Beyond Merton: Multi-Dimensional Balance Sheet in Default Modeling

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Abstract

Nearly half a century after Merton's 1974 paper, the basic framework of modeling a company's default risk in terms of one-dimensional variable, the total asset value, with fixed debt level has remain unchanged among the work by academic and quantitative modeling community. Under such simplification, the model is unable to correctly defined the state of default, which is a state of negative cash. But more importantly, Merton's one-dimensional model cannot incorporate the fundamental principle of balanced balance sheet, and thus unable to generate the rich dynamics followed by a company's multi-dimensional financial state. This article presents an example of 2-dimensional balance sheet that is still analytically solvable to demonstrate the multi-dimensional kinematics of a company's financial state, and the much more nontrivial default dynamics as a result. The idealized example is for demonstrate purpose only, but the methodology has been extended to higher-dimensional balance sheet to model actual companies and forecast default with computerizable numerical and simulative algorithms.

1. Background

In his 1974 paper, Merton modelled corporate default through the stochastic change of a firm's total asset value and a fixed debt level as the strike to default. Merton's work in predicting default was the first of its kind to model the incremental change of a corporate's financial state, and therefore the process that leads a corporate to default. Since then, much work has been done to enhance and refine Merton's approach, but the basic framework – modeling a corporate in a one-dimensional balance sheet with only changing total asset value has remained.

Despite of the conceptual breakthrough of Merton's model, it should not have come as a great surprise that the default probabilities produced do not work well in practical applications. In Merton's model, a company's financial state is vastly simplified compared to an actual company. While Merton's model sees only a company's total asset value, a company's financial state is represented in the real world by the multi-dimensional balance sheet - cash, inventory, receivable, etc. for assets, and debt, payable, etc. for liabilities. By not looking at the components of a company's total asset, Merton's model could not have correctly defined the state of default. For non-financial companies, having total assets less than total liabilities implies negative equity, but it does not imply default. Rather, default is caused by cash becoming negative, which can happen even when the equity is amply positive and may not happen even when the equity is negative. The other simplification in Merton's model, the fixed level of debt, harbors another fundamental inconsistency. In the real world, the change of a company's balance sheet must be restrained by the fundamental principle of balanced balance sheet. Thus, unless the change is equity changing (caused by either an income/cost generating event, or an equity capitalization event), a company's change in asset must equal to its change in liability. Since debt makes up the entire liability in Merton's model, the only way asset value can change while debt is fixed is if the changes are in mark-to-market, as opposed to in book values which change under a company's operating and financing activities. For non-financial operating companies, marking to market of total asset value is basically irrelevant to their operations and in fact is unobservable.

The simplifications in Merton's model does not diminish the fundamental importance of Merton's approach in modeling default through a dynamic process. The importance is even stronger today as many data centric approaches - big data, machine learning, AI, are gaining attentions in corporate default forecasting. However, to be realistically applicable, any default model must ultimately model a business realistically. Unfortunately, nearly half a century after Merton's original work, this is still poorly done. There is a collective abhorrence by the modeling community for including any real business variables – inventory, profit margin, interest rate, sales growth etc. into the models. In fact, the approaches based on company's financial reports are politely distained as the "accounting approach" while the models containing only mathematical variables are distinguished as the "structural approach". The naming convention is puzzling because the structural models actually squeeze a company's entire balance sheet into one utmost structureless single variable. It seems to me that the "dynamical approach" is a more suitable name in place of the "structural approach".

It is an elementary fact that the dynamics of any model is predicated first and most on the kinematics of the model – namely how the state of the object in the model is described. A one-dimensional state of a company leads to trivial dynamics. Considering the Merton's model as a deterministic model for a moment. If the current total asset A changes by a fixed growth rate μ while the total debt D (under single maturity M) is fixed, then Merton's default criterion is trivial: if $\mu < (D/A)^{\frac{1}{M}} - 1$, company will default.

In reality, a company's financial state is represented by the balance sheet[1], which is a multi-dimensional object. For all companies, multiple asset and liability fields are required to adequately describe the state of their financial conditions. The dynamical changes of the entire balance sheet generate incomes, other financial and operational results. The key is that the different balance sheet variables interact strongly with one another as their changes are restrained by the fundamental principle of balanced balance sheet. This principle is the equivalence of the conservation (or invariance and symmetry) law in a physical system. Default is a particular kind of balance sheet state in which cash (which must exist in all balance sheet) becomes negative. With the expanded dimensions of a company's financial state, the deterministic part of a company's default dynamics becomes much more complex and non-trivial compared to one single growth rate. Although such system normally becomes analytically intractable in producing closed form solutions, it nevertheless is capable of producing the numerical forecasts the real-world applications need.

