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Abstract

This paper re-examines the conditions under which endogenous economic growth can emerge in neoclassical models with non-renewable resources. Our analysis is based on a general production function which encompasses the Cobb-Douglas specification. We show that endogenous growth is possible only when the elasticity of substitution between effective labour input and effective resource input is constant and equal to one. If this does not hold (as some empirical studies suggested), then economic growth is solely driven by an exogenous technological factor. We also show that the assumption on this elasticity will affect the model's policy implications in regard to resource taxation.

Keywords: Non-Renewable Resources; Endogenous Growth; Knife-Edge Condition; Elasticity of Substitution.

JEL classification: O13, O41, Q32.

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1 Introduction

Economists have long been concerned with natural resource scarcity and its implications on economic growth. In a seminal paper, Stiglitz (1974) examines these issues using the now-familiar neoclassical growth model with infinitely-lived consumers. It is shown that perpetual growth in per-capita output is sustainable even when natural resources are limited in quantity but essential for production. Most importantly for the present study, Stiglitz's model is one of endogenous growth. This means the economic growth rate is not *a priori* determined by some exogenous technological factors, but rather it is derived within the model and can potentially be influenced by the choices of consumers, firms and the government. In a more recent study, Agnani, Gutiérrez and Iza (2005, henceforth AGI) show that this endogenous growth result can also be obtained in a similar neoclassical framework but with overlapping generations of finitely-lived consumers. These findings have far-reaching implications for both resource economics and economic growth theory, as they suggest that practices and policies in natural resource management can influence the long-term prospects of an economy. One such policy is resource taxation.[1] Existing studies typically focus on the microeconomic aspects of resource tax, e.g., how such a tax will affect a mining firm's exploration and extraction decisions.[2] Very few have examined the impact of resource taxation on the wider economy and economic growth.[3] For most resource-producing countries, resource taxation is a significant part of the economy. Bornhorst *et al.* (2009) report that for a sample of 30 resource-producing countries (with various degrees of economic development), resource taxation on average accounts for 49.1% of total government revenues and 16.2% of GDP over the period 1992-2005. The sheer scale of this type of taxation warrants a thorough understanding on how it will impact the wider economy.

At the same time, there is a separate strand of empirical research which estimates the elasticity of substitution between capital input, labour input and resource input (typically proxied by commercial energy consumption) in the production process.[4] While the estimates produced by this literature may vary across datasets and estimation methods, the general consensus is that the elasticity of substitution between any two of these inputs is not equal to one [see, for instance,

[1] Similar to Boadway and Keen (2010), we use the term "resource taxation" broadly to include also other types of revenues that governments collect from the extraction and utilisation of natural resources (such as royalties and equity sharing arrangements).

[2] See, for instance, Gaudet and Lasserre (2015) for a review of the theoretical literature.

[3] An exception is Groth and Schou (2007), which examine the growth effects of capital income tax and resouce tax in a model with infinitely-lived consumers. Obviously, there is a large literature that examines how pollution tax or carbon tax can curb the negative externalities (pollution) generated by resource-related economic activities. We do not consider this type of externality in the present study.

[4] See van der Werf (2008) and Henningsen *et al.* (2019) for reviews of this literature and discussions on different estimation strategies.

Kemfert (1998), Kemfert and Welsch (2000), van der Werf (2008), Su *et al.* (2012) and Henningsen *et al.* (2019)]. These findings are directly at odd with one of the key assumptions shared by Stiglitz (1974) and AGI, namely a Cobb-Douglas production function in these three inputs.[5] Motivated by these empirical findings and the real-world significance of resource taxation, the present study aims at answering two questions: First, will the endogenous growth result in Stiglitz (1974) and AGI remain valid under more general and empirically plausible specifications of production technology? Second, how will this affect the model's policy implications, especially in regard to resource taxation? These are fundamental questions in understanding the relations between natural resource management and economic growth. Yet, they have been largely overlooked by existing studies. The present study is intended to fill this gap.

