ then $\bar{\chi}_p^{SL}(\underline{\mu}^e \,|\, \underline{\varphi}_p) = \varphi_p^l$ and $\bar{\chi}_p^{SR}(\underline{\mu}^e \,|\, \underline{\varphi}_p) = \varphi_p^r$; and if $C(p \,|\, \underline{\mu}^e) > V(p \,|\, \underline{\mu}^e)$ then $\bar{\chi}_p^M(\underline{\mu}^e \,|\, \psi_p) = 1$, while if $C(p \,|\, \underline{\mu}^e) = V(p \,|\, \underline{\mu}^e)$ then $\bar{\chi}_p^M(\underline{\mu}^e \,|\, \psi_p) = \psi_p$.

We define $\underline{\varphi} \equiv (\underline{\varphi}_1, \cdots, \underline{\varphi}_P)$ with $\underline{\varphi}_p \equiv (\varphi_p^{ls}, \varphi_p^{rs}, \varphi_p^{lr}, \varphi_p^l, \varphi_p^r)$ for all $p \in \mathbb{P}$, and $\underline{\psi} \equiv (\psi_1, \cdots, \psi_P)$. We let Φ denote the set of all feasible sets of fraction vectors $\underline{\varphi}$ and Ψ denote the set of all feasible sets of fractions $\underline{\psi}$. For a selected $\underline{\varphi} \in \Phi$ and $\underline{\psi} \in \Psi$, $\bar{\chi}_p(\underline{\mu} \,|\, \underline{\varphi}_p, \psi_p) \equiv (\bar{\chi}_p^S(\underline{\mu} \,|\, \underline{\varphi}_p); \bar{\chi}_p^M(\underline{\mu} \,|\, \psi_p))$ will be the *unique* sequentially-optimal decision rule relevant for culture position p in $\mathbb{V}$ in a generation under the conjecture $\underline{\mu}$.[17] Collection of these decision-rules will constitute the *optimal migration profile* $\bar{Z}(\underline{\mu} \,|\, \underline{\varphi})$ and the *sequentially optimal culture-shift profile* $\bar{Y}(\underline{\mu} \,|\, \underline{\psi})$; taken together, they will constitute the unique *sequentially optimal decision profile* $\bar{X}(\underline{\mu} \,|\, \underline{\varphi}, \underline{\psi}) = (\bar{Y}(\underline{\mu} \,|\, \underline{\varphi}), \bar{Z}(\underline{\mu} \,|\, \underline{\psi}))$ under

[16] Here, $(1 - \chi_p^{SL} - \chi_p^{SR})$ (respectively, $(1 - \chi_p^M)$) will be the probability with which a member born at p in $\mathbb{V}$ will not culture-shift (respectively, a member situated at acquired p will stay back in $\mathbb{V}$).

[17] We clarify that starting from $\underline{\mu}^i$, under a conjecture $\underline{\mu}$, if measure m_p members are born at p and are indifferent between culture-shifting left and not shifting, then $\bar{\chi}^{LS}(.)$ will specify that $[\varphi_p^{ls}.\sigma]$ fraction of m_p will shift left; and if measure m_p^a members are situated at acquired p and are indifferent between migrating and staying back, then $\bar{\chi}^M(.)$ will specify that $[\psi_p.g(p)]$ fraction of m_p^a will migrate; etc..

$\underline{\mu}$ for the $\mathbb{V}$-minority in that generation. $\bar{X}(\underline{\mu}\,|\,\underline{\varphi},\underline{\psi})$ will generate a *unique final culture distribution* $\underline{\mu}^f(\underline{\mu},\underline{\mu}^i,\underline{\varphi},\underline{\psi}) \equiv \big(\mu^f(1\,|\,\underline{\mu},\underline{\mu}^i,\underline{\varphi},\underline{\psi}),\cdots,\mu^f(P\,|\,\underline{\mu},\underline{\mu}^i,\underline{\varphi},\underline{\psi})\big)$, where $\mu^f(p\,|\,\underline{\mu},\underline{\mu}^i,\underline{\varphi},\underline{\psi})$ is the measure of minority members who will be finally situated at p in $\mathbb{V}$ in the generational equilibrium.[18]

Before stating the equilibrium existence result, we present a thought experiment that captures its logic. In a period t given inherited $\underline{\mu}_{t-1}$, a Walrasian auctioneer announces a conjecture $\underline{\mu}_t{}'$ about the final culture distribution in $\mathbb{V}$ and asks the $\mathbb{V}$-born to report their sequentially-optimal decisions under $\underline{\mu}_t{}'$. Once they do so, the auctioneer chooses fraction sets $\{\underline{\varphi},\underline{\psi}\}$ that are used to *split* the total measures of those $\mathbb{V}$-members who are *indifferent* between (subsets of) discrete actions. Starting from $\underline{\mu}_{t-1}$, the members' decisions and the assigned fractions generate a unique final culture distribution $\underline{\mu}_t$. If $\underline{\mu}_t$ equals $\underline{\mu}_t{}'$, then an equilibrium is identified. If not, the auctioneer chooses a different set of 'splitting fractions' $\tilde{\underline{\varphi}} \in \Phi$ and $\tilde{\underline{\psi}} \in \Psi$ (keeping the announced conjecture $\underline{\mu}_t{}'$ unchanged) and repeats the process. If no set of feasible splitting fractions lead to a fixed point given $\underline{\mu}_t{}'$, then the auctioneer announces a different conjecture $\underline{\mu}_t{}''$ and repeats the process. Theorem 3.1 below assures us that the auctioneer will eventually succeed in finding a fixed point.

Formally, we fix a generation and its inherited distribution $\underline{\mu}^i \in \Delta_1^+$ and note that a culture distribution $\underline{\mu}^*$ will be an equilibrium outcome if and only if for every $p \in \mathbb{P}$, $\mu^*(p)$ equals $\mu^f(p\,|\,\underline{\mu}^*,\underline{\mu}^i,\underline{\varphi},\underline{\psi})$ for some $\underline{\varphi} \in \Phi$ and $\underline{\psi} \in \Psi$. We have the following existence result, whose proof is in Appendix A.

Theorem 3.1. *For $\underline{\mu}^i \in \Delta_+$, let $\Delta^\dagger$ be a non-empty, convex, compact subset of $\Delta_+[\underline{\mu}^i]$. Define a correspondence $\mathcal{M}^f(\cdot\,|\,\underline{\mu}^i)$ on the domain $\Delta^\dagger$ by $\mathcal{M}^f(\underline{\mu}\,|\,\underline{\mu}^i) \equiv \big\{\underline{\mu}^f(\underline{\mu},\underline{\mu}^i,\underline{\varphi},\underline{\psi}) : \underline{\varphi} \in \Phi, \underline{\psi} \in \Psi\big\}$. If $\mathcal{M}^f(\underline{\mu}\,|\,\underline{\mu}^i) \subseteq \Delta^\dagger$ for all $\underline{\mu} \in \Delta^\dagger$ given $\underline{\mu}^i$, then $\mathcal{M}^f(\cdot\,|\,\underline{\mu}^i)$ has a fixed point in $\Delta^\dagger$.*

Note that $\Delta_+[\underline{\mu}^i]$ is a non-empty, convex, compact set, and $\mathcal{M}^f(\underline{\mu}\,|\,\underline{\mu}^i) \subseteq \Delta_+[\underline{\mu}^i]$ for all $\underline{\mu} \in \Delta_+[\underline{\mu}^i]$. So, if we take $\Delta^\dagger = \Delta_+[\underline{\mu}^i]$ in Theorem 3.1 then $\mathcal{M}^f(\cdot\,|\,\underline{\mu}^i)$ will have a fixed point in $\Delta_+[\underline{\mu}^i]$ which will be an intra-generation equilibrium. We thus have the following corollary.

Corollary 3.2. *For every minority generation $t \geq 1$, with an inherited culture distribution $\underline{\mu}_{t-1} \in \Delta_+$, there exists an intra-generation equilibrium $\underline{\mu}_t^*(\underline{\mu}_{t-1}) \in \Delta_+[\underline{\mu}_{t-1}]$.*

The above results might be of independent interest in the context of research on equilibrium existence in *social interactions models* (see Brock and Durlauf (2001), Horst and Scheinkman (2006), and the references therein) where agents care about *characteristics distributions and actions* of neighbours, and take sequential decisions under uncertainty on a discrete choice set over multiple stages.

4 Long-run Assimilation Patterns

In this section, we run simulations on a set of parametric examples that satisfy our model specifications (including **[D]** and **[R]**). In different simulation regimes, we identify equilibrium trajectories

[18] The distribution $\underline{\mu}^f$ arises from the composition of the culture-shift profile $\bar{Y}(\underline{\mu}\,|\,\underline{\varphi})$ and the migration profile $\bar{Z}(\underline{\mu}\,|\,\underline{\psi})$: $\bar{Y}(\underline{\mu}\,|\,\underline{\varphi})$ maps $\underline{\mu}^i$ to a unique acquired culture distribution $\underline{\mu}^a$ and $\bar{Z}(\underline{\mu}^e\,|\,\underline{\psi})$ maps $\underline{\mu}^a$ to $\underline{\mu}^f$.

that exhibit distinct long-run outcomes. We use these findings to motivate our definitions of alternative ‘assimilation trajectories’. We then highlight some important features of intra-generation equilibria in our model that are instrumental in generating distinct long-run assimilation patterns.

4.1 Equilibrium Trajectories: Definitions and Simulations

For $P \geq 2$, we define $\mathcal{E}^{[P]}$ to be the set of parametric examples **[E]** that contain the following functional forms for $\mathbb{V}$ and $\mathbb{C}$ payoffs, and satisfy the following parameter restrictions:

[E1] $V(p \,|\, \underline{\mu}) = w + \frac{\beta.m[\underline{\mu}]}{1+\lambda.d(p,\underline{\mu})}$, with $w > 0$, $\beta > 0$, and $\lambda > 0$;

[E2] $C(p \,|\, \underline{\mu}) = w + [\delta + \theta.p] + \alpha.(1 - m[\underline{\mu}])$, with $\theta > 0$ and $\alpha > 0$; and

[E3] (i) $\lambda \leq 1$, (ii) $\delta + \theta < \beta < \delta + P\theta + \alpha$, (iii) $\lambda < \theta$.

[E1] and [E2] impose specific functional forms for social-interaction effects in the village and in the city, with $b\big(m[\underline{\mu}], d(p,\underline{\mu})\big) = \frac{\beta.m[\underline{\mu}]}{1+\lambda.d(p,\underline{\mu})}$, $[\Pi(p) - \Gamma(p)] = [\delta + \theta.p]$, and $a(m[\underline{\kappa}]) = \alpha.(m[\underline{\kappa}])$. [E3(i)] guarantees **[D]**, and [E3(ii)] implies that **[R]** holds with $C_{max} > V_{max}$. Finally, [E3(iii)] ensures that the following *global preference-monotonicity condition* **[P]** holds in every **[E]**$\in \mathcal{E}^{[P]}$.

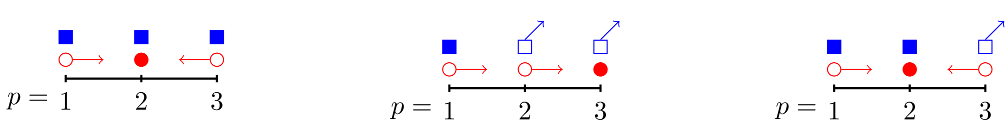
[P]: For all $\underline{\mu} \in \Delta_+$ and *for all* $p \in \mathbb{P}$, the payoff difference $[C(p|\underline{\mu}) - V(p|\underline{\mu})]$ increases in p.

Condition **[P]** states that if a member situated at *any* acquired culture position $p \in \mathbb{P}$ prefers to migrate to $\mathbb{C}$ (respectively, to stay back in $\mathbb{V}$) then all members situated at acquired positions $p' > p$ (respectively, $p'' < p$) will also prefer to migrate (respectively, to stay back). While we will assume a condition weaker than **[P]** when presenting our central ‘assimilation results’ in Sections 5 – 6, we will derive additional results about equilibrium trajectories when **[P]** holds.

We run simulations on parametric examples **[E]** belonging in the set $\mathcal{E}^{[P=3]}$. To that end, we define a set of feasible intra-generation decision profiles that specify *preferred* culture-shift and migration decisions of minority members (i.e., those decisions that they *will* take when they get the relevant opportunities) in $\mathbb{V}$ from every position $p = 1, 2, 3$. [See Figure 1.]

Figure 1: Minority Decision Profiles DP(I), DP(II.2), and DP(III)

● : prefer not to culture-shift, ○⟶: prefer to culture-shift, ■ : prefer not to migrate, □↗ : prefer to migrate



DP(I) DP(II.2) DP(III)

Profile DP(I) : members born at culture positions 1 and 3 prefer to culture-shift toward position 2, while those born at position 2 prefer not to culture-shift; and no one prefers to migrate to $\mathbb{C}$.

Profile DP(II.q) : members born at position 1 (respectively, position 2) prefer to culture-shift toward position 2 (respectively, position 3), while those born at position 3 prefer not to shift; and only those members situated at acquired position $p \geq q$, for $q = 1, 2, 3$, prefer to migrate to $\mathbb{C}$.

Profile DP(III) : members born at 1 and 3 prefer to culture-shift toward position 2, while those born at 2 do not prefer to shift; and only those situated at acquired position 3 prefer to migrate.

Next, we define the following trajectories of intra-generation outcomes in $\mathbb{V}$:

Trajectory TA : In all periods $t \geq 1$, the intra-generation outcome in $\mathbb{V}$ is generated by DP(I).

Trajectory $TB(t_1, t_2)$: In each of the periods $\{1, \cdots, t_1\}$, the intra-generation outcome in $\mathbb{V}$ is generated by DP(II.3); in each of the periods $\{t_1 + 1, \cdots, t_2\}$, the outcome is generated by DP(II.2); and in all periods after t_2, the outcome is generated by DP(II.1).[19]

Trajectory TC: In all periods $t \geq 1$, the intra-generation outcome in $\mathbb{V}$ is generated by DP(III).

Our first set of simulations study a case where there is no equilibrium migration. The following parameter values constitute our Simulation Regime [1]: $\alpha = 1.4,\ \beta = 4,\ \delta = -0.9,\ \lambda = 0.5,\ \theta = 1.2,$ $\sigma = 0.3$, and $\mathbb{G} = \{0.1, 0.3, 0.5\}$; and a 'symmetric' initial culture distribution $\underline{\mu}_0^{[1]} = \{0.3, 0.4, 0.3\}$ (there is no need to specify a numerical value for w in our simulations). In Regime [1], trajectory TA is an *equilibrium outcome trajectory*, while $TB(\cdot)$ and TC are not. Along TA, all minority members stay back in $\mathbb{V}$ in every period, and come 'culturally closer' to each other over generations. Beyond the 60^{th} period, position 2 in $\mathbb{V}$ is the livelihood-location of (almost) all minority members in every period, each of whom receive payoff arbitrarily close to $V_{max} = w + 4$. We view the long-run outcome generated by TA as one of 'complete assimilation in $\mathbb{V}$'.

Our second set of simulations identify a case where the entire minority population 'eventually resides' in $\mathbb{C}$. From Regime [1], we go to Simulation Regime [2] by making only the following changes: we set $\beta = 3$ and $\delta = -0.8$ (to achieve a uniform fall in the $\mathbb{V}$-payoff level and a uniform rise in the $\mathbb{C}$-payoff level); and take a 'more left-skewed' initial culture distribution: $\underline{\mu}_0^{[2]} = \{0.15, 0.45, 0.4\}$. In Regime [2], TB(1,2) is an *equilibrium outcome trajectory*, while TA and TC are not. Along trajectory TB(1,2), in every period, all minority members in $\mathbb{V}$ culture-shift right and some migrate to $\mathbb{C}$, and this leads to 'complete assimilation in $\mathbb{C}$'. Beyond the 50^{th} period, position 3 in $\mathbb{C}$ is the birth-position and livelihood-location of (almost) all minority members in every generation, each of whom receive payoff arbitrarily close to $C_{max} = w + 4.2$. We note that TB(0,1) is also an equilibrium trajectory in Regime [2], and generates the same long-run outcome as TB(1,2).

In our third set of simulations, the minority group does not completely assimilate in any one location. We create Simulation Regime [3] by setting $\beta = 3.5$ (which is intermediate to β-values in Regimes [1] and [2]), and by setting $\delta = -1$ and $\theta = 1.3$ (thereby making the $\mathbb{C}$-payoff rise more steeply in p as compared to that in Regimes [1] and [2]); and take an initial culture distribution that is 'right-skewed': $\underline{\mu}_0^{[3]} = \{0.4, 0.5, 0.1\}$. In Regime [3], TC is an *equilibrium outcome trajectory*, while TA and $TB(\cdot)$ are not. Along TC, in every period, all members born in $\mathbb{V}$ prefer to culture-shift to position 2 (thereby coming culturally closer), while those situated in acquired position 3

[19] $TB(0, t)$ will refer to the trajectory: In periods $\{1, \cdots, t\}$, the intra-generation outcome in $\mathbb{V}$ is generated by DP(II.2), and in all periods after t, the outcome in $\mathbb{V}$ is generated by DP(II.1); and $TB(0, 0)$ will refer to the trajectory: In all periods $t \geq 1$, the outcome in $\mathbb{V}$ is generated by DP(II.1).

prefer to migrate.[20] In every period beyond the 50^{th}, about 94.6% of all minority members are born at and live at position 2 in $\mathbb{V}$, each getting payoff close to $w + 3.312$; and about 5.4% of all members are born at and live at position 3 in $\mathbb{C}$, each getting payoff close to $w + 2.975$. We identify this long-run outcome as one of 'segmented assimilation' of the minority group across $\mathbb{V}$ and $\mathbb{C}$.

Formally, we will say that an equilibrium trajectory $\{\underline{\mu}_t^* : t \geq 1 | \underline{\mu}_0\}$ generates *complete assimilation in* $\mathbb{V}$ (respectively, *complete assimilation in* $\mathbb{C}$) if, for any $\varepsilon > 0$ however small, there exists finite $t(\varepsilon) \geq 1$ such that $m[\underline{\mu}_\tau^*] < \varepsilon$ (respectively, $m[\underline{\mu}_\tau^*] \in (1 - \varepsilon, 1)$) in all periods $\tau \geq t(\varepsilon)$. In contrast, we will say an equilibrium trajectory $\{\underline{\mu}_t^* : t \geq 1 | \underline{\mu}_0\}$ generates *segmented assimilation* if there exists some $\eta \in (0, 1)$ such that for all $\varepsilon > 0$ (however small) there exists a finite $t(\varepsilon) > 0$ such that $m[\underline{\mu}_\tau^*] \in (\eta - \varepsilon, \eta + \varepsilon)$ in all periods $\tau \geq t(\varepsilon)$ (with the rest of the minority group in $\mathbb{C}$).

Table 1: Minority Payoffs under the 'static conjecture' in Simulation Regimes [1], [2], and [3]

regimes	Regime [1]			Regime [2]			Regime [3]		
positions	1	2	3	1	2	3	1	2	3
$\underline{\mu}_0$	0.3	0.4	0.3	0.15	0.45	0.4	0.4	0.5	0.1
$C(p \mid \underline{\mu}_0)$	0.3	1.5	2.7	0.4	1.6	2.8	0.3	1.6	2.9
$V(p \mid \underline{\mu}_0)$	3.2	3.49	3.2	2.29	2.64	2.54	2.79	3.11	2.63
$U^*(p \mid \underline{\mu}_0)$	3.2	3.49	3.2	2.29	2.64	2.67	2.79	3.11	2.77

In our simulations, the structure of $\underline{\mu}_0$, and of the payoff functions in $\mathbb{V}$ and in $\mathbb{C}$, play important roles in generating specific equilibrium assimilation patterns. Table 1 presents the city, village, and expected payoffs of a minority member in Regimes [1] – [3], for different acquired culture position and under the 'static conjecture' that *all* members will stay at their birth-positions in $\mathbb{V}$ under $\underline{\mu}_0$. Note that $\underline{\mu}_0^{[3]}$ is significantly more right-skewed than $\underline{\mu}_0^{[1]}$ and $\underline{\mu}_0^{[2]}$; $C(p \mid \underline{\mu}_0^{[3]})$ rises more steeply in p as compared to $C(p \mid \underline{\mu}_0^{[1]})$ and $C(p \mid \underline{\mu}_0^{[2]})$; and while the $\mathbb{V}$-payoff peaks at the median position '2' in all three regimes, $V(3 \mid \underline{\mu}_0) < V(1 \mid \underline{\mu}_0)$ only in Regime [3]. Thus, in Regime [3], for the smaller culture positions, the $\mathbb{V}$-payoff dominates the $\mathbb{C}$-payoff more than in Regime [2]; while for the larger culture positions, the $\mathbb{C}$-payoff dominates the $\mathbb{V}$-payoff in contrast to Regime [1]. As a result, if a first-generation member made decisions holding the static conjecture, in Regime [1] she would prefer to act according to *DP*(I), in Regime [2] she would prefer to act according to *DP*(II.3), and in Regime [3] she would prefer to act according to *DP*(III). The structural differences in the parameter regimes – that generate these distinct preferences – are crucial in determining whether or not there will be segmented assimilation in the long run; we will confirm this intuition in Section 5.4.[21]

[20] This clarifies that a minority member can optimally decide to act as follows along an equilibrium trajectory: "If I get opportunities to culture-shift *and* to migrate then I will move away from the mainstream culture and stay back in $\mathbb{V}$, but if I get a chance *only* to migrate then I will move to $\mathbb{C}$".

[21] The equilibrium trajectories *TA* in Regime [1], *TB*(1, 2) and *TB*(0, 1) in Regime [2], and *TC* in Regime [3] have the following property: if each member in $\mathbb{V}$ makes decisions to maximize {own payoff + ($\nu\times$child's payoff)}, then there exists $\bar{\nu} \in (0, 1)$ such that for all $\nu < \bar{\nu}$, they remain equilibrium trajectories. Further,

In Regime [3], TC leads to a long-run outcome in which individual payoffs of the minority sub-populations in $\mathbb{V}$ and in $\mathbb{C}$ are less than $\min\{V_{max}, C_{max}\}$. This shows that minority decision-making with short-term goals can give rise to a *dynamic dilemma*: in the long run, each member's payoff (irrespective of her location) can be worse than what she would get if all members fully assimilated either in $\mathbb{V}$ or in $\mathbb{C}$. Further, under TC in Regime [3], the minority culture and payoff distributions limit to *two-point distributions*. We will confirm in Section 6.1 that segmented assimilation can generate significant poverty, inequality, and polarization across the entire minority population.

4.2 Entrenched Culture Sets and Median Culture Positions

We now present the notions of 'entrenched culture sets' and 'median culture positions' of minority (sub-)populations that will be central to our analysis of minority assimilation trajectories.

In a generation, we will say that a set of culture positions $\{p', \cdots, p''\} \subset \mathbb{P}$ is an *entrenched culture set* under some $\underline{\mu} \in \Delta_+$ if the following conditions hold when members hold the conjecture $\underline{\mu}$: [1] no member born at any $p \in \{p', \cdots, p''\}$ in $\mathbb{V}$ prefers to culture-shift out of $\{p', \cdots, p''\}$ (members born outside $\{p', \cdots, p''\}$ *can* prefer to culture-shift into it), and [2] no member situated at any acquired position $p \in \{p', \cdots, p''\}$ in $\mathbb{V}$ prefers to migrate. In contrast, we will say that a position $\tilde{p}$ is *unentrenched* under $\underline{\mu}$ in a generation if $\tilde{p}$ does not belong in any entrenched culture set under $\underline{\mu}$.

