Capital and inventory investments under quantity constraints: A microfounded Metzlerian model

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Abstract

Although modern analyses of inventory dynamics identify the implicit roles of inventory holding, they do not entirely address its primary role. This study analyzes the correlation between the current quantity constraints and choices in intertemporal optimizations by modeling inventory dynamics. The economy's quantity constraints are reflected in a representative firm's optimization of its current employment and investment based on the perceived sales constraints. Using analytical and numerical methods, the results show that the Metzlerian cycle of inventory and sales expectations survives under the canonical intertemporal optimization framework, as the stable inventory-to-sales ratio is reproduced. Additionally, this Metzlerian model illustrates the qualitative aspects of inventory dynamics in the business cycle. However, the quantitative aspects are not reproduced through the optimization framework because the firm's control is based on a stationary growth path, thereby weakening the effects of sales expectation dynamics.

Keywords: inventory dynamics, inventory holding, quantity constraints, Metzlerian dynamics

1 Introduction

According to Blinder and Maccini (1991), inventory dynamics play a prominent role in business cycles. Specifically, inventory exhibits a persistent tendency to be more sensitive to excess demand fluctuations than to production (see Figure 1). Figure 1 shows the cyclical components of the business cycle in the United States (US) for 1960Q1–2020Q4.[1] The data are regulated using a Hodrick–Prescott filter with a smoothing parameter of 1600, which is suitable for quarterly periodicity. The dashed line represents the cyclical component of real production. Sales dynamics seem to be slightly more sensitive than production dynamics.[2]

The inventory dynamics confirm this, as the cyclical component of inventory lags behind that of production (see Figure 2). This implies that during the boom, production

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[1]The datasets have been created using the method available at `https://fairmodel.econ.yale.edu/`; see Appendix B. This method was also utilized by Chiarella et al. (2005). I use it again in the numerical experiments that follow.

[2]Their variances show similar results: the variance of the cyclical component of production is $2.50 \cdot 10^{-04}$ but that of its sales counterpart is $1.80 \cdot 10^{-04}$.

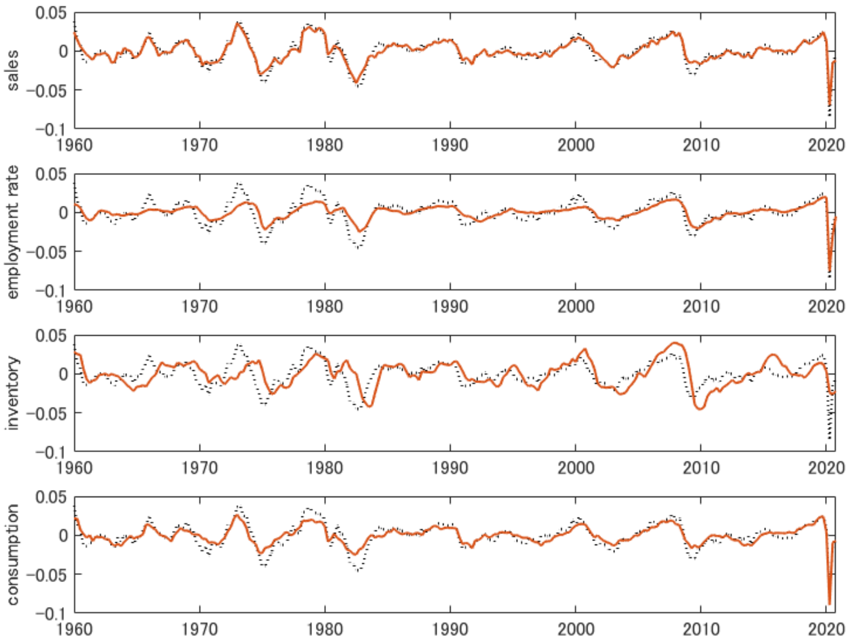

Figure 1: The cyclical components of sales, employment rate, inventory stock, and consumption

exceeds sales; thus, in the next period, unsold goods temporarily accumulate as inventory stock. Blinder and Maccini (1991) indicated that the existence of inventory may destabilize economic dynamics, contrary to intuition, and this is confirmed by the data. Although the classic study on inventory dynamics by Metzler (1941) demonstrated the destabilizing effect, Blinder claimed that the production-smoothing model, which resembled Metzler's, was unrealistic; instead, he recommended what is termed the (S, s) model, as mentioned by Fisher and Hornstein (2000).

Several modern inventory dynamics analyses, such as Bils and Kahn (2000) and Sarte et al. (2015), have utilized speculative motive in the intertemporal optimization framework to explain their findings. As inventory stock represents goods, the expected price and wage dynamics control inventory fluctuations. Assuming that capital and inventory are substitutable for production and the timing of revealing preferences and productivity parameters is distorted in decision-making, Christiano (1988) demonstrated that the extent to which the model reproduces procyclical inventory fluctuations depends on the estimation method used. Rotemberg and Saloner (1989) built a duopoly model in which the firm holds excess inventory when the goods demand is high so as to punish its possibly uncooperative rival in the next period. Although these models successfully reveal the implicit roles of inventory holding, they do not entirely address its primary role; the presence of inventory offsets the unexpected quantity constraint that emerges as a sales fluctuation.

On the contrary, according to Keynesian economists, Metzlerian dynamics play a central role in the business cycle (Franke (1996); Wegener et al. (2009); Grasselli and Nguyen-Huu (2018)). Chiarella et al. (2005) introduced the Keynes–Metzler–Goodwin (KMG) model to explain the growth cycle mechanism. In Metzlerian dynamics, the firm has an expectation regarding the current sales and determines the production level to

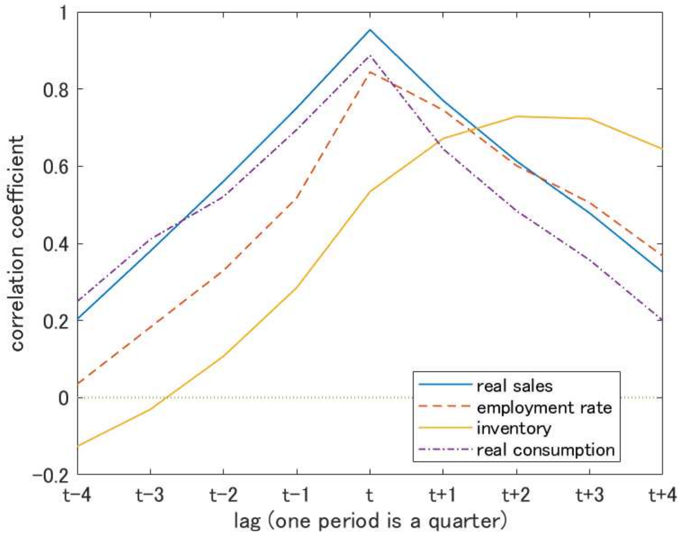

Figure 2: Correlation coefficients for the cyclical components of lagged economic variables and real production

maintain the desired inventory–sales ratio, which is usually assumed to be fixed ad hoc; the production associated with the fixed ratio is a source of the instability in Metzlerian dynamics, as the inventory holding stock is not related to production smoothing. Indeed, the inventory–sales ratio is relatively stable (see Figure 3). Although the ratio decreases with progress in inventory holding technology, the scale of the change is not large. However, traditional Keynesian models do not explain the firm's preference for fixing the ratio.

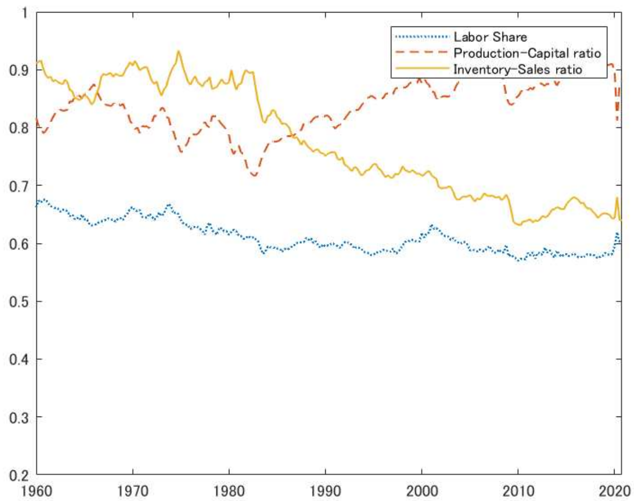

Figure 3: Labor share, production–capital ratio, and inventory–sales ratio (calculated using quarterly data; the production–capital ratio is multiplied by four)

In economic dynamics, inventory is a residual resulting from the gap between planned and actual sales. Its existence allows the inconsistency of one period's trades in model analysis, and the inventory adjustment process may explain fluctuations. Although Blinder and Maccini (1991) denied the stabilization effect of inventory in dynamics, the utility of holding inventory stock itself is undeniable; inventory dynamics can offset the flow disequilibrium between production and sales. An important question is whether the potential production-smoothing (and offsetting-disequilibrium) role of inventory and its procyclical movement can coexist in a model. Therefore, the role of inventory in disequilibrium dynamics must be further evaluated.

The (general) disequilibrium model was constructed by Barro and Grossman (1971), who were inspired by the quantity adjustment theory described by Clower (1965) and Leijonhufvud (1968). Bénassy (1975) also built a model to define the K-equilibrium, a possible interpretation of the effective demand principle proposed by Keynes (1936).[3] The disequilibrium model allows transactions under excess demand or supply, and the gap between the planned and actual transactions (quantity constraint) affects individuals' decisions. This spillover effect, termed the *dual-decision hypothesis* by Clower (1965), induces a persistent demand shortage; the low demand for goods induces a correspondingly low labor demand, and vice versa. In disequilibrium economics, the economy is characterized by excess demand or supply in the goods and labor markets. Specifically, the Keynesian unemployment regime (excess supply in both goods and labor markets) has been the focus of macroeconomic studies, such as Chiarella and Flaschel (2000) and Chiarella et al. (2000). Despite few analyses since the 1990s (Backhouse and Boianovsky, 2012), disequilibrium economics has increasingly attracted attention in macroeconomic research in the context of secular stagnation (Mankiw and Weinzierl (2011); Michaillat and Saez (2015); Eggertsson et al. (2019); Dupor et al. (2019); Ogawa (2019)).