This article presents an example of the simplest multi-dimensional model, a highly simplified two-dimensional balance sheet, with only cash and debt dynamically changing, and the debt assumed to be perpetual, leading to failure to pay interest as the only cause of default. There are only three parameters in the model: a debt borrowing ratio that controls the amount of debt borrowed in each period, a gross margin that controls gross income and an interest rate that controls the interest payment amount. The model is sufficiently simplistic so that it is solvable in closed form, allowing the default criteria to be seen clearly. Yet the model is sufficiently complicated to demonstrate the non-trivial dynamics of a multi-dimensional balance sheet.

Even in its simplicity, the model is able to demonstrate the interaction between a company's operating and financing condition in determining the future default of a company. The model shows a strong connection between default and a company's debt-to-cash ratio, as well as default and a company's growth rate. The model is even capable of reflecting the value of a company's real options by showing that a company can deter default if it plans its borrowing level to match realistic growth rates.

The work here focuses on the deterministic criteria of default but can be easily generalized to stochastic settings. Although the current model is unsuitable to be applied to actual companies, the methodology can be readily generalized to higher dimensional balance sheet with more complex dynamics, making the model calibratable to actual companies via their historical financial statements.

2. A 2-D company model with dynamic cash and debt

In order to model the cause of default (dynamics) realistically, one has to first represent the financial state of a company (kinematics) realistically. The financial state of a company is best represented by **the balance sheet**, which can be viewed as a tensor product of ${R_+}^m \otimes {R_+,}^n$, where ${R_+}^m$ is the non-negative valued subset of a m-dimensional real vector space $(a_1, a_2 \dots a_m)$, with each dimension representing an asset type and $a_i \geq 0$ for all i representing the book value of the asset type. Similarly, ${R_+,}^n$ represents the liabilities $(l_1, l_2 \dots l_n)$, with the exception that one of the dimensions can have both positive and negative values. There must be at least one asset type named cash, and one liability type named equity. Equity is the only dimension that can have both

[1] Not completely but predominately

positive and negative values. **A financial state** is a point in the ${R_+}^m \otimes {R_+,}^n$ space such that the simple sum of all asset values equals to the simple sum of all liability values $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} l_i$. This is **the principle of balanced balance sheet**, which is as central to financial forecasting as the symmetry or conservation laws to a physical system.

The dynamics of the balance sheet is defined by parameterized transformations (**actions** hereafter known as) that map financial states to other financial states. Actions correspond to a company's real operational and financial activities, such as buying inventory, borrowing debt, paying interest, etc. Defaultable action is a special kind of action for which certain financial states are undefinable following the action (see the example below). Formally, these states would be transformed by the action to non-financial states. When a company evolves into such states, default occurs following the defaultable action. A **financial model** of a company is an ordered sequence of actions that jointly generate the change of a company's financial state within one time period, corresponding to one fiscal cycle of the company, such as a fiscal year. To generate the financial changes of a company over time, the sequence of action is applied recursively to the company's financial state with the action parameters following certain time series, which can be stochastic or otherwise.

As discussed above, the principle of balanced balance sheet requires that the changes in different balance sheet fields must be interlinked, particularly across asset and liability. This paper provides an example that is the simplest and analytically solvable extension to Merton's model - a company with a two-dimensional variable balance sheet: cash and debt.

Let's first consider a ${R_+}^2 \otimes {R_+,}^2$ company for which there are 2 asset dimensions – cash and inventory (c, i) and 2 liability dimensions – debt and equity (d, e). The company runs a simple operation of buying and selling inventory at a profit margin. The company uses debt to finance the purchase of the inventory by leveraging its own cash. A starting model may be defined by the following six actions, which generate the changes of the company's financial state over a single fiscal period. The notation $(c, i) \otimes (d, e) \to (c', i') \otimes (d', e')$ is used to define an action's transformation of a financial state.