We will say that an intra-generation equilibrium outcome is *fully-entrenched* (respectively, *wholly-unentrenched*) if in that outcome, entire culture set $\mathbb{P}$ is entrenched (respectively, each p in $\mathbb{P}$ is unentrenched). In contrast, we will say that an equilibrium outcome μ^* is *fractured* if in that outcome, $\mathbb{P}$ can be partitioned into two non-empty sets $\mathbb{P}_E^*$ and $\mathbb{P}_U^*$ such that under μ^*, $\mathbb{P}_E^*$ is an entrenched set while each $p \in \mathbb{P}_U^*$ is unentrenched. Further, we will say that an equilibrium outcome $\underline{\mu}^*$ is *fractured at* p_E^* if: $p_E^* < P$, the set $\{1, \cdots, p_E^*\}$ is entrenched, and all $p > p_E^*$ are unentrenched.[22]

Note that an equilibrium trajectory will generate complete assimilation in $\mathbb{V}$ if and only if *every* intra-generation equilibrium outcome is fully-entrenched, as is the case for TA in Regime [1]. Alternatively, if *every* intra-generation equilibrium outcome is wholly-unentrenched along an equilibrium trajectory beyond some finite period $\tau \geq 1$, then it will lead to complete assimilation in $\mathbb{C}$, as is the case for $TB(1,2)$ in Regime [2] for $\tau = 1$. In contrast, the segmented-assimilation equilibrium trajectory TC in Regime [3] contains a fractured equilibrium in *every* period. We will say that an equilibrium trajectory $\{\underline{\mu}_t^* : t \geq 1 \,|\, \underline{\mu}_0\}$ exhibits *incessant fracture* from period τ (respectively, *perpetual fracture*) if $\underline{\mu}_t^*$ is fractured for all $t \geq \tau$ (respectively, for all $t \geq 1$).

We clarify, *via* our next simulation, that incessant fracture in an equilibrium trajectory, while sufficient for generating segmented assimilation, is not necessary. We create Simulation Regime [4] by changing only the following parameters in Regime [3]: $\beta = 3.55$, $\theta = 1.2$, and $\mathbb{G} = \{0.1, 0.3, 0.45\}$,

as we will explain in Section 5.2, there is no reverse-migration incentive along these trajectories.

[22] In a fractured equilibrium, multiple entrenched culture sets can exist (some subsets of others) with $\mathbb{P}_E^*$ being the largest entrenched set, and there must be migration from some $p \in \mathbb{P}_U^*$. Further, $\mathbb{P}_E^*$ must be of the 'contiguous' form $\{1, \cdots, p_E^*\}$ whenever the preference-monotonicity condition **[P]** holds. Along TC in Regime [3], each generational equilibrium is *fractured at* '2', with $\{1, 2\}$ and $\{2\}$ being entrenched sets.

and setting $\underline{\mu}_0^{[4]} = \underline{\mu}_0^{[3]}$. In Regime [4], the following trajectory TD is an equilibrium trajectory: "The $\mathbb{V}$-population follows the decision profile DP(III) in periods $t = 1, 2, 3$; and follows DP(I) forever after". Along TD, the first three generational equilibria are fractured while all subsequent ones are fully-entrenched, and that is enough to generate segmented assimilation, with 4.83% of the minority population in $\mathbb{C}$ and the rest in $\mathbb{V}$ in the long run.

Next, we study the properties of certain *median culture positions* of a minority generation in $\mathbb{V}$. For $\underline{\mu} \in \Delta_+$ and $p > 1$, we let $\mathbb{M}(p \,|\, \underline{\mu})$ (respectively, $\mathbb{M}(\underline{\mu})$) be the set of median positions of the culture subset $\{1, \cdots, p\}$ (respectively, of $\mathbb{P}$), and define $\hat{r}(p \,|\, \underline{\mu})$ to be the minimal element of the set $\{r \in \{1, \cdots, p\} : m[r \,|\, \underline{\mu}] > \frac{1}{2} m[p \,|\, \underline{\mu}]\}$. By construction, we have the following result: if $m[(\hat{r}(p \,|\, \underline{\mu}) - 1) | \underline{\mu}] < \frac{1}{2} m[p \,|\, \underline{\mu}]$ then $\mathbb{M}(p \,|\, \underline{\mu}) = \{\hat{r}(p \,|\, \underline{\mu})\}$, while if $m[(\hat{r}(p \,|\, \underline{\mu}) - 1) | \underline{\mu}] = \frac{1}{2} m[p \,|\, \underline{\mu}]$ then $\mathbb{M}(p \,|\, \underline{\mu}) = \{\hat{r}(p \,|\, \underline{\mu}) - 1, \hat{r}(p \,|\, \underline{\mu})\}$.[23] We let $\hat{p}(\underline{\mu}) := \hat{r}(P \,|\, \underline{\mu})$, and let $\hat{q}(\underline{\mu})$ denote the smaller element of the median position set $\mathbb{M}(\hat{p}(\underline{\mu}) \,|\, \underline{\mu})$. Note that if $\underline{\mu}$ is an intra-generation equilibrium outcome, then $\hat{p}(\underline{\mu})$ will be the 'larger' median culture position in $\mathbb{V}$ in that outcome; while $\hat{q}(\underline{\mu})$, satisfying $1 \leq \hat{q}(\underline{\mu}) \leq \hat{p}(\underline{\mu})$, will be the 'smaller' median culture position of the sub-population situated in the subset $\{1, \cdots, \hat{p}(\underline{\mu})\}$. In our Simulation Regimes [1], [2], and [3], $\hat{p}(\underline{\mu}_0^{[k]}) = \hat{q}(\underline{\mu}_0^{[k]}) = 2$ for $k = 1, 2, 3$.

Our next result clarifies the influence of median culture positions on minority payoffs. The proof of Lemma 4.1, and proofs all subsequent results in Sections 5–6, are contained in Appendix B.

Lemma 4.1. *For any $\underline{\mu} \in \Delta_+$, if $p^m \in \mathbb{M}(\underline{\mu})$ then $b(m[\underline{\mu}], d(p^m, \underline{\mu})) > b(m[\underline{\mu}], d(p', \underline{\mu}))$ for all $p' \notin \mathbb{M}(\underline{\mu})$. Further, $[C(p \,|\, \underline{\mu}) - V(p \,|\, \underline{\mu})]$ strictly increases in p for all $p \geq p^m$.*

Lemma 4.1 proves that in a generation, when members conjecture $\underline{\mu}$ to be the final culture distribution in $\mathbb{V}$, the social-interaction benefits in $\mathbb{V}$ will be maximal for those who stay back at a median culture position $p^m \in \mathbb{M}(\underline{\mu})$. The lemma also assures us that even if the model parameters are such that global preference monotonicity **[P]** fails to hold, the following 'partial preference monotonicity condition' will always hold in our model: if, under a conjecture $\underline{\mu}$, a member situated at some $p \geq \hat{p}(\underline{\mu})$ prefers to migrate (respectively, not to migrate) to $\mathbb{C}$, then members situated at every $p' > p$ (respectively, $p' \in \{\hat{p}(\underline{\mu}), \cdots, (p-1)\}$) will prefer to migrate (respectively, not to migrate).

5 Segmented Assimilation: Possibility Results

In our model, different kinds of assimilation patterns can emerge as long-run equilibrium outcomes under alternative parameter configurations. Note that in the different simulation regimes in Section 4.1, there exist distinct equilibrium trajectories that generate complete assimilation either in $\mathbb{V}$ or in $\mathbb{C}$, or segmented assimilation. Our aim, now, is to delineate sufficient conditions for the existence of *different forms* of equilibrium trajectories all of which generate segmented assimilation.[24]

In Section 5.1, we delineate conditions that guarantee the existence of a fractured equilibrium in a *single* generation. In Section 5.2, we find sufficient conditions for the existence of an equilibrium

[23] As equilibrium $\{\underline{\mu}_t^*\}$ arise endogenously, we cannot claim that $\mathbb{M}(\underline{\mu}_t^*)$ will be generically unique for all t.

[24] Distinct sufficient conditions for the existence of equilibrium trajectories that generate complete assimilation in $\mathbb{V}$ and in $\mathbb{C}$ are available from the authors.

trajectory that exhibits incessant fracture from a specific period. We then identify, in Section 5.3, minimal sufficient conditions for the existence of an equilibrium trajectory that leads to segmented assimilation. In each case, the sufficient conditions – that need to hold in *one* specific generation – specify a set of rankings of $\mathbb{V}$- and $\mathbb{C}$-payoffs at a particular culture position under a specific conjecture about the final culture distribution in $\mathbb{V}$. In Section 5.4, we clarify that there exist parameter configurations in our model that conform to a set of well-defined features – which we denote as Feature Set [$\ast$] – that make it likely that all stated payoff-ranking conditions will hold, thus guaranteeing that long-run segmentation of the minority group will be an equilibrium outcome.

5.1 Equilibrium Fracture in a Generation

When **[D]** and **[R]** hold, we look for conditions that guarantee that in a specific period, given the inherited culture distribution, there will exist an equilibrium whose outcome is fractured.

Define $\hat{\Delta}_+$ (respectively, $\hat{\Delta}_+[\underline{\mu}]$) to be the set of all culture distributions $\underline{\mu}' \in \Delta_+$ (respectively, $\underline{\mu}' \in \Delta_+[\underline{\mu}]$) for which $\hat{p}(\underline{\mu}') < P$.[25] For an inherited $\underline{\mu}^i \in \hat{\Delta}_+$, define $\mathbb{X}^F(\underline{\mu}^i)$ to be the set of all feasible decision profiles in $\mathbb{V}$ that satisfy the following properties: (a) the culture-subset $\{1, \cdots, \hat{p}(\underline{\mu}^i) - 1\}$ is entrenched, and within it $\{\hat{q}(\underline{\mu}^i), \cdots, \hat{p}(\underline{\mu}^i)\}$ is also entrenched, while all members born in $\{1, \cdots, \hat{q}(\underline{\mu}^i) - 1\}$ (if that set is non-empty) prefer to culture-shift toward $\hat{q}(\underline{\mu}^i)$, and (b) at least measure $[g(P).(1-\sigma).\mu^i(P)]$ of minority members migrate from acquired position P. Letting $\Delta^F(\underline{\mu}^i) \subset \hat{\Delta}_+$ be the set of culture distributions generated from $\underline{\mu}^i$ by all decision profiles in $\mathbb{X}^F(\underline{\mu}^i)$, we define the following feasible decision profiles $X^{F1}(\underline{\mu}^i)$ and $X^{F2}(\underline{\mu}^i)$ belonging in $\mathbb{X}^F(\underline{\mu}^i)$:

$X^{F1}(\underline{\mu}^i) \equiv$ members born at all $p \in \{1, \cdots, \hat{p}(\underline{\mu}^i)\}$ prefer to culture-shift toward $\hat{q}(\underline{\mu}^i)$ and stay back in $\mathbb{V}$; members born at all $p' \in \{\hat{p}(\underline{\mu}^i) + 1, \cdots, P\}$ prefer to culture-shift toward P and migrate;

$X^{F2}(\underline{\mu}^i) \equiv$ members born at all $p \in \{1, \cdots, \hat{p}(\underline{\mu}^i)\}$ (resp. $p' \in \{\hat{p}(\underline{\mu}^i) + 1, \cdots, P\}$) prefer to culture-shift toward $\hat{p}(\underline{\mu}^i)$ (resp., P); and $g(P).(1-\sigma).\mu^i(P)$ measure of members migrate from P.

Proposition 5.1. *Define the following three conditions for a given* $\underline{\mu}^i \in \hat{\Delta}_+$*:*

[F0]: *For all* $p < \hat{p}(\underline{\mu}^i)$, $[V(p \,|\, \underline{\mu}) - C(p \,|\, \underline{\mu})] \geq [V(\hat{q}(\underline{\mu}^i) \,|\, \underline{\mu}) - C(\hat{q}(\underline{\mu}^i) \,|\, \underline{\mu})]$ *for all* $\underline{\mu} \in \Delta^F(\underline{\mu}^i)$;[26]

[F1]: $V(\hat{p}(\underline{\mu}^i) \,|\, \underline{\mu}^{F1}(\underline{\mu}^i)) \;\geq\; C(\hat{p}(\underline{\mu}^i) + 1 \,|\, \underline{\mu}^{F1}(\underline{\mu}^i))$, *for* $\underline{\mu}^{F1}(\underline{\mu}^i)$ *uniquely generated by* $X^{F1}(\underline{\mu}^i)$;

[F2]: $V(P \,|\, \underline{\mu}^{F2}(\underline{\mu}^i)) \;\leq\; C(P \,|\, \underline{\mu}^{F2}(\underline{\mu}^i))$, *for* $\underline{\mu}^{F2}(\underline{\mu}^i)$ *uniquely generated by* $X^{F2}(\underline{\mu}^i)$.

In a generation with inherited $\underline{\mu}^i \in \hat{\Delta}_+$, *if* **[F0]**, **[F1]**, *and* **[F2]** *hold then there exists an equilibrium outcome* $\underline{\mu}^* \in \Delta^F(\underline{\mu}^i)$ *that is fractured at some culture position* $p^*_E \geq \hat{p}(\underline{\mu}^i)$.

Proposition 5.1 asserts that in a generation, if conditions [**F0**], [**F1**], and [**F2**] hold for an inherited $\underline{\mu}^i \in \hat{\Delta}_+$, then there exists a fractured equilibrium containing an entrenched set of 'low culture po-

[25] Thus, it is required that in each culture distribution in $\hat{\Delta}_+$ and $\hat{\Delta}_+[\underline{\mu}]$, at least half the measure of the $\mathbb{V}$-population be born at positions to the left of P. In what follows, we will focus on inherited culture distributions that are more *right-skewed*, with median culture position(s) being 'sufficiently close' to 1.

[26] Condition [**F0**], and condition [**L0**] defined in the statement of Proposition 5.2 in Section 5.2, are weaker than the global preference-monotonicity condition [**P**], and will necessarily hold if [**P**] holds.

sitions' $\{1, \cdots, p_E^*\}$ and a *migrant set* (i.e., a set of acquired culture positions from which migration occurs) of 'high culture positions' $\{p_M^*, \cdots, P\}$, with $\hat{p}(\underline{\mu}^i) \leq p_E^* < p_M^* \leq P$.

To understand Proposition 5.1, recognize that Lemma 4.1 implies that all $\underline{\mu} \in \Delta^F(\underline{\mu}^i)$ will be fractured at some $p_E(\underline{\mu}) \geq \hat{p}(\underline{\mu}^i)$. Let $\underline{\mu}^c \in \Delta^F(\underline{\mu}^i)$ be the common conjecture of the $\mathbb{V}$-born minority in some generation. Note that $V(\hat{p}(\underline{\mu}^i) \,|\, \underline{\mu}^c) > V(\hat{p}(\underline{\mu}^i) + 1 \,|\, \underline{\mu}^c)$ since $m[\hat{p}(\underline{\mu}^i) \,|\, \underline{\mu}^c] > 0.5m[\underline{\mu}^c]$. Among all conjectures in $\Delta^F(\underline{\mu}^i)$, $\underline{\mu}^{F1}(\underline{\mu}^i)$ is the *best conjecture* for someone considering migrating from $[\hat{p}(\underline{\mu}^i) + 1]$ and the *worst conjecture* for someone considering staying back at $\hat{p}(\underline{\mu}^i)$; thus $V(\hat{p}(\underline{\mu}^i) \,|\, \underline{\mu}^c) \geq V(\hat{p}(\underline{\mu}^i) \,|\, \underline{\mu}^{F1}(\underline{\mu}^i))$ and $C(\hat{p}(\underline{\mu}^i) + 1 \,|\, \underline{\mu}^c) \leq C(\hat{p}(\underline{\mu}^i) + 1 \,|\, \underline{\mu}^{F1}(\underline{\mu}^i))$, implying that when **[F1]** holds a member born at $\hat{p}(\underline{\mu}^i)$ in $\mathbb{V}$ will prefer to stay back in $\mathbb{V}$ within the culture set $\{1, \cdots, \hat{p}(\underline{\mu}^i)\}$ under $\underline{\mu}^c$. Then, under $\underline{\mu}^c$, **[F0]** ensures that no one situated in $\{1, \cdots, \hat{p}(\underline{\mu}^i) - 1\}$ will prefer to migrate to $\mathbb{C}$; all members born in $\{1, \cdots, \hat{q}(\underline{\mu}^i) - 1\}$ will prefer to culture-shift right; and those born in $\hat{q}(\underline{\mu}^i)$ will not prefer to culture-shift left (since $m[\hat{q}(\underline{\mu}^i) - 1 \,|\, \underline{\mu}^c] < 0.5m[\underline{\mu}^c]$). Further, $\underline{\mu}^{F2}(\underline{\mu}^i)$ is the *worst conjecture* in $\Delta^F(\underline{\mu}^i)$ for someone considering migrating from P and the *best conjecture* for someone considering staying back; thus $V(P \,|\, \underline{\mu}^c) \leq V(P| \,\underline{\mu}^{F2}(\underline{\mu}^i))$ while $C(P \,|\, \underline{\mu}^c) \geq C(P \,|\, \underline{\mu}^{F2}(\underline{\mu}^i))$, implying that a member at acquired position P will prefer to migrate when **[F2]** holds. For the above reasons, $\{$[**F0**], [**F1**], [**F2**]$\}$ ensure that if $\underline{\mu}^*$ is the final culture distribution in $\mathbb{V}$ generated by an optimal decision profile $X^*(\underline{\mu}^i)$ given conjecture $\underline{\mu}^c \in \Delta^F(\underline{\mu}^i)$, then $\underline{\mu}^* \in \Delta^F(\underline{\mu}^i)$.[27] Theorem 3.1 then ensures the existence of a fractured equilibrium in the generation.

Our subsequent results on assimilation patterns will build on the logic of Proposition 5.1. Specifically, to prove the existence of different forms of segmented-assimilation equilibrium trajectories, we will look for conditions under which, starting from a period τ, the culture set $\{1, \cdots, \hat{p}(\underline{\mu}_{\tau-1})\}$ will remain entrenched in the continuation equilibrium trajectory while culture position P will be unentrenched in a non-empty set of periods, precisely when $\mathbb{V}$-residents hold such conjectures.

5.2 Equilibrium Trajectories with Incessant Fracture

We now look for sufficient conditions for there to exist an equilibrium trajectory that exhibits *incessant fracture* from some period $\tau \geq 1$ onward. Recall the decision profiles $X^{F1}(\cdot)$ and $X^{F2}(\cdot)$. Note that starting from a period t with inherited $\underline{\mu}^i \in \hat{\Delta}_+$, if the $\mathbb{V}$-born members act according to $X^{F1}(\underline{\mu}^i)$ in every period from t, then in the limit, the progeny of those born in $\{\hat{q}(\underline{\mu}^i), \cdots, \hat{p}(\underline{\mu}^i)\}$ (respectively, in $\{(\hat{p}(\underline{\mu}^i) + 1), \cdots, P\}$) will be situated at $\hat{q}(\underline{\mu}^i)$ in $\mathbb{V}$ (respectively, at P in $\mathbb{C}$). Alternatively, if the $\mathbb{V}$-born members act according to $X^{F2}(\underline{\mu}^i)$ forever after from t, then in the limit, the progeny of those born in $\{\hat{q}(\underline{\mu}^i), \cdots, \hat{p}(\underline{\mu}^i)\}$ will be situated at $\hat{p}(\underline{\mu}^i)$ in $\mathbb{V}$, the culture set $\{(\hat{p}(\underline{\mu}^i) + 1), \cdots, (P - 1)\}$ will be 'empty', and the measure of the minority population in $\mathbb{C}$ will be

[27] Note that if **[F0]** does not hold for some $\underline{\mu} \in \Delta^F(\underline{\mu}^i)$ it cannot be guaranteed that there will not exist 'migration preference' from some $p < \hat{p}(\underline{\mu}^i)$; if **[F1]** does not hold for $\underline{\mu}^{F1}(\underline{\mu}^i)$ it cannot be guaranteed that there will not exist 'rightward culture-shift preference' from $\hat{p}(\underline{\mu}^i)$; and if **[F2]** does not hold for $\underline{\mu}^{F2}(\underline{\mu}^i)$ it cannot be guaranteed that there will exist 'migration preference' from P. It is in this sense that conditions $\{$[**F0**], [**F1**], [**F2**]$\}$ in Proposition 5.1 are 'tight'. It is precisely in this sense that the corresponding conditions specified in Propositions 5.2 – 5.4 are also 'tight'.

at least $[g(P).(1-\sigma).\mu^i(P)]$. Given that, we define the following *infeasible* decision profiles:[28]

$X^{L1}(\underline{\mu}^i) \equiv$ each member born at every $p \in \{1, \cdots, (\hat{q}(\underline{\mu}^i) - 1)\}$ culture-shifts one position toward $\hat{q}(\underline{\mu}^i)$ with σ probability and stays back in $\mathbb{V}$; '*all*' members born at every $p \in \{\hat{q}(\underline{\mu}^i), \cdots, \hat{p}(\underline{\mu}^i)\}$ go to $\hat{q}(\underline{\mu}^i)$ '*in one jump*' and stay back in $\mathbb{V}$; '*all*' members born in $\{(\hat{p}(\underline{\mu}^i) + 1), \cdots, P\}$ migrate.

$X^{L2}(\underline{\mu}^i) \equiv$ '*all*' born in $\{1, \cdots, \hat{p}(\underline{\mu}^i)\}$ go to $\hat{p}(\underline{\mu}^i)$ '*in one jump*'; '*all*' born in $\{(\hat{p}(\underline{\mu}^i) + 1), \cdots, P\}$ go to P '*in one jump*'; and the measure $g(P).(1-\sigma).\mu^i(P)$ of minority members migrate from P.

Then defining $\Delta^{FL}(\underline{\mu}^i)$ (for $\underline{\mu}^i \in \hat{\Delta}_+$) to be the set of all infeasible culture distributions in which at least the measure $m[\hat{p}(\underline{\mu}^i)|\underline{\mu}^i]$ of members are situated in $\{\hat{q}(\underline{\mu}^i), \cdots, \hat{p}(\underline{\mu}^i)\}$ in $\mathbb{V}$ and at least measure $(1-\sigma)g(P)\mu^i(P)$ of members reside in $\mathbb{C}$, we have the following result.

Proposition 5.2. *Define the following three conditions for a given* $\underline{\mu}^i \in \hat{\Delta}_+$*:*

[L0]: *For all* $\underline{\mu} \in \Delta^{FL}(\underline{\mu}^i)$, $[V(p\,|\,\underline{\mu}) - C(p\,|\,\underline{\mu})] \geq [V(\hat{q}(\underline{\mu}^i)\,|\,\underline{\mu}) - C(\hat{q}(\underline{\mu}^i)\,|\,\underline{\mu})]\ \forall\ p < \hat{q}(\underline{\mu}^i)$;

[L1]: $V(\hat{p}(\underline{\mu}^i)\,|\,\underline{\mu}^{L1}(\underline{\mu}^i)) \geq C((\hat{p}(\underline{\mu}^i)+1)\,|\,\underline{\mu}^{L1}(\underline{\mu}^i))$, *for* $\underline{\mu}^{L1}(\underline{\mu}^i)$ *uniquely generated by* $X^{L1}(\underline{\mu}^i)$;

[L2]: $V(P\,|\,\underline{\mu}^{L2}(\underline{\mu}^i)) \leq C(P\,|\,\underline{\mu}^{L2}(\underline{\mu}^i))$, *for* $\underline{\mu}^{L2}(\underline{\mu}^i)$ *uniquely generated by* $X^{L2}(\underline{\mu}^i)$.