In the field of disequilibrium economics, Blinder (1980, 1981) argued that the existence of a buffer stock of goods (inventory) would moderate the spillover effect in the near absence of regime switching. Existing disequilibrium analyses such as Honkapohja and Ito (1980), Simonovits (1982), and Eckalbar (1985) show that regime switching occurs only when inventory stockouts occur, as regimes are defined as the relationships among the traded quantities. However, the disequilibrium analyses in these models are incomplete as they do not account for individuals' decision-making; the disequilibrium between planned and actual transactions is relevant for economic dynamics. Thus, decision-making should be discussed based on the perception of quantity constraints rather than stockouts.

In this study, I formulate the agents' behaviors to solve intertemporal optimization problems. The agents perceive the quantity constraint as shown by Yoshikawa (1984) and Murakami (2015). In this aspect, the present model is based in the Keynesian unemployment situation of disequilibrium economics. In static disequilibrium models, agents perceive quantity constraints on their current trades. In this dynamic model, the expected future constraints are used to assess the current constraint. Despite adaptive expectations, it is the firm that decides on investment and employment and the household that plans consumption. This premise provides scope for future research on the intertemporal spillover effect.

By introducing a dynamic optimization problem into a disequilibrium model, I find that the "dynamic optimization" framework affects the disequilibrium dynamics. Stock disequilibrium accelerates the dynamics by adjusting the optimal solution. Maintaining the objective value of the sales–inventory ratio destabilizes the dynamics in ordinal Metzlerian models. Although the present model with dynamic optimization does not assume this feature, the Metzlerian dynamics partially survive. The firm chooses production and

[3] Korliras (1975) and Malinvaud (1977, 1980) emphasized the persistence of the Keynesian situation. For a survey of these economic concepts, see Backhouse and Boianovsky (2012).

investment so as to ride on the saddle path toward the objective values and keep the objective value of the sales–inventory ratio stable relative to the actual ratio (that is, the firm aims to maintain a constant ratio). However, this gradual adjustment stabilizes the Metzlerian dynamics, thereby weakening the cyclical behavior.

The remainder of this paper is organized as follows. In Section 2, the firm's and household's intertemporal optimization problems are formulated, and the perceived quantity constraints are introduced in simple form. In Section 3, each decision is synthesized using a macrodynamic model. By constructing the dynamical system, it is shown that Metzlerian dynamic feedback exists, but the feedback loop is weakened by dynamic optimization. Section 4 presents numerical experiments to confirm the model analyses. The results are qualitatively suitable, but inadequate in terms of quantitative analysis of the business cycle. Section 5 discusses the setting of expectations in disequilibrium dynamics to potentially resolve this situation. Section 6 concludes.

2 The model

The present model excludes the monetary aspects of the economy. A representative firm produces goods using capital and labor and determines its investment. Homogeneous households determine their respective consumption plans, facing a quantity constraint on employment (and realized income flow).

Time is continuous, and the flow of time at each instant is outlined as follows. (1) The firm plans its investment and production for a given expected sales constraint. (2) It produces goods using capital and labor and pays dividends and wages to households. The income flow equals the production. (3) Households receive income and plan consumption based on the expected stream of future income. (4) The realized sales are the investment decided upon by the firm *plus* the consumption decided upon by the households, and the gap between production and sales represents the current accumulation of inventory stock. (5) The firm revises the sales constraint following the adaptive adjustment rule.

2.1 The firm's employment and investment decisions

The representative firm produces goods using capital $K(\tau)$ and employed labor $E(\tau)$. The production technology is formulated as the standard production function $F(K, E)$, which satisfies

$$\frac{\partial F}{\partial X} > 0, \ \frac{\partial^2 F}{\partial X^2} < 0, F(0,0) = 0. \ \ X = K, E. \tag{1}$$

In this study, F is assumed to be linear and homogeneous.

As the firm faces a quantity constraint on sales $Y^d(\tau)$ in every period, the inventory holding $N(\tau)$ fluctuates as follows:

$$\dot{N}(\tau) = F(K(\tau), E(\tau)) - Y^d(\tau) - \delta_N N(\tau), \tag{2}$$

where $\delta_N > 0$ is the constant inventory depreciation rate.

The firm owns capital $K(\tau)$ and further accumulates capital. Capital accumulation costs and the marginal cost per existing unit of capital increase at the scale

$$\phi(z(\tau))K(\tau), \ \text{where } \phi' > 0, \ \phi'' > 0, \ \phi(0) = 0, \ \phi'(0) = 1, \tag{3}$$

where $z(\tau) = \dot{K(\tau)}/K(\tau) + \delta_K$ is the gross capital accumulation rate, and $\delta_K > 0$ is the depreciation rate of capital.

Inventory holding involves costs. This study assumes that the firm avoids inventory stockout, as argued by Metzler (1941) and Lovell (1962).[4] The potential costs of inventory stockout are excessively high. The perceived unit cost function is described as a function of the inventory per unit of capital stock, as the capital stock shows the production capacity $\tilde{\psi}(N(\tau)/K(\tau))$, such that

$$\lim_{x\to 0}\tilde{\psi}(x)x = \infty, \lim_{x\to\infty}\tilde{\psi}(x)x = 0, (\tilde{\psi}(x)x)' < 0, (\tilde{\psi}(x)x)'' > 0. \tag{4}$$

These assumptions are difficult to explain intuitively; they are imposed to ensure a solution to the firm's optimization problem. The variable is the ratio N/K and not the quantity of stock. It is evident that the stockout-avoidance motive features prominently in the firm's decision. Notably, the aggregate cost function may be U-shaped. As the present analysis is limited to the neighborhood of the economy's steady state, the above-mentioned assumption can be interpreted as a condition around the steady state.

The real profit rate per unit of capital stock, $\rho(\tau)$, is then described as follows:

$$P(\tau)\rho(\tau)K(\tau) = P(\tau)Y^d(\tau) - W(\tau)E(\tau) - P(\tau)\tilde{\psi}(N(\tau)/K(\tau))N(\tau) - P(\tau)\phi(z(\tau))K(\tau). \tag{5}$$

where $P(\tau)$ is the price of the goods, and $W(\tau)$ is the nominal wage. It should be noted that the inventory holding cost function includes the cost of possible stockout, such that ρ does not equal the actual profit rate. However, this difference is not relevant because it is the perceived profitability that controls the firm's decision, and income distribution is not key to this model.

The firm's nominal value in the initial period, $V(t)$, is defined as follows:

$$V(t)/P(t) = \int_{t=0}^{\infty} \rho(\tau)K(\tau)e^{-\int_{s=0}^{t}(r(s)-\pi(s))\,ds}\,dt \tag{6}$$

where $r(\tau)$ and $\pi(\tau)$ are the nominal interest rate and inflation rate, respectively.

Assumption 1. *The firm's expectation is static in that the sales constraint per unit of capital stock $Y^d(\tau)/K(\tau)$ is the same as the expected value in the initial period $\tilde{y}^e(t) = Y^e(t)/K(t)$, where Y^e is the expected sales amount; moreover, the nominal interest rate and the price and wage inflation rates are constant: $r(\tau) = r(t)$ and $\dot{P}(\tau)/P(\tau) = \dot{W}(\tau)/W(\tau) = \pi(t)$ for all $\tau \geq t$.*

In the following firm optimization problem, the current time t is omitted because it is assumed to be constant for the firm's time horizon.

The firm maximizes its real value by controlling employment and capital investment:

$$\max_{\{\tilde{e}(\tau),z(\tau)\}_{t=0}^{\infty}} \int_{t=0}^{\infty} [\tilde{y}^e - w\tilde{e}(\tau) - \psi(\tilde{n}(\tau)) - \phi(z(\tau))]K(\tau)e^{-(r-\pi)t}\,dt \tag{7}$$

$$\text{subject to } \dot{K}(\tau) = [z(\tau) - \delta_K]K(\tau), \tag{8}$$

$$\dot{\tilde{n}}(\tau) = f(\tilde{e}(\tau)) - \tilde{y}^e - [z(\tau) - \delta_K + \delta_N]\tilde{n}(\tau), \tag{9}$$

[4] Wen (2011) upheld the stockout-avoidance motive. Bo (2001) empirically showed that Dutch firms hold excess inventory to avoid stockout.

where $\tilde{y}^e = Y(t)/K(t)$, $\tilde{e}(\tau) = E(\tau)/K(\tau)$, $\tilde{n}(\tau) = N(\tau)/K(\tau)$, and $\psi = \tilde{\psi}\tilde{n}$. $f(\tilde{e}(\tau)) = F(K(\tau), E(\tau))/K(\tau) = F(1, \tilde{e}(\tau))$ is the production per unit of capital stock.