1. Borrows debt in an amount equal to a borrowing ratio L (positive) times the current cash level with interest rate r and maturity M: $(c, i) \otimes (d, e) \to (c + Lc, i) \otimes (d + Lc, e)$
2. Uses the entire cash to purchase inventory: $(c, i) \otimes (d, e) \to (0, i + c) \otimes (d, e)$
3. Sells a percentage S of inventory at a mark-up margin g. This action generates iSg amount of income which is added to equity: $(c, i) \otimes (d, e) \to (c + iS + iSg, i - iS) \otimes (d, e + iSg)$
4. Pays debt interest, which generates a cost that is subtracted from equity: $(c, i) \otimes (d, e) \to (c - dr, i) \otimes (d, e - dr)$
5. Pays income tax at tax rate x if the earning before tax $(iSg - dr)$ is positive: $(c, i) \otimes (d, e) \to (c - x * Max(iSg - dr, 0), i) \otimes (d, e - x * max(iSg - dr, 0))$
6. Pays back matured debt which is the amount of debt of maturity M that is borrowed $(M - 1)$ time periods ago, denoted as ∂d_{t-M+1}: $(c, i) \otimes (d, e) \to (c - \partial d_{t-M+1}, i) \otimes (d - \partial d_{t-M+1}, e)$

Note that each of the actions above preserves the balance of the balance sheet. Action 4, 5 and 6 are defaultable actions because in the cases that the cash level is less than the required amount of reduction, the formal resulting state would become negative in cash, which is not a permissible financial state, and the company defaults.

It is straightforward to generate the future financial state of this company by recursively applying the model to the company's current financial state given any values of the action parameters L, g, r, S, x, which in principle can vary in each time period. A closed analytical form of the future state is challenging to attain even at the current level of simplification. For purpose of demonstration, I will make even more simplifications so that an analytical solution can be obtained.

First, it is assumed that all action parameters are constant. Then, to simplify the balance sheet, it is assumed that the inventory is always sold in its entirety, $S = 1$. As a result, inventory i becomes 0 at the end of every time period. The tax payment action 5 is a non-linear map, so for simplicity, it is assumed that $x = 0$, and action 5 is eliminated from the model. Finally, the debt maturity is assumed to be infinite. One could have simplified the debt maturity to be 1, but then there will not be any debt outstanding at the end of each time period. In order to keep the model's variables to be non-trivially two-dimensional, the model assumes infinite debt maturity. As a result, action 6 is eliminated, and the company can default only by having insufficient cash to pay debt interest in action 4.

To summarize, the model describes a company that buys and sells inventory. At the beginning of each fiscal period, the company borrows perpetual debt to finance the inventory purchase. The amount of debt borrowed is at a borrowing ratio L to current cash on hand[2]. The company sells the entire inventory at a markup margin g. At the end of each fiscal period, the company pays debt interest, which bears a rate r. If the company has insufficient cash to pay for the interest, it defaults.

The balance sheet of this company at the end of each period is then given below. Since there is no inventory at the end of each period, and equity can be implied from cash and debt, only the level of cash and debt are variables.

(1)
$$c_t = (1+L)(1+g)c_{t-1} - Lrc_{t-1} - rd_{t-1}$$
$$d_t = Lc_{t-1} + d_{t-1}$$
$$\text{where } t = 1, 2, 3, \ldots$$

In more compact 2x2 form:

(2)
$$\vec{C}_t = \begin{pmatrix} \Delta & -r \\ L & 1 \end{pmatrix} \vec{C}_{t-1}$$
$$\text{where } \vec{C}_t \equiv \begin{pmatrix} c_t \\ d_t \end{pmatrix} \text{ and } \Delta \equiv (1+L)(1+g) - Lr$$

The key character of Equation (2) is a 2-dimensional balance sheet with interactive cash and debt. The linear equations can be solved analytically to yield:

(3)
$$c_t = \frac{(\Delta - Dr)\,(\lambda_+^t - \lambda_-^t) - \lambda_+\lambda_-(\lambda_+^{t-1} - \lambda_-^{t-1})}{\lambda_+ - \lambda_-}$$
$$d_t = \frac{(L + D)(\lambda_+^t - \lambda_-^t) - D\lambda_+\lambda_-(\lambda_+^{t-1} - \lambda_-^{t-1})}{\lambda_+ - \lambda_-}$$
$$\text{where } \lambda_\pm \equiv \frac{(\Delta+1) \pm \sqrt{(\Delta-1)^2 - 4Lr}}{2},$$

and $D \equiv \frac{d_0}{c_0}$ the initial debt-to-cash ratio. Without loss of generality, it is assumed that $c_0 = 1$.

