Sampling Dynamics and Stable Mixing in Hawk–Dove Games

Arigapudi, Srinivas and Heller, Yuval and Schreiber, Amnon

Bar-Ilan University Technion, Bar-Ilan University, Bar-Ilan University

15 July 2021

Sampling Dynamics and Stable Mixing in Hawk–Dove Games

Srinivas Arigapudi* Yuval Heller† Amnon Schreiber‡§

January 30, 2022

Abstract

The hawk–dove game admits two types of equilibria: an asymmetric pure equilibrium in which players in one population play "hawk" and players in the other population play "dove," and an inefficient symmetric mixed equilibrium, in which hawks are frequently matched against each other. The existing literature shows that populations will converge to playing one of the pure equilibria from almost any initial state. By contrast, we show that plausible sampling dynamics, in which agents occasionally revise their actions by observing either opponents' behavior or payoffs in a few past interactions, can induce the opposite result: global convergence to one of the inefficient mixed stationary states.

Keywords: Chicken game, learning, evolutionary stability, bounded rationality, payoff sampling dynamics, action sampling dynamics.

JEL codes: C72, C73.

1 Introduction

The hawk–dove game is often applied to study situations of conflict between strategic participants. As a simple motivating example, consider a situation in which a buyer

*Department of Economics, Bar-Ilan University; Faculty of Industrial Engineering and Management, Technion. email: arigapudi@campus.technion.ac.il.

†Department of Economics, Bar-Ilan University. email: yuval.heller@biu.ac.il.

‡Department of Economics, Bar-Ilan University. email: amnon.schreiber@biu.ac.il.

§We thank Daniel Friedman, Itay Kavaler, Ron Peretz, Ernst Schulte-Geers, Daniel Stephenson, Jiabin Wu, and seminar audiences at Bar-Ilan University and the University of Haifa for various helpful comments. We thank Luis R. Izquierdo and Segismundo S. Izquierdo for their help in implementing the simulations using ABED software. YH and SA gratefully acknowledge the financial support of the European Research Council (#677057) and the Israeli Science Foundation (#2443/19). SA is supported in part at the Technion by a Fine Fellowship.

Table 1: Payoff Matrix of the Standard Hawk–Dove Game ($g \in (0,1)$)

		Player 2	
		h	d
Player 1	h	$0, 0$	$1+g, 1-g$
	d	$1-g, 1+g$	$1, 1$

(Player 1) and a seller (Player 2) have to bargain over the price of an asset (e.g., a house). Each player has two possible bargaining strategies (actions): insisting on a more favorable price (referred to as being a "hawk"), or agreeing to a less favorable price in order to close the deal (being a "dove"). The payoffs of the game are presented in Table 1. Two doves agree on a price that is equally favorable to both sides, and obtain a relatively high payoff, which is normalized to 1. A hawk obtains a favorable price when being matched with a dove, which yields her an additional gain of $g \in (0,1)$, at the expense of her dovish opponent.[1] Finally, two hawks obtain the lowest payoff of 0, due to a substantial probability of bargaining failure.[2]

Observe that large values of g (close to 1) correspond to environments that are advantageous to hawks (i.e., being a hawk yields a higher expected payoff against an opponent who might play either action with equal probability), small values of g correspond to environments that are advantageous to doves, and values of g that are close to 0.5 correspond to approximately balanced environments.

The hawk–dove game (also known as the chicken game; see, e.g., Rapoport and Chammah, 1966; Aumann, 1987) has been employed in modeling various strategic situations, such as: provision of public goods (Lipnowski and Maital, 1983), nuclear deterrence between superpowers (Brams and Kilgour, 1987; Dixit et al., 2019), industrial disputes (Bornstein et al., 1997), bargaining problems (Brams and Kilgour, 2001), conflicts between countries over contested territories (Baliga and Sjöström, 2012, 2020), and task allocation among members of a team (Herold and Kuzmics, 2020).

The hawk–dove game admits three Nash equilibria: two asymmetric pure equilibria, and an inefficient symmetric mixed equilibrium. In the pure equilibria (in which one of

[1]Our formal model studies a broader class of generalized hawk-dove games, in which the gain of a hawkish player might differ from the loss of the dovish opponent (as shown in Table 2 in Section 3).

[2]Our one-parameter payoff matrix is equivalent to the commonly used two-parameter matrix (Maynard-Smith, 1982), according to which a dove obtains $\frac{V}{2}$ against another dove and 0 against a hawk, and a hawk obtains $\frac{V-C}{2}$ against another hawk and V against a dove. Specifically, our one-parameter matrix is obtained from the two-parameter matrix by the affine transformation of adding the constant $\frac{C-V}{2}$ and dividing all payoffs by $\frac{C}{2}$, followed by substituting $g \equiv \frac{V}{C}$.

the players plays hawk while the opponent plays dove), all conflicts are avoided at the cost of inequality, as the payoff of the hawkish player is substantially higher than that of the dovish opponent. By contrast, in the symmetric mixed equilibrium both players obtain the same expected payoff, yet this payoff is relatively low due to the positive probability of a conflict arising between two hawks.

A natural question is to ask which equilibrium is more likely to obtain. Standard game theory is not helpful in answering this question, as all these Nash equilibria satisfy all the standard refinements (e.g., perfection). By contrast, the dynamic (evolutionary) approach can yield sharp predictions (for textbook expositions, see Weibull, 1997; Sandholm, 2010).

Revision dynamics Consider a setup in which pairs of agents from two infinite populations are repeatedly matched at random times (each such match of an agent from population 1 is against a new opponent from population 2).[3] Agents occasionally die (or, alternatively, agents occasionally receive opportunities to revise their actions). New agents observe some information about the aggregate behavior and the payoffs, and use this information to choose the action they will play in all future encounters. We are interested in characterizing the stable rest points of such revision dynamics, which can be used as an equilibrium refinement.

Most existing models assume that the revision dynamics are *monotone* (also known as sign-preserving) with respect to the payoffs: the frequency of the strategy that yields the higher payoff (among the two feasible strategies) increases. A key result in evolutionary game theory is that in a hawk-dove game, all monotone (two-population) revision dynamics converge to the asymmetric pure equilibria from almost any initial state (henceforth, *global convergence*; see Maynard-Smith and Parker, 1976, for the classic analysis, Maynard-Smith, 1982, for the textbook presentation, Sugden, 1989, for the economic implications, and Oprea et al., 2011, for the general dynamic result.) Thus, the existing literature predicts that an efficient convention will emerge in which trade always occurs and most of the surplus goes to one side of the market. Casual observation suggests that this prediction might not fit well the behavior in situations such as the motivating example, in which the surplus of trade is typically divided relatively equally between the

[3]Our paper focuses on two-population dynamics in which players condition their play on their role in the game. In contrast, in a one-population model, players cannot condition their play on their role in the game. The predictions of the one-population model are discussed at the end of Section 3.2.

two sides of the market, and in which bargaining frequently fails.

In many applications, precise information about the aggregate behavior in the population may be difficult or costly to obtain. In such situations, new agents have to infer the aggregate behavior in the population from a small sample of other players. In what follows, we study two plausible inference procedures, both of which violate monotonicity. The first procedure is the *action-sampling dynamics* (also known as sampling best-response dynamics; Sandholm, 2001; Osborne and Rubinstein, 2003). In these dynamics, each new agent observes the behavior of k random opponents, and then adopts the action that is a best reply to her sample (with an arbitrary tie-breaking rule).

In some applications, new agents may not be able to observe opponents' actions, or they may lack information about the payoff matrix. Plausible revision dynamics in such situations are the *payoff-sampling dynamics* (also known as best experienced payoff dynamics; Osborne and Rubinstein, 1998; Sethi, 2000). In these dynamics, each new agent observes the payoffs obtained by incumbents of her own population in k interactions in which these incumbents played hawk, and in k interactions in which these incumbents played dove. Following these observations, the new agent adopts the action that yielded the higher mean payoff (with an arbitrary tie-breaking rule).

We analyze both sampling dynamics in the hawk–dove game. In our analysis, we allow agents to have heterogeneous sample sizes (i.e., each new agent is endowed with a sample size of k that is randomly chosen from an exogenous distribution). It is relatively straightforward to show that these dynamics admit at least three stationary states: two asymmetric pure states, and an inefficient symmetric state.[4, 5]

Global Convergence to Mixed States We show that sampling dynamics can yield qualitatively different results compared to monotone dynamics. Specifically, our first main result (Theorem 1 and Corollary 2) presents a simple "iff" condition for sampling dynamics to induce the opposite result (relative to monotone dynamics), namely, convergence to one of the mixed equilibria from almost any initial state. Roughly speaking, this happens iff the following three conditions hold:[6] (1) sufficiently many agents have small sample

[4] Although the symmetric stationary state does not coincide with the symmetric Nash equilibrium, they share similar qualitative properties: namely symmetry between the two populations, and inefficiency induced by frequent matching of two hawks.

[5] In some cases, the dynamics admit also asymmetric mixed states, as is demonstrated in Section 6.

[6] The quantitative versions of these three conditions are stricter for action-sampling dynamics, and less strict for payoff-sampling dynamics.

sizes, (2) the expected sample size is sufficiently large (i.e., the population should also include agents with sufficiently large sample sizes), and (3) g is not too close to 0.5.

The key to our first main result is characterizing when populations that start near one of the pure equilibria get away from this equilibrium (allowing for convergence to a mixed stationary state). Assume that initially almost all $(1-\epsilon)$ buyers are hawks, and almost all $(1-\epsilon)$ sellers are doves. Consider a new agent who bases her decision on a sample of k actions. The sample will have a single occurrence of the rare action with probability of about $k \cdot \epsilon$ (while the probability of having two or more occurrences of the rare action is negligible). The presence of agents with small samples (condition 1) is necessary for moving away from the pure equilibrium, because for dove-favorable games (i.e., $g < 0.5$), a single occurrence of a rare action can induce a new seller to be a hawk only if her sample is small (similarly, the presence of agents with small samples is necessary for inducing a new buyer to be a dove in hawk-favorable games).

In dove-favorable games a single occurrence of a rare action can induce a new buyer to be a dove even for large sample sizes (and the same holds for inducing new sellers to be hawks in hawk-favorable games). Having a large sample size k helps to increase the frequency of occurrence of rare actions, which explains why having sufficiently large expected sample size (condition 2) is necessary for moving away from the pure equilibrium. Finally, when the game is balanced between the two actions (i.e., g is close to 0.5), a single occurrence of a rare action can change the behavior of a new agent (in both populations) only if her sample is relatively small, but in this case, the total frequency in which rare actions occur in the samples might be too small. This is why having a less balanced game (i.e., g not to close to 0.5) helps moving away from a pure equilibrium.

Symmetric Mixed States In some cases, a mixed stationary state might be close to a pure stationary state: e.g., having few hawkish buyers and few dovish seller. Our next results study conditions for which the populations converge (at least from nearby initial states) to a symmetric mixed state, which is far from the pure equilibria in the sense of being (1) *egalitarian* (i.e., both populations obtain the same expected payoff), and (2) *inefficient* (i.e., the frequency of two hawks being matched together is substantial.

Theorems 2–3 show that (local) convergence to an inefficient symmetric state occurs for a relatively narrow (yet, relevant) domain of distributions of sample sizes in which most agents sample a single action, while a large share of the remaining agents have relatively large sample sizes. By contrast, Theorem 4 shows that (local) convergence to an inefficient symmetric state holds for many distributions of sample sizes, as long as the samples are not too large. The proofs of these results include *new techniques*, which apply various properties of binomial distributions, and might be of an independent interest.

Thus, taken together, our results show that when agents have limited information about the aggregate behavior (which seems plausible in various real-life applications, such as the motivating example of buyers and sellers of houses), then an egalitarian, yet inefficient, convention may arise in which bargaining frequently fails.

Structure Section 2 presents the related literature. Our model is described in Section 3. Section 4 presents a "complete" characterization for global convergence to one of the mixed stationary states. Section 5 presents various necessary and sufficient conditions for the stability of the inefficient symmetric stationary state. The analytic results of the paper are supplemented by a numeric analysis in Section 6. We conclude in Section 7. Formal proofs are presented in the appendices.

2 Related Literature and Contribution

Related theoretical literature The action-sampling dynamics were pioneered by Sandholm (2001) and Osborne and Rubinstein (2003). Oyama et al. (2015) applied these dynamics to prove global convergence results in supermodular games. Recently, Heller and Mohlin (2018) studied the conditions on the expected sample size that implies global convergence for all payoff functions and all sampling dynamics.[7]

Salant and Cherry (2020) (see also Sawa and Wu, 2021) generalized the action-sampling dynamics by allowing new agents to use various procedures to infer from their samples the aggregate behavior of the opponents. Salant and Cherry pay special attention to *unbiased* inference procedures in which the agent's expected belief about the share of opponents who play hawk coincides with the sample mean. Examples of unbiased procedures are maximum likelihood estimation, beta estimation with a prior representing

[7]Hauert and Miekisz (2018) use the term "sampling dynamics" to refer to a variant of the replicator dynamics, in which when an agent samples another agent and mimics the other agent's behavior, it is more likely for these two agents to be matched against each other. This is less related to our use of the notion of "Sampling dynamics", which is in-line of the literature cited in the main text.

complete ignorance, and a truncated normal posterior around the sample mean. In our setup, the payoffs are linear in the share of agents who play hawk, which implies that the agent's perceived best reply depends only on the expectation of her posterior belief. This implies that our results hold for any unbiased inference procedure.