The current value Hamiltonian is set as follows:

$$\begin{aligned} H(K(\tau), \tilde{e}(\tau), z(\tau), \tilde{n}(\tau), \lambda(\tau), \mu(\tau)) &= [\tilde{y}^e - w\tilde{e}(\tau) - \psi(\tilde{n}(\tau)) - \phi(z(\tau))]K(\tau) \\ &+ \lambda(\tau)[z(\tau) - \delta_K]K(\tau) + \mu(\tau)[f(\tilde{e}(\tau)) - \tilde{y}^e - [z(\tau) - \delta_K + \delta_N]\tilde{n}(\tau)] \end{aligned} \tag{10}$$

The first-order and transversality conditions are described as follows:

$$\mu(\tau) = \frac{w(\tau)K(\tau)}{f'(\tilde{e}(\tau))} \tag{11}$$

$$\lambda(\tau) = \phi'(z(\tau)) + \frac{\mu(\tau)\tilde{n}(\tau)}{K(\tau)} \tag{12}$$

$$\dot{\lambda}(\tau) = -\lambda(\tau)[z(\tau) - \delta_K - r + \pi] - \rho(\tau) \tag{13}$$

$$\dot{\mu}(\tau) = \mu(\tau)[z(\tau) - \delta_K + \delta_N - r + \pi] + \psi'(\tilde{n}(\tau))K(\tau) \tag{14}$$

$$\lim_{\tau\to\infty} \lambda(\tau)K(\tau)e^{-(r-\pi)\tau} = 0 \tag{15}$$

$$\lim_{\tau\to\infty} \mu(\tau)\tilde{n}(\tau)e^{-(r-\pi)\tau} = 0 \tag{16}$$

Combining the equations, the following three-dimensional dynamical system is obtained:

$$\dot{\tilde{e}}(\tau) = -\frac{f'(\tilde{e}(\tau))}{f''(\tilde{e}(\tau))}\left(\delta_N + r - \pi + \frac{f'(\tilde{e}(\tau))}{w}\psi'(\tilde{n}(\tau))\right) \tag{17}$$

$$\begin{aligned} -\phi''(z(\tau))\dot{z}(\tau) &= \frac{w}{f'(\tilde{e}(\tau))}\left(f(\tilde{e}(\tau)) - \tilde{y}^e + \tilde{n}(\tau)\psi'(\tilde{n}(\tau)) + \psi(\tilde{n}(\tau))\right) \\ &+ \phi'(z(\tau))(z(\tau) - \delta_K - r + \pi + \rho(\tau)) \end{aligned} \tag{18}$$

$$\dot{\tilde{n}}(\tau) = f(\tilde{e}(\tau)) - \tilde{y}^e - [z(\tau) - \delta_K + \delta_N]\tilde{n}(\tau) \tag{19}$$

The first equation represents the Euler equation for inventory control, in which the firm decides the employment level. $\delta_N + r - \pi$ can be interpreted as the depreciation rate of the inventory holding value; if the firm underevaluates the stock, the labor demand is likely to grow based on speculative motive.

The steady state of the variables $(\tilde{e}^*, z^*, \tilde{n}^*)$ is defined as the set for which the right-hand sides of all the abovementioned dynamical system equal zero. Notably, this steady state does not always represent the steady state of economy; it can be interpreted as the objective values of the variables for the firm.

For simplicity, it is assumed that the depreciation rates are common ($\delta_K = \delta_N = \delta$) and $\chi = \delta + r - \pi$ denotes the depreciation rate of capital in real terms.[5] The steady-state conditions are described as follows:

$$\chi + \frac{f'(\tilde{e}^*)}{w}\psi'(\tilde{n}(\tau)) = 0 \tag{20}$$

$$\frac{w}{f'(\tilde{e}^*)}(f(\tilde{e}^*) - \tilde{y}^e + \tilde{n}^*\psi'(\tilde{n}(\tau)) + \phi'(z^*)(z^* - \chi) + \rho^* = 0 \tag{21}$$

$$f(\tilde{e}^*) - \tilde{y}^e = z^*\tilde{n}^* \tag{22}$$

[5]The inventory stock for final sales is sometimes included in the production function in a manner similar to physical capital. For the second assumption, we consider the growth rate of the relative value of capital, which is measured as PK/B, where B is the risk-free asset that grows at a rate r. Then, the value growth rate is $\pi + z(\tau) - \delta - r$.

Substituting (22) into (21), it is concluded that $z^* - \chi < 0$ must hold. This inequality indicates that the growth rate of capital is lower than the steady-state return rate.

Proposition 1. The steady state of the system described in (17)–(19) is unique if it exists.

Proof. The steady-state conditions are reduced to the two equations for z^* and $\tilde{n}^*$ (see Appendix A). The two equations can be illustrated as upward-sloping and downward-sloping curves, respectively, on a z^* - $\tilde{n}^*$ plane. Therefore, z^* and $\tilde{n}^*$ are uniquely determined at the given $\tilde{y}^e$, w, and χ. Because $f(\tilde{e}^*)$ is a monotonic function, $\tilde{e}^*$ is also uniquely determined by (22): □

Linearizing the dynamical system around the steady state, the Jacobian matrix is attained as follows:

$$\tilde{\mathbf{J}} = \begin{pmatrix} \chi & 0 & \chi \frac{f'(\tilde{e}^*)}{f''(\tilde{e}^*)} \frac{\psi''(\tilde{n}^*)}{\psi'(\tilde{n}^*)} \\ -\frac{w}{\phi''(z^*)} \frac{f''(\tilde{e}^*) z^* \tilde{n}^*}{(f'(\tilde{e}^*))^2} & -(z^* - \chi) & -\frac{\psi''(\tilde{n}^*)\tilde{n}^*}{\phi''(z^*)} \\ f'(\tilde{e}^*) & -\tilde{n}^* & -z^* \end{pmatrix} \tag{23}$$

. The characteristic equation for $\tilde{\mathbf{J}}$ is $x^3 - \mathrm{tr}\tilde{\mathbf{J}}x^2 + (\sum \det\tilde{\mathbf{J}}_{ii})x - \det\tilde{\mathbf{J}} = 0$, where $\det\tilde{\mathbf{J}}_{ii}$ is (i, i) is a minor determinant for $\tilde{\mathbf{J}}$.

First, the signatures of the trace and the determinant of Jacobian matrix should be confirmed.

$$\begin{aligned} \mathrm{tr}\tilde{\mathbf{J}} &= -2(z^* - \chi) > 0, \\ \det\tilde{\mathbf{J}} &= \frac{\chi}{\phi''(z^*)}(\phi''(z^*)(z^* - \chi)z^* + \psi(\tilde{n}^*)(\tilde{n}^*)^2 + \frac{\psi''(\tilde{n}^*)}{\psi(\tilde{n}^*)} \frac{w}{f'(\tilde{e}^*)z^*(\tilde{n}^*)^2} \\ &\quad + \frac{(f'(\tilde{e}^*))^2}{f''(\tilde{e}^*)} \frac{\psi''(\tilde{n}^*)}{\psi'(\tilde{n}^*)} \phi''(z^*)(z^* - \chi)) < 0. \end{aligned}$$

The Routh condition states that the number of roots of the characteristic equation with positive real pairs equals the number of variations of the signs in the following schemes:

$$-1, \mathrm{tr}\tilde{\mathbf{J}}, -\sum \det\tilde{\mathbf{J}}_{ii} + \frac{\det\tilde{\mathbf{J}}}{\mathrm{tr}\tilde{\mathbf{J}}}, \det\tilde{\mathbf{J}}. \tag{24}$$

The second and fourth terms are known to be positive and negative, respectively; therefore, the dynamical system has a one-dimensional stable manifold (saddle path).

Because the variables $\tilde{e}(\tau)$ and $z(\tau)$ are controllable, the unique path that converges to the steady state is determined for a given initial value of $n(t)$. The firm's optimizes by setting the two variables of interest on the saddle path.[6]

For the calculation, the firm's decision regarding $\tilde{e}(t)$ and $z(t)$ is approximated by using the linearized system with $\tilde{\mathbf{J}}$ (see Figure 4). Consequently, the following employment rule

[6] As the objective function is concave and the constraints are convex, the transversality conditions are sufficient for optimization. Convergence to the static $\tilde{e}^*$, $\tilde{n}^*$, and $z^* < \chi$ satisfies the condition.

and investment function are obtained:[7]

$$\tilde{e}(t) = \beta_e(t)(\tilde{n}(t) - \tilde{n}^*(t)) + \tilde{e}^*(t), -\frac{\psi''(\tilde{n}^*(t))f'(\tilde{e}^*(t))}{\psi'(\tilde{n}^*(t))f''(\tilde{e}^*(t))} < \beta_e(t) < 0, \tag{25}$$

$$z(t) = \beta_z(t)(\tilde{n}(t) - \tilde{n}^*(t)) + z^*(t), 0 < \beta_z(t) < -\frac{\psi''(\tilde{n}^*(t))\tilde{n}^*(t)}{\phi(z^*(t))(z^*(t) - \chi)}. \tag{26}$$

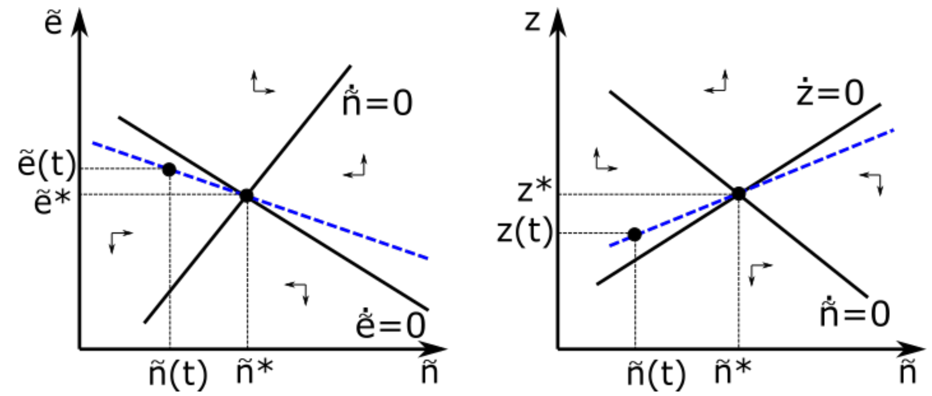

Figure 4: The saddle paths (blue dashed lines) in the linearized system

These equations indicate that the ability to determine current employment and investment is affected by the current gap between the actual and steady-state values and the steady state values themselves. In other words, stock disequilibrium causes the firm's quantity adjustment, as shown by Kawasaki et al. (1982, 1983). In every period, the firm resolves the optimization problem under the expected quantity constraint on sales $\tilde{y}^e$ and determines the employment (or production) and investment.[8] Notably, the given variables $(\tilde{y}^e, w, \chi)$ affect not only the objective values of the control variables ($\tilde{e}^*$ and z^*) but also the adjustment speeds (β_e and β_z). This nonlinearity arises from the unsteady maximization problem.