A company would default if at some future time $t = T_D$, $c_t < 0$. The criteria to default are completely determined by the values of the parameters L, g, r and the initial condition D. Even when all the parameters are held constant, the default criteria become much more non-trivial compared to the Merton's model, and able to provide practical insights on the causes of a real-world company's default.

3. Default criteria in current debt-to-cash ratio D

[2] In this paper I focus on the case $L \geq 0$, thus the company always borrows more debt. The work can be generalized to allow negative L, which can be interpreted as the company paying down (or calling) the debt.

First, the model implies a strong connection between the debt-to-cash ratio $D_t \equiv d_t/c_t$ and default.

Theorem 1. For $\boldsymbol{r} > \frac{\left(\sqrt{(1+L)(1+g)}-1\right)^2}{L}$, default occurs for all $\boldsymbol{D} \geq 0$. For $\boldsymbol{r} < \frac{\left(\sqrt{(1+L)(1+g)}-1\right)^2}{L}$, default occurs if and only if $\boldsymbol{D} > \frac{\lambda_+ - 1}{r}$.

Theorem 2. $\boldsymbol{D_t}$ can only either increase or decrease at all $\boldsymbol{t}$. In the case of no-default, $\boldsymbol{D_t}$ approaches a finite limit $\boldsymbol{D_\infty}$ that is independent of the initial debt ratio $\boldsymbol{D}$:

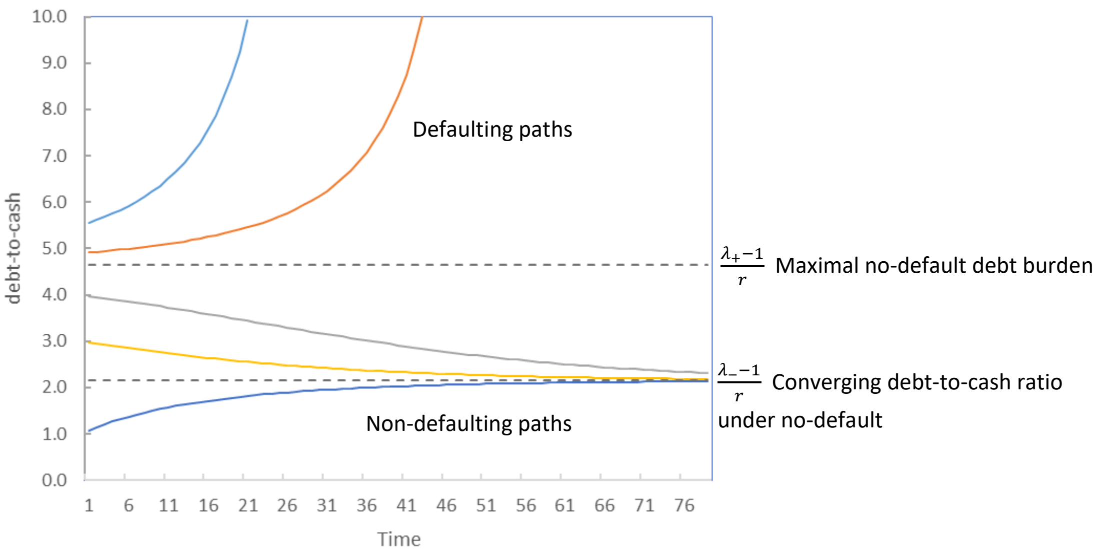
(4) $$D_\infty = \frac{L}{\lambda_+ - 1} = \frac{\lambda_- - 1}{r}$$

Note that in the case of default, D_t will obviously diverges as cash level becoming close to 0 at the time of default.

The default criteria show a "death zone" in the values of the parameters (L, g, r), within which a company will always default eventually even if it starts with no debt. Outside the "death zone", for each given set of values of (L, g, r), there is a maximal non-defaultable level of current debt burden (measured in debt-to-cash ratio), above which a company will default and below which a company will not. Interestingly, any company whose debt-to-cash ratio is below the maximal debt burden will eventually converge to the same level of debt-to-cash ratio.

Figure 1 summarizes the results with several examples of defaulting and non-defaulting solutions under a fixed set of values of (L, g, r) but different initial debt-to-cash ratio D.