In order to address these two questions in the most direct manner, we adopt the same theoretical framework as in AGI and make only two necessary changes. First, instead of assuming a Cobb-Douglas production function, we consider several more general specifications that are in line with empirical research. Second, a constant flat tax on resource input is introduced. In our benchmark model, we begin with a general class of constant-returns-to-scale (CRTS) production technology in which capital input is functionally separable from effective labour input and effective resource input.[6] The elasticity of substitution between the two effective inputs is denoted by σ_G, while the elasticity of substitution between capital input and these inputs is denoted by σ_F. In AGI's specification, both σ_G and σ_F are constant and equal to one. These restrictions are removed in our benchmark specification. Similar to AGI, we focus on characterising balanced growth equilibria. In particular, we show that two types of balanced growth equilibria are possible, depending on the value of σ_G. These results are formally stated in Theorems 1 and 2 in Section 3. Our first theorem shows that if σ_G is constant and equal to one, then the common growth factor in a balanced growth equilibrium is endogenously determined as in AGI. This does not require any restrictions on σ_F, which can be variable and different from one. Hence, our Theorem 1 subsumes and generalises the endogenous growth result in AGI. Our second theorem then considers what will happen if σ_G is *not* equal to one (in particular, it can be variable but bounded above or below by one). We show that in this case, perpetual economic growth is solely driven by an exogenous labour-augmenting technological factor as in the standard neoclassical growth model. Taken together,

[5] This is equivalent to assuming that the elasticity of substitution between any two of these inputs is constant and equal to one.

[6] Effective labour input is defined as (raw) labour input multiplied by a labour-augmenting technological factor. Similarly, effective resource input is defined as (raw) resource input multiplied by a resource-augmenting technological factor. The terminology and definition of "functional separability" are taken from Leontief (1947) and Blackorby and Russell (1976). Further details are provided in Section 2. We have also considered two other forms of functionally separability and the results are shown in Section 4.2.

these two theorems showcase the central role of σ_G in generating endogenous economic growth.[7]

The main ideas behind these results are as follows: As is well-known in the economic growth literature, perpetual growth in per-capita output requires certain factor (either exogenous or endogenous) that can counteract the diminishing marginal return of physical capital [Jones and Manuelli (1997, Section 2)]. Such a factor is dubbed as the "engine of growth." In our benchmark model, if σ_G is constant and equal to one, then total factor productivity (TFP) and resource input jointly serve as the engine of growth. While the growth rate of TFP is assumed to be exogenous, the utilisation rate of non-renewable resources is endogenously determined. This opens up a door through which other factors (such as consumers' preferences and government policies) can affect the engine of growth. But if σ_G is not equal to one, then a CRTS production technology will require effective resource input and effective labour input to grow at the same rate in balanced growth equilibria. This imposes a restriction on the utilisation rate of non-renewable resources. In particular, this rate is now pinned down by the exogenous growth rate of labour input and technological factors. As a result, the engine of growth is solely determined by exogenous forces.

The present study is also related a growing literature which shows that, in most (if not all) of the existing economic growth models, balanced growth equilibria are possible only under some "knife-edge" conditions [see, for instance, Groth and Schou (2002), Growiec (2007) and Bugajewski and Maćkowiak (2015)]. These existing studies are primarily concerned about balanced growth equilibria in general, without distinguishing between exogenous and endogenous growth. This distinction, however, is the subject of our analysis. In particular, our results suggest that even if the conditions for balanced growth equilibria are met (e.g., production function exhibits CRTS), endogenous growth will require yet another "knife-edge" condition (namely, a unitary elasticity of substitution between certain inputs).

Despite the simplicity of our benchmark model, it is able to produce a rich set of predictions regarding the effects of resource taxation. The two elasticities, σ_G and σ_F, again play a critical role in this matter. To sharpen our predictions, we adopt a two-stage, nested constant-elasticity-of-substitution (CES) production function in this part of the analysis (i.e., both σ_G and σ_F are now scalar parameters). Nested CES production functions are commonly used in models with more than two productive inputs.[8] We present two main findings regarding resource taxation. First, if

[7] These results are robust to several changes in the benchmark model. For instance, in Section 4.2 we show that the exogenous growth result will prevail under several other specifications of the production function. In a separate online Mathematical Appendix (which is available from the authors' personal website), we show that Theorem 1 and Theorem 2 can be easily extended to an environment with infinitely-lived consumers as in Stiglitz (1974).