The comparative statics of firms' decisions are shown in Table 1 (see Appendix A for the calculations). I first consider the case $w = f'(\tilde{e}^*)$, which is guaranteed in the steady state of the entire economic dynamical system. Intuitively, the increase in $\tilde{y}^e$ promotes employment because production requires a larger workforce. The increase in employment raises the marginal productivity, which resultantly increases the marginal inventory holding cost ψ' (see equation (20)). Contrary to intuition, the objective value of capital accumulation decreases when the goods demand increases. However, it should be noted that the comparative statics involve the objective values and not the actual values; the investment could increase when the goods demand increases.

2.2 The household consumption function

A representative household solves for the standard intertemporal utility maximization problem. The aggregate household size is L, which is the supply of efficient labor (a

[7]By diagonalizing $\tilde{\mathbf{J}}$, $\tilde{\mathbf{J}} = Q^{-1}\Lambda Q$ is obtained, where $Q = \{Q_{ij}\}$ is a regular matrix and Λ is a diagonal matrix with a negative factor in the first row. Then, the set $\{x \in \xi[Q_{11}Q_{21}Q_{31}] | \xi \in \mathbb{R}\}$ is a stable subspace of the linear dynamical system $\dot{x} = \tilde{\mathbf{J}}x$.

[8]Interestingly, the current production decreases in inventory stock, as in the equilibrium dynamic model of Blinder and Fischer (1981).

	$\tilde{y}^e$	w	χ
$\tilde{e}^*$	$?(\oplus)$	$\ominus$	$\ominus$
z^*	$?(\ominus)$	$\ominus$	$?$
$\tilde{n}^*$	$?(\ominus)$	$?$	$?$

Table 1: Comparative statics of the objective values of the firm's optimization problem (the figures in parentheses represent the case $w = f'(\tilde{e}^*)$)

composite of the pure workforce and labor productivity). The supply is assumed to be inelastic and growing at a constant rate, $\dot{L}(t)/L(t) = \nu > 0$. The household derives utility from consumption per unit of effective labor at time t $c(t)$. The objective utility function at time t is described using $u(c(\tau))$:

$$\int_t^\infty u(c(\tau))e^{-\sigma(\tau-t)}\,d\tau \text{ where} u' > 0,\ u'' < 0, \tag{27}$$

where $\sigma > 0$ is the constant rate of time preference. The household rations their net real income into consumption and saving to maximize equation (27). Their perceived intertemporal budget constraint is described as follows:

$$\dot{y}_N(\tau) = (r(t) - \pi(t))(y_N(\tau) - c(\tau)) \text{ where } y_N(t) = F((e(t), k(t))) - (\nu + \delta)(k(t) + n(t)), \tag{28}$$

where $y_N(t)$ is the net real income per unit of effective labor at time t, $k = K/L$, and $n = N/L = \tilde{n}k$. The abovementioned budget constraint implies that the return to saving grows at the rate $r(t) - \pi(t)$.[9]

Following Murakami (2015), I assume that the utility function is in the constant relative risk aversion (CRRA) form and includes the "social status" term $\tilde{c}(w)$:

$$u(c(\tau)) = \begin{cases} ([c(\tau) - \tilde{c}(w(t))]^{1-\theta} - 1)/(1-\theta) & \text{if } \theta \neq 1, \\ \ln(c(\tau) - \tilde{c}(w(t))) & \text{if } \theta = 1, \end{cases} \tag{29}$$

where $\tilde{c}'(w(t)) > 0$. Furthermore, the transversality condition $\lim_{\tau\to\infty} u'(c(\tau))y_N(\tau)e^{-\sigma(\tau-t)} = 0$ holds. The following consumption function with a Keynesian flavor is obtained:

$$c(t) = c_1(r(t))y_N(t) + (1 - c_1(r(t)))\hat{c}(w(t)) = c_1(r(t))y_N(t) + c_2(r(t), w(t)), \tag{30}$$

where $c_1 = 1 - \frac{r(t)-\sigma}{\theta r(t)} \in (0, 1)$.[10] The consumption demand is affected by the current net income because households perceive that the stream of future income depends on the current income (see equation (28).

3 Dynamics

3.1 Reduced form

The dynamics of the economy are now analyzed. This study explores the dynamics of the capital intensity, inventory, expected quantity constraint on goods sales per capita,

[9] This budget constraint is drawn from Uzawa (1969) and Murakami (2015).

[10] For the proof, see Murakami (2015, Proposition 2.3).

$y^e = k\tilde{y}^e$, and real wage. The first $\dot{k}$ represents capital accumulation, and the gross accumulation rate $z(t)$ is determined by the firm in each time period. Because the firm's solution to the optimization problem is characterized by the saddle path, $z(t)$ is controlled given the state variable $\tilde{n}(t) = n(t)/k(t)$:

$$z(t) = \beta_z(n(t)/k(t), \tilde{y}^e(t), w(t)) (n(t)/k(t) - \tilde{n}^*(t)) + z^*(t). \tag{31}$$

Although the adjustment process is linearized, the coefficient β_z differs with a change in the variables; the investment function exhibits a kind of nonlinearity

The second and third dynamics, $\dot{n}$ and $\dot{y}^e$, respectively, characterize (Metzlerian) inventory dynamics. The change in the inventory stock per capita equals produced goods less sales (goods demand) and total depreciation (physical depreciation and the depreciation due to population growth):

$$\dot{n}(t) = F(k(t), e(t)) - y^d(t) - (\nu + \delta)n(t) = k(t)f(\tilde{e}(t)) - y^d(t) - (\nu + \delta)n(t) \tag{32}$$

For simplicity, I assume that the production function is a Cobb–Douglas type function:

$$F(k(t), e(t)) = Ak(t)^{\alpha}e(t)^{1-\alpha} \tag{33}$$

In each period, the employment–capital ratio $\tilde{e}(t)$ is regulated to equal the gross capital accumulation rate:

$$\tilde{e}(t) = \beta_e(n(t)/k(t), \tilde{y}^e(t), w(t)) (n(t)/k(t) - \tilde{n}^*(t)) + \tilde{e}^*(t). \tag{34}$$

The goods demand per capita is the sum of consumption demand and investment per capita:

$$y^d(t) = c(t) + z(t)k(t). \tag{35}$$

Following Franke and Lux (1993) and Franke (1996), I assume that the change in expected sales per capita follows adaptive adjustment:

$$\dot{y}^e(t) = \beta_{y^e}(y^d(t) - y^e(t)). \tag{36}$$

Monetary and wage dynamics are omitted in this study for simplicity. The following assumption holds:

Assumption 2. The real wage $w > 0$, nominal interest $r > 0$, and expected inflation rate $\pi > 0$ are constant.

The reduced system is described as follows:

$$\dot{k} = \left(z\left(\frac{n}{k}, \frac{y^e}{k}, w\right) - \nu - \delta \right) k \tag{37}$$

$$\dot{n} = (1 - c_1(r)) \left(kf\left(\tilde{e}\left(\frac{n}{k}, \frac{y^e}{k}, w\right)\right) - (\nu + \delta)(n + k) \right) - c_2(r, w) - \dot{k} \tag{38}$$

$$\dot{y}^e = \beta_{y^e} \left(c_1(r) \left(kf\left(\tilde{e}\left(\frac{n}{k}, \frac{y^e}{k}, w\right)\right) - (\nu + \delta)(n + k) \right) + c_2(r, w) + z\left(\frac{n}{k}, \frac{y^e}{k}, w\right) - y^e \right) \tag{39}$$

The steady state of the full system is determined by the set of variables (k_0, n_0, y^e_0), for which the right-hand sides of all the abovementioned equations become zero.

3.2 Steady state and stability

To obtain a unique steady state, it is assumed that the real wage is set as the notional rate.

Assumption 3.

$$w = w_0 = f'(e_0/k_0)$$

The assumptions about the real wage in this model are justified using the numerical experiments that follow, as the real wage rate in the simulations is almost identical to the labor income share, which is stable, as shown in Figure 3.

Lemma 1. At the full-system steady state, the objective values of investment, employment, and inventory, $(z^*(t), \tilde{e}^*(t), \tilde{n}^*(t))$, respectively, are realized.

Proof. Because the dynamics of k and n cease, $z = \nu + \delta$ and $F(k(t), e(t)) - y^d(t) - (\nu + \delta)n(t) = 0$ must hold. Dividing the second equation by $k(t)$ yields $f(\tilde{e}) - \tilde{y}^d(t) - z(t)\tilde{n}(t) = 0$, which implies that $\tilde{n}$ no longer changes.

Notably, the firm decides upon $z(t)$ and $\tilde{e}(t)$ for the set $(z(t), \tilde{e}(t), \tilde{n}(t))$ to be on the stable manifold (saddle path) for the dynamical system in equations (17)–(19). Because the economy is also located in the set that satisfies $\dot{\tilde{n}} = 0$, the control variables $z(t)$ and $\tilde{e}(t)$ must be in the set $\dot{z} = 0$ and $\dot{\tilde{e}} = 0$. Therefore, $(z^*(t), \tilde{e}^*(t), \tilde{n}^*(t))$ is obtained. □

Proposition 2. Suppose that $\frac{\partial \tilde{n}_0}{\partial k_0} > 0$. If a steady state for the full system exists under this condition, it is unique.