The present paper, similar to the papers cited above, studies deterministic dynamics in infinite populations. When there is convergence to a stable stationary state in such dynamics, the convergence is fast (Oyama et al., 2015). By contrast, stochastic evolutionary models (see, e.g., the seminal contribution of Young, 1993, and the recent hawk–dove application in Bilancini et al., 2021), which are also based on revising agents observing a finite sample of opponents' actions, focus on the very long-run behavior of stochastic processes when players' choice rules include the possibility of rare "mistakes" (sufficient conditions for stochastic evolutionary models to yield fast convergence are studied in Kreindler and Young, 2013; Arieli et al., 2020).

The payoff-sampling dynamics were pioneered by Osborne and Rubinstein (1998) and Sethi (2000) and later generalized in various respects by Sandholm et al. (2020). It has been used in a variety of applications, including price competition with boundedly rational consumers (Spiegler, 2006), common-pool resources (Cárdenas et al., 2015), contributions to public goods (Mantilla et al., 2018), centipede games (Sandholm et al., 2019), finitely repeated games (Sethi, 2021) and the prisoner's dilemma (Arigapudi et al., 2021). The existing literature assumes that all agents have the same sample size.

A methodological contribution of the present paper is in extending the setup of payoff-sampling dynamics to analyze heterogeneous populations in which new agents differ in their sample sizes, and this heterogeneity leads to qualitatively new results

It is well-known that mixed stationary states in multiple-population games cannot be asymptotically stable under the commonly-used replicator dynamics (See, e.g., Theorem 9.1.6 of Sandholm, 2010). By contrast, we show that both classes of sampling dynamics, which are plausible in various real-life applications, can induce asymptotically stable mixed stationary states in a two-population hawk–dove game.

Related experimental literature Selten and Chmura (2008) experimentally tested the predictive power of various solution concepts in two-action, two-player games with a unique completely mixed Nash equilibrium. They show that both both the payoff-sampling equilibrium and the action-sampling equilibrium outperform the predictions of both the Nash equilibrium and the quantal-response equilibrium.

Recently, Stephenson (2019) tested the predictive validity of various evolutionary models in coordinated attacker–defender games.[8] Stephenson's experimental design is very favorable for monotone dynamics because each participant is shown the exact (population-dependent) payoff that would be obtained by each action at each point in time. Nevertheless, subjects frequently violate monotonicity: 10%–20% of the subjects switch from higher-performing strategies to lower-performing strategies.

The key prediction of monotone dynamics for hawk–dove games (in which agents from one population are randomly matched with agents from another population) is experimentally tested in Oprea et al. (2011) and Benndorf et al. (2016). Both experiments apply an interface that is favorable to monotonicity (i.e., each participant is shown the exact population-dependent payoff of each action). Both experiments show that the prediction of monotone dynamics holds in this setup, and that the populations converge to an asymmetric pure equilibrium in which one population (say, the buyers) plays hawk and the other population (say, the sellers) plays dove.

Consider a revised experimental design, where an agent observes only the behavior of her own opponent, rather than the aggregate behavior of the opposing population. An interesting testable prediction of our model is that in this experimental design, the populations are likely to converge to the symmetric stationary state in the relevant parameter domain (in particular, when g is not too far from 1; see Figure 6.1).[9]

3 Model

3.1 The Hawk–Dove Game

Let $G = \{A, u\}$ denote a symmetric two-player hawk–dove game, where:

1. $A = \{h, d\}$ is the set of actions of each player, and

[8] Experiments that directly test the dynamic predictions of evolutionary game theory are quite scarce. Two notable exceptions are the experiments showing the good fit of the dynamic predictions in the rock–paper–scissors game (Cason et al., 2014; Hoffman et al., 2015).

[9] Benndorf et al. (2016, 2021) studied a more general setup in which each participant in each round is randomly matched with an opponent from the other population with probability $\kappa \in (0, 1)$, and is randomly matched with an opponent from her own population with the remaining probability $1 - \kappa$. Our theoretical predictions fit the setup of κ close to one.

Table 2: Payoff Matrix of a Generalized Hawk–Dove Game $g, l \in (0,1)$

		Player 2	
		h	d
Player 1	h	$0, 0$	$1+g, 1-l$
	d	$1-l, 1+g$	$1, 1$

2. $u : A^2 \to \mathbb{R}$ is the payoff function of each player.

Let $i \in \{1,2\}$ be an index referring to one of the players, and let $j = \{1,2\} \setminus \{i\}$ be an index referring to the opponent. We interpret action h as the hawkish (more aggressive) action and d as the dovish action. The payoff matrix $u(\cdot,\cdot)$ of a generalized hawk–dove game is given in Table 2. When both agents are dovish, they obtain a relatively high payoff, which is normalized to 1. When both agents are hawkish, they obtain their lowest feasible payoff, which is normalized to 0. Finally, when one of the players is hawkish and her opponent is dovish, the hawkish player *gains* $g \in (0,1)$ (relative to the payoff 1 obtained by two dovish players), while her dovish opponent *looses*[10] $l \in (0,1)$. The game admits three Nash equilibria: two asymmetric pure Nash equilibria: (h,d) and (d,h), and a symmetric mixed Nash equilibrium in which each player plays h with probability $\frac{g}{1+g-l}$, and obtains a relatively low expected payoff of $\frac{(1+g)(1-l)}{1+g-l} < 1$.

An important special sub-class are the *standard* hawk–dove games in which $g = l$ (see Table 1), i.e., the gain of the hawkish player is equal to the loss of her dovish opponent.

3.2 Two-Population Evolutionary Dynamics

We assume that there are two unit-mass continuums of agents (e.g., buyers and sellers) and that agents in population 1 are randomly matched with agents in population 2. Aggregate behavior in the populations at time $t \in \mathbb{R}^+$ is described by a *state* $\mathbf{p}(t) = (p_1(t), p_2(t)) \in [0,1]^2$, where $p_i(t)$ represents the share of agents playing the hawkish action h at time t in population i. We extend the payoff function u to states (which have the same representation as mixed strategy profiles) in the standard linear way. Specifically, $u(p_i, p_j)$ denote the average payoff of population i (in which a share p_i of the population plays h) against population j (in which a share p_j plays h). With a slight abuse of notation, we use d (resp., h) to denote a degenerate population in which all of

[10] Herold and Kuzmics (2020) allow a broader domain in which the assumption of $g, l \in (0,1)$ is replaced with the weaker assumption of $g > 0$, $l < 1$, and $l + g > 0$. All of our results hold in this extended setup.

its agents play action d (resp., h). A state $\mathbf{p} = (p_i, p_j)$ is symmetric if $p_i = p_j$.

Agents occasionally die and are replaced by new agents (equivalently, agents occasionally receive opportunities to revise their actions). Let $\delta > 0$ denote the death rate of agents in each population, which we assume to be independent of the currently used actions. It turns out that δ does not have any effect on the dynamics except to multiply the speed of convergence by a constant.

The evolutionary process is represented by a continuous function $\mathbf{w} : [0,1]^2 \to [0,1]^2$, which describes the frequency of new agents in each population who adopt action h as a function of the current state. That is, $w_i(\mathbf{p})$ describes the share of new agents of population i who adopt action h, given state $\mathbf{p}$. Thus, the instantaneous change in the share of agents of population i that plays hawk is given by $\dot{p}_i = \delta \cdot (w_i(\mathbf{p}) - p_i)$.

Remark 1. Our *two-population* dynamics fit situations in which each player can condition her play on her role in the game (being Player 1 or Player 2). Common examples of such situations are (1) sellers that are matched with buyers, as in the motivating example, and (2) each player observes if she has arrived slightly earlier or slightly later at a contested resource (Maynard-Smith, 1982). By contrast, in *one-population* dynamics an agent cannot condition her play on her role. It is well known that all monotone one-population dynamics converge to the unique mixed Nash equilibrium in hawk–dove games (see, e.g., Weibull, 1997, Section 4.3.2). It is relatively straightforward to establish that one-population sampling dynamics lead to qualitatively similar results (convergence is to a somewhat different interior state than in the mixed Nash equilibrium, but the comparative statics with respect to the payoff parameters remain similar).

Monotone Dynamics The most widely studied dynamics are those that are monotone with respect to the payoffs. A dynamic is monotone if the share of agents playing an action increases iff the action yields a higher payoff than the alternative action.[11,12]

Definition 1. The dynamic $\mathbf{w} : [0,1]^2 \to [0,1]^2$ is *monotone* if for any player i, any interior $p_i \in (0,1)$, and any $p_j \in [0,1]$: $\dot{p}_i > 0 \;\Leftrightarrow\; u(h, p_j) > u(d, p_j)$.

[11]In games with more than two actions, there are various definitions that capture different aspects of monotonicity. All these definitions coincide for two-action games. In particular, Definition 1 coincides in two-action games with Weibull's (1997, Section 5.5) textbook definitions of payoff monotonicity, payoff positivity, sign preserving, and weak payoff positivity.

[12]The best-known example of payoff monotone dynamics is the standard replicator dynamic (Taylor, 1979), which is given by $\dot{p}_i = w_i(\mathbf{p}) - p_i = p_i(u(h, p_j) - u(p_i, p_j))$.

Oprea et al. (2011) showed that under monotone dynamics, from almost any initial state, the populations converge to one of the two asymmetric pure equilibria in which one population always plays h and the other population always plays d (generalizing the seminal analysis of Maynard-Smith and Parker, 1976). For completeness, we state and prove an equivalent result, using the notation of our model, in Appendix A.1.

3.3 Dynamic Stability

The following notions of stability are standard (see, e.g., Weibull, 1997, Chapter 5). A state is said to be stationary if it is a rest point of the dynamics.

Definition 2. State $\mathbf{p}^* \in [0,1]^2$ is a *stationary state* if $w_i(\mathbf{p}^*) = p_i^*$ for each $i \in \{1,2\}$.

Let $\mathcal{E}(w)$ denote the set of stationary states of w, i.e., $\mathcal{E}(w) = \{\mathbf{p}^* | w_i(\mathbf{p}^*) = p_i^*\}$. Under monotone dynamics, an interior (mixed) state $\mathbf{p}^* \in (0,1)^2$ is a stationary state iff it is a Nash equilibrium (Weibull, 1997, Prop. 4.7). By contrast, under nonmonotone dynamics (such as the sampling dynamics analyzed below) the two notions differ.

A state is Lyapunov stable if a population beginning near it remains close, and it is asymptotically stable if, in addition, it eventually converges to it. A state is unstable if it is not Lyapunov stable. It is well known (see, e.g., Weibull, 1997, Section 6.4) that every Lyapunov stable state must be a stationary state. Formally:

Definition 3. A stationary state $\mathbf{p}^* \in [0,1]^2$ is *Lyapunov stable* if for every neighborhood U of $\mathbf{p}^*$ there is a neighborhood $V \subseteq U$ of $\mathbf{p}^*$ such that if the initial state $p(0) \in V$, then $\mathbf{p}(t) \in U$ for all $t > 0$. A state is *unstable* if it is not Lyapunov stable.

Definition 4. A stationary state $\mathbf{p}^* \in [0,1]^2$ is *asymptotically stable* if it is Lyapunov stable and there is some neighborhood U of $\mathbf{p}^*$ such that all trajectories initially in U converge to $\mathbf{p}^*$, i.e., $\mathbf{p}(0) \in U$ implies $\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{p}^*$.

3.4 Sampling Dynamics

In what follows, we study two plausible nonmonotone dynamics, in which new agents base their choice on inference from small samples.

Distribution of sample sizes We allow heterogeneity in the sample sizes used by new agents. Let $\theta \in \Delta(\mathbb{Z}_+)$ denote the distribution of sample sizes of new agents. A share of $\theta(k)$ of the new agents have a sample of size k. Let $\text{supp}(\theta)$ denote the support of θ, let $\max(\text{supp}(\theta))$ denote the maximal sample size in the support of θ, and let $\max(\text{supp}((\theta)) = \infty$ if θ's support is unbounded. If there exists some k, for which $\theta(k) = 1$, then we use k to denote the degenerate (homogeneous) distribution $\theta \equiv k$.

Definition 5. An *environment* is a tuple $E = (g, l, \theta)$ where $g, l \in (0, 1)$ describe the underlying hawk-dove game, and $\theta \in \Delta(\mathbb{Z}_+)$ describes the distribution of sample sizes.

