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RECURRENCE OF A MODIFIED RANDOM WALK AND ITS APPLICATION TO AN ECONOMIC MODEL*

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Abstract. A modification of Chung and Fuchs' (Mem. Amer. Math. Soc., 6 (1951), pp. 1–12) recurrence theorem for random walks leads to an analogous result for a different discrete parameter Markov process. This latter process is applicable to an analysis of price stabilization programs involving purchases and sales from a buffer stock.

1. The stochastic model. By means of the following stochastic model, Salant (1981) analyzes how speculators maximizing expected profits under "rational expectations" would respond to a government price stabilization program involving purchases and sales from a buffer-stock. Let $\mathbb{N}$ denote the set of nonnegative integers, $\{0, 1, 2, \cdots\}$. At time $t \in \mathbb{N} \backslash \{0\}$, a harvest, H_t, of random size, is produced, drawn independently from an unchanging distribution; part of the harvest is consumed and the remainder is stored by the government or the private sector, with θ_t denoting the combined stock at the beginning of period t. Consumer demand for the commodity is assumed to increase as prices decrease. Salant shows that in any period, the market price which induces profit-seeking speculators and the government to sell exactly what consumers demand can be written as a decreasing function of the total stock at the beginning of the period. It is shown that when θ_t is in a range of $a \leqq \theta_t \leqq b$, the government can keep the market price at the official level $\bar{P}$. When θ_t falls below a, speculators "attack" the government stock, purchasing it in its entirety; the government is no longer able to defend the ceiling, and the price rises, much as it did in the gold market in 1968 when the government's stockpile was attacked (see, for example, Wolfe (1976)). If θ_t climbs above b, buffer-stock managers do not have the funds needed to support the floor and the market price falls. Letting $D(\cdot)$ denote the consumer demand function and $P^+(\cdot)$ denote the price function, the following stochastic difference equation and associated initial condition, $\theta_0 = a_0$, describe the evolution of stocks:

$$\theta_{t+1} = \theta_t + H_{t+1} - D(P^+(\theta_t)). \tag{1'}$$

Under the assumption that

$$\mu = D(\bar{P}) = E(H_t), \tag{2'}$$

we shall show, by means of a theorem proved in § 3, that for a buffer-stock manager, speculative attacks and an inability to defend the floor are recurrent events. It is almost certain that there is no way for the manager to avoid speculative attacks altogether, however large the initial stockpile, as long as the official price is set so that demand equals (or exceeds) the expected harvest.

Other aspects of the general problem warrant study as well, but are not investigated here; one such is the waiting time until the first attack. Here, we consider the case of a government whose goal is a fixed price (or "peg"), and whose policy is to maintain this price, as was done in the gold market for much of this century (refer to Wolfe (1976)). In another direction, however, we could extend our analysis to the case in which the goal is to keep the price within an interval (or "band"), with corresponding policy of appropriate intervention when the price reaches either endpoint of the

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interval. For example, such a policy is now being attempted to stabilize grain prices. The analysis of this extension requires a different approach which is being pursued separately.

2. Notation. We now examine a basic recurrence property of a specific discrete parameter Markov process, $\{X_n, n \in \mathbb{N}\}$, which we call a modified one-dimensional random walk. $\mathscr{E}_1, \mathscr{E}_2, \cdots$ are independent, identically distributed, nondegenerate, real-valued random variables on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, with finite mean μ and common distribution function F. ω denotes an element of Ω. $g:\mathbb{R}\to\mathbb{R}$ is a monotone, nondecreasing function satisfying

$$g(x)\begin{cases}\leqq \mu, & x<a,\\ =\mu, & a\leqq x\leqq b,\\ \geqq \mu, & x>b,\end{cases}$$

for some fixed $a, b \in \mathbb{R}$. The process $\{X_n\}$ is defined as

$$X_0 = a_0, \qquad a \leqq a_0 \leqq b, \quad (a_0 \text{ constant}),$$
$$X_n = X_{n-1} + \mathscr{E}_n - g(X_{n-1}), \qquad n \geqq 1.$$

and the process $\{S_n^{(M)}\}$ is defined as

$$S_0^{(M)} = X_M,$$
$$S_n^{(M)} = S_{n-1}^{(M)} + \mathscr{E}_{M+n} - \mu, \qquad n \geqq 1.$$

For convenience, $S_n = S_n^{(0)}$. We note that $\{S_n, n \in \mathbb{N}\}$ is a classical random walk, and that the process $\{X_n, n \in \mathbb{N}\}$ is identical to $\{S_n, n \in \mathbb{N}\}$ when $g(x) \equiv \mu$.