Suppose there exists an equilibrium trajectory $\{\underline{\mu}^*_t : t \geq 1 \mid \underline{\mu}_0\}$ *such that in a finite period* $\tau \geq 1$ *with inherited* $\underline{\mu}^*_{\tau-1}$ *(which equals* $\underline{\mu}_0$ *if* $\tau = 1$*):* $\underline{\mu}^*_{\tau-1} \in \hat{\Delta}_+$*, and* **[L0]***,* **[L1]***, and* **[L2]** *hold for* $\underline{\mu}^i = \underline{\mu}^*_{\tau-1}$*. Then there exists an equilibrium trajectory* $\{\underline{\mu}^\dagger_t : t \geq 1 \mid \underline{\mu}^0\}$ *(with* $\underline{\mu}^\dagger_t = \underline{\mu}^*_t\ \forall\, t < \tau$*) that exhibits incessant fracture from period* τ*, with* $\underline{\mu}^\dagger_t$ *fractured at some* $p^\dagger_t \geq \hat{p}(\underline{\mu}_{\tau-1})$ *for all* $t \geq \tau$*.*

Proposition 5.2 presents a set of payoff-ranking conditions under which an equilibrium trajectory exhibits incessant fracture beyond some period τ for the following reasons. Note that conditions **[L0]**, **[L1]**, and **[L2]** – which are modifications of **[F0]**, **[F1]**, and **[F2]** – are devised by considering the 'limit outcomes' when $\mathbb{V}$-born members act according to $X^{F1}(\cdot)$ and $X^{F2}(\cdot)$ in every period $t \geq \tau$. Thus, in period τ with inherited $\underline{\mu}^*_{\tau-1}$, if {**[L0]**, **[L1]**, **[L2]**} hold for $\underline{\mu}^*_{\tau-1}$, then, for similar reasons as in Proposition 5.1, the optimal decision profile under a conjecture in $\Delta^F(\underline{\mu}^*_{\tau-1})$ generates a final culture distribution in $\Delta^F(\underline{\mu}^*_{\tau-1})$. By Theorem 3.1, that guarantees the existence of a period τ equilibrium outcome $\underline{\mu}^\dagger_\tau$ that is fractured at some $p^\dagger_\tau \geq \hat{p}(\underline{\mu}^*_{\tau-1})$. Then the following recursion result holds in period τ+1: a set of conditions 'similar' to {**[L0]**, **[L1]**, **[L2]**} hold for $\underline{\mu}^\dagger_\tau$, and that ensures the existence of an equilibrium outcome $\underline{\mu}^\dagger_{\tau+1}$ that is fractured at some $p^\dagger_{\tau+1} \geq \hat{p}(\underline{\mu}^*_{\tau-1})$. Such a recursion result then continues to hold in *every* subsequent period, thus guaranteeing the existence of an equilibrium trajectory that exhibits incessant fracture.[29] Of course, if {**[L0]**, **[L1]**, **[L2]**} hold for $\underline{\mu}^i = \underline{\mu}_0$, then there exists an equilibrium trajectory exhibiting perpetual fracture (in all $t \geq 1$).

Note that even when the initial minority culture distribution $\underline{\mu}_0$ satisfies **[L0]**, **[L1]**, and **[L2]**, Proposition 5.2 – while guaranteeing the existence of a perpetually-fractured equilibrium trajectory

[28] Decision profiles $X^{L1}(\cdot)$ and $X^{L2}(\cdot)$ (and $X^S(\underline{\mu}^i)$ in Section 5.3) are *infeasible* because: requiring '*all*' to culture-shift in $\mathbb{V}$ requires setting $\sigma = 1$; requiring members to go to some p '*in one jump*' requires culture-shifting over many positions; and requiring '*all*' to migrate from some p requires setting $g(p) = 1$.

[29] **[L0]** (which is a weaker condition than **[F0]**), and **[L1]** and **[L2]** (which are stronger than **[F1]** and **[F2]**), are the precise modifications of **[F0]**, **[F1]**, and **[F2]** that ensure that these recursion results hold.

– *does not* predict precisely how the minority population will be 'split' between $\mathbb{V}$ and $\mathbb{C}$ in the long run. Our next result shows that if $\underline{\mu}_0$ satisfies a stronger set of conditions, then a perpetually-fractured equilibrium trajectory will exist in which the long-run measures of the minority sub-populations in $\mathbb{V}$ and $\mathbb{C}$ will be $m[\hat{p}(\underline{\mu}_0)\,|\,\underline{\mu}_0]$ and $(1 - m[\hat{p}(\underline{\mu}_0)\,|\,\underline{\mu}_0])$ respectively.

Proposition 5.3. *Suppose that the initial culture distribution* $\underline{\mu}_0 \in \hat{\Delta}_+$ *satisfies:* $1 \leq \hat{p}(\underline{\mu}_0) \leq P-2$, *conditions* **[L0]** *and* **[L1]**, *and the following conditions* **[L3]** *and* **[L4]**:

[L3]: *For infeasible two-point distribution* $\underline{\mu}^{[L3]}(\underline{\mu}_0) := \{\hat{p}(\underline{\mu}_0),\ \hat{p}(\underline{\mu}_0) + 2;\ m[\hat{p}(\underline{\mu}_0)\,|\,\underline{\mu}_0]\}$, *it is the case that* $C(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^{[L3]}(\underline{\mu}_0)) \geq V(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^{[L3]}(\underline{\mu}_0))$;[30]

[L4]: *For the infeasible distribution* $\underline{\mu}^{[L4]}(\underline{\mu}_0) := \{\hat{p}(\underline{\mu}_0),\ \hat{p}(\underline{\mu}_0) + 1;\ \hat{m}(\underline{\mu}_0) + \sigma\mu_0(\hat{p}(\underline{\mu}_0)+1)\}$ *and for the infeasible one-point distribution* $\underline{\mu}^{[L5]}(\underline{\mu}_0) := \{\hat{q}(\underline{\mu}_0);\ \hat{m}(\underline{\mu}_0)\}$, *it is the case that* $g(\hat{p}(\underline{\mu}_0)+2).C(\hat{p}(\underline{\mu}_0)+2\,|\,\underline{\mu}^{[L4]}(\underline{\mu}_0))+(1-g(\hat{p}(\underline{\mu}_0)+2)).V(\hat{p}(\underline{\mu}_0)+2\,|\,\underline{\mu}^{[L5]}(\underline{\mu}_0)) > V(\hat{p}(\underline{\mu}_0)\,|\,\underline{\mu}^{[L4]}(\underline{\mu}_0))$.

Then there exists an equilibrium trajectory $\{\underline{\mu}_t^\ddagger : t \geq 1|\underline{\mu}_0\}$ *with* $\underline{\mu}_t^\ddagger$ *fractured at* $\hat{p}(\underline{\mu}_0)$ *for all* $t \geq 1$.

To understand Proposition 5.3, note the following implications of Lemma 4.1. A generational equilibrium $\underline{\mu}^*$, given inherited $\underline{\mu}^i$, will be fractured at some p_E^* satisfying $\hat{p}(\underline{\mu}^i) \leq p_E^* \leq P-2$ only if the following preference conditions hold: [PR1] a member situated at acquired position $p_E^* + 2$ prefers to migrate, and [PR2] a member born at p_E^*+1 prefers either to culture-shift to p_E^*+2, or to migrate from $p_E^* + 1$, or both.[31] Now note that as in Proposition 5.2, the satisfaction of conditions **[L0]** and **[L1]** by $\underline{\mu}_0$ ensures that there exists an equilibrium trajectory $\{\underline{\mu}_t^\ddagger : t \geq 1\}$ in which the culture set $\{1, \cdots, \hat{p}(\underline{\mu}_0)\}$ is entrenched in every period $t \geq 1$. Then the stated payoff-ranking conditions **[L3]** and **[L4]** imply that in the period 1 equilibrium $\underline{\mu}_1^\ddagger$, conditions [PR1] and [PR2] are satisfied for $p_E^* = \hat{p}(\underline{\mu}_0)$, thus ensuring that $\underline{\mu}_1^\ddagger$ is fractured at $\hat{p}(\underline{\mu}_0)$.[32] Next, take any period $\tau \geq 2$ and assume that $\underline{\mu}_t^\ddagger$ is fractured at $\hat{p}(\underline{\mu}_0)$ in every period $t = 1, \cdots, \tau - 1$. Then conditions **[L3]** and **[L4]** (which hold for $\underline{\mu}_0$) imply that $\underline{\mu}_\tau^\ddagger$ is fractured at $\hat{p}(\underline{\mu}_0)$. Thus, there exists an equilibrium trajectory $\{\underline{\mu}_t^\ddagger : t \geq 1\}$ in which every generational equilibrium outcome is fractured at $\hat{p}(\underline{\mu}_0)$.

In Section 6.1 we will make use of Proposition 5.3 to study the extent of polarization in the minority population that a segmented-assimilation trajectory will generate in the long run. Below, we clarify how the proposition helps in identifying scenarios in which there will be no equilibrium minority incentive for reverse migration from $\mathbb{C}$ to $\mathbb{V}$, even when such migration is permitted.

We recognize that in any period τ with final $\mathbb{V}$-distribution $\underline{\mu}_\tau$, a $\mathbb{C}$-born minority member situated

[30] Here $\{p, q; m\}$ denotes a two-point distribution in which m measure of minority are located at p and $(1-m)$ measure at q in $\mathbb{V}$; and $\{r; m'\}$ is a one-point distribution with m' measure of minority at r in $\mathbb{V}$.

[31] If $\underline{\mu}^*$ is fractured at p_E^* for $\hat{p}(\underline{\mu}^i) \leq p_E^* \leq P-2$, then by Lemma 4.1, if there is no migration preference at $p_E^* + 2$, there will be none at $p_E^* + 1$, and then one born at $p_E^* + 1$ will prefer to culture-shift left. If there *is* migration preference at $p_E^* + 2$, and if one born at $p_E^* + 1$ prefers neither to culture-shift right nor to migrate, then $\{1, \cdots, p_E^* + 1\}$ must be entrenched. In either case, $\underline{\mu}^*$ cannot be fractured at p_E^*.

[32] Note that for any conjecture $\underline{\mu}^e$ that is fractured at $\hat{p}(\underline{\mu}_0)$, $C(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^e) \geq C(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^{[Lk]}(\underline{\mu}_0))$ for $k = 3, 4$, $V(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^{[L3]}(\underline{\mu}_0)) \geq V(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^e)$, $V(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^e) \geq V(\hat{p}(\underline{\mu}_0) + 2\,|\,\underline{\mu}^{[L5]}(\underline{\mu}_0))$, and $V(\hat{p}(\underline{\mu}_0)\,|\,\underline{\mu}^{[L4]}(\underline{\mu}_0)) \geq V(\hat{p}(\underline{\mu}_0)\,|\,\underline{\mu}^e)$.

at an acquired p will prefer to reverse-migrate if and only if $C(p\,|, \underline{\mu}_\tau) < [w + \rho.b\Big(m[\underline{\mu}_\tau], d(p, \underline{\mu}_\tau)\Big)]$, where $\rho \in (0, 1]$ is the fraction of the social-interactions benefit in $\mathbb{V}$ that a $\mathbb{C}$-born minority member will get if she returns to $\mathbb{V}$.[33] Given that, suppose there exists an equilibrium trajectory $\{\underline{\mu}_t^{**} : t \geq 1|\underline{\mu}_0\}$ in the 'model that does not permit reverse-migration' that satisfies the following *monotonicity in migration* condition: "if there is migration from $\mathbb{V}$ to $\mathbb{C}$ from some $p \in \mathbb{P}$ in some period τ, then there is migration from $\min\{p+1, P\}$ in every period $t > \tau$". It is clear that for all $\rho \in (0, 1]$, $\{\underline{\mu}_t^{**} : t \geq 1|\underline{\mu}_0\}$ will also be an equilibrium trajectory when reverse migration is allowed (as the conjecture that no one will reverse-migrate will be self-fulfilling). Note further that every equilibrium perpetual-fracture trajectory $\{\underline{\mu}_t^{\dagger} : t \geq 1|\underline{\mu}_0\}$ in which the generational 'fracture point' $p_t^{\dagger}$ is non-increasing in t will satisfy the stated monotonicity condition. In particular, the trajectory $\{\underline{\mu}_t^{\ddagger} : t \geq 1|\underline{\mu}_0\}$ in Proposition 5.3 satisfies the monotonicity condition, and consequently, for any $\rho \in (0, 1]$ it will remain an equilibrium trajectory even when reverse-migration is allowed.[34]

We want to emphasize that even if an equilibrium trajectory in our 'no reverse migration model' *does not* satisfy the monotonicity-in-migration condition, it can still be an equilibrium trajectory when reverse-migration is permitted, so long as the magnitude of ρ is small. In our Simulation Regime [4], this is the case for the equilibrium trajectory *TD*. Along *TD*, beyond period 3, the $\mathbb{C}$-born minority at culture position 3 *will* want to reverse-migrate to $\mathbb{V}$ if they are assured of enjoying the *same* social interactions benefit as those born in $\mathbb{V}$. Our simulation results show that if the reverse-migrants are to receive anything less than 90% of the social interactions benefit $b(\cdot, \cdot)$, then there will exist no incentive to reverse-migrate in any generation along *TD*.

Of course, equilibrium trajectory *TD* in Regime [4] also reminds us that there can exist segmented-equilibrium trajectories in our model that do not exhibit incessant fracture. Given that, in the next subsection we look for weaker sufficient conditions for the existence of such trajectories.

5.3 Minimal Sufficient Conditions for Segmented Assimilation

Note that an equilibrium trajectory will generate segmented assimilation if, along the trajectory, not all generational equilibria are fully-entrenched and not infinitely-many generational equilibria are wholly-unentrenched. Recognizing that, our next result presents condition **[S]** – that is weaker than **[L2]** in specific scenarios as indicated below – which, along with **[L0]** and **[L1]**, ensures that given an initial distribution $\underline{\mu}_0$, there will exist an equilibrium segmented-assimilation trajectory.

Proposition 5.4. *Define the following condition for a given* $\underline{\mu}^i \in \hat{\Delta}_+$*:*

---

[33] It will be quite plausible to assume that ρ is less than 1, since a city-born minority returnee will likely enjoy lesser social-interaction benefits in $\mathbb{V}$ as compared to a minority member born in $\mathbb{V}$.

[34] The claim regarding $\{\underline{\mu}_t^{\ddagger} : t \geq 1|\underline{\mu}_0\}$ is true as we have assumed that $\mathbb{C}$-born minority can always culture-shift right; thus, if a parent situated at some p migrated in period t, her offspring will be in $\min\{p+1, P\}$ in $\mathbb{C}$ in period $t+1$. If the culture-shift probability in $\mathbb{C}$ is less than 1, then to ensure 'no reverse migration', we will need to assume either a stronger monotonicity-in-migration condition where '$\min\{p+1, P\}$' is replaced by 'p', or a ρ that is 'sufficiently' small. In our simulations, the identified equilibrium trajectories in Simulations Regime [1] – [3] satisfy the stronger monotonicity-in-migration condition.

[S]: *$V(P \mid \underline{\mu}^S(\underline{\mu}^i)) < C(P \mid \underline{\mu}^S(\underline{\mu}^i))$ for the infeasible one-point distribution $\underline{\mu}^S(\underline{\mu}^i) \equiv \{\hat{p}(\underline{\mu}^i); 1\}$.*

Suppose there exists an equilibrium trajectory $\{\underline{\mu}_t^* : t \geq 1 \mid \underline{\mu}_0\}$ such that in a finite period $\tau \geq 1$ with inherited $\underline{\mu}_{\tau-1}^*$ (which equals $\underline{\mu}_0$ if $\tau = 1$): $\underline{\mu}_{\tau-1}^* \in \hat{\Delta}_+$, and **[L0]**, **[L1]**, *and* either **[L2]** or **[S]** *(or both) hold for $\underline{\mu}^i = \underline{\mu}_{\tau-1}^*$. Then there exists an equilibrium trajectory $\{\underline{\mu}_t^S : t \geq 1 \mid \underline{\mu}^0\}$ (with $\underline{\mu}_t^S = \underline{\mu}_t^*$ for $t < \tau$) that generates segmented assimilation, with the long-run measure of the minority population in $\mathbb{V}$ being at least $m[\hat{p}(\underline{\mu}_{\tau-1}^*) \mid \underline{\mu}_{\tau-1}^*]$.*

Recognize that when (a) there are many culture positions in $\mathbb{P}$, (b) the initial culture distribution $\underline{\mu}_0$ is sufficiently right-skewed (so that $\hat{p}(\underline{\mu}_0)$ is 'close enough' to 1), and (c) $[g(P).(1-\sigma).\mu_0(P)]$ is small, condition **[S]** is weaker than **[L2]** in that there can exist an equilibrium trajectory $\{\underline{\mu}_t^* : t \geq 1 \mid \underline{\mu}_0\}$ such that **[S]** holds for the equilibrium inherited $\underline{\mu}_{\tau-1}^*$ in some period τ even though no generational equilibrium $\underline{\mu}_t^*$ satisfies **[L2]**. When that is the case, if $\underline{\mu}_{\tau-1}^*$ also satisfies **[L0]** and **[L1]**, then Proposition 5.2 implies that there exists a continuation equilibrium trajectory $\{\underline{\mu}_t^{**} : t \geq \tau \mid \underline{\mu}_{\tau-1}^*\}$ from period τ onward in which $\{1, \cdots, \hat{p}(\underline{\mu}_{\tau-1}^*)\}$ is an entrenched culture set in every intra-generation equilibrium $\underline{\mu}_t^{**}$ for $t \geq \tau$. Note that if $\underline{\mu}_{t'}^{**}$ is not fully-entrenched in some period $t' \geq \tau$ then $\{\underline{\mu}_t^{**} : t \geq \tau \mid \underline{\mu}_{\tau-1}^*\}$ generates segmented assimilation, while if $\underline{\mu}_{t'}^{**}$ *is* fully-entrenched then all $\mathbb{V}$-residents born in $\{\hat{p}(\underline{\mu}_{\tau-1}^*)+1, \cdots, P\}$ must culture-shift left in period t'. Then the fact that **[S]** holds for $\underline{\mu}_{\tau-1}^*$ rules out the possibility that *every* generational equilibrium in $\{\underline{\mu}_t^{**} : t \geq \tau \mid \underline{\mu}_{\tau-1}^*\}$ will be fully-entrenched. That is because **[S]** ensures that if, along an equilibrium trajectory, there exists a sequence of fully-entrenched intra-generation equilibria in which all $\mathbb{V}$-residents born in $\{\hat{p}(\underline{\mu}_{\tau-1}^*)+1, \cdots, P\}$ culture-shift left, then that sequence will necessarily be followed by a fractured intra-generation equilibrium (with migration from P). The conclusion of Proposition 5.4 then follows. Note that the result requires only a 'one time check' of whether the model parameters satisfy {**[L0]**, **[L1]**, **[S]**} for the 'initial' distribution $\underline{\mu}_{\tau-1}^*$.

The conditions in Proposition 5.4 are *not necessary* for equilibrium segmented-assimilation trajectories to exist, as is shown by the equilibrium trajectory *TD* in Simulation Regime [4] which violates both **[L2]** and **[S]**.[35] Given that, we now establish that our stated sufficient conditions are *minimal* in the following specific sense: there exist particular model parameter regimes, that satisfy our maintained assumptions as well as condition **[P]**, in which an equilibrium segmented-assimilation trajectory will exist *if and only if* the initial $\underline{\mu}_o$ satisfies conditions **[L1]** and **[S]**. Recall the set of parametric examples **[E]** $\in \mathcal{E}^{[P]}$ described in Section 4.1, and note that since each **[E]** $\in \mathcal{E}^{[P]}$ satisfies condition **[P]** it also satisfies **[L0]**. For this class of models, we have the following result.

Proposition 5.5. *Consider a model* **[E]** $\in \mathcal{E}^{[P=2]}$. *There exists a positive measurable set of values of the parameter set $\{w, \alpha, \beta, \lambda, \delta, \theta, \sigma, \mathbb{G}; \underline{\mu}_0\}$ of* **[E]** *such that if either* **[L1]** *or* **[S]** *is violated, then there does not exist an equilibrium trajectory that generates segmented assimilation.*

Proposition 5.5 clarifies the sense in which conditions **[L1]** and **[S]** are *minimal* sufficient conditions

[35] Our stated sufficient conditions are indeed tight, in that if the conditions are relaxed then existence of equilibrium segmentation cannot be guaranteed. But we *cannot* claim in general that if a set of sufficient conditions fail to hold, that will imply non-existence of an equilibrium segmented-assimilation trajectory.

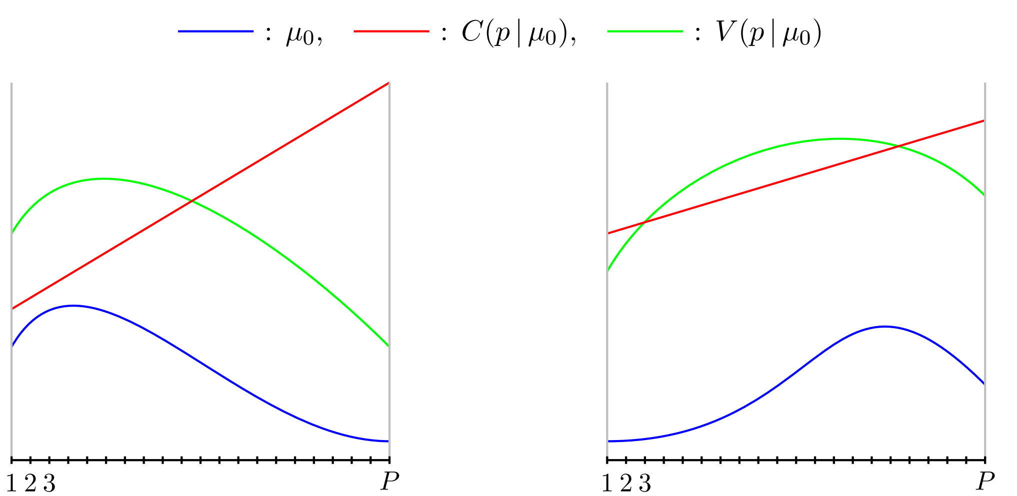
for the existence of equilibrium segmented-assimilation trajectories. Its proof establishes that in a model **[E]** $\in \mathcal{E}^{[P=2]}$, there exist parameter values such that if **[L1]** (respectively, **[S]**) is violated then all equilibrium trajectories will generate complete assimilation in $\mathbb{C}$ (respectively, in $\mathbb{V}$).

5.4 Model Structures for Segmented Assimilation

In Sections 5.2 – 5.3, we have identified specific sets of payoff-ranking conditions that are (minimally) sufficient for the existence of intra- and inter-generational equilibria that generate segmented assimilation in general, and incessant fracture in particular. Drawing upon the intuition gained from our simulations in Section 4.1, we now clarify that there exist particular parameter structures in our model that will ensure the satisfaction of those payoff ranking conditions, given that the rankings need to be guaranteed only for a specific 'initial generation'. To that end, we define the following 'feature set' – that we denote as Feature Set [∗] – of our model.

Feature Set [∗]: The initial culture distribution $\underline{\mu}_0$ is sufficiently right-skewed, with $\hat{p}(\underline{\mu}_0)$ much closer to culture position 1 than to P. For all $\underline{\mu} \in \Delta_+$, $C(p\,|\,\underline{\mu})$ rises steeply in p, with $C(1\,|\,\underline{\mu})$ taking a low value and $C(P\,|\,\underline{\mu})$ taking a high value. For all sufficiently right-skewed $\underline{\mu} \in \hat{\Delta}_+$ that have $m[\underline{\mu}] \geq 0.5$, $V(p\,|\,\underline{\mu})$ is single-peaked achieving a maximum at a p' close to 1, and $V(P\,|\,\underline{\mu})$ is significantly smaller than $V(1\,|\,\underline{\mu})$, so that the following is true: $V(p\,|\,\underline{\mu})$ is sufficiently larger than $C(p\,|\,\underline{\mu})$ for all $p \leq \hat{p}(\underline{\mu})$ while the opposite is true for all p close to P. [See Figure 2.]