Action-sampling dynamics The action-sampling dynamics (Sandholm, 2001) fit situations in which agents do not know the exact distribution of actions being played in the opponent's population, but know the payoffs of the underlying game. Agents estimate the unknown distribution of actions by sampling a few opponents' actions. Specifically, each new agent with sample size k (henceforth, a k-agent) samples k randomly drawn agents from the opponent's population and then adopts the action that is the best reply against the sample. To simplify notation, we assume that in case of a tie, the new agent plays action d. Our results are independent of the tie-breaking rule.

Let $X_k(p_j) \sim Bin(k, p_j)$ denote a random variable with binomial distribution with parameters k (number of trials) and p_j (probability of success in each trial), which is interpreted as the number of h-s in the sample. Observe that the sum of payoffs of playing action h against the sample is $(1 + g) \cdot (k - X_k(p_j))$ and the sum of payoffs of playing action d against the sample is $(1 - X_k(p_j)) + X_k(p_j) \cdot (1 - l) = 1 - lX_k(p_j)$.

This implies that action h is the unique best reply to a sample of size k iff $(1 + g)(k - X_k(p_j) > k - lX_k(p_j) \Leftrightarrow \frac{X_k(p_j)}{k} < \frac{g}{1+g-l}$. This, in turn, implies that the action-sampling dynamic in environment (g, l, θ) is given by:

$$w^A(p_j) \equiv w_i^A(\mathbf{p}) = \sum_{k \in \text{supp}(\theta)} \theta(k) \cdot \Pr\left(\frac{X_k(p_j)}{k} < \frac{g}{1+g-l}\right). \tag{3.1}$$

Payoff-sampling dynamics The payoff-sampling dynamics (Osborne and Rubinstein, 1998) fit situations in which agents either do not know the payoff matrix or do not have feedback about the actions being played in the opponent's population. Specifically, a new k-agent observes for each of her feasible actions the mean payoff obtained by playing this action in k interactions (with each play of each action being against a newly drawn opponent), and then chooses the action whose mean payoff was highest. One possible interpretation for these observations is that each new k-agent tests each of the available actions k times, and then adopts for the rest of her life the action with the highest mean payoff during the testing phase. As above, we assume that a tie induces a new agent to play action d (and the results are independent of the tie-breaking rule).

We refer to the sample against which action h (resp., d) is tested as the h-sample (resp., d-sample). Let $X_k(p_j), Y_k(p_j) \sim Bin(k, p_j)$ denote two iid random variables with a binomial distribution with parameters k and p_j. Random variable X_k (resp., Y_k) is interpreted as the number of times in which the opponents have played action h in the h-sample (resp., d-sample). Observe that the sum of payoffs of playing action h (resp., d) against its h-sample (resp., d-sample) is $(1+g)(k - X_k(p_j))$ (resp., $1 - lY_k(p_j)$). This implies that action h has the highest mean payoff iff $(1+g)(k - X_k(p_j) > k - lY_k(p_j) \Leftrightarrow (1+g)X_k(p_j) < gk + lY_k(p_j)$. Thus the payoff-sampling dynamic in environment (g, l, θ) is given by:

$$w^P(p_j) \equiv w_i^P(\mathbf{p}) = \sum_{k \in \operatorname{supp}(\theta)} \theta(k) \cdot \Pr((1+g)X_k(p_j) < gk + lY_k(p_j)) \tag{3.2}$$

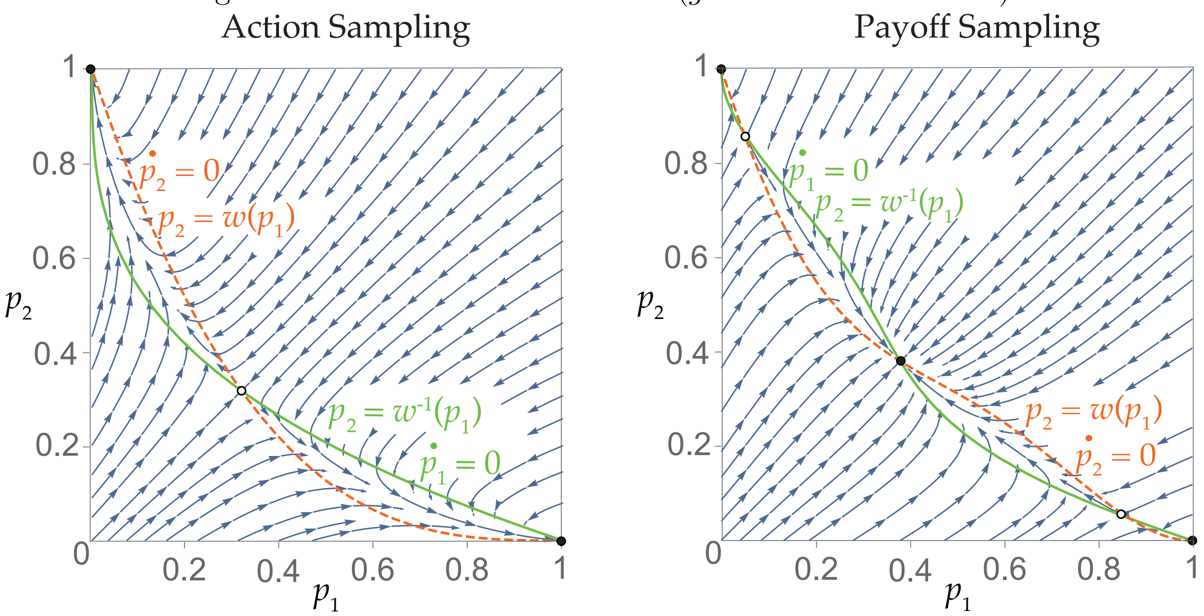
Henceforth, we omit the superscript A / P (i.e., we write $w(p_j)$) when referring to properties that hold for both classes of dynamics. The following fact is immediate from the basic properties of X_k and Y_k as binomial variables.

Fact 1. *$w(p_j)$ is a decreasing polynomial function that satisfies $w(0) = 1$ and $w(1) = 0$.*

3.5 Generic Environments

Figure 3.1 illustrates the phase plots of the sampling dynamics for the environment in which $g = l = 0.25$ and $\theta \equiv 3$. The green solid curve is the polynomial $p_2 = w(p_1)$, which describes the states in which $\dot{p}_1 = 0$. The orange dashed curve is the polynomial $p_2 = w^{-1}(p_1)$. The stationary states are the points where the two curves $w(\cdot)$ and $w^{-1}(\cdot)$ intersect. Due to Fact 1, the two curves always intersect at $(0, 1)$, $(1, 0)$ and at a unique symmetric state (p, p). Observe that stationary states $\mathbf{p}$ in which the slope of the curve $\dot{p}_1 = 0$ is larger than the slope of the curve $\dot{p}_2 = 0$ (i.e., $w'(p_1) > (w^{-1})'(p_1)$) are asymptotically stable, while the states with the inverse inequality are unstable.

Figure 3.1: Illustrative Phase Plots ($g = l = 0.25$ and $\theta \equiv 3$)



The figure illustrates the phase plots of the action-sampling dynamics (left panel) and payoff-sampling dynamics (right panel) for the environment in which $g = l = 0.25$ and $\theta \equiv 3$. The solid green (resp., dashed orange) curved line shows the states in which $\dot{p}_1 = 0$. The intersection points of these curves lines are the stationary states. A solid (resp., hollow) dot represents an asymptotically stable (resp., unstable) stationary state.

In order to simplify the presentation of our results, and to allow these results to be independent of the specific tie-breaking rule, we assume throughout the paper that the environment is generic in the sense of satisfying the following assumption:

Assumption 1 (Generic environment)**.**

1. *For any $p \in [0, 1]$, if $w(p) = w^{-1}(p)$, then $w'(p) \neq (w^{-1})'(p)$.*

2. *For any two samples of opponents' actions $\boldsymbol{s}, \boldsymbol{s}' \in A^k$ of the same size $k \in \mathbb{Z}$ and different number of occurrences of action h, the mean payoff of action h against sample $\boldsymbol{s}$ is different than the mean payoff of action d against sample $\boldsymbol{s}'$.*

Part (1) requires that the slopes of the curves $\dot{p}_1 = 0$ and $\dot{p}_2 = 0$ differ at their intersection points. This condition rules out non-generic cases in which (i) the two curves coincide on an interval (rather than on isolated points), and (ii) a stationary state in an intersection point of the two curves is neither asymptotically stable nor it is unstable.

The second part of Assumption 1 requires that there does not exist two (equal-size) samples for which the mean payoff of action h is exactly the same as the mean payoff of action g. This rules out cases of ties between the mean payoffs of the two actions against the samples, which implies that the tie-breaking rule will not play any role in our results.

Observe that both properties hold in almost all environments in the sense that if one chooses the parameters g and l and the values of the distribution θ from an arbitrary full-support continuous (atomless) distribution, then Assumption 1 holds with probability one. *Henceforth, we assume in all our results that the environment is **generic** (i.e., it satisfies Assumption 1).*

4 Global Convergence to Mixed Equilibria

Recall, that in monotone dynamics the population converges from almost any initial state to one of the pure equilibria. In this section, we characterize the conditions for which the opposite results holds for sampling dynamics, i.e., the populations converge from almost any initial state to one of the mixed (interior) stationary states.

4.1 Convergence Result for Generic Environments

Our preliminary result characterizes general convergence properties of both sampling dynamics in any generic hawk-dove environment. Specifically, it shows that:

1. all stationary states are isolated points;
2. in each pair of neighboring stationary states, one of these states is asymptomatically stable and its neighbor is unstable;
3. the population converges to one of the stationary states from any initial state; and
4. the population converges to one of the asymptotically stable states from almost any initial states. We refer to such a convergence to a set of states from almost all initial states as *global convergence*.

This is formalized as follows. We say that two stationary states $\mathbf{p}, \mathbf{p}'$ are *neighboring* if for any stationary state $\mathbf{p}'' \neq \mathbf{p}, \mathbf{p}'$, either $p_1'' > \max(p_1, p_1')$ or $p_1'' < \min(p_1, p_1')$.

Proposition 1. *1. For any stationary state s, there exists an open neighborhood U around s, such that s is the unique stationary state in U;*

2. *If $\boldsymbol{p}, \boldsymbol{p}'$ are neighboring stationary states, then either $\boldsymbol{p}$ is asymptotically stable and $\boldsymbol{p}'$ unstable, or $\boldsymbol{p}'$ is asymptotically stable and $\boldsymbol{p}$ is unstable;*

3. $\lim_{t\to\infty} \mathbf{p}(t)$ *exists for any $\boldsymbol{p}(0)$, and it is a stationary state; and*

4. *There exists a null (measure-zero) set of states $\hat{S}$ such that if $\boldsymbol{p}(0) \notin \hat{S}$ then* $\lim_{t\to\infty} \mathbf{p}(t)$ *is an asymptotically stable state.*

The intuition of Proposition 1 is illustrated in Figure 3.1. Part 1 of Assumption 1 requires that the two curves ($\dot{p}_1 = 0$ and $\dot{p}_2 = 0$ have different slopes in any interaction point. This implies that these intersection points must be isolated, and that in any pair of neighboring intersection points, the curve $\dot{p}_1 = 0$ is higher than the curve $\dot{p}_2 = 0$ in the left side of one of these intersection points (which must be asymptotically stable as all the nearby arrows in the phase plot converge to this point), and in the right side of the remaining intersection point (which must be unstable as all the nearby arrows in the phase plot converge away from this point). The proof is presented in Appendix A.2.

4.2 Single Appearance of a Rare Action

The following lemma characterizes when a single appearance of a rare action in a new agent's sample can change the agent's behavior (see the proof in Appendix A.3).

Lemma 1. *Consider a new agent in population i with a sample size of k.*

1. *Action-sampling dynamics: (I) Action h induces a higher payoff against a sample with a single opponent's action d iff*[13] $k \leq \left\lfloor \frac{1+g-l}{1-l} \right\rfloor$*; and (II) Action d induces a higher payoff against a sample with a single h iff* $k \leq \left\lfloor \frac{1+g-l}{g} \right\rfloor$.

2. *Payoff-sampling dynamics: (I) a h-sample with a single action d induces a higher mean payoff than a d-sample with no d-s iff* $k \leq \left\lfloor \frac{1+g}{1-l} \right\rfloor$*; and (II) a d-sample with no h-s induces a higher mean payoff than a h-sample with a single h iff* $k \leq \left\lfloor \frac{1+g}{g} \right\rfloor$.

Lemma 1 allows us to define the maximal sample sizes in which a single appearance of a rare action can change the behavior of a new agent.

Definition 6. Let $m_h^A = \left\lfloor \frac{1+g-l}{1-l} \right\rfloor$, $m_d^A = \left\lfloor \frac{1+g-l}{g} \right\rfloor$, $m_h^P = \left\lfloor \frac{1+g}{1-l} \right\rfloor$, $m_d^P = \left\lfloor \frac{1+g}{g} \right\rfloor$.