Figure 1. Debt-to-cash ratios of defaulting and non-defaulting paths ($\boldsymbol{L = 0.2, g = -5\%, r = 2\%}$)



4. Default criteria in debt borrowing ratio L and profit margin g

Theorem 3. Given $\boldsymbol{D}$ **and** $\boldsymbol{r}$, default occurs if and only if $\boldsymbol{g} < \boldsymbol{g_D(L)}$

(5) $$g_D(L) = \begin{cases} \frac{L+D}{L+1}\left(r + \frac{1}{D}\right) - 1 & \text{for } 0 \leq L < D^2 r \\ \frac{(1+\sqrt{Lr})^2}{1+L} - 1 & \text{for } L \geq D^2 r \end{cases}$$

(6) Furthermore, in the case of default, the time to default is given by

$$T_D = \frac{\ln(\delta - \lambda_-) - \ln(\delta - \lambda_+)}{ln\lambda_+ - ln\lambda_-}, \quad \text{where } \delta \equiv \frac{\Delta + Lr}{\Delta - Dr}$$

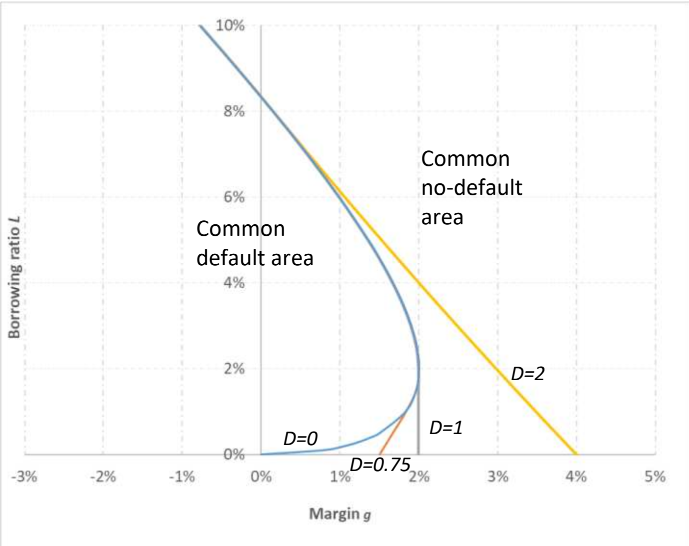
Figure 2 illustrates four default boundaries $g_D(L)$ in the $(g, L+)$ plane. The four boundaries correspond to four different values of D $(0, 0.75, 1, \text{and } 2)$ and $r = 2\%$. The area lying on the left side of each boundary is the default area for that particular D while the right side the no-default area. Interestingly, default boundaries $g_D(L)$ under different D eventually all merge to the boundary $g_{D=0}(L)$ for $D = 0$ at $L = D^2 r$.

The default criteria reveal the bifurcating effect of whether a company currently has more debt or more cash $(D = 1)$. For $D > 1$, $g_D(L)$ is monotonic in L. For $D < 1$, $g_D(L)$ is not monotonic and reaches a maximal value $g_{max} = r$ at $L = r$. In other words, if a company currently has more cash than debt, it can avoid default by either increasing or decreasing L sufficiently (still keeping $L > 0$). But if a company currently has more debt than cash, it can avoid default only by increasing L sufficiently.

Note that $g_D(L) = Dr$ at $L = 0$ for all values of D. This implies that if a company suddenly loses the ability to borrow any new debt, it will not default if and only if its profit margin is greater than Dr.

The figure also shows the "death zone" of default in (L, g) space: the area to the left of the $D = 0$ boundary labeled as the "common default area" for all values of $D \geq 0$.

Figure 2. Default boundaries in (g, L) space $(r = 2\%)$



The next 2 tables give some examples of default outcome for different values of (L, g). The shaded cells represent no-default, while the others represent default, and the numbers are equal to the time of default (number of time steps in equation (1)).

Table 1. Time to default $(D = 0.5, r = 2\%)$

L \ g	-2.0%	-1.0%	0.0%	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
0.60									
0.50									
0.40									
0.30									
0.20									
0.10	108	286							
0.05	61	78	113	254					
0.00	54	69	100	538					

Table 2. Time to default $(D = 5, r = 2\%)$

L \ g	-2.0%	-1.0%	0.0%	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
0.60									
0.50									
0.40									
0.30									
0.20									
0.10	16	18	22	28					
0.05	11	12	13	15	16	19	22	30	
0.00	9	9	10	11	11	12	13	14	16

Within the current model, a company can always avoid default if the borrowing ratio L is sufficiently high, even if the company sells inventory permanently at a loss ($g < 0$). Realistically, this cannot happen because as the next section shows the higher leverage must be supported by higher growth if the margin remains unchanged.