[8] The influential work of Krusell *et al.* (2000) has inspired a large literature that uses nested CES functions to study issues related to capital-skill complementary. Ready (2018) and Hassler *et al.* (2021) are two recent studies that use nested CES function to study issues related to energy economics and environmental economics.

σ_G is one and σ_F is no less than one, then a unique balanced growth equilibrium exists under some additional conditions (Proposition 1) and resource taxation is growth-enhancing (Proposition 2). The intuition is as follows: A higher resource tax rate means that it is now more costly to use resource input for production. This will deter the utilisation of non-renewable resources. As a result, a large stock of resources will be available for future use. By the complementarity between capital input and resource input in the production process, this will raise the marginal product of capital and the return from capital holdings. This will then promote capital accumulation and economic growth. If σ_G is one but σ_F is *strictly less* than one, then multiple balanced growth equilibria may emerge and resource tax is either growth-enhancing or growth-prohibiting depending on the equilibrium in question. This is shown by means of a numerical example. Our second major finding is that if σ_G is *not* equal to one, then any changes in resource tax will only affect the level of per-capita variables but not their growth rate (Proposition 3). There are again two possible scenarios, depending on the value σ_G. If σ_G is strictly less than one, then an increase in the resource tax rate will raise the after-tax price for resource input and discourage its use. By the complementarity between capital input and resource input, this will then lower the level of physical capital in the balanced growth equilibrium. The opposite results hold if σ_G is strictly greater than one. These results demonstrate the importance of σ_G in analysing the effects of resource taxation.

Whether σ_G is equal to one is ultimately an empirical question. In the aforementioned empirical literature, studies typically report a less-than-unity elasticity of substitution between labour input and energy consumption [see, for instance, Kemfert (1998), Kemfert and Welsch (2000) and van der Werf (2008)]. When combining with these estimates, our benchmark model suggests that (i) any changes in the tax rate on resource input will have no effect on economic growth rate, and (ii) a higher resource tax rate will negatively impact capital formation and aggregate output. These predictions are in stark contrast to those obtained under a Cobb-Douglas production function.

The rest of the paper is organised as follows: Section 2 describes the setup of the benchmark model. Section 3 presents the baseline results concerning the balanced growth equilibria of the model. Section 4 provides some discussions and robustness checks on our baseline results. Section 5 concludes.

2 The Benchmark Model

2.1 Consumers

Our benchmark model is built upon the two-period overlapping-generation model in AGI, but with a more general specification of production function and a flat tax on resource input. Unless otherwise stated, we will adopt the same notation as in AGI to facilitate comparison between the two works.

Time is discrete and is indexed by $t \in \{0, 1, 2, ...\}$. In each time period, a new generation of identical consumers is born. The size of generation t is given by $N_t = (1 + n)^t$, where $n \geq 0$ is the population growth rate. Each consumer lives two periods, which we will refer to as the young age and the old age. All young consumers have one unit of time which is supplied inelastically to work. The market wage rate at time t is denoted by w_t. All consumers are retired when old. There are two types of commodities in this economy: a composite good which can be used for consumption and capital accumulation, and non-renewable natural resources which are primarily used as input of production. All prices are expressed in units of the composite good.

Consider a consumer who is born at time $t \geq 0$. Let $c_{1,t}$ and $c_{2,t+1}$ denote his young-age and old-age consumption, respectively. The consumer's lifetime utility is given by

$$U(c_{1,t}, c_{2,t+1}) \equiv \ln c_{1,t} + \frac{1}{1+\theta} \ln c_{2,t+1}, \tag{1}$$

where $\theta > 0$ is the rate of time preference. The consumer can accumulate wealth by investing in physical capital and natural resources. Let s_t and m_t denote, respectively, the consumer's holdings of physical capital and natural resources. The rate of return from physical capital is denoted by r_{t+1}, and the price of natural resources at time t is p_t.