[13] $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

We omit the superscript ^{A,P} when stating a result that is true for both dynamics; for example, we write m_h, which denotes m_h^P when the underlying dynamics is payoff sampling, and which denotes m_h^A when the underlying dynamics is action sampling.

The parameter m_h is the maximal sample size for which a single appearance of d in the sample, when all other sampled actions are h, can induce a new agent to adopt action h. Similarly, m_d is the maximal sample size for which a single appearance of h in the sample, when all other sampled actions are d, can induce a new agent to adopt action d.

We conclude this subsection by presenting a definition of *m-bounded expectation* of a probability distribution with support on the set of positive integers. It is the expected value of the probability distribution by restricting its support to m. Formally, we have:

Definition 7. The *m-bounded expectation* $\mathbb{E}_{\leq m}$ of distribution θ with support on positive integers is defined as $\mathbb{E}_{\leq m}(\theta) = \sum_{k=1}^{m} \theta(k) \cdot k$.

4.3 Asymptotic Stability of Pure Equilbria

Our next result characterizes the asymptotic stability of the pure states. It shows that the asymptotic stability depends only on whether the product of the bounded expectations of the distribution of sample sizes is larger or smaller than one, where the bound of each distribution is the maximal sample size for which a single appearance of a rare action can change the behavior of a new agent. Formally:

Proposition 2. *1.* $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) > 1 \Leftrightarrow$ *both pure stationary states are unstable.*

2. $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) < 1 \Leftrightarrow$ *both pure stationary states are asymptotically stable.*

Sketch of Proof. Consider a slightly perturbed state $(\epsilon, 1-\epsilon)$ near the pure equilibrium $(0,1)$. Observe that almost all agents in population 1 (resp., 2) play d (resp., h). The event of two rare actions appearing in a sample of a new agent has a negligible probability of $O(\epsilon^2)$. If a new agent has a sample size of k, then the probability of a rare action appearing in the sample is approximately $k \cdot \epsilon$. This rare appearance changes the perceived best reply of a new agent of population 1 iff k is smaller than m_h. Thus, the total probability that a new agent of population 1 (resp., 2) adopts a rare action is equal to $\mathbb{E}_{\leq m_h}(\theta)$ (resp., $\mathbb{E}_{\leq m_d}(\theta)$). This implies that the product of the share of new agents adopting a rare action in each population is $\epsilon \cdot \mathbb{E}_{\leq m_h}(\theta) \cdot \epsilon \cdot \mathbb{E}_{\leq m_d}(\theta)$. This shows that the share of agents playing rare actions gradually increases (resp., decreases) if $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) > 1$ (resp., $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) < 1$), which implies instability (resp., asymptotic stability). See Appendix A.4 for a formal proof. $\square$

Observe that the fact that $l > 0$ immediately implies that $m_h^A < m_h^P$ and $m_d^A < m_d^P$, which, in turn, implies that instability under the action-sampling dynamics holds in a strictly smaller set of distributions than under the payoff-sampling dynamics.

Corollary 1. *If the pure stationary states are unstable under the action-sampling dynamics, then they are also unstable under the payoff-sampling dynamics.*

4.4 Global Convergence to A Mixed Equilibrium

Combining Propositions 1 and 2 yields the main result of this section. It shows that the population converges from any initial state to a mixed stationary state iff the product of the bounded expectations of the distribution of sample sizes is larger than 1.

Theorem 1. *The population converges to a mixed (interior) stationary state from almost any initial state iff* $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) > 1$.

Proof. Assume that $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) > 1$. Proposition 2 implies that both pure stationary states are unstable. Part (4) of Proposition 1 implies that from almost initial state the population converges to an asymptotically stable state, which must be mixed. Due to the environment being generic, if $\mathbb{E}_{\leq m_h}(\theta) \cdot \mathbb{E}_{\leq m_d}(\theta) \leq 1$, then this inequality must be strict, which due to Proposition 2 implies that both pure stationary states are asymptotically stable, and, thus a positive mass of initial states converge to pure states. $\square$

Next we apply Theorem 1 to standard hawk-dove games in which $g = l$ describes both the gain of a hawkish player and the loss of her dovish opponent.

Corollary 2. *Assume that $g = l$. The population converges to a mixed stationary state from almost any initial state iff:*

1. *Action-sampling dynamics:* $\mathbb{E}_{\leq \max\left(\frac{1}{g}, \frac{1}{1-g}\right)}(\theta) > \frac{1}{\theta(1)}$.
2. *Payoff-sampling dynamics: either* $g < \dfrac{1}{3}$ *and* $\mathbb{E}_{\leq \frac{1+g}{g}}(\theta) > \dfrac{1}{\theta(1)}$

   or $g \geq \dfrac{1}{3}$ *and* $\mathbb{E}_{\leq \max\left(3, \frac{1+g}{1-g}\right)}(\theta) > \dfrac{1}{\theta(1) + 2 \cdot \theta(2)}$.

The proof is immediately implied from substituting $g = l$ in Definition 1. Corollary 2 implies that global convergence to one of the mixed stationary states holds iff: (1) sufficiently many agents have a sample size of 1, (2) the expected sample size is sufficiently large, and (3) g is not too close to 0.5. Under the payoff-sampling dynamics (and assuming $g > \frac{1}{3}$), global convergence holds also if one relaxes condition (1) by requiring that (1') sufficiently many agents have a sample size of at most 2. As demonstrated in Section 6 these conditions hold for many environments under the payoff-sampling dynamics, and for a somewhat limited set of environments under the action-sampling dynamics.

5 Stability of the Symmetric Stationary State

In Section 4 we characterized the conditions for global convergence to one of the mixed stationary states. In some cases, a mixed stationary state $\mathbf{p}$ might be close to a pure stationary state: i.e., p_1 is close to 1, p_2 is close to 0, there is a large difference between the payoffs of the two populations, and two hawks are rarely matched together. In this section we study conditions for which the populations converge (at least from nearby initial states) to the symmetric mixed stationary state, which is far from the pure equilibria in the follows sense: (1) both populations obtain the same expected payoff, and (2) the frequency of the inefficient matching of two hawks is substantial (~25% or higher).

5.1 Action-Sampling Dynamics

Our first result shows that any symmetric stationary state is unstable under the action-sampling dynamics if all agents have the same sample size.

Theorem 2. *For* [14] $\theta \equiv k > 1$, *the unique symmetric stationary state is unstable under the action-sampling dynamics.*

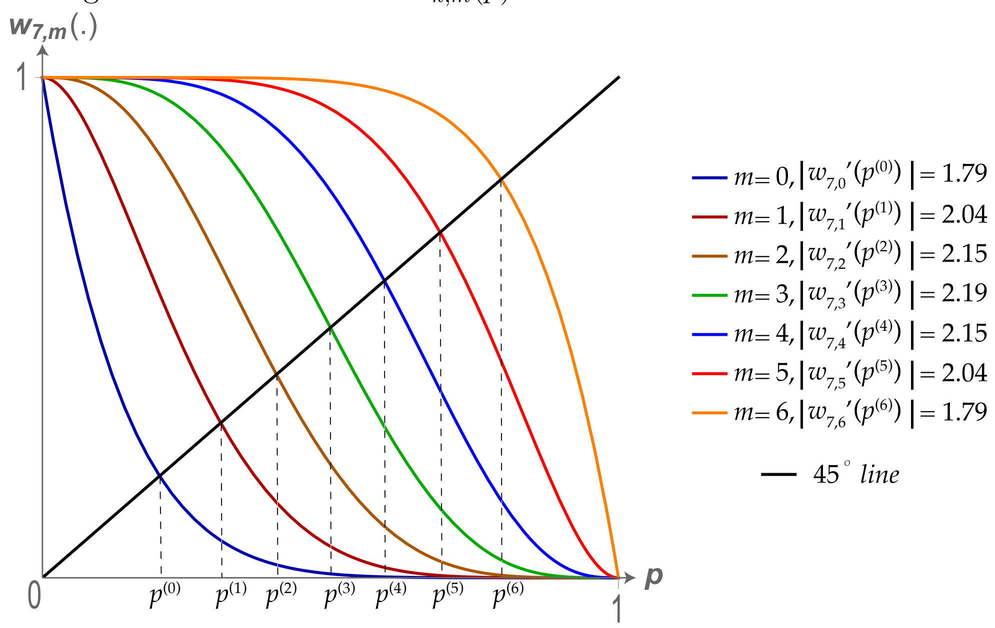
Sketch of Proof. Each pair of parameters g, l induces a threshold $0 \leq m < k$, such that playing hawk is the best reply against a sample of size k iff the sample includes at most m hawkish actions. This implies that a symmetric state $\left(p^{(m)}, p^{(m)}\right)$ is stationary iff $p^{(m)}$ is a fixed point of the function $w_{k,m}(p) \equiv P(X_k(p) \leq m)$, where $X_k(p)$ is a binomial distribution with parameters k (number of trials) and p (probability of success). The fact

[14] The case in which all agents have sample size 1 induces a non-generic environment in which the symmetric stationary state is neither asymptotically stable nor unstable.

$w_{k,m}(p)$ is decreasing in p, $w_{k,m}(0) = 1$ and $w_{k,m}(1) = 0$ implies that there exists a unique symmetric stationary state.

Consider the perturbed state $\left(p^{(m)} - \epsilon, p^{(m)} + \epsilon\right)$ in which population 1 (resp., 2) has slightly more dovish (resp., hawkish) players. The share of new agents of population 1 (resp., 2) who play the hawkish action is approximately equal to $p^{(m)} - |w'_{k,m}(p^{(m)})| \cdot \epsilon$ (resp., $p^{(m)} + |w'_{k,m}(p^{(m)})| \cdot \epsilon$). This implies that the perturbation will gradually increase iff the absolute value of the derivative $|w'_{k,m}(p^{(m)})|$ is greater than 1. It is easy to verify that the function $|w'_{k,m}(p)|$ is unimodal with a peak at $\frac{m}{k-1}$. The formal proof shows that the fixed point $p^{(m)}$ is sufficiently close to the peak, such that $|w'_{k,m}(p^{(m)})| > 1$. See Appendix A.5 for a formal proof. Figure 5.1 illustrates the functions $w_{k,m}(p)$ and their rest points for $k = 7$ and for all values of m. $\square$

Figure 5.1: The Function $w_{k,m}(p)$ for $k = 7$ and All Values of m

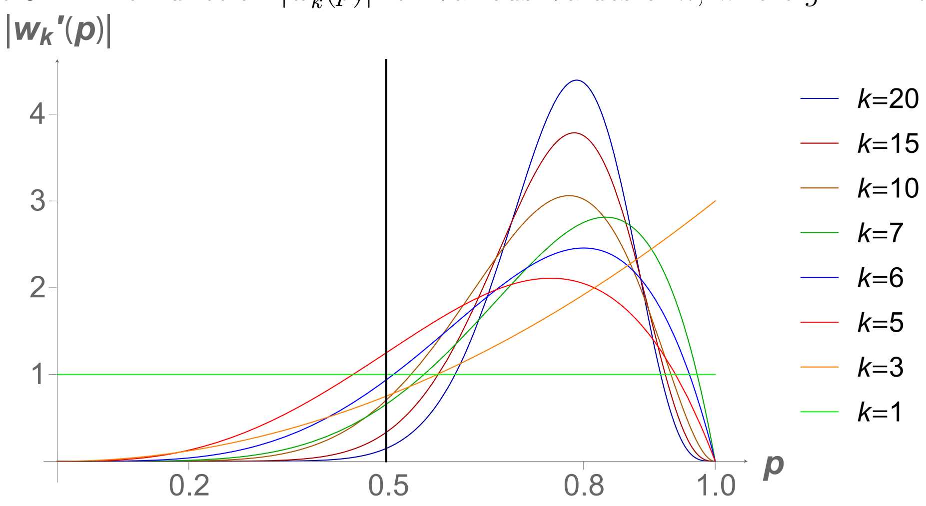


Thus, the dynamic behavior under the action–sampling dynamics is similar to monotone dynamics when all agents have the same sample size: the symmetric stationary state is unstable (and for standard games with $g = l$, due to $\theta(1) = 0$ in part (1) of Corollary 2, the asymmetric pure stationary states are stable). By contrast, our next result shows that the converse is true if most agents have a small sample size of one, and most of the remaining few agents have sufficiently large sample sizes.

Theorem 3. *For any $\epsilon > 0$, there exists $\hat{k} \in \mathbb{N}$ and $q_1, q_{\geq \hat{k}} \in (0, 1)$ such that the unique symmetric stationary state is asymptotically stable under the action-sampling dynamics* *for any distribution θ satisfying: (a) $\theta(1) \geq q_1$, (b) $\sum_{k \geq \hat{k}} \theta(k) \geq q_{\geq \hat{k}}$. Moreover, the probability of two hawks being matched together in this state is at-least* $25\% - \epsilon$.