5. Default criteria in growth rate

The results so far show that the company in the model can always avoid default if it borrows sufficiently high level of debt. This is because the model assumes that company sells the entire inventory at a fixed margin g no matter how much the inventory is. Realistically, there is a natural limit on how much sales can grow in a given market. The sales during period t is given by $(1 + L)(1 + g)c_{t-1}$, implying that the growth rate of sales follows the growth rate of cash under constant L and g.

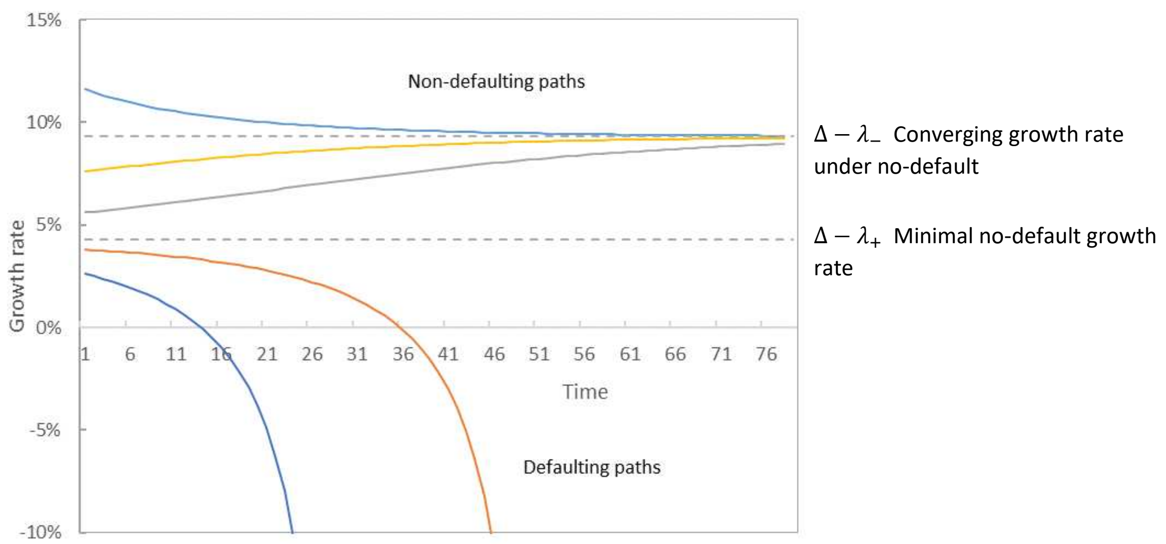
Define the growth rate of cash $\delta_t \equiv \frac{c_t}{c_{t-1}} - 1$, then it follows from equation (2) that

$$\delta_t = (\Delta - 1) - rD_{t-1} \quad (7)$$

It implies an equivalent Theorem 2 in terms of the growth rate. Specifically, there must also be a default boundary in the growth rate as well as a finite future limit of the growth rate in the case of no-default. Figure 3 summarizes the default boundary and the limit in the growth rate.

Even If a company borrows in a high borrowing ratio to cash, it still has to grow its sales without declining margin in order to avoid default. In the real world, there is a natural limit on how much a market can grow without profit starting to decline. If the natural growth rate is below the theoretical growth rate, a company still can default even with high level of borrowing[3]. In fact, defaults do occur more often during periods of low economic growth than high economic growth.

Figure 3. Growth rates of defaulting and non-defaulting paths ($L = 0.2, g = -5\%, r = 2\%$)



6. Real options: default criteria under growth-targeting borrowing ratio L

The current model implies complex deterministic dynamics of default even under the constant values of the model parameters. One certainly can extend the model so that these parameters become stochastic variables following some independent processes. However, it is worth considering first a deeper connection among these variables.

[3] One way to picture the situation is: if growth reaches maximal limit, higher borrowing must lead to reduced margin.

For one, the borrowing ratio L is a value that in reality is planned by a company to maximize certain goals or targets. In this sense, L is an option variable that reflects the value of a company's real options. One possible goal for which a company plans is the growth rate of sales $\delta_{s,t} \equiv \frac{s_t}{s_{t-1}} - 1$ where $s_t = (1 + L_t)(1 + g)c_{t-1}$ and L_t indicates that the borrowing ratio is no longer constant. Assuming $\delta_{s,t}$ is a constant target δ_s, then the value of L_t needed to achieve the target is given by

$$(8) \qquad 1 + L_t = \frac{(1 + \delta_s)(1 + L_{t-1})}{c_{t-1}/c_{t-2}} = \frac{(1 + \delta_s)^{t-1}(1 + L_1)}{c_{t-1}}$$

L_1 being an arbitrary initial constant[4].