Taking $\{w_t, r_{t+1}, p_t, p_{t+1}\}$ as given, the consumer's problem is to choose a consumption profile $\{c_{1,t}, c_{2,t+1}\}$ and an investment portfolio $\{s_t, m_t\}$ so as to maximise his lifetime utility in (1), subject to the budget constraints:

$$c_{1,t} + s_t + p_t m_t = w_t, \qquad \text{and} \qquad c_{2,t+1} = (1 + r_{t+1}) s_t + p_{t+1} m_t. \tag{2}$$

The first-order conditions of this problem can be expressed as

$$c_{2,t+1} = \left(\frac{1 + r_{t+1}}{1 + \theta}\right) c_{1,t}, \tag{3}$$

$$\frac{p_{t+1}}{p_t} = 1 + r_{t+1}. \tag{4}$$

Equation (3) is the familiar Euler equation of consumption, which determines the growth rate of individual consumption between young and old ages. Equation (4) is the Hotelling rule, which can be interpreted as a no-arbitrage condition. It states that in order for the consumer to invest in both types of assets, the capital gain from natural resources must be equal to the gross return from physical capital. Using (2)-(4), we can derive the optimal level of consumption,

$$c_{1,t} = \left(\frac{1+\theta}{2+\theta}\right) w_t \qquad \text{and} \qquad c_{2,t+1} = \left(\frac{1+r_{t+1}}{2+\theta}\right) w_t, \tag{5}$$

and the optimal level of investment in physical capital,

$$s_t = \frac{w_t}{2+\theta} - p_t m_t. \tag{6}$$

2.2 Production

On the supply side of the economy, there is a large number of identical firms that produce the composite good. In every period $t \geq 0$, each firm hires labour (N_t), rents physical capital (K_t) and purchases extracts of natural resources (X_t) from the competitive factor markets, and produces output (Y_t) according to the production technology

$$Y_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right). \tag{7}$$

In the above expression, Q_t is a resource-augmenting technological factor and A_t is a labour-augmenting technological factor.[9] Both are assumed to grow at some constant exogenous rate, so that $Q_t = (1+q)^t$ and $A_t = (1+a)^t$, with $q > 0$ and $a \geq 0$, for all $t \geq 0$.

The production function in (7) is a composition of two functions, $F(\cdot)$ and $G(\cdot)$. Intuitively, one can interpret this as a two-stage production process: In the first stage, effective labour input $(A_t N_t)$ and effective resource input $(Q_t X_t)$ are combined using an aggregator function $G(\cdot)$. The resultant is then combined with physical capital under another aggregator function $F(\cdot)$ to produce the final output. To use the terminology of Leontief (1947) and Blackorby and Russell (1976, p.286), the subset of inputs $\{Q_t X_t, A_t N_t\}$ is *functionally separable* from K_t. The assumption of functional separability is desirable because it can help simplify the analysis but without sacrificing

[9] The notion of resource-augmenting (or resource-saving) technological factor is not new. See, for instance, Stiglitz (1974, p.128), and more recently, Hassler *et al.* (2021).

much generality. In the empirical literature that estimates the elasticity of substitution between capital input, labour input and resource input, researchers typically adopt a nested CES production function which embodies the assumption of functional separability.

With three productive inputs, there is more than one way to define functional separability. Another possibility is to assume that $\{K_t, Q_t X_t\}$ is functionally separable from $A_t N_t$. A third possibility is to assume that $\{K_t, A_t N_t\}$ is functionally separable from $Q_t X_t$. Among these three possibilities, only the one in (7) can generate endogenous economic growth. Hence, we choose this as our benchmark specification. The other two possibilities are analysed in Section 4.2.

The main properties of $F(\cdot)$ and $G(\cdot)$ are summarised in Assumptions A1 and A2. Recall that an input is deemed *essential for production* if output cannot be produced without this input [Dasgupta and Heal (1974) and Solow (1974, p.34)]. Throughout this paper, we will use $F_i(\cdot)$ to denote the partial derivative of $F(\cdot)$ with respect to its ith argument, $i \in \{1, 2\}$. The partial derivatives of $G(\cdot)$ are similarly represented.

Assumption A1 Both $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ are twice continuously differentiable, strictly increasing, strictly concave and exhibit constant returns to scale (CRTS) in their arguments.

Assumption A2 Each input $I \in \{K, X, N\}$ is either essential for production or its marginal product is unbounded when I is arbitrarily close to zero.