Sketch of Proof. In populations in which all agents have a sample size of one, the unique symmetric stationary state is $p^{(1)} = 0.5$. By continuity, this implies that the unique symmetric stationary state $p^{(\theta)}$ is very close to 0.5 if $\theta(1)$ is sufficiently close to one. An analogous argument to the sketch of proof of Theorem 2 shows that a sufficient condition for the symmetric stationary state to be asymptotically stable is $|w'_\theta(p^{(\theta)})| < 1$.

Figure 5.2: The Function $|w'_k(p)|$ for Various Values of k, where $g = 1 - l = 0.8$



The function $|w'_\theta(p)|$ is a mixture of the functions $|w'_k(p)|$ for the various k-s in the support of θ_q (which are illustrated in Figure 5.2). Observe that $|w'_1(p) \equiv 1|$. The formal proof applies the central limit theorem to show that as k increases, $|w'_k(p)|$ converges to a normal distribution with mean $\frac{g}{g+l}$ and variance $\frac{1}{4k}$. In generic environments, $\frac{g}{g+l} \neq \frac{1}{2}$, which implies that $|w'_k(0.5)|$ converges to zero. This, in turn, implies that $|w'_{\theta_q}(0.5)| < 1$ if $\hat{k}$ is sufficiently large. By continuity, $|w'_{\theta_q}(p^{(\theta_q)})| < 1$, which implies that the symmetric stationary state is asymptotically stable. See Appendix A.6 for a formal proof. □

5.2 Payoff-Sampling Dynamics

For tractability in the analysis of payoff-sampling dynamics, we focus on the cases where the gain of a hawkish player and the loss of her dovish opponent are large, namely, $l, g > \frac{1}{\max(\text{supp}(\theta))}$. Our result shows that in this domain, the symmetric stationary state is asymptotically stable in the following common cases:

1. for any homogeneous distribution of sample sizes $\theta \equiv k < 20$; or

2. for any heterogeneous distribution with a maximal sample size of at most 5.

The threshold of $k = 20$ is binding. The symmetric stationary state becomes unstable if the sample size $k \geq 20$. By contrast, the bound of a maximal size of 5 for heterogeneous distributions of sample sizes is only a constraint of our proof technique. Numeric analysis (see Section 6) suggests that the stability of the symmetric stationary state:

1. holds for many distributions of types with larger maximal sample sizes (in particular, it holds for uniform distributions over $\{1, \ldots, k\}$ for any $k \leq 20$); and

2. is often global (i.e., in many cases almost all initial states converge to this state).

Theorem 4. *Assume that* $l, g \in \left(\frac{1}{\max(\mathrm{supp}(\theta))}, 1\right)$*, and either (1)* $\theta \equiv k < 20$*, or (2)* $\max(\mathrm{supp}(\theta)) \leq 5$*. Then, the game admits an asymptotically stable symmetric stationary state under the payoff-sampling dynamics. Moreover, the frequency of two hawks being matched together in this state is at-least* 25%*.*

Sketch of Proof. When l and g are sufficiently large, the payoff of action h is slightly below twice the number of d-s in the h-sample, and the payoff of action d is slightly above the number of d-s in the d-sample. This implies that action h has a higher mean payoff than action d iff the number of d-s in the h-sample is strictly greater than half the number of d-s in the d-sample.

Thus, we can write $w_k(p)$ as follows:

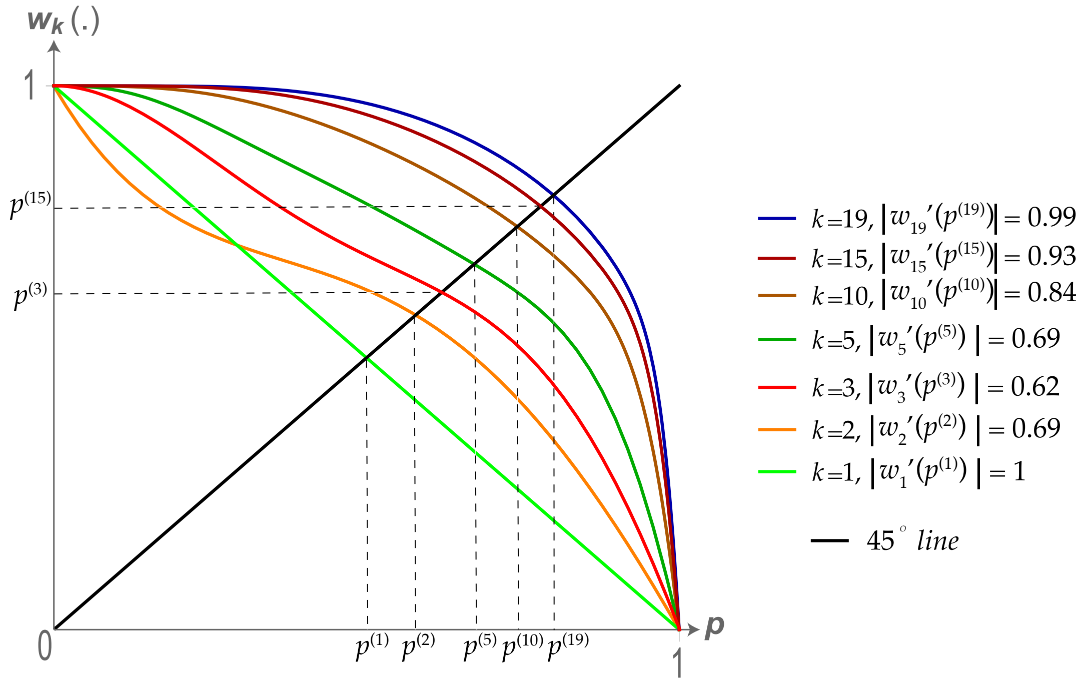
$$w_k(p) = P\left(\underbrace{k - X_k(p)}_{\#d_j \text{ in } h\text{-sample}} > \frac{1}{2}\underbrace{(k - Y_k(p))}_{\#d_j \text{ in } d\text{-sample}}\right) = P(2X_k(p) - Y_k(p) < k), \tag{5.1}$$

where $X_k(p)$ and $Y_k(p)$ are iid binomial random variables with parameters k and p.

In the formal proof (see Appendix A.7), we show that for any $k < 20$, $w_k(p)$ has a unique fixed point $p^{(k)}$ such that $|w'_k(p^{(k)})| < 1$ (see Figure 5.3). This implies, by the same argument as in the sketch of proof of Theorem 2, that the symmetric stationary state is asymptotically stable. (By contrast, one can verify that $|w'_k(p^k)| > 1$ for $k \geq 20$, which implies that the symmetric stationary state is unstable for large $k \geq 20$.)

Next, we verify in the formal proof that for any $k \in \{1, 2, 3, 4, 5\}$ it holds that (I) the fixed points are all in the interval $(0.5, 0.68)$, and (II) $|w'_k(p)| < 1$ for any $k \in \{1, ..., 5\}$ and any $p \in (0.5, 0.68)$. Let θ be any distribution with $\max(\mathrm{supp}(\theta)) \leq 5$. The fact that

Figure 5.3: The Function $w_k(p)$ for Various Values of k



$w_\theta(p)$ is a weighted average of the various $w_k(p)$ implies that (I) the fixed point $p^{(\theta)}$ of $w_\theta(p)$ is in $(0.5, 0.68)$, and (II) $|w'_\theta(p^{(\theta)})| < 1 \Rightarrow (p^{(\theta)}, p^{(\theta)})$ is asymptotically stable. □

6 Numeric Analysis

We present numeric results that complement the analytic results of the previous sections.

Methodology and Parameter Values The analysis focuses on the standard hawk–dove games, in which the gain of a hawkish player is equal to the loss of her dovish opponent, i.e., $g = l$ for each $i \in \{1, 2\}$. We have tested the following $270 = 10 \times 27$ combinations of parameter values for each of the two sampling dynamics:

1. 10 values for the ratio g: 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.85, 0.95.

2. 27 distributions of sample sizes:

   (a) 9 *degenerate distributions*, in which all agents have sample size k, for each $2 \leq k \leq 10$ (the case of $k = 1$ is analytically analyzed in Appendix **??**).

   (b) 9 *uniform distributions* over $\{1, .., k\}$, for each $2 \leq k \leq 10$.

(c) 9 1-*biased distributions*, in which a share $q \in \{10\%, 20\%, ..., 90\%\}$ of the new agents have sample size one, while the sample sizes of the remaining agents are distributed uniformly over $\{1,2,\ldots, 10\}$. That is, the frequency of sample size 1 is $\frac{1-q}{10} + q$ and the frequency of each $k \in \{2, \ldots, 10\}$ is $\frac{1-q}{10}$.

For each set of parameters, we have numerically calculated the phase portrait and the curves for which $\dot{p}_1 = 0$ and $\dot{p}_2 = 0$, and used this to determine the dynamic behavior. The code is provided in the online supplementary material.[15]

Results The numeric results are summarized in Figure 6.1. The action-sampling dynamics typically yield convergence to the pure equilibria from almost all initial states (orange shaded region in Figure 6.1), henceforth called *global convergence*. The exceptions are consistent with Proposition 2 and Theorem 3. Specifically, the action-sampling dynamics admit (almost) global convergence to the symmetric stationary state (green shaded region) if (1) most agents have sample size 1, and (2) g is sufficiently far from 0.5.

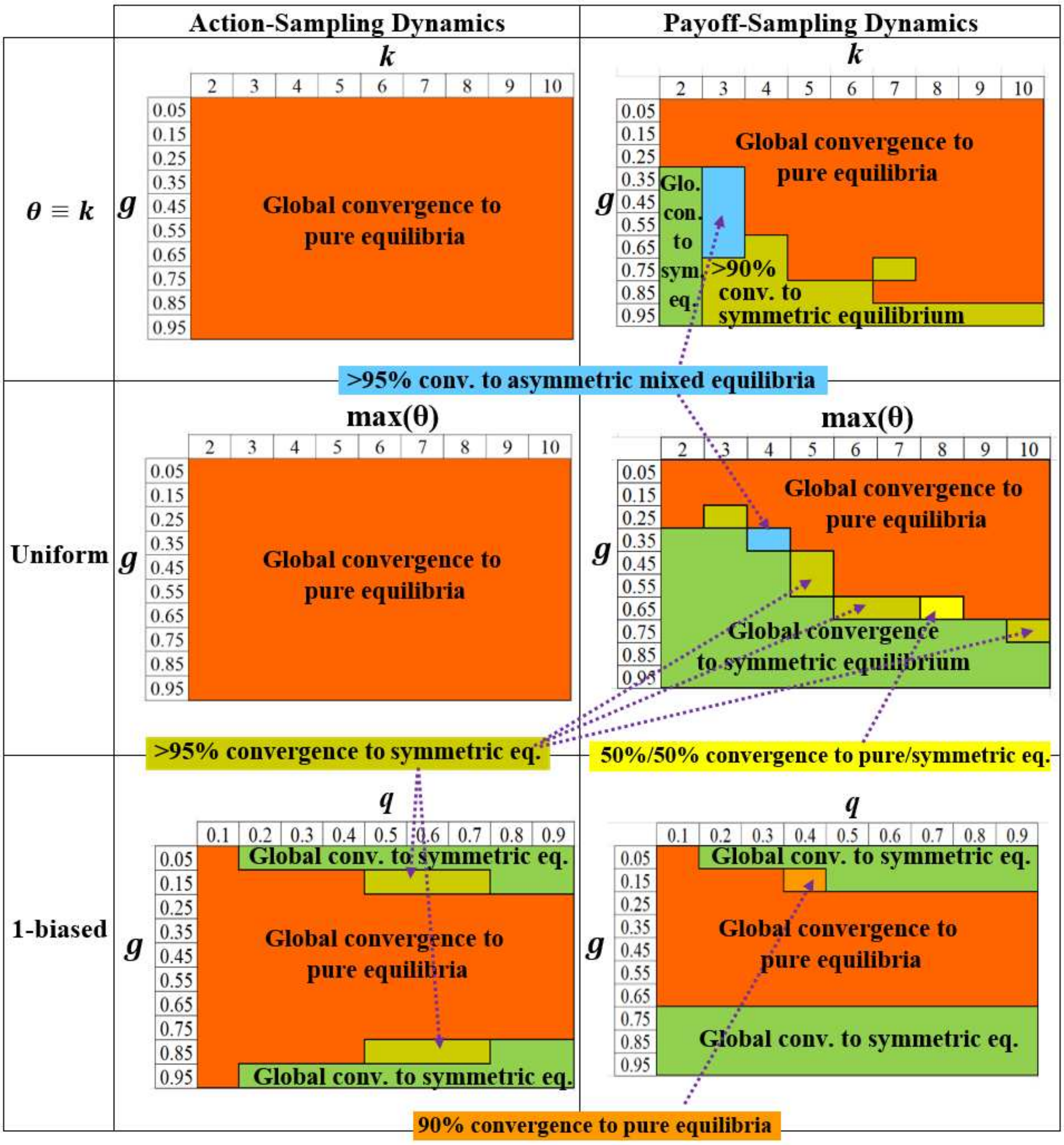
The payoff-sampling dynamics typically yield global (or almost global) convergence to either the asymmetric pure equilibria or the symmetric mixed stationary state, where each kind of convergence holds, roughly, in half of the parameter combinations. Global convergence to the symmetric mixed stationary state occurs for all parameter values for which it occurs under the action-sampling dynamics. In addition, the payoff-sampling dynamics globally converge to the symmetric mixed stationary state for all distributions of types, provided that the ratio g is sufficiently close to one.