Substituting (8) into (2) leads to the following nonhomogeneous linear equations:

$$(9) \qquad \vec{C}_t = \begin{pmatrix} r & -r \\ -1 & 1 \end{pmatrix} \vec{C}_{t-1} + (1 + L_1)(1 + \delta_s)^{t-1} \begin{pmatrix} 1 + g_t - r \\ 1 \end{pmatrix}$$

g_t being time-dependent in general.

It is interesting to see first the solution of (9) under a constant profit margin again ($g_t = g$). In this case, the solutions for $t \geq 1$ can be written as:

$$(10) \qquad \begin{aligned} c_t &= \left(r(1 - D) + \frac{(1 + L_1)r(g - r)}{r - \delta_s} \right)(1 + r)^{t-1} \\ &\quad + \left((1 + g - r)(1 + L_1) + \frac{(1 + L_1)r(g - r)}{r - \delta_s} \right)(1 + \delta_s)^{t-1} \\ d_t &= \left((D - 1) + \frac{(1 + L_1)(r - g)}{r - \delta_s} \right)(1 + r)^{t-1} \\ &\quad + \left((1 + L_1) - \frac{(1 + L_1)(r - g)}{r - \delta_s} \right)(1 + \delta_s)^{t-1} \end{aligned}$$

Without loss of much generality in the real world, it is always assumed in the following that $1 + g - r > 0$ and default does not occur immediately at $t = 1$. Then the following default criteria imply:

Default Criteria. If $\boldsymbol{g > r}$, default occurs iff $\boldsymbol{D > 1}$ and $\boldsymbol{\delta_s < r - \frac{(1+L_1)(g-r)}{D-1}}$. If $\boldsymbol{g < r}$ and $\boldsymbol{D < 1}$, default occurs iff $\boldsymbol{r - \frac{(1+L_1)(r-g)}{1-D} < \delta_s < r + \frac{r(r-g)}{1+g-r}}$. If $\boldsymbol{g < r}$ and $\boldsymbol{D > 1}$, default occurs iff $\boldsymbol{\delta_s < r + \frac{r(r-g)}{1+g-r}}$.

The time of default whenever applies is then given by

$$(11) \qquad T_D = \log_{\frac{1+\delta_s}{1+r}} \left(-\frac{A}{B} \right)$$

where A, B are respectively the coefficient to the term $(1 + r)^{t-1}$ and $(1 + \delta_s)^{t-1}$ in the solution (10) for c_t.

Figure 4 illustrates in the (g, δ_s) space the default boundaries in minimal margin g under a given value of r and several values of D. Within $g < r$, all values of D shares a common default boundary shown by the dashed lie (slightly uprising under decreasing g). The default boundary then bifurcates under different values of D as a straight line emanating from $(g = r, \delta_s = r)$. From $D \to \infty$ to $D = 0$, the boundary sweeps clockwise from the horizontal line $(g \geq r, \delta_s = r)$ to the point $(g = 0, \delta_s = -Lr)$, becoming the vertical $(g = r, 0 \leq \delta_s < r)$ at $D = 1$.

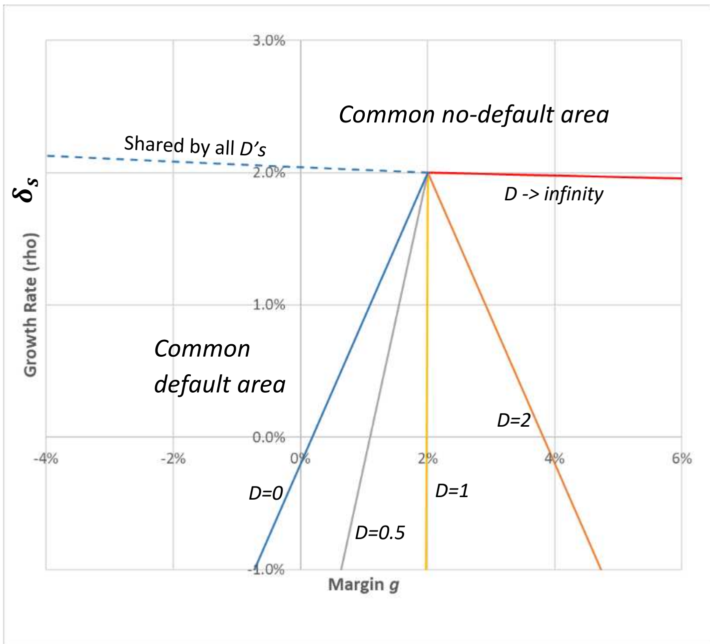
The result implies that under growth-planning, default is determined by the relative value of the three pairs of variables: g to r, D to 1, and δ_s to r. When $g > r$, if the current cash is more than debt, default never occurs