Figure 2: Models structures that do / do not conform to Feature Set [∗]



Panel A: conforms to [∗] Panel B: does not conform to [∗]

Recognize that **[L0]** and **[L1]** are likely to be satisfied for $\underline{\mu}_0$ when $\hat{p}(\underline{\mu}_0)$ is close to 1, $C(p\,|\,\underline{\mu}_0)$ rises steeply in p, and $V(p\,|\,\underline{\mu}_0)$ is sufficiently larger than $C(p\,|\,\underline{\mu}_0)$ for all $p \in \{1, \cdots, \hat{p}(\underline{\mu}_0)\}$.[36] Further, **[L2]** as well as **[S]** are likely to be satisfied for a sufficiently right-skewed $\underline{\mu}_0$ when $C(p|\underline{\mu}_0)$

[36] Rewriting **[L0]** as: $[C(\hat{q}(\underline{\mu}^i)|\underline{\mu}^i) - C(p|\underline{\mu}^i)] \geq [V(\hat{q}(\underline{\mu}^i)|\underline{\mu}^i) - V(p|\underline{\mu}^i)]$ for all $p < \hat{q}(\underline{\mu}^i)$, note that LHS is likely to be larger than RHS for $\underline{\mu}^i = \underline{\mu}_0$ when $C(p|\underline{\mu}_0)$ rises steeply in p and $\hat{q}(\underline{\mu}_0)$ $(\leq \hat{p}(\underline{\mu}_0))$ is close to 1.

is substantially bigger than $V(p|\underline{\mu}_0)$ for p close to P. These observations allow us to conclude that when our model structure conforms to Feature Set [∗], long-run segmentation of the minority population – with or without perpetual / incessant fracture – will likely be an equilibrium outcome.

6 Dilemmas of Segmented Assimilation

In this section, we focus on three features of equilibrium segmented-assimilation trajectories: (*a*) the impact of segmentation on long-run minority payoff-poverty, inequality, and polarization; (*b*) the possibility of hysteresis in the evolution of minority lineages, and (*c*) the possibility of adverse impact of affirmative-action policies on minority welfare in the long-run.

6.1 Poverty, Inequality, and Polarization

Our simulations have shown that along an equilibrium segmented-assimilation trajectory – *viz.* trajectory TC in Regime [3] – minority decisions over generations can lead to a long-run scenario in which each member's payoff (irrespective of her location) is less than $\min\{V_{max}, C_{max}\}$.[37] To further explore the issue of long-run minority payoff poverty under segmentation, suppose that $C_{max} > V_{max}$, and consider a segmented-assimilation trajectory that generates a long-run outcome in which $m^S \in (0,1)$ measure of minority members remain $\mathbb{V}$-residents. Defining $\tilde{m} > 0$ as the measure of $\mathbb{V}$-residents that satisfies: $b(1,0) = \Pi(P) - \Gamma(P) + a(1-\tilde{m})$, note that if $m^S \in (\tilde{m}, 1)$, then the limit payoff to every minority member in $\mathbb{V}$ *and* in $\mathbb{C}$ will be lower than V_{max}.[38] Alternatively, if $m^S \in (0, \tilde{m})$ then there will certainly be payoff-inequality in the long run with $\mathbb{C}$-residents getting higher payoffs than $\mathbb{V}$-residents, and this long-run payoff-inequality will be greater the smaller is m^S.

Under any complete-assimilation outcome, the entire minority population will get substantially homogenized in terms of overt culture traits and payoffs, and that can be beneficial for them in many socio-economic scenarios. In contrast, a segmented-assimilation outcome can generate significant polarization across the minority population in $\mathbb{V}$ and $\mathbb{C}$ – with respect to culture traits, and concomitantly with respect to payoffs – if the equilibrium trajectory limits to 'two-point distributions' (as do the trajectories TC and TD in our simulations).[39]

Formally, we will say that an equilibrium segmented-assimilation trajectory $\{\underline{\mu}_t^* : t \geq 1 \,|\, \underline{\mu}_0\}$ leads to *extreme culture bi-modality* in the long-run if there exists some $\eta \in (0,1)$ and some position $\bar{p} < P$ such that for all $\varepsilon > 0$ (however small) there exists a period $t(\varepsilon) > 0$ such that $\mu_t^*(\bar{p}) \in (\eta - \varepsilon, \eta + \varepsilon)$ and $\mu_t^*(p) \in (0, \varepsilon)$ for all $p \in \mathbb{P} \setminus \{\bar{p}\}$ and for all $t \geq t(\varepsilon)$. We then have the following result:

[37] In contrast, 'complete assimilation in $\mathbb{V}$' trajectory TA in Regime [1] leads to long-run payoff V_{max}, and 'complete assimilation in $\mathbb{C}$' trajectories TB(1,2) and TB(0,1) in Regime [2] generate long-run payoff C_{max}.

[38] In the case where $C_{max} < V_{max}$, the limit payoff to every minority member (irrespective of livelihood location) will be lower than C_{max} when m^S is smaller than a critical value.

[39] If a minority group is polarized in different socio-economic dimensions, it may be unable to effectively resist ethnicity-targeted attacks on it; more generally, it might be very costly for a polarized minority group to take various kinds of collective actions necessary to protect and sustain group members.

Proposition 6.1. *Suppose there exists an equilibrium trajectory* $\{\underline{\mu}_t^* : t \geq 1 \mid \underline{\mu}_0\}$ *such that the hypotheses of Proposition* 5.4 *hold for an inherited culture distribution* $\underline{\mu}_{\tau-1}^*$ *in some finite period* $\tau \geq 1$ *(which equals* $\underline{\mu}_0$ *if* $\tau = 1$*). Then there exists an equilibrium trajectory* $\{\underline{\mu}_t^B : t \geq 1 \mid \underline{\mu}^0\}$ *(with* $\underline{\mu}_t^B = \underline{\mu}_t^*$ *for* $t < \tau$*) that exhibits extreme culture bi-modality.*

Along a segmented-assimilation trajectory, the desire to raise wage premium and to limit discrimination costs induce the minority sub-population in $\mathbb{C}$ to 'converge' to culture-position P over time; while in $\mathbb{V}$, the presence of local social interactions benefits also encourage the $\mathbb{V}$-residents to culture-shift 'closer to each other'. But whether the agglomeration incentives in $\mathbb{V}$ will induce the residents to converge to a *unique* culture position in $\mathbb{V}$ depends on whether they succeed in resolving a dynamic coordination problem. Remarkably, Proposition 6.1 assures us that the conditions that ensure the existence of a segmented-assimilation equilibrium trajectory also guarantee that there will exist at least one equilibrium trajectory in which the minority sub-population in $\mathbb{V}$ will successfully coordinate and converge to a common culture position, thereby ensuring that the trajectory will generate extreme culture bi-modality in the long run.

The class of polarization measures proposed by Esteban and Ray (1994), hereafter referred to as the *E-R class*, allows us to determine the extent of culture (and payoff) polarization in the limit outcomes of equilibrium trajectories that exhibit extreme bi-modality. Consider two such trajectories $\{\underline{\mu}_t^{B(J)} : t \geq 1 \,|\, \underline{\mu}^0\}$ for $J = 1, 2$, that generate the limit outcomes $\{p^{B(J)}, m^{B(J)}\}$ for $J = 1, 2$, where $p^{B(J)}$ is the limit culture position of all $\mathbb{V}$-residents under trajectory J and $m^{B(J)}$ is their measure (with $[1 - m^{B(J)}]$ measure of minority residing at P in $\mathbb{C}$). If $p^{B(1)} \leq p^{B(2)}$ and $m^{B(1)}$ is (weakly) closer to 0.5 than $m^{B(2)}$, with at least one condition strict, then for all measures in the *E-R class*, trajectory $J = 1$ will generate strictly greater 'culture polarization' (i.e., polarization with respect to overt culture-traits) for the minority population in the long run than trajectory $J = 2$.

For an equilibrium trajectory $\{\underline{\mu}_t^{B(J)} : t \geq 1 \,|\, \underline{\mu}^0\}$ that leads to the limit outcome $\{p^{B(J)}, m^{B(J)}\}$, we can say more about the location of $p^{B(J)}$ and the size of $m^{B(J)}$ if $\underline{\mu}_0$ satisfies the hypotheses of Proposition 5.3. In that case, we are assured that $1 \leq p^{B(J)} \leq \hat{p}(\underline{\mu}_0)$ and that $m^{B(J)} = m[\hat{p}(\underline{\mu}_0) \,|\, \underline{\mu}_0]$, implying that the closer are $\hat{p}(\underline{\mu}_0)$ to 1 and $m[\hat{p}(\underline{\mu}_0) \,|\, \underline{\mu}_0]$ to 0.5 the more like it is that long-run culture polarization will be close to its maximal value. In an additional simulation exercise, we create Simulation Regime [5] by changing only the values of α and $\underline{\mu}_0$ in Regime [3]: we set $\alpha = 0.9$, and $\underline{\mu}_0^{[5]} = \{0.5, 0.25, 0.25\}$. We find that in Regime [5], there exists an equilibrium trajectory TE in which there is perpetual fracture at culture position 1. As a result, the long-run outcome exhibits *maximum* culture polarization, with 50% of the minority having position 1 in $\mathbb{V}$ as their birth- and livelihood-location, and the remaining population being born at and residing at position P in $\mathbb{C}$.

Next, consider an equilibrium trajectory that generates extreme culture bi-modality, with the measure of long-run $\mathbb{V}$-population being $m^B \in (0, 1)$. Then the limit minority payoff in $\mathbb{V}$ will be $[w + b(m^B, 0)]$ and that in $\mathbb{C}$ will be $[w + \Pi(P) - \Gamma(P) + a(1 - m^B)]$. Thus, except in the special case where these two payoffs are equal, the trajectory will also generate extreme payoff bi-modality. Will an equilibrium trajectory that exhibits greater culture polarization also exhibit greater payoff polarization? Not necessarily, as our simulations reveal. In Simulation Regime [5], the individual limit payoffs are $(w + 1.75)$ for the $\mathbb{V}$-minority and $(w + 3.35)$ for the $\mathbb{C}$-minority. We contrast this outcome to that in a modified Simulation Regime [5′] where we set $\underline{\mu}_0^{[5']} = \{0.475, 0.275, 0.25\}$. In Regime [5′], there is an equilibrium trajectory TE' that is perpetually fractured at position 1, but with a long-run outcome in which 47.5% of all minority members reside at position 1 in $\mathbb{V}$, each getting payoff $w + 1.6625$, while the remaining 52.5% reside at position 3 in $\mathbb{C}$, receiving payoff $w + 3.3725$. Following the 'two-point distribution analysis' in Esteban and Ray (1994), it is easily verified that for any polarization measure in the *E-R class*, trajectory TE in Regime [5] leads to greater culture polarization but to lesser payoff polarization than does TE' in Regime [5′].

In essence, given the social interactions effects in $\mathbb{V}$ and $\mathbb{C}$, it is the fracturing of the minority population in a segmented-assimilation trajectory that creates the following dilemmas for the minority community: payoff poverty and/or inequality, and substantial culture (and payoff) polarization.

6.2 Hysteresis in Minority Lineages

In our simulations, if the minority population is on trajectory *TA* in Regime [1] (respectively, on *TB*(1,2) or *TB*(0,1) in Regime [2]) leading to complete homogenization in $\mathbb{V}$ (respectively, in $\mathbb{C}$), then for every minority member i, the culture location and payoff of her distant progeny will be independent of the generation and the culture position in which member i was born, and also independent of her realized opportunities to culture-shift and to migrate. It is in this sense that minority lineages are *path-independent* in these complete-assimilation trajectories.

In contrast, there can be different kinds of path dependence in the evolution of minority lineages in segmented-assimilation trajectories. Below, we focus on two kinds of *hysteresis*. In Scenario 1, two minority members i and j, born in the same period at the same culture-position in $\mathbb{V}$, are subject to different opportunity realizations; as a result, their distant progeny have very different life experiences. In Scenario 2, a minority parent i and her offspring j, born at a common culture position in $\mathbb{V}$, have very different life outcomes even when experiencing identical opportunity realizations, due to altered intertemporal incentives regarding culture-shifting.

***Scenario* 1**: Consider an equilibrium trajectory in which every generational equilibrium is fractured at some p^E. Assume that in every period, the preferences of a member born at $(p^E + 1)$ are: culture-shift left to p^E, but migrate if stuck at $(p^E + 1)$ (as is the case for members at '3' in $\mathbb{V}$ along trajectory TC in Simulation Regime [3]). When that is the case, consider members i and j born in $\mathbb{V}$ at $(p^E + 1)$ in some period τ, where i gets the chance to culture-shift, while j only gets the opportunity to migrate. It is immediate that i's distant progeny will be situated within the culture set $\{1, \cdots, p^E\}$ in $\mathbb{V}$, while j's distant progeny will be situated at position P in $\mathbb{C}$.

***Scenario* 2**: Next, we consider a case in which equilibrium entrenched culture sets *expand* over time. We create Simulation Regime [6] from Regime [2] by changing the values of α, σ, and $\mathbb{G}$ thus: $\alpha = 0.6$, $\sigma = 0.2$, $\mathbb{G} = \{0.1, 0.2, 0.46\}$, and by setting $\underline{\mu}_0^{[6]} = \{0.55, 0.35, 0.1\}$. The following trajectory *TF* is an equilibrium trajectory in Regime [6]: "The $\mathbb{V}$-population follows the decision profile *DP*(II.3) in period 1, and follows *DP*(III) forever after". Along *TF*, the (largest) generational entrenched set expands from the null set in period 1 to the set $\{1, 2\}$ in period 2. Along *TF*, consider a parent born in period 1 at position 3 in $\mathbb{V}$, who gets the chance to culture-shift but not to migrate. Then the parent will refuse to culture-shift, and so her child will be born at position 3 in $\mathbb{V}$. When the offspring gets a chance to culture-shift, she will shift to position 2 in $\mathbb{V}$ and reside there. Given that the parent and her offspring receive similar opportunities (i.e., only to culture-shift), it is the change in minority incentives over the two generations that causes such change in their worldviews, and thus in their life experiences, with the offspring becoming 'more minority-immersed' than the parent.

6.3 Affirmative Action Policy Dilemma

In our model, it is natural to contemplate policy interventions aimed at improving the long-run well-being of the entire minority community. To that end, we simulate a simple affirmative action policy intervention, and uncover a particular *policy dilemma*.

We consider the country's government instituting an affirmative action policy in some period τ that aims to raise minority welfare by reducing mainstream discrimination of minority workers in $\mathbb{C}$. In the class of models **[E]**, this can be achieved by permanently raising parameter θ from some period τ, so that $\Gamma(p)$ is 'lowered and made steeper', thereby reducing the extent of discrimination for all $\mathbb{C}$-minority and more for those culturally closer to the mainstream. We reconsider Simulation Regime [1], where TA is an equilibrium trajectory in the absence of any policy intervention (while $TB(\cdot)$ and TC are not), and study the impact of a policy that permanently raises θ from 1.2 to 1.32 in period τ. We take the government's aim to be to 'induce' an equilibrium trajectory of the form $TB(\cdot)$ that will lead to complete assimilation in $\mathbb{C}$ ensuring maximum minority payoff $C_{max} = (w + 4.46)$ in the long-run, with the minority being 'nudged' to shift to such a trajectory from TA if need be.

Our first simulation result is that if the policy intervention comes late – in period 10 – then it will be *ineffective*. Specifically, the rise in θ will fail to induce a trajectory of the form $TB(\cdot)$ or trajectory TC to be an equilibrium continuation trajectory, while TA will remain an equilibrium trajectory.

Our second simulation result considers the policy to be implemented earlier – in period 4. Then the following problem arises. Along TA, when θ is raised to 1.32 in period 4, TA remains an equilibrium trajectory and TC becomes one, but no trajectory of the form $TB(\cdot)$ does so. If the government's 'migration-promoting nudges' cause the minority to switch from TA to TC from period 4, then while some migrants will benefit in the immediately following periods, the economy will move to a new path that will lead to long-run minority payoffs of $(w + 3.778)$ for the $\mathbb{V}$-residents and $(w + 3.138)$ for the $\mathbb{C}$-residents, both lower than $\min\{V_{max}, C_{max}\}$. That is because the segmented-assimilation trajectory TC 'pulls' only a small fraction of the minority to $\mathbb{C}$. Thus, an affirmative-action policy introduced at an intermediate time period can have the long-run effect of *lowering* the payoffs of *all* minority members. Further, TC will lead to significant culture (and payoff) polarization (in contrast to TA that generates none), and that can impose additional welfare costs on the minority.

In contrast, our final simulation result shows that if the policy is implemented soon enough along TA, then it can be possible to achieve complete minority assimilation in $\mathbb{C}$. Specifically, if the intervention happens in period 2, then TA, $TB(2{,}3)$ and TC will all become continuation equilibrium trajectories. If appropriate nudging can induce the minority to shift from TA to $TB(2{,}3)$, then, beyond the 40^{th} generation, the entire minority population will be born at position 3 in $\mathbb{C}$, live there, and enjoy the optimal payoff $C_{max} = (w + 4.46)$.

In our simulation model, the timing of an affirmative action policy's implementation matters precisely because the policy's long-run impact depends on the prevailing culture distribution in the minority sub-population in $\mathbb{V}$ at the time of implementation. Further, determining the optimal timing can be a non-trivial exercise. It is *not necessarily* the case that the long-run impact of the policy will more beneficial the earlier it is implemented – policy implementation in the 4^{th} period in our example can generate a worse long-run outcome than implementation in the 10^{th} period. More generally, the above-described policy dilemma clarifies that in many real world scenarios, policy makers cannot simply focus on the immediate impact of a policy change, but have to consider its long-run impact that results from a change in the dynamic equilibrium path. This also means that a 'past policy mistake' – that adversely altered the long-run equilibrium trajectory of the society – might be more difficult to rectify by a policy correction in the future.[40]

7 Concluding Remarks

This paper has considered a formalization of the sociological theory of 'segmented assimilation' in order to study dynamic assimilation patterns of minority populations with a country's mainstream. We have aimed to understand the causes and the impacts of segmentation of a minority community into subgroups – some of which 'assimilate upward' with the mainstream while others remain dissociated. Our analysis has delineated the following logic behind minority segmentation: when a minority community is initially heterogeneous with respect to overt culture traits, and when there are benefits to being close to the dominant local culture, minority members with birth culture traits 'close enough' to the mainstream feel encouraged to take the risk of culture-shifting away from the 'median' minority culture in order to improve mainstream returns, while those born with more traditional culture traits are induced to become more entrenched in the minority subculture.

We recognize that it will be worthwhile to further explore the nature of minority assimilation trajectories in an environment in which some of our critical modeling assumptions are relaxed. In the course of our analysis, we have discussed the import of our assumptions of 'minority myopia' and 'no reverse migration' on our segmentation results. In this concluding section, we briefly comment on two other restrictions in our model: (a) that minority members can culture-shift only by one position, and (b) that they cannot concurrently 'reside' at multiple culture positions.

Note that if the $\mathbb{V}$-born minority could *culture-shift globally* and jump to any culture position p' from any other $p \in \mathbb{P}$ whenever they got the chance to culture-shift, and it was the case that $C_{max} > V_{max}$ and $g(P) \approx 1$, then in every generation there would exist an equilibrium in which all

[40] Our simulation model considers an affirmative action policy that aims to reduce discrimination by raising θ from 1.2 to 1.32. Suppose that this policy was implemented in period 4, and that forced the society to shift from TA to TC. Then in a future period, a 'stronger' policy that raises θ even higher might fail to redirect the society to a desired trajectory $TB(\cdot)$, while it could have done so if it was implemented in period 4.

$\mathbb{V}$-born minority would prefer to jump to culture position P and then migrate to $\mathbb{C}$. That might suggest that the ability to culture-shift globally would severely limit the occurrence of equilibrium segmented assimilation. But recall our Simulation Regime [3] (where $C_{max} > V_{max}$ but $g(P) = 0.5$) and note that even if 'global culture-shifting' was possible in $\mathbb{V}$ with probability $\sigma = 0.3$ in every generation, TC would remain an equilibrium trajectory.

Next, recognize that if a minority member could be *fully multi-cultural* and could concurrently locate at every $p \in \mathbb{P}$ whenever she got the chance to culture-shift, then it would be a dominant strategy for her to do so. Again, if it was the case that $C_{max} > V_{max}$ and $g(P) \approx 1$, then complete assimilation in $\mathbb{C}$ would be a long-run equilibrium outcome. In contrast, consider the following case of 'limited multiculturalism': when a member born at some p in $\mathbb{V}$ got the chance to culture-shift, she could concurrently locate at p and at an adjacent position. Note that even if such limited multiculturalism was possible in Simulation Regime [3], TC would still be an equilibrium trajectory.

The above arguments indicate that while relaxing the assumptions of 'limited culture-shifting' and 'no multiculturalism' would likely expand the set of possible equilibrium assimilation trajectories, segmented assimilation of the minority community would still remain an equilibrium possibility. A formal analysis identifying conditions that would guarantee the emergence of long-run minority segmentation given more sophisticated minority culture-shifting possibilities awaits further research.

Appendix A

In the text, we have defined the culture distribution sets Δ_+, $\Delta_+(\underline{\mu})$, $\hat{\Delta}_+$, and $\hat{\Delta}_+(\underline{\mu})$. Let Δ, $\Delta(\underline{\mu})$, $\hat{\Delta}$, and $\hat{\Delta}(\underline{\mu})$ be the closure of Δ_+, $\Delta_+(\underline{\mu})$, $\hat{\Delta}_+$, and $\hat{\Delta}_+(\underline{\mu})$, respectively. For any $\underline{\mu} \in \Delta$, we define $\Delta(p|\underline{\mu}) = \{\underline{\mu}' \in \Delta(\underline{\mu}) : m[p|\underline{\mu}'] \geq m[p|\underline{\mu}]\}$, and extend this definition in obvious ways to define $\Delta_+(p|\underline{\mu})$ and $\hat{\Delta}_+(p|\underline{\mu})$. Further, for any $p \in \mathbb{P}$ and $0 < \eta < 1$, we define $\Delta^*(p, \eta) = \left\{\underline{\mu} \in \Delta : \sum_{p'=p+1}^{P} \mu(p') < \eta\right\}$.

We now define the culture-shift map (and hence the acquired culture distribution) under a given (common) conjecture $\underline{\mu}^e$.[41] To do this, we now introduce a few notations:

(i) $A_p^l = \left\{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p-1|\underline{\mu}^e) > U^*(p+j|\underline{\mu}^e), j = 0, 1\right\}$;

(ii) $A_p^s = \left\{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p|\underline{\mu}^e) > U^*(p+j|\underline{\mu}^e), j = -1, 1\right\}$;

(iii) $A_p^r = \left\{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p+1|\underline{\mu}^e) > U^*(p+j|\underline{\mu}^e), j = -1, 0\right\}$;

(iv) $A_p^{ls} = \left\{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p-1|\underline{\mu}^e) = U^*(p|\underline{\mu}^e) > U^*(p+1|\underline{\mu}^e)\right\}$;

(v) $A_p^{rs} = \left\{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p|\underline{\mu}^e) = U^*(p+1|\underline{\mu}^e) > U^*(p-1|\underline{\mu}^e)\right\}$;

[41] In order to describe decision rules compactly, we extend the culture position set to $\mathbb{P}^+ \equiv \mathbb{P} \cup \{0, P+1\}$. This allows us to specify culture-shift decisions of all members in a consistent manner without having to worry about members born at the ends of $\mathbb{P}$. We exogenously set $U^*(0|\underline{\mu}^e) = U^*(P+1|\underline{\mu}^e) = 0$ for all $\underline{\mu}^e \in \Delta_1^+$, so that no member born at culture-position 1 (respectively, P) will want to culture-shift left (respectively, right) since $U^*(p|\underline{\mu}^e)$ is strictly positive for all $p \in \mathbb{P}$.

(vi) $A_p^{lr} = \{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p-1|\underline{\mu}^e) = U^*(p+1|\underline{\mu}^e) > U^*(p|\underline{\mu}^e)\}$;

(vii) $A_p^{lsr} = \{\underline{\mu}^e \in \Delta_+[\underline{\mu}^i] : U^*(p-1|\underline{\mu}^e) = U^*(p|\underline{\mu}^e) = U^*(p+1|\underline{\mu}^e)\}$.

For any $p \in \mathbb{P}$, we can see that $A_p^j \cap A_p^k = \emptyset$ if $j \neq k$. Furthermore, for any (common) conjecture $\underline{\mu}^e$ of final distribution of individuals in $\mathbb{V}$ and any $p \in \mathbb{P}$, there is some j such that $\underline{\mu}^e \in A_p^j$. Let $\mathcal{A}(p) = \{A_p^l, A_p^s, A_p^r, A_p^{ls}, A_p^{rs}, A_p^{lr}, A_p^{lsr}\}$. Hence, any conjecture $\underline{\mu}^e$ belongs to exactly one element of $\mathcal{A}(p)$ for each culture position $p \in \mathbb{P}$.