Proof. Given the condition $\dot{n} = 0$, the goods demand per capita in the steady state is as follows:

$$y_0^d = y_0^e = \frac{c_2(r, w_0)}{1 - c_1} + (\delta + \nu)k_0 = \hat{c}(w_0) + (\delta + \nu)k_0. \tag{40}$$

By Lemma 1, the steady-state condition can be described as follows: equations (20)—(22), (40), $z^* = z_0 = \delta + \nu$, and $f'(e_0/k_0) = w$. Because the change in $\tilde{n}$ stops, its steady-state value is obtained:

$$\tilde{n}_0 = \frac{f(e_0/k_0) - y_0^d/k_0}{\delta + \nu} \tag{41}$$

Substituting the values of z^*, w, and $\tilde{n}$ into equations (20) and (21), the steady-state conditions for e_0 and k_0 are as follows:

$$\chi + \psi'(\tilde{n}(e_0, k_0)) = 0 \tag{42}$$

$$\tilde{n}_0(e_0, k_0)\left(\psi'(\tilde{n}(e_0, k_0)) + \delta + \nu\right) + (\delta + \nu - \chi)\phi'(\delta + \nu) + \rho_0(e_0, k_0) = 0, \tag{43}$$

where $\rho_0 = y_0^d(e_0, k_0) - w(e_0, k_0)(e_0/k_0) - \psi(\tilde{n}_0(e_0, k_0))\tilde{n}_0(e_0, k_0) - \phi(\delta + \nu)$. Using total differentials yields the two equations $e_0 = F_1(k_0)$ and $e_0 = F_2(k_0)$ for each condition. The functions are set as $A(e_0, k_0) = \tilde{n}_0(\psi'(\tilde{n}_0) + \delta + \nu) - \psi(\tilde{n}_0)$ and $B(e_0, k_0) = \hat{c}(w_0)/k_0 - w_0(e_0/k_0)$, where $\tilde{n}_0 = \tilde{n}_0(e_0, k_0)$ and $w_0 = f'(e_0/k_0)$. When $A_{e0}/A_{k0} \neq B_{e0}/B_{k0}$ holds, where the subscript indicates the partial differential, it implies that $F_1' \neq F_2'$. In fact, $A_{e0}/A_{k0} > B_{e0}/B_{k0}$ as long as $\frac{\partial \tilde{n}_0}{\partial k_0} > 0$. Therefore, the two conditions intersect only once on the k_0 - e_0 plane. □

I now consider the stability conditions of the system. The Jacobian around the steady state is

$$\mathbf{J} = \begin{pmatrix} -(\gamma_1\tilde{n}_0 + \gamma_2\tilde{y}^e_0) & \gamma_1 & \gamma_2 \\ (1-c_1)X_1 + (\gamma_1\tilde{n}_0 + \gamma_2\tilde{y}^e_0) & (1-c_1)X_2 - \gamma_1 & (1-c_1)X_3 - \gamma_2 \\ \beta_{y^e}(c_1X_1 + (\delta+\nu) - (\gamma_1\tilde{n}_0 + \gamma_2\tilde{y}^e_0)) & \beta_{y^e}(c_1X_2 + \gamma_1) & \beta_{y^e}(c_1X_3 + \gamma_2 - 1) \end{pmatrix}, \tag{44}$$

where $\gamma_1 = \frac{\partial z}{\partial \tilde{n}}$, $\gamma_2 = \frac{\partial z}{\partial \tilde{y}^e}$, $X_1 = kf'(\tilde{e})\frac{\partial \tilde{e}}{\partial k} + f(\tilde{e}) - (\delta+\nu)$,$X_2 = kf'(\tilde{e})\frac{\partial \tilde{e}}{\partial \tilde{n}} - (\delta+\nu)$, and $X_3 = kf'(\tilde{e})\frac{\partial \tilde{e}}{\partial \tilde{y}^e}$. The trace and determinant are given as follows:

$$\mathrm{tr}\mathbf{J} = -(\gamma_1\tilde{n}_0 + \gamma_2\tilde{y}^e_0) + (1-c_1)X_2 - \gamma_1 + \beta_{y^e}(c_1X_2 + \gamma_1) + \beta_{y^e}(c_1X_3 + \gamma_2 - 1) \tag{45}$$

$$\det\mathbf{J} = (1-c_1)\beta_{y^e}\left(\gamma_1(X_1 + \tilde{n}_0X_2 + (\delta+\nu)X_3) + \gamma_2X_2(\tilde{y}^e - (\delta+\nu))\right) \tag{46}$$

As the system includes unsteady optimization problems and their linearizations, it is difficult to specify the stability conditions. Before substituting the estimated values, it is essential to examine how the variables affect each other's dynamics.

3.3 Feedback mechanism

Figure 5 illustrates the three loops of dynamics with fixed wages. The first loop shows the dynamics of k, which is directly determined by the investment decision z. Notably, the stability of k is not ensured. First, a change in k affects the variables expressed per unit of capital stock, $\tilde{n}$ and $\tilde{y}^e$. The firm estimates the objective values $(\tilde{e}^*, z^*, \tilde{n}^*)$ on $\tilde{y}^e$ and controls z so that $(\tilde{e}, z, \tilde{n})$ lies on the saddle path. The current gap between $\tilde{n}$ and $\tilde{n}^*$ also affects the magnitude of z (see Figure 4).

The second and third feedback loops in the figure depict what are called Metzlerian feedbacks in the literature (see Franke and Lux (1993), Franke (1996), and Chiarella et al. (2005, Section 7.2) for the working models).

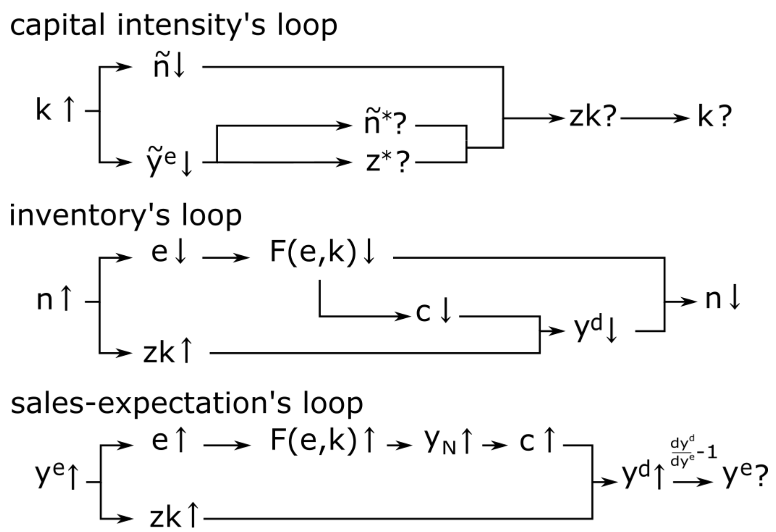

Figure 5: The feedback loops

Chiarella et al. (2005, Section 7.2) stated that Metzlerian feedback comprises two feedback loops: the unstable dynamics of expectation and the stable adjustment of inventory. The first destabilizing loop appears as the $(3,3)$ factor of $\mathbf{J}$, or $\beta_{y^e}\left(\frac{\partial y^d}{\partial y^e} - 1\right)$.

When $\frac{\partial y^d}{\partial y^e}$ is larger than unity, the actual sales quantity is higher than the increase in expected sales. In this case, sales expectation has a positive feedback, that is, instability is induced.

In the present model, the firm revises its sales expectation and "controls" employment and investment in the dynamic optimization problem. A characteristic of this model is the saddle-path adjustment process (see equations (31) and (34)). The first terms in the two equations correspond to the gradual adjustment processes on the saddle path. Suppose that sales expectation y^e (and $\tilde{y}^e$) increases. The firm's objective employment $\tilde{e}^*$ would increase and the objective investment z^* would decrease accordingly. However, the control process remains incomplete. The firm must be on the saddle path for the change in $\tilde{n}^*$ to affect the control. Furthermore, given the decrease in the objective inventory–capital ratio $\tilde{n}^*$, the firm would reduce employment and increase investment. This secondary effect is likely to stabilize sales expectation dynamics, as the decrease in current employment directly lowers consumption demand. The eventual result is ambiguous due to the complicated interconnections between the variables; thus, numerical experiments are presented in the following section.

Analyzing the stable feedback for n is relatively less complicated because the firm's optimization is unaffected by the current quantity of inventory. The increase in n lowers the current production to adjust for the excess inventory holding. The decrease in consumption demand is offset by increased investment. Eventually, the inventory holding is properly adjusted.

4 Numerical experiments

In this section, I describe the implementation of the numerical experiments that simulate the dynamics. I first formulate the unidentified functions and parameters following Fair's (2018) empirical surveys on macroeconomics in the US. Although the available data are quarterly, one period in the model is assumed to represent one year so as to adjust the estimated values accordingly. The steady state values are set based on the US' empirical data:

$k_0 = 1.13,\ e_0 = 0.94,\ \tilde{n}_0 = 0.16,\ w_0 = 0.61,\ A = 0.93,\ \tilde{y}_0 = 0.83,\ \nu = 0.04,\ \delta = 0.10,$
$\pi = 0.03$

The steady-state values of the variables are estimated as the average value for the US for 1960–2020. Notably, the values of α, A, and w_0 are determined by the assumption of a Cobb–Douglas production function and a real wage pegged to the marginal product of labor. The physical capital depreciation rate—which persistently and slowly grows—is set using the data for the last 20 years. The endogenous expected inflation rate is excluded; the expected inflation rate is assumed to be pegged at the average rate.