[4] So far, (8) is derived under constant g. In real life, g is a stochastic variable $\tilde{g}_t$, the value of which at time t is unknown to a company when it decides on the borrowing ratio L_t. However, if a company always assumes $\tilde{g}_t$ to equal to the latest observed value: $\tilde{g}_t = g_{t-1}$, then (8) again is valid.

regardless the growth rate δ_s, otherwise the growth rate δ_s has to be below r by some value for default to occur. When $g < r$, if the current debt is more than cash, default occurs if the growth rate δ_s is not sufficiently higher than r, but if the current cash is more than debt, default can occur if the growth rate δ_s is either not sufficiently higher or not sufficiently lower than r. It is interesting to see that the default boundary in δ_s is not exactly r, but above or below r by a distance determined by the difference of g to r and D to 1.

A default "death zone" again exists in (δ_s, g) within the region $(g < r)$ which is shown by the "common default area" in Figure 4. Overall, the default criteria under growth-targeting bear strong similarities to the default criteria discussed in section 6.

Figure 4. Default boundaries in the (g, δ_s) space under growth-targeting $(r = 2\%, L_1 = 0)$



The next 2 tables provide examples of default outcome for a table of (δ_s, g) values. The shaded cells represent no-default, and the others default with time to default provided. Although not completely comparable to the case of constant L, the much longer time to default below compared to **Table 1** and **Table 2** suggest that a company can deter default significantly by planning its borrowing to match the market growth rate.

Table 3. Time to default under growth targeting $(D = 0.5, r = 2\%)$

		g								
		-2.0%	-1.0%	0.0%	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
	4.0%									
rho	3.5%									
	3.0%									
	2.5%									
	2.0%	1,237	1,666	2,524	5,099					
	1.5%	409	467	553	720					
	1.0%	274	307	358	468					
	0.5%	213	237	277	385					
	0.0%	177	198	233						

Table 4. Time to default under growth targeting $(D = 5, r = 2\%)$

		g								
		-2.0%	-1.0%	0.0%	1.0%	2.0%	3.0%	4.0%	5.0%	6.0%
	4.0%									
rho	3.5%									
	3.0%									
	2.5%									
	2.0%	1,122	1,513	2,295	4,640	25,762,431				
	1.5%	313	345	385	437	514	649	2,703		
	1.0%	190	203	217	235	256	285	325	394	1,421
	0.5%	137	144	152	160	170	183	197	217	244
	0.0%	107	111	116	122	128	134	142	151	163

7. Stochastic profit margin g

If the profit margin g and the growth rate δ_s are time-dependent variables, but r is constant, the solution to growth-targeting model (9) is given by:

$$\begin{aligned}\vec{C}_t &= \begin{pmatrix} r & -r \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ D \end{pmatrix}(1+r)^{t-1} + (1+L_1)(1+\delta_{s,t})^{t-1}\begin{pmatrix} 1+g_t-r \\ 1 \end{pmatrix} \\ &+ \frac{1+L_1}{1+r}(1+\delta_{s,t})^{t-1}\begin{pmatrix} r & -r \\ -1 & 1 \end{pmatrix}\sum_{i=1}^{t-1}\left(\frac{1+r}{1+\delta_{s,t}}\right)^i\begin{pmatrix} 1+g_{t-i}-r \\ 1 \end{pmatrix}\end{aligned} \quad (12)$$

Both $\delta_{s,t}$ and g_t can be considered as stochastic variables following certain random distributions.

8. Conclusion

This article presented a simple example of redefining default modeling through realistic representation, the multi-dimensional balance sheet, of a company's financial state and dynamics of the balance sheet restrained by the principle of balancing. The example is simplistic but already capable of producing default criteria connectable to real-world experiences. The methodology shown here have been generalized to higher-dimensional balance sheet of actual companies with models calibrated to actual historical financial statements. The current work is a first step in bringing fundamental credit analysis into the quantitative modeling of default.