For any $D \subseteq \Delta_+[\underline{\mu}^i]$, we now define the *indicator function* $\mathbf{1}_D$ on $\Delta_+[\underline{\mu}^i]$ by letting

$$\mathbf{1}_D(\underline{\mu}^e) = \begin{cases} 1, & \text{if } \underline{\mu}^e \in D; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\bar{\chi}_p^{SL}(\underline{\mu}^e|\underline{\varphi}_p) = \mathbf{1}_{A_p^l}(\underline{\mu}^e) + \varphi_p^{ls}\mathbf{1}_{A_p^{ls}}(\underline{\mu}^e) + \varphi_p^{lr}\mathbf{1}_{A_p^{lr}}(\underline{\mu}^e) + \varphi^l(p)\mathbf{1}_{A_p^{lsr}}(\underline{\mu}^e)$, and $\bar{\chi}_p^{SR}(\underline{\mu}^e|\underline{\varphi}_p) = \mathbf{1}_{A_p^r}(\underline{\mu}^e) + \varphi_p^{rs}\mathbf{1}_{A_p^{rs}}(\underline{\mu}^e) + (1-\varphi_p^{lr})\mathbf{1}_{A_p^{lr}}(\underline{\mu}^e) + \varphi_p^r\mathbf{1}_{A_p^{lsr}}(\underline{\mu}^e)$.

From each culture position p and a (common) conjecture $\underline{\mu}^e$, let $\mathbf{S}^l(p|\underline{\mu}^e, \underline{\varphi})$, $\mathbf{S}^s(p|\underline{\mu}^e, \underline{\varphi})$ and $\mathbf{S}^r(p|\underline{\mu}^e, \underline{\varphi})$ be the measures of individuals who culture-shift left (i.e., to $p-1$), the measure of individuals who stay back (i.e., at p) and the measure of individuals who culture-shift right (i.e., to $p+1$), respectively. Consequently, $\mathbf{S}^l(p|\underline{\mu}^e, \underline{\varphi}) = \sigma\mu^i(p)\bar{\chi}_p^{SL}(\underline{\mu}^e|\underline{\varphi}_p)$ and $\mathbf{S}^r(p|\underline{\mu}^e, \underline{\varphi}) = \sigma\mu^i(p)\bar{\chi}_p^{SR}(\underline{\mu}^e|\underline{\varphi}_p)$. A similar argument yields

$$\begin{aligned} \mathbf{S}^s(p|\underline{\mu}^e, \underline{\varphi}) &= \sigma\mu^i(p)[\mathbf{1}_{A_p^s}(\underline{\mu}^e) + (1-\varphi_p^{ls})\mathbf{1}_{A_p^{ls}}(\underline{\mu}^e) + (1-\varphi_p^{rs})\mathbf{1}_{A_p^{rs}}(\underline{\mu}^e) \\ &+ (1-\varphi_p^l-\varphi_p^r)\mathbf{1}_{A_p^{lsr}}(\underline{\mu}^e)]. \end{aligned}$$

It is worth to pointing out that $\mathbf{S}^l(p|\underline{\mu}^e, \underline{\varphi}) + \mathbf{S}^s(p|\underline{\mu}^e, \underline{\varphi}) + \mathbf{S}^r(p|\underline{\mu}^e, \underline{\varphi}) = \sigma\mu^i(p)$. Moreover, if all minority individuals in $\mathbb{V}$ holding the conjecture $\underline{\mu}^e$ of the final culture distribution of minority individuals in $\mathbb{V}$, the measure of individuals with acquired culture position p in $\mathbb{V}$ for any given $\underline{\varphi}$ is given by $\mu^a(p|(\underline{\mu}^e, \underline{\varphi})) = (1-\sigma)\mu^i(p) + \mathbf{S}^s(p|\underline{\mu}^e, \underline{\varphi}) + \mathbf{S}^r(p-1|\underline{\mu}^e, \underline{\varphi}) + \mathbf{S}^l(p+1|\underline{\mu}^e, \underline{\varphi})$.

We denote by $\underline{\mu}^a(\underline{\mu}^e, \underline{\varphi}) := (\mu^a(1|(\underline{\mu}^e, \underline{\varphi})), \cdots, \mu^a(P|(\underline{\mu}^e, \underline{\varphi})))$ the acquired culture distribution in $\mathbb{V}$ under the (common) conjecture $\underline{\mu}^e$ and $\underline{\varphi} \in \Phi$. In order to prove the next lemma, we now introduce some notations: consider two sets $\mathscr{C}(p)$ and $\mathscr{D}(p)$ for every $p \in \mathbb{P}$, defined as follows:

$$\mathscr{C}(p) = \left\{A_{p-1}^r, A_p^s, A_{p+1}^l\right\} \text{ and } \mathscr{D}(p) = \left\{A_{p-1}^{rs}, A_{p-1}^{lr}, A_{p-1}^{lsr}, A_p^{ls}, A_p^{rs}, A_p^{lsr}, A_{p+1}^{ls}, A_{p+1}^{lr}, A_{p+1}^{lsr}\right\}.$$

For any $\underline{\mu}^e \in \Delta_+[\underline{\mu}^i]$ and $p \in \mathbb{P}$, define

$$\Upsilon_p(\underline{\mu}^e) = \left\{(p', j) : A_{p'}^j \in \mathscr{C}(p) \text{ and } \underline{\mu}^e \in A_{p'}^j\right\}$$

and

$$\Lambda_p(\underline{\mu}^e) = \left\{(p', j) : A_{p'}^j \in \mathscr{D}(p) \text{ and } \underline{\mu}^e \in A_{p'}^j\right\}.$$

Clearly, $|\Lambda_p(\underline{\mu}^e)| \leq 3$ for all $\underline{\mu}^e \in \Delta_+[\underline{\mu}^i]$, where $|\Lambda_p(\underline{\mu}^e)|$ is the number of elements in $\Lambda_p(\underline{\mu}^e)$. Moreover, for any $p \in \mathbb{P}$, the value of $\mu^a(p|(\underline{\mu}^e, \underline{\varphi}))$ is independent of $\underline{\varphi}$ if $\Lambda_p(\underline{\mu}^e) = \emptyset$. For all $p \in \mathbb{P}$, let

$$\mathbf{\Theta}(p,x)=\begin{cases}\{ls,lr,lsr\}, & \text{if } x=l;\\ \{ls,rs,lsr\}, & \text{if } x=s;\\ \{lr,rs,lsr\}, & \text{if } x=r.\end{cases}$$

For any $y\in\{ls,rs,lr,lsr\}$, we now define a set $\mathbf{Z}(y)$ by

$$\mathbf{Z}(y)=\begin{cases}\{l,s,ls\}, & \text{if } y=ls;\\ \{s,r,rs\}, & \text{if } y=rs;\\ \{l,r,lr\}, & \text{if } y=lr;\\ \{l,s,r,ls,rs,lr,lsr\}, & \text{if } y=lsr.\end{cases}$$

Lemma 7.1. *Let $\widetilde{\Delta}$ be a non-empty subset of $\Delta_+[\underline{\mu}^i]$. Suppose that $\{\underline{\mu}^k:k\geq 1\}$ is a sequence in $\widetilde{\Delta}$ converging to a point $\underline{\mu}^0\in\widetilde{\Delta}$. For any sequence $\{\underline{\varphi}^k:k\geq 1\}$ of elements in Φ, if the sequence $\{\underline{\mu}^a(\underline{\mu}^k,\underline{\varphi}^k):k\geq 1\}$ converges to some $\underline{\mu}^a$ then $\underline{\mu}^a=\underline{\mu}^a(\underline{\mu}^0,\underline{\varphi}^0)$ for some $\underline{\varphi}^0\in\Phi$.*

Proof. Since $U^*(p|\underline{\mu})$ is continuous in $\underline{\mu}$, we obtain $\Upsilon_p(\underline{\mu}^0)\subseteq\Upsilon_p(\underline{\mu}^k)$ for all large k and $\Lambda_p(\underline{\mu}^0)=\emptyset\Rightarrow\Lambda_p(\underline{\mu}^k)=\emptyset$ for all large k and all $p\in\mathbb{P}$. We need to consider the following cases.

Case 1. $\bigcup_{p\in\mathbb{P}}\Lambda_p(\underline{\mu}^0)=\emptyset$**.** In this case, for each $p\in\mathbb{P}$, the value of $\mu^a(p|(\underline{\mu}^0,\underline{\varphi}))$ is the same for every $\underline{\varphi}$, and

$$\mu^a(p|(\underline{\mu}^0,\underline{\varphi}))=(1-\sigma)\mu^i(p)+\sum_{(p',j)\in\Upsilon_p(\underline{\mu}^0)}\sigma\mu^i(p').$$

Hence, for all large k, $\mu^a(p|(\underline{\mu}^k,\underline{\varphi}^k))$ is also independent of $\underline{\varphi}^k$ and

$$\mu^a(p|(\underline{\mu}^k,\underline{\varphi}^k))=(1-\sigma)\mu^i(p)+\sum_{(p',j)\in\Upsilon_p(\underline{\mu}^k)}\sigma\mu^i(p').$$

Note that, in this case, we must have $\Upsilon_p(\underline{\mu}^0)=\Upsilon_p(\underline{\mu}^k)$ for all $p\in\mathbb{P}$ and for all large k. Otherwise, there must exist a culture position p, a subsequence $\{\underline{\mu}^s:s\geq 1\}$ of $\{\underline{\mu}^k:k\geq 1\}$ and a (p_0,j_0) such that $(p_0,j_0)\in\bigcap_{s=1}^{\infty}\Upsilon_p(\underline{\mu}^s)$ and $(p_0,j_0)\notin\Upsilon_p(\underline{\mu}^0)$. Thus, $(p_0,z)\in\Lambda_p(\underline{\mu}^0)$ for some $z\in\mathbf{\Theta}(p_0,j_0)$, which contradicts with the fact that $\bigcup_{p\in\mathbb{P}}\Lambda_p(\underline{\mu}^0)=\emptyset$. Hence, $\underline{\mu}^a(\underline{\mu}^k,\underline{\varphi}^k)=\underline{\mu}^a(\underline{\mu}^0,\underline{\varphi}^0)$ for all large k and for any $\varphi^0\in\Phi$. It follows that for every $\underline{\varphi}^0\in\Phi$, we have $\underline{\mu}^a=\underline{\mu}^a(\underline{\mu}^0,\underline{\varphi}^0)$.

Case 2. $\bigcup_{p\in\mathbb{P}}\Lambda_p(\underline{\mu}^0)\neq\emptyset$**.** Without loss of generality, we assume that $\{\underline{\varphi}^k:k\geq 1\}$ converges to $\underline{\varphi}^*$ (if that is not the case, we will work with one of its convergent sub-sequences). Define $\mathbb{Q}=\left\{p\in\mathbb{P}:\Lambda_p(\underline{\mu}^0)\neq\emptyset\right\}$. Thus, the value of $\mu^a(p|(\underline{\mu}^0,\underline{\varphi}))$ is the same for every $\underline{\varphi}\in\Phi$ if and only if $p\notin\mathbb{Q}$. Moreover, $\mu^a(p|(\underline{\mu}^k,\underline{\varphi}^k))$ is also independent of $\underline{\varphi}^k$ for all large k and all $p\notin\mathbb{Q}$, and $\{\mu^a(p|(\underline{\mu}^k,\underline{\varphi}^k)):k\geq 1\}$ converges to $\mu^a(p|(\underline{\mu}^0,\underline{\varphi}^0))$ for all $p\notin\mathbb{Q}$. Let

$$\mathscr{K}=\left\{(p',j):(p',j)\in\Lambda_p(\underline{\mu}^0)\text{ for some }p\in\mathbb{Q}\right\}.$$

For convenience, we rewrite $\mathscr{K}$ as $\mathscr{K}=\{(p_1,j_1),\cdots,(p_K,j_K)\}$ with $p_1<\cdots<p_K$ (recall that, for every $p'\in\mathbb{P}$, $\underline{\mu}^0$ belongs to exactly one element of $\mathcal{A}(p')$). Thus, for every $p\in\mathbb{Q}$, there must

exist at least one element $(p_m, j_m) \in \mathscr{K}$ such that $A_{p_m}^{j_m} \in \mathscr{D}(p)$, and the family of sets associated with elements of $\mathscr{K}$ can be summarized as $\mathscr{M}(\underline{\mu}^0) = \left\{A_{p_1}^{j_1}, \cdots, A_{p_K}^{j_K}\right\}$.

For every $A_{p_m}^{j_m} \in \mathscr{M}(\underline{\mu}^0)$, we let $\Sigma_m = \left\{z \in \mathbf{Z}(j_m) : \underline{\mu}^k \in A_{p_m}^z \text{ for infinitely many } k\right\}$. Choose a sub-sequence $\{\underline{\mu}^s : s \geq 1\}$ of $\{\underline{\mu}^k : k \geq 1\}$ such that for every $(p_m, j_m) \in \mathscr{K}$ there is some $z_m \in \Sigma_m$ such that

$$\left\{\underline{\mu}^s : s \geq 1\right\} \subseteq \left\{\underline{\mu}^k : k \geq 1\right\} \cap A_{p_m}^{z_m}.$$

It is given that the sequence $\{\underline{\mu}^a(\underline{\mu}^s, \underline{\varphi}^s) : s \geq 1\}$ converges to $\underline{\mu}^a$. We now calculate the value of $\underline{\varphi}_p^0 := (\varphi_p^{ls}, \varphi_p^{rs}, \varphi_p^{lr}, \varphi_p^l, \varphi_p^r)$, for all $p \in \mathbb{P}$, such that $\underline{\mu}^a = \underline{\mu}^a(\underline{\mu}^0, \underline{\varphi}^0)$, where $\underline{\varphi}^0 := (\underline{\varphi}_1^0, \cdots, \underline{\varphi}_P^0) \in \Phi$. Firstly, note that $\varphi_{p_0}^j$ can be chosen to be any member of $[0,1]$ if $j \in \{ls, rs, lr\}$ and $(p_0, j) \notin \mathscr{K}$; and $(\varphi_{p_0}^l, \varphi_{p_0}^r)$ can be chosen to be any member of Γ^2 if $(p_0, lsr) \notin \mathscr{K}$. For $(p_m, j_m) \in \mathscr{K}$ and $j_m = ls$, we have the following: (i) if $z_m = l$ then $\varphi_{p_m}^{j_m} = 1$; (ii) if $z_m = s$ then $\varphi_{p_m}^{j_m} = 0$; and (iii) if $z_m = ls$ then $\varphi_{p_m}^{j_m} = \varphi_{p_m}^{*j_m}$ ($\varphi_{p_m}^{*j_m}$ is the sl-component of $\underline{\varphi}_{p_m}^*$ for $\underline{\varphi}^* \in \Phi$). Likewise, one can easily calculate the value of $\varphi_{p_m}^{j_m}$ if $j_m \in \{rs, lr\}$ and $(p_m, j_m) \in \mathscr{K}$. If $(p_m, lsr) \in \mathscr{K}$, then we have the following: (i) if $z_m = l$ then $\varphi_{p_m}^l = 1$; (ii) if $z_m = r$ then $\varphi_{p_m}^r = 1$; (iii) if $z_m = s$ then $\varphi_{p_m}^l = \varphi_{p_m}^r = 0$; (iv) if $z_m = ls$ then $\varphi_{p_m}^l = \varphi_{p_m}^{*ls}$ and $\varphi_{p_m}^r = 0$; (v) if $z_m = rs$ then $\varphi_{p_m}^l = 0$ and $\varphi_{p_m}^r = \varphi_{p_m}^{*rs}$; (vi) if $z_m = lr$ then $\varphi_{p_m}^l = \varphi_{p_m}^{*lr}$ and $\varphi_{p_m}^r = 1 - \varphi_{p_m}^{*lr}$; and (vii) if $z_m = lsr$ then $\varphi_{p_m}^l = \varphi_{p_m}^{*l}$ and $\varphi_{p_m}^r = \varphi_{p_m}^{*r}$. This completes the proof. □

For any (common) conjecture $\underline{\mu}^e$, acquired culture distribution $\underline{\mu}^a$ and any $\underline{\psi} \in \Psi$, the measure of individuals not migrating from p is denoted by $\mathbf{R}(p|\underline{\mu}^e, \underline{\mu}^a, \underline{\psi})$ and is defined as

$$\mathbf{R}(p|\underline{\mu}^e, \underline{\mu}^a, \underline{\psi}) = \begin{cases} \mu^a(p), & \text{if } V(p|\underline{\mu}^e) > C(p|\underline{\mu}^e); \\ (1 - g(p))\mu^a(p), & \text{if } V(p|\underline{\mu}^e) < C(p|\underline{\mu}^e); \\ (1 - g(p) + \psi_p g(p))\mu^a(p), & \text{if } V(p|\underline{\mu}^e) = C(p|\underline{\mu}^e). \end{cases}$$

Remark 7.2. Let $\{\underline{\mu}_k^a : k \geq 1\}$ be a sequence in $\Delta_+[\underline{\mu}^i]$ converging to $\underline{\mu}^a \in \Delta_+[\underline{\mu}^i]$ and let $\{\underline{\psi}_k : k \geq 1\} \subseteq [0,1]^P$ be a sequence converging to $\underline{\psi} \in [0,1]^P$. Assume further that $\{\underline{\mu}_k^e : k \geq 1\} \subseteq \Delta_+[\underline{\mu}^i]$ converges to $\underline{\mu}^e \in \Delta_+[\underline{\mu}^i]$. From the continuity of $V(p|\cdot)$ and $C(p|\cdot)$, we conclude that if $V(p|\underline{\mu}^e) > C(p|\underline{\mu}^e)$ (resp. $V(p|\underline{\mu}^e) < C(p|\underline{\mu}^e)$) then $V(p|\underline{\mu}_k^e) > C(p|\underline{\mu}_k^e)$ (resp. $V(p|\underline{\mu}_k^e) < C(p|\underline{\mu}_k^e)$) for all large k, which immediately implies that $\lim_{k\to\infty} \mathbf{R}(p|\underline{\mu}_k^e, \underline{\mu}_k^a, \underline{\psi}_k) = \mathbf{R}(p|\underline{\mu}^e, \underline{\mu}^a, \underline{\psi})$. So, we assume that $V(p|\underline{\mu}^e) = C(p|\underline{\mu}^e)$, and let ψ_p^1, ψ_p^0 be arbitrary elements of Ψ such that the p^{th}-coordinate of ψ_p^1 and ψ_p^0 are 1 and 0, respectively. Then we have the following possibilities:

(O1) $\lim_{k\to\infty} \mathbf{R}(p|\underline{\mu}_k^e, \underline{\mu}_k^a, \psi_k) = \mathbf{R}(p|\underline{\mu}^e, \underline{\mu}^a, \psi_p^1)$ if $V(p|\underline{\mu}_k^e) > C(p|\underline{\mu}_k^e)$ for all large k.

(O2) $\lim_{k\to\infty} \mathbf{R}(p|\underline{\mu}_k^e, \underline{\mu}_k^a, \psi_k) = \mathbf{R}(p|\underline{\mu}^e, \underline{\mu}^a, \psi_p^0)$ if $V(p|\underline{\mu}_k^e) < C(p|\underline{\mu}_k^e)$ for all large k.

(O3) $\lim_{k\to\infty} \mathbf{R}(p|\underline{\mu}_k^e, \underline{\mu}_k^a, \psi_k) = \mathbf{R}(p|\underline{\mu}^e, \underline{\mu}^a, \psi)$ if $V(p|\underline{\mu}_k^e) = C(p|\underline{\mu}_k^e)$ for all large k.

Proof of Theorem 3.1: Clearly, $\mathcal{M}^f(\underline{\mu}^e|\underline{\mu}^i)$ is a non-empty convex set for each $\underline{\mu}^e \in \Delta^\dagger$. It is claimed that $\mathcal{M}^f(\cdot|\underline{\mu}^i)$ is closed. To that end, let $\{\underline{\mu}_k^e : k \geq 1\}$ be a sequence of minority culture distributions in $\Delta^\dagger$ converging to a distribution $\underline{\mu}_0^e \in \Delta^\dagger$ and let $\underline{\mu}^f(\underline{\mu}_k^e, \underline{\mu}^i, \underline{\varphi}_k, \underline{\psi}_k) \in \mathcal{M}^f(\underline{\mu}_k^e|\underline{\mu}^i)$ be such that $\{\underline{\mu}^f(\underline{\mu}_k^e, \underline{\mu}^i, \underline{\varphi}_k, \underline{\psi}_k) : k \geq 1\}$ converges to $\underline{\mu}_0^f$. We will show that $\underline{\mu}_0^f \in \mathcal{M}^f(\underline{\mu}_0^e|\underline{\mu}^i)$. To that end, without any loss of generality, we assume that $\{\underline{\psi}_k : k \geq 1\}$ converges to $\underline{\psi}_*$. First, note that

$$\mu^f(p|\underline{\mu}_k^e, \underline{\mu}^i, \underline{\varphi}_k, \underline{\psi}_k) = \mathbf{R}(p|\underline{\mu}_k^e, \mu^a(p|(\underline{\mu}_k^e, \underline{\varphi}_k)), \underline{\psi}_k), \tag{7.1}$$

for all $k \geq 1$. We now choose a subsequence $\{\underline{\mu}^s : s \geq 1\}$ of $\{\underline{\mu}^k : k \geq 1\}$ that satisfies the following property: for each $p \in \mathbb{P}$, either $V(p|\underline{\mu}^s) > C(p|\underline{\mu}^s)$ for all s or $V(p|\underline{\mu}^s) < C(p|\underline{\mu}^s)$ for all s or $V(p|\underline{\mu}^s) = C(p|\underline{\mu}^s)$ for all s (for a different value of p a different inequality or equality may occur). Choose an arbitrary $p \in \mathbb{P}$. Consequently, by (7.1), $\{\underline{\mu}^a(\underline{\mu}_s^e, \underline{\varphi}_s) : s \geq 1\}$ converges to $\underline{\mu}_0^a$, where

(i) $\mu_0^a(p) = \mu_0^f(p)$ if $V(p|\underline{\mu}_s^e) > C(p|\underline{\mu}_s^e)$ for all $s \geq 1$;

(ii) $\mu_0^a(p) = \frac{\mu_0^f(p)}{1-g(p)}$ if $V(p|\underline{\mu}_s^e) < C(p|\underline{\mu}_s^e)$ for all $s \geq 1$; and

(iii) $\mu_0^a(p) = \frac{\mu_0^f(p)}{1-g(p)+\psi_{*p}g(p)}$ if $V(p|\underline{\mu}_s^e) = C(p|\underline{\mu}_s^e)$ for all $s \geq 1$.

By Lemma 7.1, we conclude that there is some $\underline{\varphi}_0$ such that $\underline{\mu}_0^a = \underline{\mu}^a(\underline{\mu}_0^e, \underline{\varphi}_0)$. Define $\underline{\psi}_0 := (\psi_{01}, \cdots, \psi_{0P}) \in \Psi$ by letting[42]

$$\psi_{0p} = \begin{cases} 1, & \text{if } V(p|\underline{\mu}_s^e) > C(p|\underline{\mu}_s^e) \text{ for all } s \geq 1; \\ 0, & \text{if } V(p|\underline{\mu}_s^e) < C(p|\underline{\mu}_s^e) \text{ for all } s \geq 1; \\ \psi_{*p}, & \text{if } V(p|\underline{\mu}_s^e) = C(p|\underline{\mu}_s^e) \text{ for all } s \geq 1. \end{cases}$$

It can be easily verified that $\underline{\mu}_0^f = \underline{\mu}^f(\underline{\mu}_0^e, \underline{\mu}^i, \underline{\varphi}_0, \underline{\psi}_0)$.Thus, $\underline{\mu}_0^f \in \mathcal{M}^f(\underline{\mu}_0^e|\underline{\mu}^i)$. Consequently, by Kakatuni's fixed point theorem, we infer that there is a point $\underline{\mu}_{00}^e \in \Delta^\dagger$ such that $\underline{\mu}_{00}^e \in \mathcal{M}^f(\underline{\mu}_{00}^e|\underline{\mu}^i)$.