7 Conclusion

A key result in evolutionary game theory is that two populations that are matched to play a hawk-dove game converge to one of the pure equilibria from almost any initial state. We demonstrate that this result crucially depends on the revision dynamics being monotone. Specifically, we show that two plausible classes of dynamics, in which new agents base their chosen actions on sampling the actions of a few agents in the opponent population (action-sampling dynamics) or on sampling the payoffs of a few agents in their

[15] Our numeric analysis is based on deterministic dynamics in a continuum population. We have randomly chosen 10 of these combinations of parameter values, and tested each of them by running it 100 times in the stochastic dynamics induced by a finite population of 1,000 agents, using ABED software (Izquierdo et al., 2019). The results for finite populations are qualitatively the same.

Figure 6.1: Summary of Results of the Numeric Analysis



own population (payoff-sampling dynamics) can lead to the opposite prediction: global convergence to a mixed stationary state.

Our results might help to explain why in bargaining situations, such as the motivating example of buying and selling houses, players in both populations tend to play hawkish strategies, and bargaining frequently fails.

Our model assumes that all players in each population have the same payoff matrix. Heterogeneity in the payoffs, and private information regarding one's payoff, are important aspects of many real-life bargaining situations. An interesting direction for future research is to apply the analysis of sampling dynamics in more complicated models that incorporate heterogeneous payoffs.

A Appendix

A.1 Benchmark Result: Stability under Monotone Dynamics

For completeness we adapt Oprea et al.'s 2011 result to our setup and notation.

Proposition 3 (Adaptation of Oprea et al., 2011, Prop. 1)**.** *From almost all initial states, the populations converge to a pure stationary state under monotone dynamics.*

Proof. Observe that $u(h,p_j) > u(d,p_j)$ iff $p_j < p_j^N = \frac{g}{l+g}$. Due to payoff monotonicity this implies that $\dot{p}_i > 0$ iff $p_j < p_j^N$. This implies that one can divide the unit square into four rectangles (as illustrated in Figure A.1 below):

1. Upper-left rectangle ($p_1 < p_1^N$, $p_2 > p_2^N$) in which the dynamics move upward and to the left (i.e., $\dot{p}_1 < 0 < \dot{p}_2$) until converging to $(0,1)$.

2. Upper-right rectangle ($p_1 > p_1^N$, $p_2 > p_2^N$) in which the dynamics move downward and to the left (i.e., $\dot{p}_1, \dot{p}_2 < 0$) until converging to either the upper-left rectangle, the lower-right rectangle, or the unstable equilibrium $\left(p_1^N, p_2^N\right)$.

3. Lower-right rectangle ($p_1 > p_1^N$, $p_2 < p_2^N$) in which the dynamics move downward and to the right (i.e., $\dot{p}_2 < 0 < \dot{p}_1$) until converging to $(1,0)$.

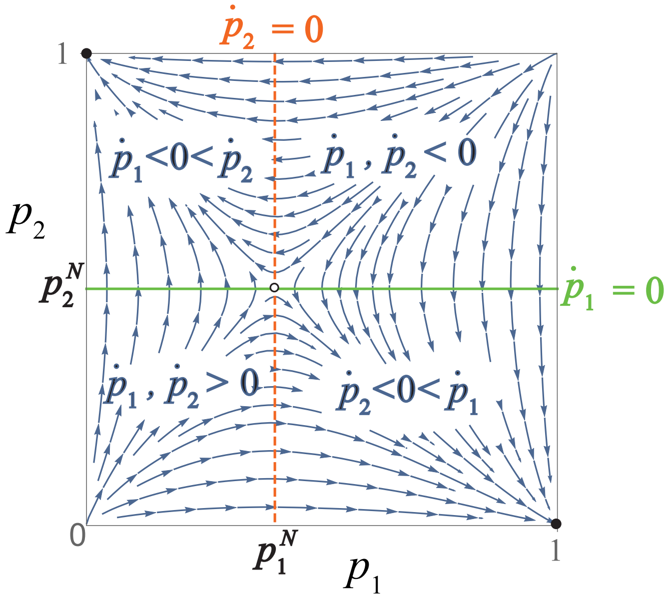
4. Lower-left rectangle ($p_1 > p_1^N$, $p_2 < p_2^N$) in which the dynamics move upward and to the right (i.e., $\dot{p}_1, \dot{p}_2 > 0$) until converging to either the upper-left rectangle, the lower-right rectangle, or the unstable equilibrium $\left(p_1^N, p_2^N\right)$.

This implies that $\{(d,h),(h,d)\}$ is globally stable. □

A.2 Proof of Proposition 1 (Generic Convergence Properties)

1. The stationary states are those that satisfy $w(p) = w^{-1}(p)$. Part (1) of Assumption 1 implies that $w(p') \neq w^{-1}(p')$ for any sufficiently close $p' \neq p$, which implies that all stationary states are isolated.

Figure A.1: Global Stability of $\{(0,1),(1,0)\}$ for Monotone Dynamics.



The figure illustrates the four rectangles described in the proof of Proposition 3. A solid (resp., hollow) dot represents an asymptotically stable (resp., unstable) equilibrium.

2. Part (1) of Assumption 1 implies that $w(p_1) \neq w^{-1}(p_1)$. Assume WLOG that $w'(p_1) > (w^{-1})'(p_1)$ (the argument in the case of t $w'(p_1) < (w^{-1})'(p_1)$ is analogous). This inequality implies that the curve $w(p)$ is higher (resp., lower) than the curve $w^{-1}(p)$ in the area to the left (resp., right) of p_1 (as is the case for the interior stationary state in the left panel of Figure 3.1). This implies that stationary state $\mathbf{p}$ must be unstable (as populations starting at an initial state slightly to the left (resp., right) of $\mathbf{p}$ goes further to the left (resp., right)). Due to Part (1) of Assumption 1, the opposite relation between the curves must hold for the neighboring stationary state $\mathbf{p}'$: the curve $w(p)$ is lower (resp., higher) than the curve $w^{-1}(p)$ in the area to the left (resp., right) of p_1'. This implies that stationary state $\mathbf{p}'$ must be asymptotically stable (as any nearby populations converge to **p'**, as demonstrated by the symmetric equilibrium in the right panel of Figure 3.1.

3+4. Observe that any populations starting at an initial state that is above both curves of $w(p)$ and $w'(p)$ move south-west (i.e., the share of hawks in both populations decrease) till the state reaches one of the two curves (see Figure 3.1). An analogous argument holds for populations starting below both curves. A measure-zero of such populations converge exactly to an unstable intersection point of the two curves. In all other cases, the populations either converge to an asymptotically stable state, or they cross one of the curves and enter a region that is between the two curves (i.e., a region above one of the curves and below the other curve). Further observe, that populations in any state that is between the two curves always converge to the nearby asymptotically stable stationary state.

A.3 Proof of Lemma 1 (Maximal Sample Sizes)

1. (I) The sum of payoffs of action h (resp., d) against a sample with a single d is $1+g$ (resp., $1+(k-1)(1-l)$). The mean payoff of h is greater than the mean payoff of d iff[16] $k<\frac{1+g-l}{1-l}$. (II) The sum of payoffs of action d (resp., h) against a sample with a single h is $k-1+1-l$ (resp., $(k-1)(1+g)$). The mean payoff of h is greater than the mean payoff of d iff $k<\frac{1+g-l}{g}$.

2. (I) The sum of payoffs of a h-sample with a single d is equal to $1+g$. The sum of payoffs of a d-sample with no d is equal to $k\cdot l$. The former sum is greater than the latter iff $k<\frac{1+g}{1-l}$. (II) The sum of payoffs of a d-sample with no h_j-s is equal to k. The sum of payoffs of an h-sample with a single h_j is equal to $(k-1)(1+g)$. The former sum is greater than the latter iff $k<\frac{1+g}{g}$.

A.4 Proof of Proposition 2 (Stability of Pure Equilibria)

We are interested in deriving conditions for the stability of the pure stationary states. In what follows, we compute the Jacobian of the sampling dynamics at the state $(0,1)$. For this, we consider a state $(\epsilon_i, 1-\epsilon_j)$ which is infinitesimally close to the state $(0,1)$ i.e., $0<\epsilon_i,\epsilon_j<<1$. In words, we consider the state with a "very small" ϵ_i share of hawks in population i and a "very small" ϵ_j share of doves in population j. By "very small," we mean that higher-order terms of ϵ are neglected.

Consider a new agent of population i with a sample size of k_i. Action h has a higher mean payoff against a sample size of k_i iff (neglecting rare events of having multiple d-s in the sample): (1) the sample includes the single action d of an opponent, and (2) $k_i \le m_h$ (due to Lemma 1). The probability of (1) is $k_i\cdot\epsilon_j+o(\epsilon_j)$, where $o(\epsilon_j)$ denotes terms that are sublinear in ϵ_j, and, thus, it will not affect the Jacobian as $\epsilon_j\to 0$. This implies that the probability that a new agent of population i (with a random sample size

[16] A tie between the two samples is impossible due to Assumption 1 of having a generic environment.

distributed according to θ) has a higher mean payoff for action h against her sample is $w_\theta(1-\epsilon_j) = \epsilon_j \cdot \sum_{k_i=1}^{m_h} \theta(k_i) \cdot k_i + o(\epsilon_j)$. An analogous argument implies that the probability that a new agent of population j has a higher mean payoff for action d_j against her sample is $w_{\theta_j}(\epsilon_i) = \epsilon_i \cdot \sum_{k_j=1}^{m_d} \theta_j(k_j) \cdot k_j + o(\epsilon_i)$. Therefore, the sampling dynamics at $(\epsilon_i, 1-\epsilon_j)$ can be written as follows (ignoring the higher-order terms of ϵ_i and ϵ_j):

$$\dot{\epsilon}_i = \epsilon_j \cdot \sum_{k_i=1}^{m_h} \theta(k_i) \cdot k_i - \epsilon_i, \qquad \dot{\epsilon}_j = \epsilon_i \cdot \sum_{k_j=1}^{m_d} \theta_j(k_j) \cdot k_j - \epsilon_j. \tag{A.1}$$

Define: $a_\theta = \sum_{k_i=1}^{m_h} \theta(k_i) \cdot k_i$ and $b_\theta = \sum_{k_j=1}^{m_d} \theta_j(k_j) \cdot k_j$. The Jacobian of the above system of Equations (A.1) is then given by $J = \begin{pmatrix} -1 & a_\theta \\ b_\theta & -1 \end{pmatrix}$. The eigenvalues of J are $-1 - \sqrt{a_\theta b_\theta}$ and $-1 + \sqrt{a_\theta b_\theta}$. Observe that: (1) if $a_\theta b_\theta < 1$ then both eigenvalues are negative, which implies that the state (d, h) is asymptotically stable, and (2) if $a_\theta b_\theta > 1$ then one of the eigenvalues is positive, which implies that the state (d, h) is unstable (see, e.g., Perko, 2013, Theorems 1–2 in Section 2.9).

A.5 Proof of Theorem 2 (Instability of Symmetric State: ASD)

Recall that the action-sampling dynamics in state (p_1, p_2) are given by

$$\dot{p}_1 = \delta(w_k(p_2) - p_1) \qquad \text{and} \qquad \dot{p}_2 = \delta(w_k(p_1) - p_2), \tag{A.2}$$

where, for brevity we omit the superscript A, i.e., we write $w_k \equiv w_k^A$.

Observe that a symmetric state (r, r) is a stationary state of the dynamics iff r is a fixed point of the function $w_k(p)$. Let $X_k(p)$ denote a binomial distribution with parameters k and p. Let $m = \left\lfloor \frac{kg}{1+g-l} \right\rfloor$. Note that the possible values of m are $\{0, 1, \ldots, k-1\}$. To make the dependence of the function $w_k(p)$ on m explicit, we write as follows:

$$w_k(p) \equiv w_{k,m}(p) = P(X_k(p) \le m) = F(m; k, p), \tag{A.3}$$

where $F(\cdot\ ; k, p)$ is the cumulative distribution function of a binomial distribution with parameters k and p. For all m, it follows that $w_{k,m}(0) = 1, w_{k,m}(1) = 0$, and $w_{k,m}(p)$ is decreasing in p, which implies that $w_{k,m}(p)$ has a unique interior fixed point $p^{(m)}$.