Assumption A1 is a list of commonly used conditions in the economic growth literature. These conditions imply that the composite function in (7) is also twice continuously differentiable, strictly increasing, strictly concave and exhibits CRTS in all three inputs. In neoclassical growth models (without natural resources), it is also common to impose two other assumptions on the production function: First, both physical capital and labour are essential for production. Second, the marginal product of these inputs are unbounded as their quantity approach zero (aka the Inada conditions). These assumptions, however, are rather restrictive. For instance, within the class of CES production functions, only Cobb-Douglas production functions satisfy both of these assumptions.[10] Our Assumption A2 gets around this problem by requiring only one of these properties to hold. This is sufficient to ensure that in equilibrium all three inputs are used in every time period.[11] The

[10] The same point has been made by Dasgupta and Heal (1974, p.14) and Solow (1974, p.34) in natural resource economics. Solow (1974) cites this as the main reason for using Cobb-Douglas production function in his work.

[11] Since resource input is used (or needed) for production in every period, this means the non-renewable resource stock will never be exhausted in equilibrium. This assumption is not only used in Stiglitz (1974) and Agnani, Gutiérrez and Iza (2005), but also in other growth models with non-renewable resources, such as Scholz and Ziemes (1999), Grimaud and Rougé (2003) and Groth and Schou (2007). If resource input is non-essential for production and the resource stock is completely exhausted at some time $T < \infty$, then from time $T + 1$ onwards our model is

argument goes as follows: As suggested by Solow (1974), it is natural and reasonable to focus on equilibria that have a strictly positive amount of final output in every period. If an input is deemed essential for production, then a strictly positive amount must always be used in this kind of equilibria. On the other hand, since both factor markets and goods markets are perfectly competitive, the price of any input must be equated to its marginal product in equilibrium. If the marginal product of an input is unbounded at or around zero, then the marginal benefit of using an infinitesimal amount will certainly outweigh the marginal cost. Hence, it is never optimal to use a zero quantity of this input.

Assumptions A1 and A2 are satisfied by the two-stage CES production functions proposed by Sato (1967).[12] This class of functions can be obtained by setting

$$F\left(K_t, Z_t\right) = \left[\alpha K_t^{\eta} + \left(1 - \alpha\right) Z_t^{\eta}\right]^{\frac{1}{\eta}}, \qquad \text{with } \alpha \in (0,1) \text{ and } \eta < 1, \tag{8}$$

$$G\left(Q_t X_t, A_t N_t\right) \equiv \left[\phi \left(Q_t X_t\right)^{\psi} + \left(1 - \phi\right)\left(A_t N_t\right)^{\psi}\right]^{\frac{1}{\psi}}, \qquad \text{with } \phi \in (0,1) \text{ and } \psi < 1. \tag{9}$$

The production function in AGI corresponds to the special case in which $\eta = \psi = 0$. Under this "double Cobb-Douglas" specification, the two technological factors A_t and Q_t are observationally equivalent to a single Hicks neutral technological factor (or total factor productivity). For this reason, the separate effects of A_t and Q_t are not considered by AGI but will be discussed later in the present study. We want to stress that our main results (Theorems 1 and 2) are valid for any $F(\cdot)$ and $G(\cdot)$ that satisfy Assumptions A1 and A2, not just the CES functions in (8) and (9).

Since the production function exhibits CRTS in all three inputs, we can focus on the profit-maximisation problem faced by a single representative firm. Let R_t be the rental price of physical capital at time t and $\delta \in (0,1)$ be the depreciation rate. Expenditures on natural resource input are subject to a constant flat tax $\mu \geq 0$, which is of finite value.[13] Taking $\{R_t, w_t, p_t, \mu\}$ as given, the representative firm solves the following problem:

$$\max_{K_t, X_t, N_t} \left\{F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) - R_t K_t - \left(1 + \mu\right) p_t X_t - w_t N_t\right\}.$$

equivalent to the standard neoclassical growth model without natural resources. It follows that long-term economic growth will be solely driven by A_t and resource taxation will be completely irrelevant.

[12] In Section A of the online Mathematical Appendix, we verify that Assumption A2 is satisfied by all the nested CES production functions mentioned in this paper.

[13] This is also referred to as an *ad valorem* severence tax in resource economics. As explained in Groth and Schou (2007, p.83), this type of tax is closely related to the royalties collected by the government from resource extraction.