The consumption function is then estimated. In this study, I simplify $\hat{c}(w) = bw_0, b > 0$ in equation (30). From equation (40), the value of the coefficient $b = 1.24$. Because the multiplier $1 - c_1$ is not functional under steady-state conditions, the consumption propensity c_1 is estimated from the data. Utilizing ordinary least squares (OLS) regression, the results show that $c_1 = 0.73$ with a standard error of $9.37 \cdot 10^{-4}$ and revised $R^2 = 0.995$. Therefore, the consumption function obtained is

$$c = 0.73y_N + 0.33w_0 \tag{47}$$

Next, I determine the function parameters. The inventory holding cost ψ and the cost of capital accumulation ϕ are not formulated. For simplicity, it is assumed that the first is the sum of the linear and hyperbolic functions and the second is a quadratic function that satisfies $\phi'(0) = 1$. The parameter values are set so as to ensure that the steady-state values of e, k, and n satisfy the empirical data. The functions are given as follows.

$$\psi(\tilde{n}) = 0.0035/\tilde{n}^2 - 0.012\tilde{n} \tag{48}$$

$$\phi(z) = 5.82z^2 + z \tag{49}$$

For the simulation of dynamics, the Jacobian matrix $\mathbf{J}$ is first verified:

$$\mathbf{J} = \begin{pmatrix} 0.0029 & 0.0170 & -0.008 \\ 0.0459 & -0.4207 & 0.2508 \\ 0.2748\beta_{y^e} & -1.0747\beta_{y^e} & -0.3516\beta_{y^e} \end{pmatrix} \tag{50}$$

This result implies that both the Metzlerian feedback loops are stabilized in these parameters. The dynamic property changes with a change in their signatures, necessitating a numerical experiment.

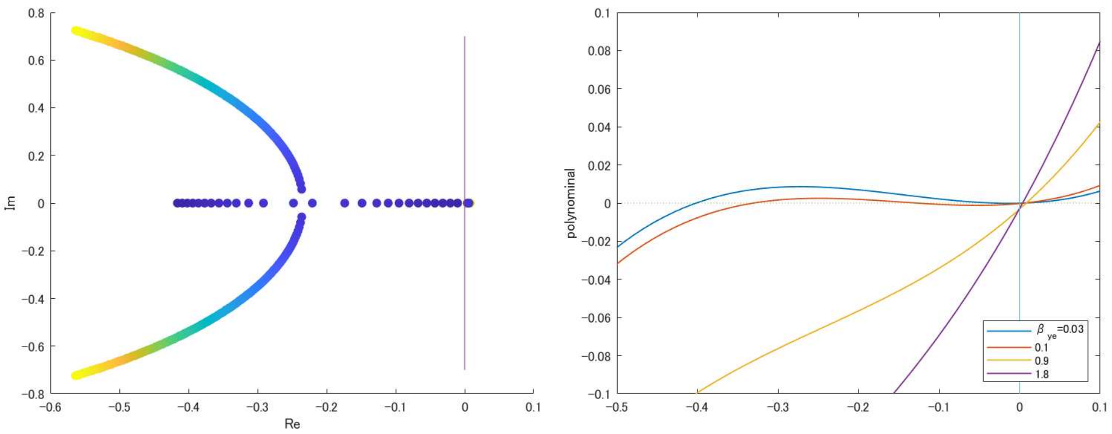

Figure 6: Eigenvalues of $\mathbf{J}_3$: $\beta_{y^e} \in [0.01, 2.00]$ (left) and characteristic equation (right)

Figure 6 (right) shows the trajectories of three eigenvalues on the complex plane whose colors change from blue to yellow as the value of β_{y^e} increases from 0.01 to 2.00. To the right of the vertical line $x = 0$, in fact, the points are accumulated; for every value, a positive real root exists. As β_{y^e} increases from 0.01, the two negative real roots change into complex pairs with negative real parts. This implies that the cyclical behavior occurs when the adjustment coefficient is not too small. The right side of Figure 7 shows the two manifolds at $\beta_{y^e} = 1.00$. The blue cyclical path and red line thus correspond to the 2-D stable manifold and unstable manifold, respectively.

Although the unstable Metzlerian feedback loop is stabilized, the cyclical dynamics remain in the present model with dynamic optimization.

The simulated dynamics for $[k, n, y^e]$ and $w = w_0$ are now presented. Figure 7 shows the variables' trajectories in the case of a boom as an initial point, where the initial point is $(k, n, y^e) = (1.1250, 0.1825, 0.9209)$ and $\beta_{ye} = 0.30$. The dashed lines on the left correspond to the steady states. The economy converges cyclically to the unstable manifold. There would be divergence but for the fact that the instability of capital accumulation is not aggressive enough; thus, the dynamics of k are slow relative to those of y^e and n. These dynamics are treated simultaneously, but the model seems to reproduce the differences in the scale of changes. There are medium-term changes in inventory and sales expectation, but capital accumulation occurs in the long term.

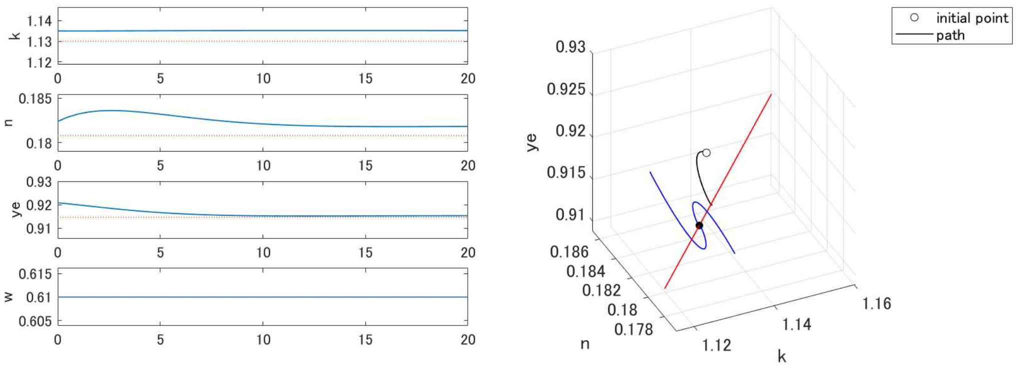

Figure 7: The dynamics of $[k, n, y^e, w]$ (left) and the trajectory and manifolds (right)

In this simulation, the initial situation is characterized by the existence of ample capital stock and the optimistic expectation of a boom, which expands employment; however, the increase in aggregate goods demand is not as high as that in production. Despite the large consumption demand, investment is not as sensitive in this model; the investment decision is affected by the variables expressed per unit of capital stock. As the firm revises its expectation, employment, production, and goods demand decrease. Production recovers slightly (around $t = 6$) due to the sale of the excess inventory holding. As capital can substitute for the workforce, employment remains low despite sufficient production.

The simultaneous movement of production, employment, and consumption $(y^d - zk)$ and the lagged movement of inventory with respect to production are consistent with the results shown in Figure 2.

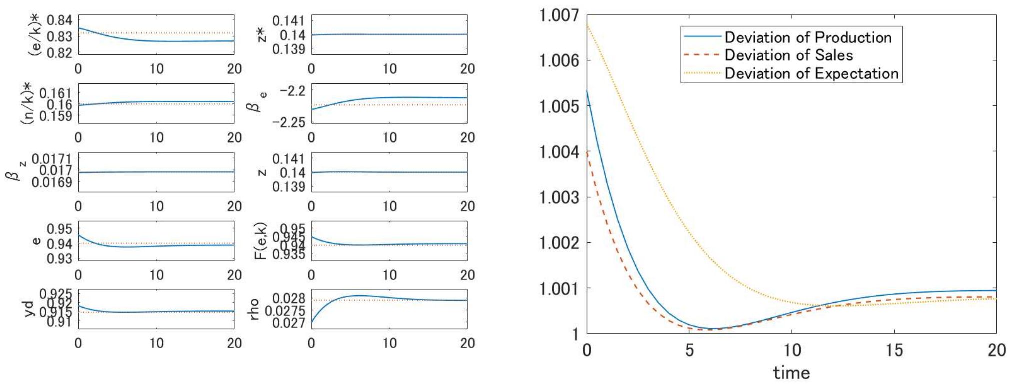

Figure 8: Dynamics of the other variables (left) and the deviation from steady-state value(right)

Notably, the deviation from steady-state production is (a little) larger than that from steady-state sales. When the steady-state value is set as unity, the production and sales slightly exceed it (see the right side of Figure 8). The three lines show that the deviation from the steady state is greater for production than for sales, which implies that production dynamics are more sensitive than sales dynamics. This bears out empirical facts. As the consumption demand under dynamic optimization has Keynesian elements, the change in goods demand (sales) is moderate than that in production. In addition, the firm's optimization also contributes to the fluctuation in goods demand because consumption demand depends on current production and investment demand is determined by the firm. Therefore, intertemporal optimizations under perceived quantity constraints seem to reproduce a qualitative feature of inventory dynamics: aggressive production fluctuations.[11] Furthermore, the firm's control partially follows Metzlerian dynamics (see figure 9). The change in the objective value of the inventory-to-sales ratio n^*/y^e, which is often assumed to be constant, is gradual. The firm determines its production and indirectly controls n/y^e. At the initial point, the inventory stock is excessive ($n > n^*$), but there is inertia in inventory dynamics. Owing to an optimistic sales expectation, the firm continues to produce an excessive quantity of goods. This instability induces cyclical dynamics but is weakened in this model, as observed in the previous subsection.