In order to assess the asymptotic stability, we compute the Jacobian J of Eq. (A.2) at the symmetric rest point $(p^{(m)}, p^{(m)})$ (ignoring the constant δ, which plays no role in the dynamics, except multiplying the speed of convergence by a constant):

$$J = \begin{pmatrix} -1 & w'_{k,m}(p^{(m)}) \\ w'_{k,m}(p^{(m)}) & -1 \end{pmatrix}.$$

The eigenvalues of J are $-1 + w'_{k,m}(p^{(m)})$ and $-1 - w'_{k,m}(p^{(m)})$. A sufficient condition for instability at $(p^{(m)}, p^{(m)})$ is that $|w'_{k,m}(p^{(m)})| > 1$.

From Eq. (A.3), we now compute as follows:

$$\begin{aligned} w_{k,k-m-1}(1-p) + w_{k,m}(p) &= P(X_k(1-p) \leq k-m-1) + P(X_k(p) \leq m) \\ &= P(X_k(p) \geq k-(k-m-1)) + P(X_k(p) \leq m) \\ &= P(X_k(p) \geq m+1) + P(X_k(p) \leq m) = 1. \end{aligned}$$

The fact that $w_{k,k-m-1}(1-p) = 1 - w_{k,m}(p)$ implies that $p^{(k-m-1)} = 1 - p^{(m)}$ and $w'_{k,k-m-1}(p^{(k-m-1)}) = w'_{k,m}(p^{(m)})$. Without loss of generality, we can therefore focus on analyzing the cases of m for which $m \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. To ease notation, we fix $k \geq 2$ and define a new function $f_m : [0,1] \to [0,1]$ as $f_m(p) \equiv w_{k,m}(p)$ and let $r_m = p^{(m)}$ be the fixed point of the function $f_m(p)$, i.e., $f_m(r_m) = r_m$. Since $f_m(0) = 1$, $f_m(1) = 0$, and $f_m(\cdot)$ is a strictly decreasing function, it follows that the fixed point $r_m \in (0,1)$ and that it is unique. To complete the proof we need to show that $|f'_m(r_m)| > 1$ for nonnegative integer values of m such that $m \leq \frac{k-1}{2}$. In what follows, we show this.

It is well known (see, e.g., Green, 1983) that

$$f_m(p) \equiv w_{k,m}(p) = \sum_{i=0}^{m} \binom{k}{i} p^i (1-p)^{k-i} = \binom{k}{m}(k-m)\int_0^{1-p} t^{k-m-1}(1-t)^m dt.$$

We now compute as follows:

$$\begin{aligned} f'_m(p) &= -\binom{k}{m}(k-m)p^m(1-p)^{k-m-1} = -k\binom{k-1}{m}p^m(1-p)^{k-m-1} \qquad \text{(A.4)} \\ f''_m(p) &= \binom{k}{m}(k-m)p^{m-1}(1-p)^{k-m-2}((k-m-1)p - m(1-p)). \end{aligned}$$

Fix an m such that $1 \leq m \leq \left\lfloor \frac{k-1}{2} \right\rfloor$. From the above computations, it follows that the function $f_m(p)$ is concave for values of $p \leq p^*$ and convex for values of $p \geq p^*$ where

$p^* = \frac{m}{k-1}$. This is because $f''_m(p^*) = 0$. Either the concave part or the convex part of the function $f_m(p)$ intersects the 45° line. Suppose that the concave part of the function $f_m(p)$ intersects the 45° line from the origin, i.e., $r_m \leq \frac{m}{k-1}$. In this case, $f_m(p)$ intersects the $-45°$ line joining the points (1,0) and (0,1) at q^*, where $q^* < r_m$. This is because $m \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ implies that $r_m \leq 0.5$ as $f_m(0.5) \leq 0.5$ for such values of m. Since the function $f_m(p)$ intersects the $-45°$ line from above, we can conclude that $f'_m(q^*) < -1$. The function $f_m(p)$ is concave between q^* and r_m; therefore, we have $f'_m(r_m) < f'_m(q^*) < -1$, i.e., $|f'_m(r_m)| > 1$. Therefore we are done in cases where $r_m \leq \frac{m}{k-1}$.

If the convex part of the function intersects the 45° line from the origin, then $r_m > m/(k-1)$. By definition, we have

$$r_m = (1 - r_m)^k + \binom{k}{1} r_m (1 - r_m)^{k-1} + \cdots + \binom{k}{m} r_m^m (1 - r_m)^{k-m} \tag{A.5}$$

For $j = 0, 1, 2, \ldots, m$, let a_j denote the j^{th} term of the sum on the RHS of Eq. (A.5). For $j = 1, 2, \ldots, m$, we compute as follows:

$$\frac{a_j}{a_{j-1}} = \frac{\binom{k}{j} r_m^j (1 - r_m)^{k-j}}{\binom{k}{j-1} r_m^{j-1} (1 - r_m)^{k-j+1}} = \left(\frac{k - j + 1}{j}\right)\left(\frac{r_m}{1 - r_m}\right)$$

$$\frac{a_j}{a_{j-1}} \geq 1 \iff (k - j + 1) r_m \geq j(1 - r_m) \iff r_m \geq \frac{j}{k+1}.$$

Since $\frac{m}{k-1} > \frac{j}{k+1}$ for $j = 1, 2, \ldots, m$, we have $r_m \geq \frac{j}{k+1}$ and thus $a_j \geq a_{j-1}$. This implies that $a_j \leq a_m$ for $j = 1, 2 \ldots, m - 1$. From Eq.(A.5), we can thus conclude the following:

$$r_m \leq (m + 1) \binom{k}{m} r_m^m (1 - r_m)^{k-m}. \tag{A.6}$$

From Eqs. (A.4) and (A.6), we have

$$|f'_m(r_m)| = (k - m) \binom{k}{m} r_m^m (1 - r_m)^{k-m-1} \geq \left(\frac{k - m}{m + 1}\right) \frac{r_m}{1 - r_m}.$$

From the above set of equations, a sufficient condition for $|f'_m(r_m)| > 1$ can be written as follows:

$$\left(\frac{k - m}{m + 1}\right) \frac{r_m}{1 - r_m} > 1 \iff r_m > \frac{m + 1}{k + 1}. \tag{A.7}$$

We will now establish that $|f'_m\left(\frac{m+1}{k+1}\right)| > 1$. From Eq. (A.4), we have

$$|f'_m(p)| = k\binom{k-1}{m}p^m(1-p)^{k-m-1} = k \cdot \Pr(X_{k-1}(p) = m).$$

where $X_{k-1}(p)$ is a binomial distribution with parameters $k-1$ and p. It is a known fact that the binomial distribution's mode with parameters $k-1$ and p is attained at $\lfloor kp \rfloor$. For $p = \frac{m+1}{k+1}$, we have

$$m \leq k \cdot \left(\frac{m+1}{k+1}\right) < m + 1.$$

The above inequalities imply that $\left\lfloor k \cdot \left(\frac{m+1}{k+1}\right)\right\rfloor = m$. The binomial distribution $X_{k-1}(\cdot)$ has k possible values and thus the probability of the occurrence of the mode has to be greater than $\frac{1}{k}$, i.e.,

$$\Pr\left(X_{k-1}\left(\frac{m+1}{k+1}\right) = m\right) > \frac{1}{k} \implies \left|f'_m\left(\frac{m+1}{k+1}\right)\right| > 1.$$

We need to consider the following two possible cases:

Case 1: $\frac{m}{k-1} < r_m \leq \frac{m+1}{k+1}$. For $1 \leq m \leq \frac{k-1}{2}$, we know that $|f'_m(\cdot)|$ attains its maximum value at $\frac{m}{k-1}$ and that it is strictly decreasing for $p > \frac{m}{k-1}$. For $m = 0$, $|f'_m(\cdot)|$ is strictly decreasing for $p > 0$. Thus, we have:

$$|f'_m(r_m)| > \left|f'_m\left(\frac{m+1}{k+1}\right)\right| > 1.$$

Case 2: $r_m > \frac{m+1}{k+1}$. Here, we are done by the sufficient condition of Eq. (A.7).

A.6 Proof of Theorem 3 (Stability of Symmetric State: ASD)

Let $a = \frac{g}{1+g-l}$. Under the assumption of generic environment, $a \neq \frac{1}{2}$. From Eq. (A.4), we have

$$|w'_{k,m}(p)| \equiv |f'_m(p)| = k\binom{k-1}{m}p^m(1-p)^{k-m-1} \Rightarrow |w'_{k,m}(p)| = k \cdot \Pr\left(X_{k-1} = m\right),$$

where $m = \lfloor ka \rfloor$ and $X_{k-1} \sim Bin\left(k-1, p\right)$.

Our first step is to show that if $p = 0.5$ then $|w'_{k,m}(p)|$ converges to zero as k goes to infinity. So let V be a sequence defined by: $V_k = k \cdot \Pr\left(X_{k-1} = m\right)$, and $p = 0.5$.

We assume that a is a rational number,[17] and specifically, $a = n_1/n_2$, where n_1 and n_2 are integers. By assumption $a < 1$ and therefore $n_1 < n_2$. Our strategy is to divide V into n_2 disjoint sub-sequences whose union is exactly V, and show that each one of them convergences to zero. This completes the proof that V convergences to zero.

We define these sub-sequences as follows: for each value of $i \in \{1, \dots n_2\}$, the elements of a sub-sequence V^i are $V^i_k = V_{i+k\cdot n_2}$. We would like to show that for each such i,

$$\lim_{k\to\infty} \frac{V^i_k}{V^i_{k+1}} = \lim_{k\to\infty} \prod_{j=0}^{n_2-1} \frac{V_{i+k\cdot n_2+j}}{V_{i+k\cdot n_2+j+1}} < 1.$$

Let m and j be two integers, and let

$$U_{m+j} \equiv \frac{V_{m+j}}{V_{m+j+1}} = \frac{(m+j)\binom{m+j-1}{\lfloor (m+j)a\rfloor}0.5^{m+j-1}}{(m+j+1)\binom{m+j}{\lfloor (m+j+1)a\rfloor}0.5^{m+j}}.$$

Given some integer m, let

$$S = \left\{ \lfloor (m+j+1)a\rfloor - \lfloor (m+j)a\rfloor \,\middle|\, j \in \{1, 2, \dots, n_2\} \right\},$$

be a sequence of n_2 numbers, and note that S contains only zeros and ones. We claim that S contains n_1 ones and $n_2 - n_1$ zeros. To see that, take a sequence of n_2+1 numbers, of $n_1 + 1$ different consecutive integers (with duplicates) ordered by size, and consider the sequence of differences between any two adjacent numbers. This sequence contains n_2 elements, with n_1 out of them equal to one, and $n_2 - n_1$ elements equal to zero. The sequence S can be defined exactly this way. Indeed, for values of j in $(1, 2, \dots, n_2, n_2+1)$, the expression $\lfloor (m+j)a\rfloor$ takes each value between $\lfloor (m+1)a\rfloor$ to $\lfloor (m+n_2+1)a\rfloor$, i.e., n_1+1 different values, and the n_2 elements of S are exactly the differences between each two adjacent elements.

For values of j for which the corresponding value of S is zero we have

$$U_{m+j} = 2\frac{m+j}{m+j+1} \cdot \frac{\lfloor (m+j+1)a\rfloor}{m+j} = 2\frac{\lfloor (m+j+1)a\rfloor}{m+j+1},$$

[17] If a is irrational, we can apply the same method of proof by taking a rational number that is close enough to a.

and when the corresponding value is one we have

$$U_{m+j} = 2\frac{m+j}{m+j+1} \cdot \frac{m+j+1-\lfloor (m+j+1)a \rfloor}{m+j} = 2\frac{m+j+1-\lfloor (m+j+1)a \rfloor}{m+j+1}.$$

Since $\lim_{m\to\infty} \frac{\lfloor (m+j+1)a \rfloor}{m+j+1} = a$ and $\lim_{m\to\infty} \frac{m+j+1-\lfloor (m+j+1)a \rfloor}{m+j+1} = 1-a$, we have

$$\begin{aligned}\lim_{k\to\infty} \frac{V_k^i}{V_{k+1}^i} = \lim_{k\to\infty} \prod_{j=0}^{n_2-1} U_{i+k\cdot n_2+j} &= 2^{n_2} \cdot a^{(1-a)\cdot n_2} \cdot (1-a)^{a\cdot n_2} \\ &= \left(2 \cdot a^{1-a} \cdot (1-a)^a\right)^{n_2} < 1.\end{aligned}$$

It is smaller than one because the function $a^{1-a} \cdot (1-a)^a$ takes its maximum at $a = 0.5$ in which its value is 0.5. By the assumption of generic environment, $a \neq 0.5$. This completes the first stage of the proof.