The first-order conditions are given by

$$R_t = r_t + \delta = F_1 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right), \tag{10}$$

$$(1+\mu) \, p_t = Q_t F_2 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right) G_1 \left(Q_t X_t, A_t N_t \right), \tag{11}$$

$$w_t = A_t F_2 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right) G_2 \left(Q_t X_t, A_t N_t \right). \tag{12}$$

Equation (11) states that the representative firm will choose a level of X_t so that its marginal product equals the after-tax price. The tax rate μ thus drives a wedge between the marginal product of X_t and the price received by the owners of natural resources (i.e., the consumers).

2.3 Natural Resources

The economy has a fixed and known stock of non-renewable natural resources which can be costlessly extracted in each time period. The initial size of the stock is denoted by $M_0 > 0$.[14] Let M_t be the stock available at the beginning of time t, and X_t be the quantity extracted and sold in the factor market at time t.[15] Define the utilisation rate at time t as $\tau_t \equiv X_t / M_t$. The stock of natural resources then evolves according to

$$M_{t+1} = M_t - X_t = (1 - \tau_t) \, M_t. \tag{13}$$

2.4 Competitive Equilibrium

All the revenues collected from the resource tax are spent on some "unproductive" government purchases. Let H_t be the level of unproductive government spending at time t.[16] The government's budget is balanced in every period, so that

$$H_t = \mu p_t X_t, \qquad \text{for all } t \geq 0. \tag{14}$$

Given a set of initial conditions, $K_0 > 0$ and $M_0 > 0$, and a constant tax rate $\mu \geq 0$, a competitive equilibrium of this economy includes sequences of allocation $\{c_{1,t}, c_{2,t+1}, s_t, m_t\}_{t=0}^{\infty}$,

[14] At time 0, the initial stock of physical capital and natural resources are owned by a group of "initial old" consumers. The decision problem of these consumers is trivial and does not play any role in our main results.

[15] This notation is slightly different from the one in AGI. Specifically, these authors define M_t as the stock remaining at the end of time t (after extraction). This difference is immaterial since we both focus on balanced growth paths along which M_t depletes at a constant rate.

[16] This spending is deemed unproductive because it has no direct impact on the consumers' utility and the production of goods. Our main results remain valid if the tax revenues are redistributed evenly to the young consumers through a lump-sum transfer. The details of this are shown in Section 4.1.

aggregate inputs $\{K_t, N_t, X_t\}_{t=0}^{\infty}$, natural resources $\{M_t\}_{t=0}^{\infty}$, government spending $\{H_t\}_{t=0}^{\infty}$ and prices $\{w_t, R_t, p_t, r_{t+1}\}_{t=0}^{\infty}$ such that,

(i) Given prices, $\{c_{1,t}, c_{2,t+1}, s_t, m_t\}$ solves the consumer's problem at any time $t \geq 0$.

(ii) Given prices and the tax rate, $\{K_t, N_t, X_t\}$ solves the representative firm's problem at any time $t \geq 0$.

(iii) The stock of natural resources evolves according to (13).

(iv) The government's budget is balanced in every period, i.e., (14) holds.

(v) All markets clear in every period, which means $K_{t+1} = N_t s_t$ and $M_{t+1} = N_t m_t$ for all $t \geq 0$.

Using (6) and $M_{t+1} = N_t m_t$, we can write the capital market clearing condition as

$$K_{t+1} = \frac{w_t N_t}{2+\theta} - p_t M_{t+1}. \tag{15}$$

This shows that capital accumulation (i.e., $K_{t+1} > 0$) is possible only if $w_t N_t > (2+\theta)\, p_t M_{t+1} \geq 0$. Using (13) and the definition of τ_t, we can get

$$M_{t+1} = (1-\tau_t)\frac{M_t}{X_t} \cdot X_t = \left(\frac{1-\tau_t}{\tau_t}\right) X_t.$$

Substituting this, (11) and (12) into (15) gives

$$K_{t+1} = F_2\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right)\left[\frac{1}{2+\theta} A_t N_t G_2\left(Q_t X_t, A_t N_t\right) - \frac{1}{1+\mu}\left(\frac{1-\tau_t}{\tau_t}\right) Q_t X_t G_1\left(Q_t X_t, A_t N_t\right)\right]. \tag{16}$$

We will use this version of capital market clearing condition repeatedly in the proof of our results.