Appendix B

Proof of Lemma 4.1: Choose some $\underline{\mu} \in \Delta_+$. It can be verified that

$$d(p+1, \underline{\mu}) - d(p, \underline{\mu}) = \frac{1}{(P-1)} \left[\sum_{p'=1}^{p} \mu(p') - \sum_{p'=p+1}^{P} \mu(p') \right],$$

for all $p \in \mathbb{P} \setminus \{P\}$. It follows that if $\mathbb{M}(\underline{\mu})$ contains two elements $\bar{p}, \hat{p}$ then $d(\bar{p}, \underline{\mu}) = d(\hat{p}, \underline{\mu})$. Let p^m be an element of $\mathbb{M}(\underline{\mu})$. Recognized that, for any $p \notin \mathbb{M}(\underline{\mu})$ and $p < p^m$, we have $d(p, \underline{\mu}) > d(p+1, \underline{\mu})$. Similarly, for any $p \notin \mathbb{M}(\underline{\mu})$ and $p > p^m$, we have $d(p, \underline{\mu}) < d(p+1, \underline{\mu})$. Combining these, we get $d(p^m, \underline{\mu}) < d(p', \underline{\mu})$ for all $p' \notin \mathbb{M}(\underline{\mu})$, which implies that $b(m[\underline{\mu}], d(p^m, \underline{\mu})) > b(m[\underline{\mu}], d(p', \underline{\mu}))$ for all

[42] Note that $V(p|\underline{\mu}_s^e) > C(p|\underline{\mu}_s^e)$ for all $s \geq 1$ imply $V(p|\underline{\mu}_s^0) \geq C(p|\underline{\mu}_s^0)$. In case the strict inequality occurs one can take any value of ψ_{0p} as nobody will be migrating from p and thus the exact value of ψ_{0p} is irrelevant. However, if the inequality occurs with equality then any fraction of $g(p)\underline{\mu}^a(\underline{\mu}_0^e, \underline{\varphi}_0)$ can migrate to $\mathbb{C}$. As nobody is migrating from p under the conjecture $\underline{\mu}_s^e$ for all $s \geq 1$ and $\underline{\mu}_0^e$ is the limit of $\{\underline{\mu}_s^e : s \geq 1\}$, we choose $\psi_{0p} = 1$ in order to maintain the fact that $\{\underline{\mu}^f(\underline{\mu}_s^e, \underline{\mu}^i, \underline{\varphi}_s, \underline{\psi}_s) : s \geq 1\}$ converges to $\underline{\mu}^f(\underline{\mu}_0^e, \underline{\mu}^i, \underline{\varphi}_0, \underline{\psi}_0)$. A similar argument applies when $V(p|\underline{\mu}_s^e) < C(p|\underline{\mu}_s^e)$ for all $s \geq 1$.

$p' \notin \mathbb{M}(\underline{\mu})$. For the second part, note that $d(p, \underline{\mu}) < d(p+1, \underline{\mu})$ for all $p \geq p^m$ with $p+1 \notin \mathbb{M}(\underline{\mu})$ implies $V(p|\underline{\mu}) > V(p+1|\underline{\mu})$ for all $p \geq p^m$ with $p+1 \notin \mathbb{M}(\underline{\mu})$; while $C(p|\underline{\mu}) < C(p+1|\underline{\mu})$ for all $p \geq 1$. Thus, $C(p|\underline{\mu}) - V(p|\underline{\mu})$ strictly increases in p for all $p \in \{p^m, \cdots, P\}$. □

Let $\underline{\mu}^i \in \hat{\Delta}_+$ be the initial distribution of minority individuals in some generation τ. Pick an element $\tilde{p}_0 \in \mathbb{P}$ such that $P > \tilde{p}_0 \geq \hat{p}(\underline{\mu}^i)$. Then, let $\widetilde{\mathbb{X}}$ denote the set of all feasible decision profiles that satisfy the following properties: when relevant culture-shifting and/or migration opportunities arise, (i) all minority members born in the culture-subset $\{1, \cdots, \hat{q}(\underline{\mu}^i)-1\}$ (if the set is non-empty) culture-shift towards $\hat{q}(\underline{\mu}^i)$ and do not migrate, (ii) all born in $\{\hat{q}(\underline{\mu}^i), \cdots, \tilde{p}_0\}$ culture-shift within the subset and do not migrate, and (iii) all born at $\hat{p}(\underline{\mu}^i)$ do not culture-shift right. Let $\widetilde{\Delta}(\underline{\mu}^i)$ be the set of all culture distribution generated from $\underline{\mu}^i$ by all decision profiles in $\widetilde{\mathbb{X}}$. Recognize that the set $\widetilde{\Delta}(\underline{\mu}^i)$ is non-empty, convex and compact.

Lemma 7.3. *Consider some generation τ with the inherited culture distribution $\underline{\mu}^i \in \hat{\Delta}_+$. Select two culture positions $\tilde{p}_0, \tilde{q}_0 \in \mathbb{P}$ satisfying $P > \tilde{p}_0 \geq \hat{p}(\underline{\mu}^i)$ and $\tilde{q}_0 \leq \hat{q}(\underline{\mu}^i)$. If $\{\tilde{p}_0, \tilde{q}_0\}$ satisfy either conditions* **[A0]-[A1]** *or conditions* [**B0**]-[**B1**] *stated below, then there exist an equilibrium outcome $\underline{\mu}^*$ in generation τ that belongs in $\widetilde{\Delta}(\underline{\mu}^i)$ and satisfies $\hat{q}(\underline{\mu}^i) \leq \hat{q}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^i)$.*

[**A0**]*: For all $\underline{\mu} \in \widetilde{\Delta}(\underline{\mu}^i)$, $[V(p|\underline{\mu}) - C(p|\underline{\mu})] \geq [V(\tilde{p}_0|\underline{\mu}) - C(\tilde{p}_0|\underline{\mu})]$ for all $p < \tilde{p}_0$;*

[**A1**]*: $V(\tilde{p}_0|\underline{\mu}^{A1}(\underline{\mu}^i)) \geq C(\tilde{p}_0+1|\underline{\mu}^{A1}(\underline{\mu}^i))$, where $\underline{\mu}^{A1}(\underline{\mu}^i)$ is uniquely generated from $\underline{\mu}^i$ by the following feasible decision profile: "whenever relevant opportunities arise, members born at $p \in \{1, \cdots, \tilde{p}_0\}$ culture-shift toward $\tilde{q}_0$ and stay back in $\mathbb{V}$, and members born at $p' \in \{\tilde{p}_0+1, \cdots, P\}$ culture-shift toward P and migrate to $\mathbb{C}$".*

[**B0**]*: For all $\underline{\mu} \in \widetilde{\Delta}(\underline{\mu}^i)$, $[V(p|\underline{\mu}) - C(p|\underline{\mu})] \geq [V(\tilde{q}_0|\underline{\mu}) - C(\tilde{q}_0|\underline{\mu})]$ for all $p < \tilde{q}_0$;*

[**B1**]*: $V(\tilde{p}_0|\underline{\mu}^{B1}(\underline{\mu}^i)) \geq C(\tilde{p}_0+1|\underline{\mu}^{B1}(\underline{\mu}^i))$, where $\underline{\mu}^{B1}(\underline{\mu}^i)$ is uniquely generated from $\underline{\mu}^i$ by the following infeasible decision profile: "members born at $p \in \{1, \cdots, \tilde{p}_0\}$ culture-shift toward $\tilde{q}_0$ and stay back in $\mathbb{V}$, and all members born in $\{\tilde{p}_0+1, \cdots, P\}$ migrate to $\mathbb{C}$ in one shot".*

Proof. Letting $\underline{\mu}^e \in \widetilde{\Delta}(\underline{\mu}^i)$ be the conjecture of the minority members born in $\mathbb{V}$, we begin by proving that under $\underline{\mu}^e$, members' optimal decision profiles will belong in $\widetilde{\mathbb{X}}$.

First, we establish that a member born at $\tilde{p}_0$ will neither culture-shift right under $\underline{\mu}^e$ nor migrate to $\mathbb{C}$ from $\tilde{p}_0$. Since $m[\tilde{p}_0|\underline{\mu}^e] \geq m[\tilde{p}_0|\underline{\mu}^i] > \frac{1}{2}m[\underline{\mu}^i] \geq \frac{1}{2}m[\underline{\mu}^e]$, we have $b\left(m[\underline{\mu}^e], d(\tilde{p}_0, \underline{\mu}^e)\right) > b\left(m[\underline{\mu}^e], d(\tilde{p}_0+1, \underline{\mu}^e)\right)$. Consequently, $V\left(\tilde{p}_0|\underline{\mu}^e\right) > V\left(\tilde{p}_0+1|\underline{\mu}^e\right)$, implying that under $\underline{\mu}^e$, a member born at $\tilde{p}_0$ in $\mathbb{V}$ will not culture shift right and stay back in $\mathbb{V}$.

It follows from [**D**] that $V\left(\tilde{p}_0|\underline{\mu}^e\right) \geq V\left(\tilde{p}_0|\underline{\mu}^{A1}(\underline{\mu}^i)\right) \geq V\left(\tilde{p}_0|\underline{\mu}^{B1}(\underline{\mu}^i)\right)$. On the other hand, $C(\tilde{p}_0+1|\underline{\mu}^{B1}(\underline{\mu}^i)) \geq C(\tilde{p}_0+1|\underline{\mu}^{A1}(\underline{\mu}^i)) \geq C(\tilde{p}_0+1|\underline{\mu}^e)$. Consequently, by [**A**1] or [**B**1], we have $V\left(\tilde{p}_0|\underline{\mu}^e\right) \geq C(\tilde{p}_0+1|\underline{\mu}^e) > C(\tilde{p}_0|\underline{\mu}^e)$. Therefore, $V\left(\tilde{p}_0|\underline{\mu}^e\right) > (1-g(p))V(\tilde{p}_0+1|\underline{\mu}^e) + g(p)C(\tilde{p}_0+1|\underline{\mu}^e)$. These inequalities imply that under $\underline{\mu}^e$, a member born at $\tilde{p}_0$ will neither culture-shift right (either to stay in $\mathbb{V}$ at $\tilde{p}_0+1$ or to migrate to $\mathbb{C}$ from $\tilde{p}_0+1$) nor migrate to $\mathbb{C}$ from $\tilde{p}_0$.

Second, given the above conclusion, the following arguments prove that the culture set $\{1, \cdots, \tilde{p}_0\}$ will be entrenched under $\underline{\mu}^e$, with there being no migration from any of these culture positions. If

[**A**0] is satisfied, then the result is immediate. So, consider the case where [**A**0] is not satisfied but [**B**0] and [**B**1] are. [**B**0] ensures that no one will migrate from any position in $\{1, \cdots, \tilde{q}_0 - 1\}$ under $\underline{\mu}^e$ whenever $V\left(\tilde{q}_0|\underline{\mu}^e\right) > C(\tilde{q}_0|\underline{\mu}^e)$. Thus, in order to guarantee that $\{1, \cdots, \tilde{p}_0\}$ will be entrenched under $\underline{\mu}^e$, we need to show that no one will migrate from $\{\tilde{q}_0, \cdots, \tilde{p}_0 - 1\}$. This conclusion follows from the fact that when [**B**1] holds (with $\underline{\mu}^{B1}(\underline{\mu}^i)$ having zero measure of members at every $p > \tilde{p}_0$), we have $V\left(p|\underline{\mu}^e\right) \geq V\left(\tilde{p}_0|\underline{\mu}^{B1}(\underline{\mu}^i)\right)$ and $C(\tilde{p}_0 + 1|\underline{\mu}^{B1}(\underline{\mu}^i)) \geq C(\tilde{p}_0 + 1|\underline{\mu}^e) \geq C(p|\underline{\mu}^e)$ for every $p \in \{\tilde{q}_0, \cdots, \tilde{p}_0 - 1\}$.

Third, the following arguments prove that optimal decision profiles under $\underline{\mu}^e$ satisfy conditions (i)-(iii) in the definition of $\widetilde{\mathbb{X}}$. If $\hat{q}(\underline{\mu}^i) > 1$ then $\underline{\mu}^e \in \widetilde{\Delta}(\underline{\mu}^i)$ implies that

$$\sum_{p=1}^{\hat{q}(\underline{\mu}^i)-1} \mu^e(p) < \sum_{p=1}^{\hat{q}(\underline{\mu}^i)-1} \mu^i(p) < \frac{1}{2}m[\tilde{p}_0|\underline{\mu}^i] \leq \frac{1}{2}m[\tilde{p}_0|\underline{\mu}^e] \leq \frac{1}{2}m[\underline{\mu}^e].$$

Thus, under conjecture $\underline{\mu}^e$, a member born at p in $\mathbb{V}$ will culture-shift right for all $p < \hat{q}(\underline{\mu}^i)$, thus ensuring that the optimal decision profile will satisfy condition (i). Further, condition (i) also implies that no member born at $\hat{q}(\underline{\mu}^i)$ will culture-shift left; and that, together with the fact that $\{1, \cdots, \tilde{p}_0\}$ will be entrenched under $\underline{\mu}^e$ will imply the satisfaction of condition (ii). Finally, note that

$$m[\hat{p}(\underline{\mu}^i)|\underline{\mu}^e] \geq m[\hat{p}(\underline{\mu}^i)|\underline{\mu}^i] > \frac{1}{2}m[\underline{\mu}^i] \geq \frac{1}{2}m[\underline{\mu}^e].$$

Consequently, $V(\hat{p}(\underline{\mu}^i)|\underline{\mu}^e) > V(\hat{p}(\underline{\mu}^i)+1|\underline{\mu}^e)$. Further, note that it will be the case that $V(\hat{p}(\underline{\mu}^i)|\underline{\mu}^e) \geq C(\hat{p}(\underline{\mu}^i) + 1|\underline{\mu}^e)$ irrespective of whether $\hat{p}(\underline{\mu}^i) = \tilde{p}_0$ or $\hat{p}(\underline{\mu}^i) < \tilde{p}_0$. As a result, one born at $\hat{p}(\underline{\mu}^i)$ will not culture-shift right under $\underline{\mu}^e$. Thus, condition (iii) will also be satisfied by optimal decision profiles under $\underline{\mu}^e$.

We have thus proved that for any conjecture $\underline{\mu}^e \in \widetilde{\Delta}(\underline{\mu}^i)$, an optimal decision profile of minority members in $\mathbb{V}$ will belong to $\widetilde{\mathbb{X}}^F$, and will thus generate a final distribution $\underline{\mu}^f(\underline{\mu}^e, \underline{\mu}^i, \underline{\varphi}, \underline{\psi}) \in \widetilde{\Delta}(\underline{\mu}^i)$ for any $\underline{\varphi} \in \Phi$ and $\underline{\psi} \in \Psi$. Given that, we now define a correspondence $\mathcal{M}^f(\cdot|\underline{\mu}^i) : \widetilde{\Delta}(\underline{\mu}^i) \rightrightarrows \widetilde{\Delta}(\underline{\mu}^i)$ such that

$$\mathcal{M}^f(\underline{\mu}^e|\underline{\mu}^i) = \left\{\underline{\mu}^f(\underline{\mu}^e, \underline{\mu}^i, \underline{\varphi}, \underline{\psi}) : (\underline{\varphi}, \underline{\psi}) \in \Phi \times \Psi\right\}.$$

Then by Theorem 3.1, we conclude that there exists a fixed point $\underline{\mu}^*$ of $\mathcal{M}^f(\cdot|\underline{\mu}^i)$. Consequently, $\underline{\mu}^* = \underline{\mu}^f(\underline{\mu}^*, \underline{\mu}^i, \underline{\varphi}, \underline{\psi}) \in \widetilde{\Delta}(\underline{\mu}^i)$ for some $(\underline{\varphi}, \underline{\psi}) \in \Phi \times \Psi$. As $\{1, \cdots, \tilde{p}_0\}$ is entrenched under $\underline{\mu}^*$ and all minority members born in the culture-subset $\{1, \cdots, \hat{q}(\underline{\mu}^i) - 1\}$ (if the set is non-empty) culture-shift towards $\hat{q}(\underline{\mu}^i)$ under the conjecture $\underline{\mu}^*$, we must have $\hat{q}(\underline{\mu}^i) \leq \hat{q}(\underline{\mu}^*)$. Further, it is immediate that $\hat{p}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^i)$, since otherwise, members born at $\hat{p}(\underline{\mu}^i)$ would strictly prefer to culture-shift right (to improve both $\mathbb{V}$ and $\mathbb{C}$ payoffs) under the conjecture $\underline{\mu}^*$. □

Let $\widetilde{\mathbb{X}}^F$ be the set of all feasible decision profiles in $\widetilde{\mathbb{X}}$ such that at least the measure $(1-\sigma)g(P)\mu^i(P)$ of members migrate from P under any decision profile in $\widetilde{\mathbb{X}}^F$, and let $\widetilde{\Delta}^F(\underline{\mu}^i)$ denote the set of all culture distributions generated from $\underline{\mu}^i$ by all decision profiles in $\widetilde{\mathbb{X}}^F$.

Lemma 7.4. *Consider some generation τ with the inherited culture distribution $\underline{\mu}^i \in \hat{\Delta}_+$. Select two culture positions $\tilde{p}_0, \tilde{q}_0 \in \mathbb{P}$ satisfying $P > \tilde{p}_0 \geq \hat{p}(\underline{\mu}^i)$ and $\tilde{q}_0 \leq \hat{q}(\underline{\mu}^i)$. If $\{\tilde{p}_0, \tilde{q}_0\}$ satisfy either conditions* [**A0**]-[**A1**] *or conditions* [**B0**]-[**B1**] *stated in Lemma* 1.1, *and the condition* [**A2**] *stated below, then there exist an equilibrium outcome $\underline{\mu}^*$ in generation τ that belongs in $\widetilde{\Delta}^F(\underline{\mu}^i)$, satisfies $\hat{q}(\underline{\mu}^i) \leq \hat{q}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^i)$.*

[**A2**]*: $V(P|\underline{\mu}^{A2}(\underline{\mu}^i)) \leq C(P|\underline{\mu}^{A2}(\underline{\mu}^i))$, where $\underline{\mu}^{A2}(\underline{\mu}^i)$ is uniquely generated from $\underline{\mu}^i$ by the following feasible decision profile: "whenever relevant opportunities arise, members born at $p \in \{1, \cdots, \tilde{p}_0\}$ culture-shift toward $\tilde{p}_0$ and stay back in $\mathbb{V}$, members born at $p' \in \{\tilde{p}_0+1, \cdots, P\}$ culture-shift toward P, and the measure $(1-\sigma)g(P)\mu^i(P)$ of members situated at acquired position P migrate to $\mathbb{C}$".*

Proof. For any $\underline{\mu}^e \in \widetilde{\Delta}^F(\underline{\mu}^i)$, we have $C(P|\underline{\mu}^e) \geq C\left(P|\underline{\mu}^{A2}(\underline{\mu}^i)\right)$. We now introduce a distribution $\underline{\mu}''$, which is not necessarily feasible, as follows: $\mu''(p) = \mu^e(p)$ for all $p \neq \tilde{p}_0 + 1$; and $\mu''(\tilde{p}_0 + 1) = \mu^e(\tilde{p}_0 + 1) + m[\underline{\mu}^{A2}(\underline{\mu}^i)] - m[\underline{\mu}^e]$. It follows that $m[\underline{\mu}''] = m[\underline{\mu}^{A2}(\underline{\mu}^i)]$. By the construction of $\underline{\mu}''$, we have $V\left(P|\underline{\mu}^{A2}(\underline{\mu}^i)\right) > V(P|\underline{\mu}'')$. Moreover, in the light of [**D**], $V(P|\underline{\mu}'') \geq V(P|\underline{\mu}^e)$. Combining all these inequalities with [**A**2], we get $C(P|\underline{\mu}^e) > V(P|\underline{\mu}^e)$.

Hence, applying an argument similar to that in Lemma 7.3, we can guarantee the existence of an equilibrium outcome $\underline{\mu}^*$ that belongs in $\widetilde{\Delta}^F(\underline{\mu}^i)$, satisfies $\hat{q}(\underline{\mu}^i) \leq \hat{q}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^*) \leq \hat{p}(\underline{\mu}^i)$. □

Proof of Proposition 5.1: The result follows from Lemma 1.1 and Lemma 1.2 for $\tilde{p}_0 = \hat{p}(\underline{\mu}^i)$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}^i)$. □

Proof of Proposition 5.2: Note that when [**L**0]-[**L**2] hold for $\underline{\mu}^i = \underline{\mu}_{\tau-1}$, conditions [**B**0], [**B**1] and [**A**2] hold for $\underline{\mu}^i = \underline{\mu}_{\tau-1}$, $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1})$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}_{\tau-1})$. Thus, by Lemmas 7.3 and 7.4, there exists an equilibrium outcome $\underline{\mu}_\tau^*$ in generation τ that belongs in $\widetilde{\Delta}(\underline{\mu}_{\tau-1})$, satisfies $\hat{q}(\underline{\mu}_{\tau-1}) \leq \hat{q}(\underline{\mu}_\tau^*) \leq \hat{p}(\underline{\mu}_\tau^*) \leq \hat{p}(\underline{\mu}_{\tau-1})$, and exhibits the property that all minority members situated in acquired culture position P migrate to $\mathbb{C}$ whenever they get the opportunity. Choose such an equilibrium $\underline{\mu}_\tau^*$ for generation τ and note that it will be inherited distribution in generation $\tau + 1$. We now prove the following result: If conditions [**L0**], [**L1**], and [**L2**] hold for $\underline{\mu}^i = \underline{\mu}_{\tau-1}$, then conditions [**B0**], [**B1**], and [**A2**] hold for $\underline{\mu}^i = \underline{\mu}_\tau^*$, $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1})$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}_{\tau-1})$. As above, [**L0**] hold for $\underline{\mu}^i = \underline{\mu}_{\tau-1}^i$ implies that [**B0**] holds for $\underline{\mu}^i = \underline{\mu}_\tau^*$, $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1})$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}_{\tau-1})$. Second, note that [**L**1] and [**D**] imply that the distribution $\underline{\mu}^{B1}(\underline{\mu}_\tau^*)$ satisfies [**B**1] for $\underline{\mu}^i = \underline{\mu}_\tau^*$, $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1})$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}_{\tau-1})$. We now claim that if [**L2**] holds for $\underline{\mu}^i = \underline{\mu}_{\tau-1}$, then [**A2**] hold for $\underline{\mu}^i = \underline{\mu}_\tau^*$, $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1})$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}_{\tau-1})$. Since the measure $(1-\sigma)g(P)\mu_{\tau-1}(P)$ of minority individuals have already migrated in generation τ , we have

$$C\left(P|\underline{\mu}^{A2}(\underline{\mu}_\tau^*)\right) > C\left(P|\underline{\mu}^{L2}(\underline{\mu}_{\tau-1})\right).$$

We now introduce two distributions $\underline{\mu}'$ and $\underline{\mu}''$ as follows: $\underline{\mu}'$ and $\underline{\mu}''$ both have the same measure of members in every culture position in the set $\{1, \cdots, \hat{p}(\underline{\mu}_{\tau-1})\}$ as does $\underline{\mu}^{A2}(\underline{\mu}_\tau^*)$, and both have zero measure of members in every culture position in the set $\{\hat{p}(\underline{\mu}_{\tau-1}) + 1, \cdots, P-1\}$; but while the measure of members at P (in $\mathbb{V}$) under $\underline{\mu}'$ is the same as under $\underline{\mu}^{L2}(\underline{\mu}_{\tau-1})$, the corresponding measure under $\underline{\mu}''$ is the aggregate measure of $\underline{\mu}^{A2}(\underline{\mu}^*_\tau)$ over $\{\hat{p}(\underline{\mu}_{\tau-1})+1,\cdots,P\}$. Then, by **[D]**, we have $V\left(P|\underline{\mu}^{A2}(\underline{\mu}^*_\tau)\right) \leq V\left(P|\underline{\mu}''\right) \leq V\left(P|\underline{\mu}'\right) \leq V\left(P|\underline{\mu}^{L2}(\underline{\mu}_{\tau-1})\right)$. This proves our claim.