3. A recurrence property. Now, we present the following result.

THEOREM. *If $\mathscr{E}_1, \mathscr{E}_2, \cdots$ are i.i.d. real-valued random variables with common distribution function F and finite mean μ, in the sense that*

$$\int_{-\infty}^{\infty} |x|\, dF < \infty,$$

then

$$\mathbb{P}\{X_n < a \text{ i.o.}\} = 1,$$

and

$$\mathbb{P}\{X_n > b \text{ i.o.}\} = 1.$$

In view of (1′) and (2′), setting

$$X_t = \theta_t, \quad \mathscr{E}_t = H_t, \quad g(x) = D(P^+(x)),$$

we reach the conclusion that, indeed (in the context of § 1), speculative attacks and an inability to defend the floor are recurrent events.

Proof. Let

$$A = \left\{ -\infty = \varliminf_{n\to\infty} S_n < \varlimsup_{n\to\infty} S_n = +\infty \right\}. \tag{1}$$

By a slight modification of a theorem of Chung and Fuchs (1951), to be discussed in § 4,

(2) $$\mathbb{P}(A)=1.$$

For $\omega \in A$, (1) guarantees the existence of a stopping time $N_0 < \infty$ such that

$$S_{N_0} < a$$

and

$$S_i \geqq a \quad \text{for all } i < N_0.$$

Certainly, if $X_n \geqq a$ for all $n < N_0$

$$X_{N_0} \leqq S_{N_1} < a.$$

Therefore, there exists $N_1 \leqq N_0$ such that $X_{n_1} < a$.
Suppose

$$X_n \leqq b \text{ for all } n \geqq N_1.$$

Then

$$S^{(N_1)}_{n-N_1} \leqq X_n \leqq b \qquad \text{for all } n \geqq N_1.$$

But this contradicts $\omega \in A$, since a translation of the S-process at the N_1th step does not affect its recurrence. Therefore, there exists $N_2 > N_1$ such that

$$X_{N_2} > b,$$

and

$$X_n \leqq b \quad \text{for all } N_1 \leqq n < N_2.$$

Continuing in this fashion, employing an induction argument, we obtain

$$A \subset \{X_n < a \text{ i.o.}\} \quad \text{and} \quad A \subset \{X_n > b \text{ i.o.}\}.$$

An appeal to (2) establishes our result.

4. Remarks. Given a process $\{Y_n\}$, and $\omega \in \Omega$, the value $r \in \mathbb{R}$ is *recurrent* for ω with respect to $\{Y_n\}$ if for every $\delta > 0$

$$|Y_n(\omega) - r| < \delta \quad \text{i.o.}$$

Let $\mathscr{E}_1$ be a nondegenerate random variable that assumes values other than integral multiples of a fixed number. Chung and Fuchs (1951), and again, Chung and Ornstein (1962) have shown that under the above assumption, for every $r \in \mathbb{R}$, for every $\delta > 0$,

(3) $$\mathbb{P}\{|S_n - r| < \delta \text{ i.o.}\} = 1.$$

However, in addition, it follows from (3) and a standard argument employing the separability of $\mathbb{R}$ and the Archimedian principle that, for almost every $\omega \in \Omega$, all real values are recurrent with respect to $\{S_n^{(M)}, n \in \mathbb{N}\}$. More precisely, for any M,

(4) $$\mathbb{P}\Big(\bigcap_{\delta} \bigcap_{r \in \mathbb{R}} \{|S_n^{(M)} - r| < \delta \text{ i.o.}\}\Big) = 1.$$

Of course, (2) follows from (4) for the case of nondegenerate, nonlattice $\mathscr{E}_1$; in the lattice case, (2) follows from the appropriate analogue of (4).

An immediate consequence of (4) is that the process $\{S_n; n \in \mathbb{N}\}$ almost surely produces a countable, dense set of real numbers (see Chung (1974, p. 272)).

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