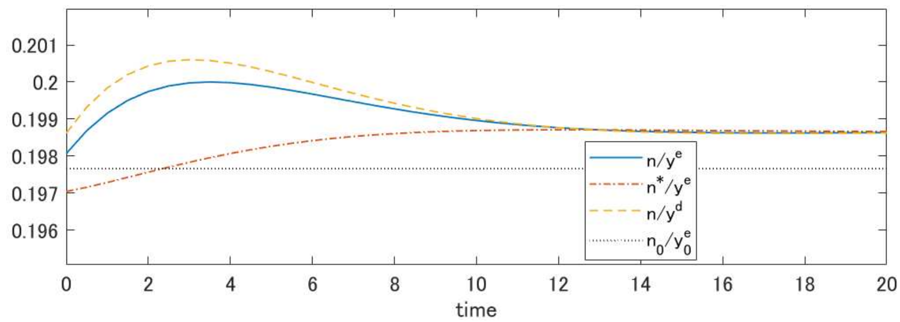

Figure 9: Inventory to sales ratio

Although the results of the numerical experiments qualitatively reproduce the estimated time series in Section 1, the movements of the variables in this simulation are but moderate. Unlike Figure 1, which shows that the cyclical components of sales and production fluctuate around the 5% range, the results show that this value is approximately 1%. This moderated production drives the slow movement of inventory; the scale of the lag is several years, although the data shows it to be several quarters. These inconsistencies persist even when a larger value of β_{ye} is used.

Evidently, the moderation effect arises from the dynamics of z and $F(e, k)$: the firm's decisions about investment and employment (production), respectively. As the firm in this model performs dynamic optimization for a particular formulation of expectation, the next section considers the treatment of expectation in the quantity constrained model.

5 Notes on the firm's control

As the model's dynamics are invoked by the determination of current employment (and subsequently, current production and income), the most important assumption is regarding the firm's optimization. The stabilization effect in gradual adjustment due to dynamic optimization manifests excessively in the dynamics, inducing extremely moderate reactions to stock disequilibrium. This is natural for an ordinary growth model with

[11]For the case of an initial depression, see Appendix C. This simulation also shows that the lagged adjustment of inventory and the production deviation are larger than the sales deviation.

dynamic optimization, such as the Ramsey model, in which the economy converges to the steady state in a straightforward manner.[12] Moving beyond the simple assumptions about expectation considered earlier, other possible formulations are presented here.

The critical assumption is that of the static $\tilde{y}^e$ in the firm's optimization problem; the firm believes that goods sales would grow at the same rate as the scale of production, which is characterized by capital stock. In the steady state (or balanced growth path), this proposition holds true. However, sales can trace various paths in transition.

For instance, the following example is considered:

$$\tilde{y}^e(\tau) = \tilde{q}\left(\tilde{y}^e(t), \tau, (z(s))_t^\tau\right), \tag{51}$$

where τ is the instant of time in the planning horizon, and t represents the timing of planning. $(z(s))_t^\tau$ is the set of planned investments from t to τ. If separability is imposed, this can be further simplified as follows:

$$\tilde{y}^e(\tau) = \tilde{y}^e(t) q(z(\tau), t) e^{-\int_t^\tau z(s)\, ds}. \tag{52}$$

Function q corresponds to the expected integrated fluctuations of sales at τ. If the function q is well behaved, the Hamilton—Jacobi-–Bellman equation solves the optimization problem. The most crucial question pertains to the form of q and how sales change in the future.

A canonical example is the "rational expectation" assumption with complete information.[13] Suppose that the firm knows the current and the steady-state values of goods demand. The first implies that $y^e(t) = y^d(t)$ always holds. The second presents a complication. Goods demand comprises firm investment and household consumption. As consumption is affected by the firm's production, the goods demand is directly affected by the firm's control. When it is assumed that the firm is aware of this structure, recursiveness emerges; y_0^d is affected by the investment schedule, and vice versa. The firm can then completely control the dynamics to exploit profitability, and the model is reduced to a game-theoretic environment, such as proposed by Ginsburgh et al. (1985), in that rigidity and stochastic fluctuation are the main problems. In this case, the firm is never surprised at the disequilibrium between the planned and actual trades. This implies that disequilibrium dynamics are but equilibrium dynamics with quantity rationing.[14] Therefore, the rational expectation model cannot be a starting point for disequilibrium dynamics. A jump is always needed to describe the persistent disequilibrium (or downturn), such that model rationality and consistency are not desirable. This is one reason for using a limited formulation of expectation in the present analysis.

The abovementioned example describes the key complexity in disequilibrium dynamics: the intertemporal recursive structures of demand and expectation that represent the

[12] Stiglitz (2018) and Guzman and Stiglitz (2020) criticized the dynamic stochastic general equilibrium (DSGE) model for its inability to explain the persistent downturn. They presented disequilibrium dynamics as an alternative, but it does not ensure stationarity. However, in this section, I examine the destabilization effect within the dynamic optimization framework, which indicates a preference to converge to the steady state.

[13] The definition of rational expectation in the analysis that follows goes beyond the original concept by Muth (1961). The present concept resembles the ones in canonical equilibrium dynamic models in that the firm perceives that sales fluctuate in the absence of stochastic disturbances.

[14] If sunspot dynamics are introduced into this canonical model, inventory holding would be attained. However, sunspot dynamics cannot be perceived as disequilibrium dynamics because they provide no information about why the firm underestimates sales.

"bootstrap effect," as mentioned by Neary and Stiglitz (1983). A decision in one period directly affects future events, and an individual who has determined their action (demand and supply signals) forms an expectation about the future, which should be considered while determining the action. To build a complete model, this positive feedback should be factored in. The present model avoids this structure by using static evolutions of sales per unit of capital stock. Although this evasion may constitute a shortcoming toward building a model consistent with existing equilibrium dynamics models, the results of the numerical experiments imply that adaptive adjustment modeling can approximate the macrodata.

If the modelling of q were to be improved in the abovementioned equation, the numerical experiments would present more desirable results. For instance, a firm is characterized by sentiment dynamics.[15] It could be pessimistic during a depression and optimistic during a boom; therefore, the expected dynamics of y^e heavily depend on the firm's evaluation of the current state of the business cycle. By solving for the recursive structure, a reference point about the firm's expectation is obtained; for instance, if the sales lie above the reference point, the firm tends to be optimistic. These sentiment dynamics induce greater changes in production and explain the persistence of the business cycle despite the existence of inventory.

6 Concluding remarks

This study develops an inventory dynamics model in which agents solve unsteady optimization problems. As Metzlerian feedback loops are found to exist in dynamics, this model can be interpreted as a microfounded Metzlerian model. The numerical experiments substantiate the qualitative findings about business cycles regarding aggressive production fluctuations and lagged inventory dynamics. However, the quantitative results seem inadequate because the firm's reactions are too mild. The model analysis shows that the firm's optimized control of employment and investment decisions moderates inventory dynamics, as the two variables must be chosen to lie on the saddle path. The firm gradually adjusts them to their objective levels, which lie around their steady-state values. As the Metzlerian unstable feedback loop results from an aggressive reaction to the firm's sales expectation, the microfounded framework (rather, intertemporal optimization) tends to weaken the Metzlerian cycle.

Stability mainly arises because of the reactions to the perceived quantity constraints that feature in the firm's decisions. The expected sales constraint is assumed to grow at the same rate as the capital. However, a firm experiencing a serious lack of goods demand is likely to have a pessimistic sales expectation in a discontinuous manner; sales expectation below the criterion threshold should be qualitatively different from that above it.

A possible way is to change the path of $\tilde{y}^e$ in the firm's optimization. In Section 5, this is discussed as the properties of the function q. The shape of this time-dependent function may explain the excessive changes in investment, which characterize the business cycle. Although this approach may be insufficient for dual-decision discontinuity, it enables its nonlinear approximation.

[15] This problem is related to the issue of animal spirits (see Franke and Westerhoff (2017) for heterodox macroeconomics).

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A Comparative statics of the firm's decision

The conditions of $(z^*, \tilde{e}^*, \tilde{n}^*)$ and equations (20)–(22), are reduced to the following two equations for $(z^*, \tilde{n}^*)$:

$$(z^* - \chi)\left(\phi'(z^*) - \frac{\tilde{n}^*\psi'(\tilde{n}^*)}{\chi}\right) + \tilde{y}^e - wf^{-1}(\tilde{y}^e + z^*\tilde{n}^*) - \psi(\tilde{n}^*) - \phi(z^*) = 0 \tag{53}$$

$$\chi + \frac{\psi'(\tilde{n}^*)}{w} f'\left(f^{-1}(\tilde{y}^e + z^*\tilde{n}^*)\right) = 0 \tag{54}$$

Using the total differential, the linear equation of the comparative statics of $(z^*, \tilde{n}^*)$ is obtained.

$$Dy_1 \begin{pmatrix} dz^* \\ d\tilde{n}^* \end{pmatrix} = Dx_1 \begin{pmatrix} d\tilde{y}^e \\ dw \\ d\chi \end{pmatrix} \tag{55}$$

$$\text{where } Dy_1 = \begin{pmatrix} (z^* - \chi)\phi''(z^*) & -\frac{z^*-\chi}{\chi}\psi''(\tilde{n}^*)\tilde{n}^* \\ \tilde{n}^* & z^* + f'(\tilde{e}^*)\frac{\psi''(\tilde{n}^*)}{\psi'(\tilde{n}^*)}\frac{f'(\tilde{e}^*)}{f''(\tilde{e}^*)} \end{pmatrix},$$

$$Dx_1 = \begin{pmatrix} -1 + \frac{w}{f'(\tilde{e}^*)} & \tilde{e}^* & \phi'(z^*) + \frac{w}{f'(e^*)}\frac{z^*\tilde{n}^*}{\chi} \\ -1 & \frac{f'(\tilde{e}^*)}{w}\frac{f'(\tilde{e}^*)}{f''(\tilde{e}^*)} & \frac{f'(\tilde{e}^*)}{\chi}\frac{f'(\tilde{e}^*)}{f''(\tilde{e}^*)} \end{pmatrix}$$

Using Cramer's rule, the comparative statics of $(z^*, \tilde{n}^*)$ are obtained. Note that z^* is always decreasing in w and χ; the objective capital accumulation rate decreases when the real wage and the depreciation rate of real capital increase. The first is intuitive because low profitability reduces investment.