Hence, by Lemmas 7.3 and 7.4, there exists an equilibrium outcome $\underline{\mu}^*_{\tau+1}$ in generation $\tau+1$ such that (i) all minority members born in the culture-subset $\{1,\cdots,\hat{q}(\underline{\mu}^*_\tau)-1\}$ (if the set is non-empty) culture-shift towards $\hat{q}(\underline{\mu}^*_\tau)$ and do not migrate, and (ii) all born in $\{\hat{q}(\underline{\mu}^*_\tau),\cdots,\hat{p}(\underline{\mu}_{\tau-1})\}$ culture-shift within the subset and do not migrate; (iii) all situated at acquired culture position P migrate whenever they get opportunity; and (iv) $\hat{q}(\underline{\mu}^*_\tau) \leq \hat{q}(\underline{\mu}^*_{\tau+1}) \leq \hat{p}(\underline{\mu}^*_{\tau+1}) \leq \hat{p}(\underline{\mu}^*_\tau)$. Continuing in this way, one can find a continuation equilibrium trajectory $\{\underline{\mu}^*_t : t \geq \tau | \underline{\mu}_{\tau-1}\}$ such that for all $t \geq \tau$, $\underline{\mu}^*_t$ is a fractured equilibrium outcome such that $\{1,\cdots,\hat{p}(\underline{\mu}_{\tau-1})\}$ is entrenched and there is migration from P under $\underline{\mu}^*_t$. □

Proof of Proposition 5.3: As in Proposition 4.2, one can find an equilibrium trajectory $\{\underline{\mu}^*_t : t \geq 1 | \underline{\mu}_0\}$ such that $\{1,\cdots,\hat{p}(\underline{\mu}_0)\}$ will be entrenched under $\underline{\mu}^*_t$ and all born in $\{1,\cdots,\hat{q}(\underline{\mu}_0)-1\}$ culture shift towards $\hat{q}(\underline{\mu}_0)$ whenever relevant opportunities arise, for all $t \geq 1$. We now show that the set $\{\hat{p}(\underline{\mu}_0)+1,\cdots,P\}$ is unentrenched under $\underline{\mu}^*_t$, for all $t \geq 1$. Consider an arbitrary generation t, and note that

$$C(\hat{p}(\underline{\mu}_0)+2|\underline{\mu}^*_t) \geq C(\hat{p}(\underline{\mu}_0)+2|\underline{\mu}^{[L3]}(\underline{\mu}_0)) \geq V(\hat{p}(\underline{\mu}_0)+2|\underline{\mu}^{[L3]}(\underline{\mu}_0)) > V(\hat{p}(\underline{\mu}_0)+2|\underline{\mu}^*_t).$$

Thus, all minorities with acquired culture position $\hat{p}(\underline{\mu}_0)+2$ migrate to $\mathbb{C}$ whenever relevant opportunities arise, which implies that minority individuals will migrate from any position $p \geq \hat{p}(\underline{\mu}_0)+2$ whenever relevant opportunities arise. Furthermore, by **[L4]**, $U^*(\hat{p}(\underline{\mu}_0)+2|\underline{\mu}^*_1) > V(\hat{p}(\underline{\mu}_0)|\underline{\mu}^{[L4]}(\underline{\mu}_0)) > V(\hat{p}(\underline{\mu}_0)|\underline{\mu}^*_1) = U^*(\hat{p}(\underline{\mu}_0)|\underline{\mu}^*_1)$. This means that all born at $\hat{p}(\underline{\mu}_0)+1$ do not culture-shift left in generation 1. We now prove by the principle of mathematical induction that all born at $\hat{p}(\underline{\mu}_0)+1$ do not culture shift left in any generation. To this end, suppose that all born at $\hat{p}(\underline{\mu}_0)+1$ do not culture shift left in generations $t=1,\cdots,t_0$. We show that the same holds in generation $t=t_0+1$. As nobody culture shift left under $\underline{\mu}^*_t$ for all $t=1,\cdots,t_0$, we have

$$U^*(\hat{p}(\underline{\mu}_0)+2|\underline{\mu}^*_{t_0+1}) > V(\hat{p}(\underline{\mu}_0)|\underline{\mu}^{[L4]}(\underline{\mu}_0)) > V(\hat{p}(\underline{\mu}_0)|\underline{\mu}^*_{t_0+1}) = U^*(\hat{p}(\underline{\mu}_0)|\underline{\mu}^*_{t_0+1}).$$

Thus, all individuals with acquired culture position $\hat{p}(\underline{\mu}_0)+1$ will not culture shift left whenever relevant opportunities arise. Consequently, by the principle of mathematical induction, we conclude that nobody culture shift left from $\hat{p}(\underline{\mu}_0)+1$ under $\underline{\mu}^*_t$ in any generation t. Thus, individuals at $\hat{p}(\underline{\mu}_0)+1$ migrate from $\hat{p}(\underline{\mu}_0)+1$ or culture shift right whenever opportunities arise. □

Proof of Proposition 5.4: Given the initial culture distribution $\underline{\mu}_0$, suppose that there exists an equilibrium trajectory $\{\underline{\mu}^*_t : t \geq 1|\underline{\mu}_0\}$ such that $\underline{\mu}^*_{\tau-1}$ (for some $\tau \geq 1$) belongs in $\hat{\Delta}_+$ and satisfies **[L0]** and **[L1]**. Then, by our previous results, there must exist a continuation equilibrium trajectory $\{\underline{\mu}''_t : t \geq \tau|\underline{\mu}^*_{\tau-1}\}$ in which$\{1,\cdots,\hat{p}(\underline{\mu}^*_{\tau-1})\}$ will be an entrenched set in the current and every subsequent intra-generation equilibrium. Further, if there is no migration to $\mathbb{C}$ in any specific intra-generation equilibrium $\underline{\mu}''_t$, then $\{1,\cdots,P\}$ will be an entrenched set in that equilibrium, and consequently, all members born in the culture set $\{\hat{p}(\underline{\mu}^*_{\tau-1})+1,\cdots,P\}$ will culture-shift left whenever they get the chance.

So, consider the following equilibrium trajectory $\{\underline{\mu}_t^{**} : t \geq 1 | \underline{\mu}_0\}$ in which $\underline{\mu}_t^{**} = \underline{\mu}_t^*$ for all $t < \tau$ and $\underline{\mu}_t^{**} = \underline{\mu}_t''$ for all $t \geq \tau$. Recognize that $\{\underline{\mu}_t^{**} : t \geq 1 | \underline{\mu}_0\}$ cannot exhibit complete assimilation in $\mathbb{C}$. Alternatively, for $\{\underline{\mu}_t^{**} : t \geq 1 | \underline{\mu}_0\}$ to exhibit complete assimilation in $\mathbb{V}$ it must be that $m[\underline{\mu}_{\tau-1}^*] = 1$ as well as that $\{1, \cdots, P\}$ is an entrenched set in every equilibrium $\underline{\mu}_t''$ with $t \geq \tau$.

Now recognize that if **[S]** holds for $\underline{\mu}^i = \underline{\mu}_{\tau-1}^*$, then irrespective of the magnitude of $m[\underline{\mu}_{\tau-1}^*]$, it cannot be that $\{1, \cdots, P\}$ is an entrenched set in every $\underline{\mu}_t''$ for $t \geq \tau$ for the following reason (the case of **[L2]** has already been discussed in Proposition 5.2). If that is indeed the case, then

$$\mu_t^{**}(p) = \begin{cases} (1-\sigma)\mu_{t-1}^{**}(p) + \sigma\mu_{t-1}^{**}(p+1), & \text{if } \hat{p}(\underline{\mu}_{\tau-1}^*) < p < P; \\ (1-\sigma)\mu_{t-1}^{**}(P), & \text{if } p = P, \end{cases}$$

for all $t \geq \tau$. It can be checked that, for any $\hat{p}(\underline{\mu}_{\tau-1}^*) < p \leq P$ and $t \geq \tau$,

$$\mu_t^{**}(p) = \sum_{r=0}^{t-\tau+1} \binom{t-\tau+1}{r} (1-\sigma)^{t-\tau+1-r} \sigma^r \mu_{\tau-1}^{**}(p+r),$$

where $\mu_{\tau-1}^{**}(p+r)$ is taken as 0 for all $r > P - p$. As a consequence, for every $\epsilon > 0$, there will exist a T large enough such that in every $\underline{\mu}_t^{**}$ for $t > T$, $V(P|\underline{\mu}_t^{**})$ will be strictly less than $[V(P|\underline{\mu}^S(\underline{\mu}_{\tau-1}^*)) + \epsilon]$. Since **[S]** requires that $V(P|\underline{\mu}^S(\underline{\mu}_{\tau-1}^*)) < C(P|\underline{\mu}^S(\underline{\mu}_{\tau-1}^*))$, by suitable choice of ϵ and T we will be able to find at least one generation $\hat{t}$ far enough into the future in which every member in acquired position P will prefer to migrate to $\mathbb{C}$ under the conjecture $\underline{\mu}_{\hat{t}}^{**}$.

The above arguments imply the following conclusion: Given $\underline{\mu}_0$, if there exists an equilibrium trajectory $\{\underline{\mu}_t^* : t \geq 1 | \underline{\mu}_0\}$ such that in some finite generation $\tau \geq 1$, $\underline{\mu}_{\tau-1}^* \in \hat{\Delta}_+$ satisfies **[L0]**, **[L1]**, and **[S]**, there will exist an equilibrium trajectory $\{\underline{\mu}_t^S : t \geq 1 | \underline{\mu}^0\}$ that exhibits segmented assimilation in the long-run with the long-run measure of the minority population in $\mathbb{V}$ being at least $m[\hat{p}(\underline{\mu}_{\tau-1}^*)|\underline{\mu}_{\tau-1}^*]$. $\square$

Recall that $\mathcal{E}^{[2]}$ is the set of models **[E]** satisfying restrictions [E1] - [E3] and the additional restriction $P = 2$; and let $\hat{\Delta}_0^{[2]}$ be the set of all feasible *initial* culture distributions in $\hat{\Delta}_+$ for the case $P = 2$. When $P = 2$, note that for any $\underline{\mu} = (\mu(1), \mu(2)) \in \hat{\Delta}_0^{[2]}$ (which requires $m[\underline{\mu}] = 1$ and $\mu(1) > 0.5$), $\underline{\mu}^{L1}(\underline{\mu}) = (\mu(1), 0)$ and $\underline{\mu}^S(\underline{\mu}) = (1, 0)$. Consequently, for a model **[E]** $\in \mathcal{E}^{[2]}$ and for a culture distribution $\underline{\mu} \in \hat{\Delta}_0^{[2]}$, condition **[L1]** is: $V(1\,|\,(\mu(1), 0)) = w + \beta.\mu(1) \geq C(2\,|\,(\mu(1), 0)) = w + \delta + 2\theta + \alpha.\mu(2)$, and condition **[S]** is: $V(2\,|\,(1, 0)) = w + \frac{\beta}{1+\lambda} < C(2\,|\,(1, 0)) = w + \delta + 2\theta$. Here, **[S]** is independent of $\underline{\mu}$, while **[L1]** is not. Further, note that **[S]** is violated if and only if $\frac{\beta}{1+\lambda} \geq \delta + 2\theta$, which is a parameter condition that is neither required by the payoff ranking condition **[R]** nor is in violation of **[R]**; and **[L1]** is violated for all $\underline{\mu}_0 \in \hat{\Delta}_0^{[2]}$ if and only if $\delta + 2\theta > \beta$, which is also a parameter condition that is neither required by **[R]** nor is in violation of **[R]**.

Restricting attention to the set of models in $\mathcal{E}^{[2]}$, we now study the *necessity* of condition **[S]** and of condition **[L1]** in order for there to exist an equilibrium trajectory that exhibits segmented assimilation. To that end, we present the following result that considers the case where a model **[E]** $\in \mathcal{E}^{[2]}$ violates **[S]**.

Lemma 7.5. *Consider a model* $[\mathbf{E}] \in \mathcal{E}^{[2]}$ *with parameter values of* $\{\beta, \lambda, \delta, \theta\}$ *such that* $\frac{\beta}{1+\lambda} \geq \delta + 2\theta$*, implying that* $[\mathbf{S}]$ *is violated. In that case,for any initial culture distribution* $\underline{\mu}_0 \in \hat{\Delta}_0^{[2]}$*, there exists* $\bar{\sigma} \in (0,1)$ *and* $\bar{g} \in (0,1)$*, such that if the culture-shift probability is any* $\sigma < \bar{\sigma}$ *and if the migration probability vector* $\mathbb{G}$ *satisfies the restriction* $0 < g(1) \leq g(2) < \bar{g}$*, there does not exist an equilibrium trajectory that exhibits segmented assimilation.*

Proof. Given that the parameter values of $\{\beta, \lambda, \delta, \theta\}$ in the given model $[\mathbf{E}] \in \mathcal{E}^{[2]}$ are such that $\frac{\beta}{1+\lambda} \geq \delta + 2\theta$. Suppose that $\underline{\mu}_0$ is the inital distribution in generation 1. Choose $\bar{\sigma}$ so small that $(1-\bar{\sigma})\mu_0(1) > 0.5$ and $\bar{g}$ very small such that $\min\{V(2\,|\,\underline{\mu}^1), V(2\,|\,\underline{\mu}^2)\} > V(2\,|\,(1,0))$, where $\underline{\mu}^1 = ((1-\bar{g})(\mu_0(1)+\bar{\sigma}\mu_0(2)), (1-\bar{g})(1-\bar{\sigma})\mu_0(2))$, and $\underline{\mu}^2 = ((1-\bar{g})(1-\bar{\sigma})\mu_0(1), (1-\bar{g})(\bar{\sigma}\mu_0(1)+\mu_0(2)))$. Take any σ and $g(2)$ such that $\sigma \leq \bar{\sigma}$ and $g(2) < \bar{g}$. We now show that every equilibrium trajectory must exhibit complete assimilation in $\mathbb{V}$.

In the first generation, consider a conjecture $\underline{\mu}^e$ that is generated by a decision profile under which there is culture-shifting in any direction (i.e., either from 1 to 2 or from 2 to 1). Then, it must be that $V(2\,|\,\underline{\mu}^e) > V(2\,|\,(1,0)) \geq C(2\,|\,(1,0))$, implying that under conjecture $\underline{\mu}^e$, members in acquired position 2 will strictly prefer to stay back in $\mathbb{V}$. $[\mathbf{P}]$ implies then that the same will hold for position 1. Hence, the culture set $\{1,2\}$ must be fully entrenched in every equilibrium in generation 1. Since $(1-\sigma)\mu_0(1) > 0.5$, there will be culture-shifting from position 2 to position 1 in equilibrium.

The generation 1 results imply that an equilibrium inherited distribution for generation 2 must be of the form $\underline{\mu}_1 = (\mu_1(1) > \mu_0(1), \mu_1(2) < \mu_0(2))$ with $m[\underline{\mu}_1] = 1$. Then the above arguments can be repeated to establish that $\{1,2\}$ will be fully entrenched in every generation 2 equilibrium. Repeating this set of arguments over successive generations yields the result that in the suitably structured model $[\mathbf{E}]$ (that violates $[\mathbf{S}]$), starting from *any* $\underline{\mu}_0 \in \hat{\Delta}_0^{[2]}$, every equilibrium trajectory must exhibit complete assimilation in $\mathbb{V}$. □

Our next result considers the case where a model $[\mathbf{E}] \in \mathcal{E}^{[2]}$ violates $[\mathbf{L1}]$ for all $\underline{\mu}_0 \in \hat{\Delta}_0^{[2]}$.

Lemma 7.6. *Consider a model* $[\mathbf{E}] \in \mathcal{E}^{[2]}$ *with parameter values of* $\{\beta, \delta, \theta, \sigma, g(1), g(2)\}$*. There exist a positive measurable set of parameter values such that if* $[\mathbf{L1}]$ *is violated for a positive measurable set of values of* $\underline{\mu}_0 \in \hat{\Delta}_0^{[2]}$ *then all equilibrium trajectories must exhibit complete assimilation in* $\mathbb{C}$*.*

Proof. Define

$$\Upsilon = \left\{(\lambda, \sigma, \kappa) \in (0,1]^3 : \kappa > \frac{1}{1+\lambda(1-\sigma)\kappa}\right\}.$$

Thus, for $\bar{\lambda}$ sufficiently close to 1 and small $\bar{\sigma}$, there is an $\bar{\kappa} \in (0,1)$ such that $(\bar{\lambda}, \bar{\sigma}, \bar{\kappa}) \in \Upsilon$. Fix such a $(\bar{\lambda}, \bar{\sigma}, \bar{\kappa})$. Note also that $(\bar{\lambda}, \sigma, \kappa) \in \Upsilon$ for all $\sigma \leq \bar{\sigma}$ and $\kappa \geq \bar{\kappa}$. Define $\underline{\mu}(\kappa, \sigma) = ((1-\sigma)\kappa, 1-\kappa+\sigma\kappa)$ for all $\sigma \leq \bar{\sigma}$ and $\kappa \geq \bar{\kappa}$. Recognize that

$$\mathbb{V}(1|(\kappa,0)) = w + \beta\kappa > w + \frac{\beta}{1+\bar{\lambda}(1-\sigma)\kappa} = \mathbb{V}(2|\underline{\mu}(\kappa,\sigma))$$

for all $\kappa \geq \kappa_0$ and all $\sigma \leq \bar{\sigma}$. Moreover, $\mathbb{V}(1|(\kappa,0)) - \mathbb{V}(2|\underline{\mu}(\kappa,\bar{\sigma}))$ increases as κ tends to 1, and $\mathbb{C}(2|(\kappa,0)) - \mathbb{C}(2|\underline{\mu}(\kappa,\bar{\sigma}))$ decreases and converges to 0 as κ approaches to 1. Let $\kappa_0 \in [\bar{\kappa}, 1)$ be such that for all $\kappa \geq \kappa_0$, we have $\mathbb{V}(1|(\kappa,0)) - \mathbb{V}(2|\underline{\mu}(\kappa,\bar{\sigma})) > \mathbb{C}(2|(\kappa,0)) - \mathbb{C}(2|\underline{\mu}(\kappa,\bar{\sigma}))$. For all $\sigma \leq \bar{\sigma}$, as $\mathbb{V}(2|\underline{\mu}(\kappa,\sigma)) \leq \mathbb{V}(2|\underline{\mu}(\kappa,\bar{\sigma}))$ and $\mathbb{C}(2|\underline{\mu}(\kappa,\sigma)) = \mathbb{C}(2|\underline{\mu}(\kappa,\bar{\sigma}))$, we have

$$\mathbb{V}(1|(\kappa,0)) - \mathbb{V}(2|\underline{\mu}(\kappa,\sigma)) > \mathbb{C}(2|(\kappa,0)) - \mathbb{C}(2|\underline{\mu}(\kappa,\sigma)). \tag{7.2}$$

Let $\Delta^{\Upsilon} := \{\underline{\mu} \in \Delta_+ : \mu_0(1) \geq \kappa_0\}$. Choose an initial distribution $\underline{\mu}_0 \in \Delta^{\Upsilon}$. Since **[L1]** is not satisfied, there is some $\sigma_0 \leq \bar{\sigma}$ small enough so that $\mathbb{V}(1|\underline{\mu}_0^{\sigma}) < \mathbb{C}(2|\underline{\mu}_0^{\sigma})$ for all $\sigma \leq \sigma_0$, where $\underline{\mu}_0^{\sigma} := (\mu_0(1) + \sigma\mu_0(2), (1-\sigma)\mu_0(2))$. We can choose σ_0 in such a way that $\sigma_0\mu_0(1) < \frac{\mu_0(2)}{2}$. Take any $\sigma \in (0, \sigma_0]$, and let $\widetilde{\underline{\mu}_0^{\sigma}} = \underline{\mu}(\mu_0(1), \sigma)$. Thus, by (7.2), we have

$$\mathbb{V}(1|\underline{\mu}^{L1}(\underline{\mu}_0)) - \mathbb{V}(2|\widetilde{\underline{\mu}_0^{\sigma}}) > \mathbb{C}(2|\underline{\mu}^{L1}(\underline{\mu}_0)) - \mathbb{C}(2|\widetilde{\underline{\mu}_0^{\sigma}}).$$

This implies that

$$\mathbb{C}(2|\widetilde{\underline{\mu}_0^{\sigma}}) - \mathbb{V}(2|\widetilde{\underline{\mu}_0^{\sigma}}) > \mathbb{C}(2|\underline{\mu}^{L1}(\underline{\mu}_0)) - \mathbb{V}(1|\underline{\mu}^{L1}(\underline{\mu}_0)) > 0.$$

Let $\{\underline{\mu}_t^* : t \geq 1\}$ be an equilibrium trajectory. Note that $\mathbb{C}(2|\underline{\mu}_1^*) \geq \mathbb{C}(2|\widetilde{\underline{\mu}_0^{\sigma}})$ and $\mathbb{V}(2|\underline{\mu}_1^*) \leq \mathbb{V}(2|\widetilde{\underline{\mu}_0^{\sigma}})$. It follows that

$$\mathbb{C}(2|\underline{\mu}_1^*) - \mathbb{V}(2|\underline{\mu}_1^*) \geq \mathbb{C}(2|\widetilde{\underline{\mu}_0^{\sigma}}) - \mathbb{V}(2|\widetilde{\underline{\mu}_0^{\sigma}}) > 0.$$

Hence, minorities with acquired culture position 2 migrate to $\mathbb{C}$ in generation 1. In order to show $\mathbb{P}$ is unentrenched under $\underline{\mu}_t^*$ for all $t \geq 1$, we now define lower bounds on the migration probabilities $g(1)$ and $g(2)$. Take any $\eta \in (0, \frac{\mu_0(2)}{2})$ satisfying $\mathbb{V}(1|\underline{\mu}^{\sigma}) < (1-\eta)\mathbb{C}(2|\underline{\mu}^{\sigma})$. Choose $g(1)$ and $g(2)$ such that $g(1) \geq \sigma_0$ and $g(2) > 1-\eta$. Note that $\mathbb{V}(1|\underline{\mu}_1^*) \leq \mathbb{V}(1|\underline{\mu}_0^{\sigma}) < g(2)\mathbb{C}(2|\underline{\mu}_0^{\sigma}) \leq U^*(2|\underline{\mu}_1^*)$.

Thus, under the equilibrium $\underline{\mu}_1^*$, either a positive measure of individual with acquired culture position 1 will culture shift to 2 or migrate to $\mathbb{C}$ from position 1. So, $\mathbb{P}$ is unentrenched under $\underline{\mu}_1^*$.