The first step implies that there is $\hat{k}$ such that $|w_k'(0.5)| < 0.5$ for any $k \geq \hat{k}$. By continuity, there exists $\epsilon > 0$, such that $|w_k'(p)| < 1$ for any $p \in [0.5-\epsilon, 0.5+\epsilon]$ and any $k \geq \hat{k}$. This implies that $|w_\theta{}'(p)| < 1$ for any $p \in [0.5-\epsilon, 0.5+\epsilon]$ and any distribution θ satisfying $\theta(k) = 0$ for each $1 < k < \hat{k}$ and $\theta(1) < 1$. The fact that the fixed point of $w_1(p)$ is 0.5 implies that there exists $\hat{q} < 1$ such that the fixed point of $w_\theta(p)$ is in the interval $[0.5-\epsilon, 0.5+\epsilon]$ for any distribution θ satisfying $\theta(1) > \hat{q}$. This, in turn, implies that $|w_k'(p)| < 1$ at the fixed point, which implies that the symmetric stationary state is asymptotically stable.

A.7 Proof of Theorem 4 (Stability of Symmetric State: PSD)

Recall that the payoff-sampling dynamics in state (p_1, p_2) are given by

$$\dot{p}_1 = \delta(w_\theta(p_2) - p_1) \qquad \text{and} \qquad \dot{p}_2 = \delta(w_\theta(p_1) - p_2), \tag{A.8}$$

where, for brevity we omit the superscript P, i.e., we write $w_\theta \equiv w_\theta^P$.

Observe that a symmetric state $(p^{(\theta)}, p^{(\theta)})$ is a stationary state of the dynamics iff $p^{(\theta)}$ is a rest point of w_θ, i.e., if $w_\theta(p^{(\theta)}) = p^{(\theta)}$. Such a rest point $p^{(\theta)}$ exists because $w_\theta(1) = 0$, $w_\theta(0) = 1$, and $w_\theta(\cdot)$ is continuous on $[0,1]$.

In order to assess the asymptotic stability, we compute the Jacobian J of Eq. (A.8) at the symmetric rest point $(p^{(\theta)}, p^{(\theta)})$ (ignoring the constant δ, which plays no role in the dynamics, except multiplying the speed of convergence by a constant):

$$J = \begin{pmatrix} -1 & w'_\theta(p^{(\theta)}) \\ w'_\theta(p^{(\theta)}) & -1. \end{pmatrix}.$$

The eigenvalues of J are $-1 + w'_\theta(p^{(\theta)})$ and $-1 - w'_\theta(p^{(\theta)})$. A sufficient condition for the asymptotic stability at $(p^{(\theta)}, p^{(\theta)})$ is therefore that $|w'_\theta(p^{(\theta)})| < 1$.

We now establish some properties of the payoff-sampling dynamics and the symmetric rest points for symmetric distributions of types $\theta \equiv k$. If $l, 1-g \in \left(0, \frac{1}{\max(\mathrm{supp}(\theta))}\right)$, action h_i has a higher mean payoff iff the number of d_j-s in the h_i-sample is strictly greater than half the number of d_j-s in the d_i-sample. To express $w_k(p)$ concisely in this case, we define $X_k(p)$ and $Y_k(p)$ to be independent and identically distributed binomial random variables with parameters k and p. We can then write $w_k(p)$ as follows:

$$w_k(p) = P\left(k - X_k(p) > \frac{1}{2}(k - Y_k(p))\right) = P(2X_k(p) - Y_k(p) < k). \tag{A.9}$$

Observe that $w_k(p)$ is a polynomial in p of degree at most $2 \cdot k$.

We have verified the following facts about these polynomials for $k < 20$ (for an illustration see Figure 5.3; the Mathematica code is given in the online supplementary material, and the explicit values of the rest points and the derivatives are presented in Table 3):

- For $k \in \{1, 2, \ldots, 18, 19\}$, $w_k(p)$ is decreasing in p.
- For $k \in \{1, 2, \ldots, 18, 19\}$, $w_k(p)$ has a unique fixed point $p^{(k)}$.
  Moreover, $0.5 < p^{(k)} < 0.68$ for any $k \in \{1, 2, 3, 4, 5\}$.
- $|w'_1(p)| \equiv 1$, and $|w'_k(p)| < 1$ for any $k \in \{2, 3, 4, 5\}$ and $0.5 < p < 0.68$.

Recall that $w_\theta(p)$ is a convex combination of the $w_k(p)$ for the k-s in its support (i.e., $w_\theta(p) = \sum_k \theta(k) \cdot w_k(p)$). From the above facts, it follows that:

1. For $\theta \equiv k < 20$, the function $w_k(p)$ has a unique fixed point $p^{(k)}$ such that $|w'_k(p^{(k)})| < 1$, which implies that $(p^{(k)}, p^{(k)})$ is asymptotically stable.
2. For $\max(\mathrm{supp}(\theta)) \leq 5$, the function $w_\theta(p)$ has a unique fixed point $p^{(\theta)}$ such that $p^{(\theta)} \in (0.5, 0.68)$ and $|w'_\theta(p^{(\theta)})| < 1$ if $\theta(1) \neq 1$, which implies that $(p^{(\theta)}, p^{(\theta)})$ is

asymptotically stable.

Table 3: Fixed Points of the Function $w_k(p)$ in the Proof of Theorem 4

k	1	2	3	4	5	6	7	8	9	10
$p^{(k)}$	0.500	0.579	0.620	0.649	0.672	0.690	0.706	0.720	0.731	0.741
$\lvert w'_k(p^{(k)})\rvert$	1	0.690	0.618	0.645	0.690	0.730	0.763	0.793	0.818	0.840

k	11	12	13	14	15	16	17	18	19	20
$p^{(k)}$	0.750	0.758	0.765	0.773	0.778	0.784	0.789	0.794	0.799	0.803
$\lvert w'_k(p^{(k)})\rvert$	0.861	0.88	0.899	0.916	0.932	0.948	0.963	0.978	0.991	1.001

Table 4: Values of $|w'_k(p^{(j)})|$ for $k, j \in \{1, 2, 3, 4, 5\}$.

$k \backslash p^{(j)}$	$p^{(1)}$	$p^{(2)}$	$p^{(3)}$	$p^{(4)}$	$p^{(5)}$
1	1	1	1	1	1
2	0.5	0.690	0.812	0.905	0.981
3	0.562	0.560	0.618	0.687	0.759
4	0.625	0.616	0.623	0.645	0.679
5	0.605	0.642	0.659	0.673	0.690

References

Arieli, I., Y. Babichenko, R. Peretz, and H. P. Young (2020). The speed of innovation diffusion in social networks. *Econometrica 88*(2), 569–594.

Arigapudi, S., Y. Heller, and I. Milchtaich (2021). Instability of defection in the prisoner's dilemma under best experienced payoff dynamics. *Journal of Economic Theory 197*, 105174.

Aumann, R. J. (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica 55*(1), 1–18.

Baliga, S. and T. Sjöström (2012). The strategy of manipulating conflict. *American Economic Review 102*(6), 2897–2922.

Baliga, S. and T. Sjöström (2020). The strategy and technology of conflict. *Journal of Political Economy 128*(8), 3186–3219.

Benndorf, V., I. Martinez-Martinez, and H.-T. Normann (2016). Equilibrium selection with coupled populations in hawk–dove games: Theory and experiment in continuous time. *Journal of Economic Theory 165*, 472–486.

Benndorf, V., I. Martínez-Martínez, and H.-T. Normann (2021). Games with coupled populations: An experiment in continuous time. *Journal of Economic Theory*, 105281.

Bilancini, E., L. Boncinelli, S. Ille, and E. Vicario (2021). Memory length and harshness of conflict in the hawk–dove game. *Mimeo.*

Bornstein, G., D. Budescu, and S. Zamir (1997). Cooperation in intergroup, n-person, and two-person games of chicken. *Journal of Conflict Resolution 41*(3), 384–406.

Brams, S. J. and D. M. Kilgour (1987). Threat escalation and crisis stability: A game-theoretic analysis. *American Political Science Review 81*(3), 833–850.

Brams, S. J. and D. M. Kilgour (2001). Fallback bargaining. *Group Decision and Negotiation 10*(4), 287–316.

Cárdenas, J., C. Mantilla, and R. Sethi (2015). Stable sampling equilibrium in common pool resource games. *Games 6*(3), 299–317.

Cason, T. N., D. Friedman, and E. Hopkins (2014). Cycles and instability in a rock–paper–scissors population game: A continuous time experiment. *Review of Economic Studies 81*(1), 112–136.

Dixit, A., D. McAdams, and S. Skeath (2019). "We haven't got but one more day": The Cuban missile crisis as a dynamic chicken game. *Available at SSRN 3406265*.

Green, D. N. (1983). Fixed point properties of the binomial function. *Journal of the Franklin Institute 316*(1), 51–62.

Hauert, C. and J. Miekisz (2018). Effects of sampling interaction partners and competitors in evolutionary games. *Physical Review E 98*(5), 052301.

Heller, Y. and E. Mohlin (2018). Social learning and the shadow of the past. *Journal of Economic Theory 177*, 426–460.

Herold, F. and C. Kuzmics (2020). The evolution of taking roles. *Journal of Economic Behavior & Organization 174*, 38–63.

Hoffman, M., S. Suetens, U. Gneezy, and M. A. Nowak (2015). An experimental investigation of evolutionary dynamics in the rock–paper–scissors game. *Scientific Reports 5*(1), 1–7.

Izquierdo, L. R., S. S. Izquierdo, and W. H. Sandholm (2019). An introduction to ABED: Agent-based simulation of evolutionary game dynamics. *Games and Economic Behavior 118*, 434–462.

Kreindler, G. E. and H. P. Young (2013). Fast convergence in evolutionary equilibrium selection. *Games and Economic Behavior 80*, 39–67.

Lipnowski, I. and S. Maital (1983). Voluntary provision of a pure public good as the game of "chicken". *Journal of Public Economics 20*(3), 381–386.

Mantilla, C., R. Sethi, and J. C. Cárdenas (2018). Efficiency and stability of sampling equilibrium in public goods games. *Journal of Public Economic Theory 22*(2), 355–370.

Maynard-Smith, J. (1982). *Evolution and the Theory of Games.* Cambridge University Press.

Maynard-Smith, J. and G. A. Parker (1976). The logic of asymmetric contests. *Animal Behaviour 24*(1), 159–175.

Oprea, R., K. Henwood, and D. Friedman (2011). Separating the hawks from the doves: Evidence from continuous time laboratory games. *Journal of Economic Theory 146*(6), 2206–2225.

Osborne, M. J. and A. Rubinstein (1998). Games with procedurally rational players. *American Economic Review 88*(4), 834–847.

Osborne, M. J. and A. Rubinstein (2003). Sampling equilibrium, with an application to strategic voting. *Games and Economic Behavior 45*(2), 434–441.

Oyama, D., W. H. Sandholm, and O. Tercieux (2015). Sampling best response dynamics and deterministic equilibrium selection. *Theoretical Economics 10*(1), 243–281.

Perko, L. (2013). *Differential Equations and Dynamical Systems*, Volume 7. Berlin: Springer Science & Business Media.

Rapoport, A. and A. M. Chammah (1966). The game of chicken. *American Behavioral Scientist 10*(3), 10–28.

Salant, Y. and J. Cherry (2020). Statistical inference in games. *Econometrica 88*(4), 1725–52.

Sandholm, W. H. (2001). Almost global convergence to p-dominant equilibrium. *International Journal of Game Theory 30*(1), 107–116.

Sandholm, W. H. (2010). *Population Games and Evolutionary Dynamics.* Cambridge, MA: MIT Press.

Sandholm, W. H., S. S. Izquierdo, and L. R. Izquierdo (2019). Best experienced payoff dynamics and cooperation in the centipede game. *Theoretical Economics 14*(4), 1347–85.

Sandholm, W. H., S. S. Izquierdo, and L. R. Izquierdo (2020). Stability for best experienced payoff dynamics. *Journal of Economic Theory 185*, 104957.

Sawa, R. and J. Wu (2021). Statistical inference in evolutionary dynamics. *SSRN 3767635*.

Selten, R. and T. Chmura (2008). Stationary concepts for experimental 2x2-games. *American Economic Review 98*(3), 938–66.

Sethi, R. (2000). Stability of equilibria in games with procedurally rational players. *Games and Economic Behavior 32*(1), 85–104.

Sethi, R. (2021). Stable sampling in repeated games. *Journal of Economic Theory 197*, 105343.

Spiegler, R. (2006). Competition over agents with boundedly rational expectations. *Theoretical Economics 1*(2), 207–231.

Stephenson, D. (2019). Coordination and evolutionary dynamics: When are evolutionary models reliable? *Games and Economic Behavior 113*, 381–395.

Sugden, R. (1989). Spontaneous order. *Journal of Economic Perspectives 3*(4), 85–97.

Taylor, P. D. (1979). Evolutionarily stable strategies with two types of player. *Journal of Applied Probability 16*, 76–83.

Weibull, J. W. (1997). *Evolutionary Game Theory*. Cambridge, MA:MIT press.

Young, H. P. (1993). The evolution of conventions. *Econometrica 61*(1), 57–84.