For $(\tilde{e}^*, \tilde{n}^*)$, equations (20)—(22) are reduced as follows:

$$\left(\frac{f(\tilde{e}^*) - \tilde{y}^e}{\tilde{n}^*} - \chi\right)\left(\frac{\tilde{n}^* w}{f'(\tilde{e})^*} + \phi'\left(\frac{f(\tilde{e}^*) - \tilde{y}^e}{\tilde{n}^*}\right)\right) + \tilde{y}^e - w\tilde{e}^* - \psi(\tilde{n}^*) - \phi\left(\frac{f(\tilde{e}^*) - \tilde{y}^e}{\tilde{n}^*}\right) = 0 \tag{56}$$

$$f(\tilde{e}^*) - \tilde{y}^e - z^*\tilde{n}^* = 0 \tag{57}$$

, and

$$Dy_2 \begin{pmatrix} d\tilde{e}^* \\ d\tilde{n}^* \end{pmatrix} = Dx_2 \begin{pmatrix} d\tilde{y}^e \\ dw \\ d\chi \end{pmatrix} \tag{58}$$

$$\text{where } Dy_2 = \begin{pmatrix} (z^* - \chi)\left(-\tilde{n}^*\frac{w}{f'(\tilde{e}^*)}\frac{f''(\tilde{e}^*)}{f'(\tilde{e}^*)} + \phi''(z^*)\frac{f'(\tilde{e}^*)}{\tilde{n}^*}\right) & -z^*\frac{z^*-\chi}{\tilde{n}^*}\phi''(z^*) \\ \frac{f''(\tilde{e}^*)\psi'(\tilde{n}^*)}{w} & \frac{f'(\tilde{e}^*)}{w}\psi''(\tilde{n}^*) \end{pmatrix},$$

$$Dx_2 = \begin{pmatrix} -1 + \frac{w}{f'(\tilde{e}^*)} + \frac{z^-\chi}{\tilde{n}^*}\phi''(z^*) & \tilde{e}^* - (z^* - \chi)\frac{\tilde{n}^*}{f'(\tilde{e}^*)} & \tilde{n}^*\frac{w}{f'(e^*)} + \phi'(z^*) \\ 0 & \frac{f'(\tilde{e}^*)\psi'(\tilde{n}^*)}{w^2} & -1 \end{pmatrix}$$

B On the empirical data

The data supplied by Ray C. Fair (`https://fairmodel.econ.yale.edu/`) are utilized for the numerical experiments. The time series data for the US for 1960—2020 have been used. The abbreviations below follow Fair's.

- CD : real consumption expenditures for durable goods
- CN : real consumption expenditures for nondurable goods
- CS : real consumption expenditures for services
- E : total employment
- HF : average number of hours paid per job
- JF : number of jobs
- KK : capital stock
- PF : output price index for X
- SIFG : employer social insurance contributions (paid to the US government)
- SIFS : employer social insurance contributions (paid to state and local governments)
- U : total unemployment
- V : inventory stock, the change of it is Y-X
- WF : average hourly earnings excluding overtime of workers
- X : real sales
- Y : real production

The following points should be noted regarding variable construction: First, the labor input can be calculated as $\mathtt{JF} \cdot \mathtt{HF}$. However, this labor input index is not convenient; hourly input and labor productivity are excluded. As Fair (2018, Figure 4) showed, $\mathtt{Y}/(\mathtt{JF} \cdot \mathtt{HF})$ continues to grow, although the present model does not include a specific technical change. Therefore, employment is interpreted as an efficient workforce, which includes labor productivity. Second, the real wage rate w and efficient labor E are reevaluated from the data because WF represents hourly earnings and is thus too low for long-run model construction.

Let a denote the labor productivity index such that workforce E equals $a\hat{E}$, where $\hat{E}$ is the labor input, and $\hat{E} = \mathtt{JF} \cdot \mathtt{HF}$. The production function is described as follows:

$$F(K, E) = F(K, \hat{E}; a) = AK^{\alpha}(a\hat{E})^{1-\alpha}. \tag{59}$$

This production function implies that

$$1 - \alpha = \frac{wE}{Y} = \frac{\hat{w}\hat{E}}{Y}, \tag{60}$$

where $\hat{w} = \frac{\partial F}{\partial \hat{E}}$ and $w = \frac{\partial F}{\partial E}$, respectively. In the remainder, W is estimated using equation (60).

First, the labor share is almost stable: $(\mathtt{WF} \cdot \mathtt{JF} \cdot \mathtt{HF} + \mathtt{SIFG} + \mathtt{SIFS})/(\mathtt{PF} \cdot \mathtt{Y})$ fluctuates around the average value of 0.61.[16] This implies that the paid real wage $\hat{w}$ fluctuates around the marginal value of labor input and both grow almost constantly.

[16] In this study, the wage rate and workforce values are calculated from their averages, which implies that the over-hourly working people are excluded. If they are included, the wage rate and the wage share would increase slightly.

Second, the production–capital ratio is almost stable at approximately 0.21.[17] The ratio is assumed to be constant over the long term. Because the production function is the Cobb-Douglas type, it can be concluded that $F(K, \hat{E}; a)$, K, and E grow at a common rate in the long term.

Therefore, a is calculated as follows: $a = F(K, \hat{E}; a)/\hat{E} = \mathtt{Y}/(\mathtt{JF} \cdot \mathtt{HF})$. Using (60), the real wage rate is evaluated as the workforce that includes productivity, $w = \hat{w}/a = 1 - \alpha$. This implies that $Y/E = 1$ such that the capital intensity is calculated as $k = e/(Y/K) = (\mathtt{E}/(\mathtt{E}+\mathtt{U})) \cdot (\mathtt{KK}/\mathtt{Y})$. This value is stable, and its average value is taken as the steady-state value, $k_0 = 1.13$ for a one-year efficient labor supply.

The variables in this model are specified as follows:

$$\begin{aligned}
F(K, E)/K &= \mathtt{Y}/\mathtt{KK} \\
C &= \mathtt{CD} + \mathtt{CN} + \mathtt{CS} \\
\tilde{n} &= \mathtt{V}/\mathtt{KK} \\
e &= \mathtt{E}/(\mathtt{E} + \mathtt{U}) \\
E &= \mathtt{JF} \cdot \mathtt{HF} \\
\text{Labor Share} &= (\mathtt{WF} \cdot \mathtt{JF} \cdot \mathtt{HF} + \mathtt{SIFG} + \mathtt{SIFS})/(\mathtt{PF} \cdot \mathtt{Y})
\end{aligned}$$

Using these variables, the correlation coefficients of the time series data in the text are calculated. Table 2 corresponds to Figure 2.

Table 2: Correlation coefficients of the cyclical components of lagged economic variables and real production

Variable	$t-4$	$t-3$	$t-2$	$t-1$	t	$t+1$	$t+2$	$t+3$	$t+4$
real sales	0.2034	0.3799	0.5608	0.7510	0.9535	0.7696	0.6132	0.4781	0.3245
employment rate	0.03500	0.1824	0.3297	0.5182	0.8440	0.7452	0.6014	0.5052	0.3668
inventory	−0.1263	−0.03030	0.1075	0.2853	0.5339	0.6716	0.7287	0.7231	0.6439
consumption	0.2491	0.4105	0.5210	0.6948	0.8871	0.6442	0.4842	0.3566	0.1997

C Simulation of a depression

Figure 10 shows the case of a recession, which is characterized by a smaller sales expectation. The initial values are $(k, n, y^e) = (1.1250, 0.1650, 0.9329)$. This simulation also presents the lagged adjustment of inventory and more sensitive production dynamics in comparison with sales dynamics; see Figure 11.

[17] Other Keynesian models such as that by Chiarella et al. (2005) use annual production; their ratio is about four times that in the present model; 0.70.

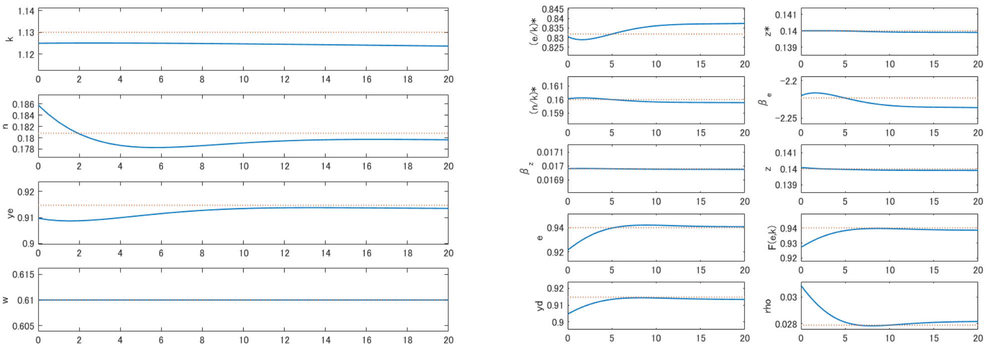

Figure 10: Dynamics of $[k, n, y^e]$ (left) and other variables (right)

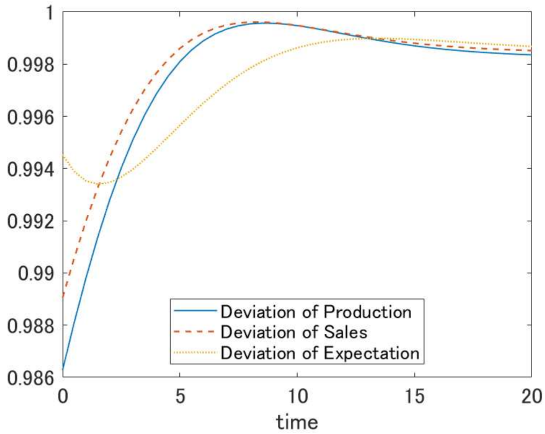

Figure 11: Deviation from the steady-state value in the case of a depression