We prove by the principle of mathematical induction that $\mathbb{P}$ is unentreched under $\underline{\mu}_t^*$ for all $t \geq 1$. To do this, suppose that $\mathbb{P}$ is unentrenched under $\underline{\mu}_t^*$ for all $t = 1, \cdots, t_0$. It remains to verify that $\mathbb{P}$ is unentrenched under $\underline{\mu}_{t_0+1}^*$. The maximum village payoff will be attained at position 2 in generation t_0+1 whenever σ-fraction of $\mu_{t_0}^*(1)$ will culture-shift right and nobody migrate to the city. Thus, we consider the distribution $\widetilde{\underline{\mu}_{t_0}^{\sigma}} = \underline{\mu}(\mu_{t_0}(1), \sigma)$. Since $\mathbb{P}$ is unentrenched under μ_t^* for all $t = 1, \cdots, t_0$ and $g(1) \geq \sigma$, we have $\mu_{t_0}^*(1) \leq (1-\sigma)\mu_0(1)$[43] and $\mu_{t_0}^*(2) < \eta$. Thus, $\mathbb{V}(2|\widetilde{\underline{\mu}_{t_0}^{\sigma}}) < \mathbb{V}(2|\widetilde{\underline{\mu}_0^{\sigma}})$. In view of $\mathbb{C}(2|\widetilde{\underline{\mu}_{t_0}^{\sigma}}) \geq \mathbb{C}(2|\widetilde{\underline{\mu}_0^{\sigma}})$, it follows that

$$\mathbb{C}(2|\widetilde{\underline{\mu}_{t_0}^{\sigma}}) - \mathbb{V}(2|\widetilde{\underline{\mu}_{t_0}^{\sigma}}) > \mathbb{C}(2|\widetilde{\underline{\mu}_0^{\sigma}}) - \mathbb{V}(2|\widetilde{\underline{\mu}_0^{\sigma}}) > 0. \tag{7.3}$$

By Equation (7.3), as $\mathbb{C}(2|\underline{\mu}_{t_0+1}^*) \geq \mathbb{C}(2|\widetilde{\underline{\mu}_{t_0}^{\sigma}})$ and $\mathbb{V}(2|\underline{\mu}_{t_0+1}^*) \leq \mathbb{V}(2|\widetilde{\underline{\mu}_{t_0}^{\sigma}})$, we have $\mathbb{C}(2|\underline{\mu}_{t_0+1}^*) > \mathbb{V}(2|\underline{\mu}_{t_0+1}^*)$. Hence, minorities with acquired culture position 2 migrate to $\mathbb{C}$ in generation $t = t_0+1$. Furthermore, as $\mu_{t_0}^*(1) \leq (1-\sigma)\mu_0(1)$ and $\mu_{t_0}^*(2) < \eta$, we have

$$\mathbb{V}(1|\underline{\mu}_{t_0+1}^*) \leq \mathbb{V}(1|\underline{\mu}_0^{\sigma}) < g(2)\mathbb{C}(2|\underline{\mu}_0^{\sigma}) \leq U^*(2|\underline{\mu}_{t_0+1}^*).$$

[43] If $U^*(1|\underline{\mu}_1^*) < U^*(2|\underline{\mu}_1^*)$ then σ-fraction of $\mu_0(1)$ will culture shift right and thus, $\mu_{t_0}^*(1) \leq (1-\sigma)\mu_0(1)$; if $U^*(1|\underline{\mu}_1^*) > U^*(2|\underline{\mu}_1^*)$ then $g(1)$-fraction of $\mu_0(1)$ will migrate from position 1 and so $\mu_{t_0}^*(1) \leq (1-g(1))\mu_0(1) \leq (1-\sigma)\mu_0(1)$; and if $U^*(1|\underline{\mu}_1^*) = U^*(2|\underline{\mu}_1^*)$ then $\alpha\sigma$-fraction of $\mu_0(1)$ will culture shift right for some $\alpha \in [0,1]$ and thus, $\mu_{t_0}^*(1) = (1-g(1))(1-\alpha\sigma)\mu_0(1) \leq (1-\sigma)\mu_0(1)$.

Consequently, as above, $\mathbb{P}$ is unentrenched under the equilibrium distribution $\underline{\mu}^*_{t_0+1}$. Thus, by the principle of mathematical induction, we conclude that $\mathbb{P}$ is unentrenched under $\underline{\mu}^*_t$ for all $t \geq 1$. $\square$

Proof of Proposition 5.5: The result follows from Lemma 7.5 and Lemma 7.6. $\square$

Suppose there exists an equilibrium trajectory $\{\underline{\mu}^*_t : t \geq 1 \mid \underline{\mu}_0\}$ such that in a finite period $\tau \geq 1$ with inherited $\underline{\mu}^*_{\tau-1}$ (which equals $\underline{\mu}_0$ if $\tau = 1$): $\underline{\mu}^*_{\tau-1} \in \hat{\Delta}_+$, and **[L0]**, **[L1]**, and either **[L2]** *or* **[S]** (or both) hold for $\underline{\mu}^i = \underline{\mu}^*_{\tau-1}$. For any $\tau \geq 1$, let $\mathcal{A}_\tau$ denote the set of all equilibrium trajectories $\{\underline{\mu}^S_t : t \geq 1|\underline{\mu}_0\}$ (with $\underline{\mu}^S_t = \underline{\mu}^*_t$ for $t < \tau$) that exhibits segmented assimilation in the long-run such that $\{1, \cdots, \hat{p}(\underline{\mu}^S_{\tau-1})\}$ is entrenched under $\underline{\mu}^S_t$ for all $t \geq \tau$ and $\{\hat{p}(\underline{\mu}^S_t) : t \geq \tau\}$ is a monotonically decreasing sequence of elements in $\{\hat{q}(\underline{\mu}^S_{\tau-1}), \cdots, \hat{p}(\underline{\mu}^S_{\tau-1})\}$, where $\underline{\mu}^S_{\tau-1}$ is the inherited distribution for generation τ ($\underline{\mu}_{\tau-1} = \underline{\mu}_0$ if $t = 1$). Under the hypothesis of Proposition 5.4, we have $\mathcal{A}_\tau \neq \emptyset$.

Lemma 7.7. *For any $\{\underline{\mu}^S_t : t \geq 1|\underline{\mu}_0\} \in \mathcal{A}_\tau$ for some $\tau \geq 1$ with $\lim_{t\to\infty} \hat{p}(\underline{\mu}^*_t)$ exists, if the set $\mathbb{P}_0 = \{p \in \mathbb{P} : p < \lim_{t\to\infty} \hat{p}(\underline{\mu}^*_t)\}$ is non-empty then $\{\mu^*_t(p) : t \geq 1\}$ converges to 0 for all $p \in \mathbb{P}_0$.*

Proof. It follows from the fact $\{\underline{\mu}^S_t : t \geq 1|\underline{\mu}_0\} \in \mathcal{A}_\tau$ that $\{1, \cdots, \hat{p}(\underline{\mu}^S_{\tau-1})\}$ is entrenched under $\underline{\mu}^S_t$ for all $t \geq \tau$, which means that no individual will migrate from each of these positions in all generations $t \geq \tau$. Furthermore, all minority individuals at any $p \in \mathbb{P}_0$ culture-shift right in all large generations whenever relavant opportunities arise. Take such a period $t \geq \tau$. It follows that

$$\mu^S_t(p) = \begin{cases} (1-\sigma)\mu^S_{t-1}(p) + \sigma\mu^S_{t-1}(p-1), & \text{if } p \in \mathbb{P}_0 \setminus \{1\}; \\ (1-\sigma)\mu^S_{t-1}(1), & \text{if } p = 1. \end{cases}$$

Therefore,

$$\mu^S_t(p) = \sum_{m=0}^{t-\tau+1} \binom{t-\tau+1}{m} (1-\sigma)^{t-\tau+1-m} \sigma^m \mu_\tau(p-m),$$

where $\mu_\tau(p-m)$ is taken as 0 for all $m \geq p$. As a consequence, we conclude that $\{\mu^S_t(p) : t \geq 1\}$ converges to 0 for all $p \in \mathbb{P}_0$. $\square$

Lemma 7.8. *Given the initial distribution $\underline{\mu}_0$, for any equilibrium trajectory $\{\underline{\mu}^S_t : t \geq 1|\underline{\mu}^0\} \in \mathcal{A}_\tau$ satisfying the condition* **[S]**, *$\{\mu^S_t(p) : t \geq 1\}$ converges to 0 for all $p \in \mathbb{P}$ with $p > \lim_{t\to\infty} \hat{p}(\mu^S_t)$.*

Proof. Let $\{\underline{\mu}^S_t : t \geq 1|\underline{\mu}_0\} \in \mathcal{A}_\tau$ for some $\tau \geq 1$. For $t \geq \tau$, define $\mathbf{\Psi}(\underline{\mu}^S_t) = \left\{p \in \mathbb{P} : V(p|\mu^S_t) < C(p|\mu^S_t)\right\}$. It is immediate from the proof of Proposition 5.4 that $P \in \mathbf{\Psi}(\underline{\mu}^S_t)$ for some $t \geq \tau$. Define t_1 to be the minimum of all those t's. We claim that there must exist a generation $t_2 > t_1$ such that $P \in \mathbf{\Psi}(\underline{\mu}^S_{t_2})$. If not, all minority individuals move from p to $p-1$ for all $\hat{p}(\underline{\mu}^S_{\tau-1}) < p \leq P$ in each generation $t > t_1$ and thus, as in the proof of Proposition 5.2, one can show that there is some generation $t > t_1$ such that $P \in \mathbf{\Psi}(\underline{\mu}^S_t)$, which is a contradiction. So, the claim is verified. By invoking the same argument, we can generate a sequence $\{t_1, t_2, \cdots\}$ of generations such that $t_1 < t_2 < \cdots$ and $P \in \mathbf{\Psi}(\underline{\mu}^S_{t_i})$ for all $i \geq 1$.

Next, we define $\mathbb{P}^\infty = \left\{p \in \mathbb{P} : p \in \mathbf{\Psi}(\underline{\mu}^*_t) \text{ for infinitely many } t\right\}$. From above, we have $\mathbb{P}^\infty \neq \emptyset$. Let $p^{\min} = \min \mathbb{P}^\infty$. It thus follows that $p^{\min} > \hat{p}(\mu^S_{\tau-1})$ and there is some $\tilde{t} \geq \tau$ such that $p^{\min} - 1 \notin \mathbf{\Psi}(\underline{\mu}^S_t)$ for all $t \geq \tilde{t}$. Define $p^\infty = p^{\min} - 1$, and consider the following two cases.

Case 1. $p^\infty = \hat{p}(\mu^S_{\tau-1})$. Recognize that $\{1, \cdots, p^\infty\}$ is entrenched under $\underline{\mu}^S_t$ for all $t \geq \tilde{t}$. Thus, $\{\sum_{p\in\mathbb{P}^\infty} \mu^S_t(p) : t \geq \tilde{t}\}$ is a monotonically decreasing sequence of real numbers. Let b be the limit of this sequence.We claim that $b = 0$. Let $\{t_1, t_2, \cdots\}$ be a sequence of generations such that $\tilde{t} \leq t_1 < t_2 < \cdots$ and $p \in \mathbf{\Psi}(\underline{\mu}^S_{t_k})$ for all $p \in \mathbb{P}^\infty$ and all $k \geq 1$. Since $\{\mu^*_{t_k}(p) : k \geq 1\}$ is a bounded sequence, it has a convergent sub-sequence for all $p \in \mathbb{P}^\infty$. Without any loss of generality, we assume that $\{\mu^S_{t_k}(p) : k \geq 1\}$ converges to b_p for all $p \in \mathbb{P}^\infty$. Consequently, $b = \sum_{p\in\mathbb{P}^\infty} b_p$. By the way of contradiction, we assume that $b_{p_0} > 0$ for some $p_0 \in \mathbb{P}^\infty$. Choose some $c_{p_0} > 0$ such that $c_{p_0} < b_{p_0} < \frac{c_{p_0}}{1-g(p_0)}$. Let $\varepsilon > 0$ be such that $\varepsilon < g(p_0)c_{p_0}$. Let K be a positive integer such that for all $k \geq K$, we have: (i) $\sum_{p\in\mathbb{P}^\infty} \mu^S_{t_k}(p) < b + \varepsilon$; and (ii) $\mu^S_{t_k}(p_0) \in \left(c_{p_0}, \frac{c_{p_0}}{1-g(p_0)}\right)$.

Choose some $k' > K$ and let $d_{p_0} > 0$ be the number such that $d_{p_0}(1 - g(p_0)) = \mu^S_{t_{k'}}(p_0)$. Thus, d_{p_0} is the measure of minority individuals with acquired culture position p_0 in generation $t_{k'}$ in $\mathbb{V}$ following the culture-shift decisions by minority residents, but before the migration decisions under the equilibrium $\underline{\mu}_{t_{k'}}$. It follows from (ii) that $d_{p_0} > c_{p_0}$. Therefore, the measure $g(P)d_P$ of individuals will migrate in generation $t_{k'}$, and hence, it is evident from the definition of ε that $\sum_{p\in\mathbb{P}^\infty} \mu^S_t(p) < b$, which contradicts with the fact that $\sum_{p\in\mathbb{P}^\infty} \mu^S_t(p) \geq b$ for all $t \geq \tilde{t}$. Therefore, we conclude that $b_p = 0$ for all $p \in \mathbb{P}^\infty$, which means $b = 0$. Hence, the claim is verified. As in the proof of Lemma 7.7, if the set $\mathbb{Q} = \{p \in \mathbb{P} : \lim_{t\to\infty} \hat{p}(\mu^S_t) < p \leq p^\infty\}$ is non-empty, then one can easily show that $\{\mu^S_t(p) : t \geq 1\}$ converges to 0 for all $p \in \mathbb{Q}$.

Case 2. $p^\infty > \hat{p}(\mu^S_{\tau-1})$. It yields that $V(p^\infty - 1|\underline{\mu}^S_t) > V(p^\infty|\underline{\mu}^S_t) \geq C(p^\infty|\underline{\mu}^S_t)$ for all $t \geq \tilde{t}$, which further implies that $\{1, \cdots, p^\infty - 1\}$ is entrenched and the σ-fraction of minority individuals in p^∞ in $\mathbb{V}$ move to either $p^\infty - 1$ or $p^{\min}$ under $\underline{\mu}^S_t$, in all generations $t \geq \tilde{t}$. Thus, the sequence $\{\sum_{p\in\mathbb{P}^\infty\cup\{p^\infty\}} \mu^S_t(p) : t \geq \tilde{t}\}$ is a monotonically decreasing sequence of real numbers, which is assumed to converge to some non-negative real number b. We claim that $b = 0$. Suppose by the way of contradiction that $b > 0$. As in **Case 1**, there is a sequence $\{t_1, t_2, \cdots\}$ of generations such that (i) $\tilde{t} \leq t_1 < t_2 < \cdots$ and $p \in \mathbf{\Psi}(\underline{\mu}^S_{t_k})$ for all $p \in \mathbb{P}^\infty$ and all $k \geq 1$; and (ii) $\{\mu^S_{t_k}(p) : k \geq 1\}$ converges to some non-negative real number b_p for all $p \in \mathbb{P}^\infty \cup \{p^\infty\}$. An argument similar to that in **Case 1** guarantees that $b_p = 0$ for all $p \in \mathbb{P}^\infty$. As a consequence, we get $b = b_{p^\infty}$. Choose c_{p^∞} such that $c_{p^\infty} < b_{p^\infty}$ and then pick some K such that

$$\sum_{p\in\mathbb{P}^\infty} \mu^S_{t_k}(p) < \min\left\{\frac{(1-\sigma)(1-g(p^{\min}))c_{p^\infty}}{2}, \frac{\sigma(1-g(p^{\min}))c_{p^\infty}}{4}\right\}$$

for all $k \geq K$. Hence, the total measure of minority individuals with inherited culture positions in $\mathbb{P}^\infty$ is less than $\frac{c_{p^\infty}}{2}$ in generation t_k for all $k \geq K$. Thus, the total measure of individuals with inherited culture position p^∞ in $\mathbb{V}$ in generation t_k must be at least $\frac{c_{p^\infty}}{2}$, for all $k \geq K$. Therefore, at least the measure $\frac{\sigma c_{p^\infty}}{2}$ of individuals move to either $p^\infty - 1$ or $p^{\min}$ from p^∞ in generation t_k, for all $k \geq K$. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{\sigma c_{p^\infty}}{4}$. Choose some $K_0 \geq K$ such that $\sum_{p\in\mathbb{P}^\infty\cup\{p^\infty\}} \mu^S_{t_k}(p) < b + \varepsilon$ for all $k \geq K_0$. Pick an element $k' > K_0$. If at least the measure $\frac{\sigma c_{p^\infty}}{4}$ of minority individuals in p^∞ in $\mathbb{V}$ move to $p^\infty - 1$ in generation $t_{k'}$ then $\sum_{p\in\mathbb{P}^\infty\cup\{p^\infty\}} \mu^S_{t_{k'}}(p) < b$, which is a contradiction. On the other hand, if at least the measure $\frac{\sigma c_{p^\infty}}{4}$ of minority individuals in p^∞ in $\mathbb{V}$ move to $p^{\min}$ in generation $t_{k'}$ then $\mu^S_{t_{k'}}(p^{\min}) \geq \frac{\sigma(1-g(p^{\min}))c_{p^\infty}}{4}$, which is again a contradiction. Hence, we conclude that $b = 0$. Therefore, $\{\mu_t^S(p) : t \geq 1\}$ converge to 0 for all $p \geq p^\infty$. Similar to **Case 1**, if the set $\mathbb{Q} = \{p \in \mathbb{P} : \lim_{t\to\infty} \hat{p}(\mu_t^S) < p < p^\infty\}$ is non-empty, then one can easily show that $\{\mu_t^S(p) : t \geq 1\}$ converges to 0 for all $p \in \mathbb{Q}$. □

Proof of Proposition 6.1: Let $\{\underline{\mu}_t^S : t \geq 1\} \in \mathcal{A}_\tau$. Since $\{\hat{q}(\underline{\mu}_{\tau-1}^S), \cdots, \hat{p}(\underline{\mu}_{\tau-1}^S)\}$ has only finitely many elements, there is some $p' \in \{\hat{q}(\underline{\mu}_{\tau-1}^S), \cdots, \hat{p}(\underline{\mu}_{\tau-1}^S)\}$ such that $\hat{p}(\underline{\mu}_t^S) = p'$ for all $t \geq t_0$ for some $t_0 \geq \tau$, which implies that the social interaction is maximized on $\{p'-1, p'\}$ in generation $t \geq t_0$. By Lemmas 7.7 and 7.8, we conclude that $\{\mu_t^S(p) : t \geq 1\}$ converges to 0 for all $p \in \mathbb{P} \setminus \{p'-1, p'\}$.

We now show that there exist an element $\{\underline{\mu}_t^B : t \geq 1\} \in \mathcal{A}_\tau$ and some $\tilde{p} \in \mathbb{P}$ such that all village residents will eventually move to $\tilde{p}$ under this equlibrium trajectory. To this end, we define

$$\mathbb{Q} = \left\{p \in \mathbb{P} : p = \lim_{t\to\infty} \hat{p}(\underline{\mu}_t^S) \text{ for some } \{\underline{\mu}_t^S : t \geq 1|\underline{\mu}_0\} \in \mathcal{A}_\tau\right\}.$$

As $\mathcal{A}_\tau \neq \emptyset$, from above it follows that $\mathbb{Q} \neq \emptyset$. Let p_0 denote the smallest element of $\mathbb{Q}$. It follows that $p_0 = \hat{p}(\underline{\mu}_t^{**})$ for all large t and for some $\{\underline{\mu}_t^{**} : t \geq 1|\underline{\mu}_0\} \in \mathcal{A}_\tau$. We claim that $\{\mu_t^{**}(p_0 - 1) : t \geq 1\}$ converges to 0. Recognized that if $p_0 = \hat{q}(\underline{\mu}_{\tau-1}^*)$ then there is nothing remains to prove. Thus, we assume that $p_0 > \hat{q}(\underline{\mu}_{\tau-1}^*)$ and $\{\mu_t^{**}(p_0 - 1) : t \geq 1\}$ does not converge to 0. Then there must exist a subsequence $\{t_k : k \geq 1\}$ of $\{t : t \geq 1\}$ such that $d(p_0, \underline{\mu}_{t_k}^{**}) = d(p_0 - 1, \underline{\mu}_{t_k}^{**})$ for all $k \geq 1$. Note that, in the light of [**L2**] or [**S**], there is some $\eta_0 \in (0, 1)$ satisfying: $\underline{\mu} \in \Delta^*(\hat{p}(\underline{\mu}_{\tau-1}^*), \eta_0)$ implies $V(P|\underline{\mu}) < C(P|\underline{\mu})$. Recogized that $\underline{\mu}_t^{**} \in \Delta^*(\hat{p}(\underline{\mu}_{\tau-1}^*), \eta_0)$ for all large t. Pick an integer $k_0 \geq 1$ such that $t_{k_0} \geq \tau$ be such a generation, and note that $\underline{\mu}^i = \underline{\mu}_{t_{k_0}}^{**}$ is the inherited distribution in generation $t_{k_0} + 1$. We now construct a continuation equilibrium trajectory $\{\underline{\mu}_t'' : t \geq t_{k_0} + 1|\underline{\mu}_{t_{k_0}}^{**}\}$ that exhibits incessant fracture with $\{1, \cdots, \hat{p}(\underline{\mu}_{\tau-1}^*)\}$ being entrenched and $\hat{p}(\underline{\mu}_t'') \leq p_0 - 1$ in each generation $t > t_{k_0}$. First note that when [**L**0]-[**L**1] hold for $\underline{\mu}^i = \underline{\mu}_{\tau-1}^*$, conditions [**B**0]-[**B**1] hold for $\underline{\mu}^i = \underline{\mu}_{t_{k_0}}^{**}$, $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1}^*)$ and $\tilde{q}_0 = \hat{q}(\underline{\mu}_{\tau-1}^*)$. For $\underline{\mu}^i = \underline{\mu}_{t_{k_0}}^{**}$ and $\tilde{p}_0 = \hat{p}(\underline{\mu}_{\tau-1}^*)$, we define by $\mathbb{X}^P$ the set of decision profiles in $\widetilde{\mathbb{X}}$ such that all born at $p_0 - 1$ do not culture-shift right. Let $\Delta^P(\underline{\mu}_{t_{k_0}}^{**})$ be the set of all culture distribution generated from $\underline{\mu}^i = \underline{\mu}_{t_{k_0}}^{**}$ by all decision profiles in $\mathbb{X}^P$. Recognize that the set $\Delta^P(\underline{\mu}_{t_{k_0}}^{**})$ is non-empty, convex and compact. For any conjecture $\underline{\mu}^e \in \Delta^P(\underline{\mu}_{t_{k_0}}^{**})$, as the set $\{1, \cdots, \hat{p}(\underline{\mu}_{\tau-1}^*)\}$ is entrenched under $\underline{\mu}^e$, we have $\underline{\mu}^e \in \Delta^*(\hat{p}(\underline{\mu}_{\tau-1}^*), \eta_0)$, which yields $V(P|\underline{\mu}^e) < C(P|\underline{\mu}^e)$. This implies that $m[p_0 - 1|\underline{\mu}^e] \geq m[p_0 - 1|\underline{\mu}_{t_{k_0}}^{**}] = \frac{1}{2}m[\underline{\mu}_{t_{k_0}}^{**}] > \frac{1}{2}m[\underline{\mu}^e]$. Consequently, $V(p_0 - 1|\underline{\mu}^e) > V(p_0|\underline{\mu}^e) > C(p_0|\underline{\mu}^e)$. As a result, one born at $p_0 - 1$ will not culture-shift right under $\underline{\mu}^e$. Thus, there exists an equilibrium outcome $\underline{\mu}_{t_{k_0}+1}''$ in generation $t_{k_0} + 1$ that belongs in $\Delta^P(\underline{\mu}_{t_{k_0}}^{**})$, satisfies $\hat{q}(\underline{\mu}_{t_{k_0}}^{**}) \leq \hat{q}(\underline{\mu}_{t_{k_0}+1}'') \leq \hat{p}(\underline{\mu}_{t_{k_0}+1}'') \leq \hat{p}(\underline{\mu}_{t_{k_0}}^{**})$, and exhibits the property that all minority members situated at acquired culture position P migrate to $\mathbb{C}$ whenever they get the opportunity. Invoking a similar argument, we can construct a continuation equilibrium trajectory $\{\underline{\mu}_t'' : t \geq t_{k_0} + 1|\underline{\mu}_{t_{k_0}}^{**}\}$ that exhibits incessant fracture with $\{1, \cdots, \hat{p}(\underline{\mu}_{\tau-1}^*)\}$ being entrenched and $\hat{p}(\underline{\mu}_t'') \leq p_0 - 1$ in each generation $t > t_{k_0}$. Thus, the equilibrium trajectory $\{\underline{\mu}_t^{00} : t \geq 1|\underline{\mu}_0\}$ in which $\underline{\mu}_t^{00} = \underline{\mu}_t^{**}$ for all $t \leq t_{k_0}$ and $\underline{\mu}_t^{00} = \underline{\mu}_t''$ for all $t > t_{k_0}$, belongs in $\mathcal{A}_\tau$. This implies that p_0 is not the smallest element of $\mathbb{Q}$, which is a contradiction. □

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