3 Baseline Results

In order to provide a valid comparison with the findings of AGI, we restrict our attention to the same type of equilibrium paths that they have considered. These equilibrium paths have the following properties: (a) All variables grow at a constant rate over time. (b) The rate of return from physical capital remains constant over time. (c) Certain "big ratios", such as the ratio between physical capital and aggregate output, are constant over time. (d) All factor income shares are strictly positive and unchanged over time. (e) The utilisation rate of non-renewable resources is strictly positive and constant over time.

Using the "double Cobb-Douglas" specification, AGI show that (a) implies (b), (c) and (e), while (d) follows immediately from the Cobb-Douglas assumption. This approach, however, is invalid under the general production function in our benchmark model. To see this, consider the first-order condition (11) from the representative firm's problem. Since both $F(\cdot)$ and $G(\cdot)$ exhibit CRTS, we can rewrite this condition as

$$(1+\mu)\, p_t = Q_t F_2\left(\widehat{k}_t, G\left(\widehat{x}_t, 1\right)\right) G_1\left(\widehat{x}_t, 1\right),$$

where $\widehat{k}_t \equiv K_t / \left(A_t N_t\right)$ and $\widehat{x}_t \equiv \left(Q_t X_t\right) / \left(A_t N_t\right).$ The Hotelling rule in (4) then implies

$$\frac{p_{t+1}}{p_t} = \frac{Q_{t+1}}{Q_t} \frac{F_2\left(\widehat{k}_{t+1}, G\left(\widehat{x}_{t+1}, 1\right)\right) G_1\left(\widehat{x}_{t+1}, 1\right)}{F_2\left(\widehat{k}_t, G\left(\widehat{x}_t, 1\right)\right) G_1\left(\widehat{x}_t, 1\right)} = 1 + r_{t+1}.$$

If both $F(\cdot)$ and $G(\cdot)$ are Cobb-Douglas, e.g., by setting $\eta = \psi = 0$ in (8) and (9), then the above equation can be simplified to become

$$(1+q)\left(\frac{\widehat{k}_{t+1}}{\widehat{k}_t}\right)^{\alpha}\left(\frac{\widehat{x}_{t+1}}{\widehat{x}_t}\right)^{\phi(1-\alpha)-1} = 1 + r_{t+1},$$

where $q > 0$ is the exogenous growth rate of Q_t. Hence, if $\widehat{k}_t$ and $\widehat{x}_t$ are growing at some constant rate, then the rate of return from physical capital (r_{t+1}) must be constant.[17] This line of argument, however, is invalid without the "double Cobb-Douglas" assumption. There is also no guarantee that all factor income shares are strictly positive and time-invariant under a general production function, *even if* aggregate output and all the inputs are growing at some constant rate.

In order to circumvent these problems, we have devised a novel approach in characterising the solution of this model. Specifically, we focus on *any equilibrium paths* that satisfy the following properties [in addition to (i)-(v) mentioned above]:

(vi) Per-worker output (Y_t/N_t) grows at a constant rate $\gamma^* - 1$, for some $\gamma^* > 0$, in every period.

(vii) The rate of return from physical capital is constant over time, i.e., $r_t = r^*$, for some $r^* > -\delta$.

(viii) The utilisation rate of non-renewable resources is strictly positive and constant over time, i.e., $\tau_t = \tau^*$, for some $\tau^* \in (0, 1)$.

Conditions (vi) and (vii) are consistent with the empirical observations made by Kaldor (1963) and many subsequent studies in the economic growth literature. Condition (viii) is commonly

[17] See Appendix A of Agnani, Gutiérrez and Iza (2005, p.401).

used in economic growth models with natural resources.[18] We show in Theorems 1 and 2 that any equilibrium paths that satisfy these conditions will also satisfy properties (a), (c) and (d) mentioned above, hence belong to the same type of *balanced growth equilibria* considered by AGI. Our approach can be summarised as follows: First, given the simple linear structure of (13), condition (viii) implies that X_t and M_t must be decreasing at the same constant rate, i.e.,

$$\frac{X_{t+1}}{X_t} = \frac{M_{t+1}}{M_t} = 1 - \tau^*.$$

Second, a constant growth rate of p_t is implied by the Hotelling rule in (4) and a constant r^*. Third, under Assumptions A1 and A2, a constant interest rate implies a constant ratio between K_t and Y_t. This result is formally established in Lemma 1. The proof of this and other baseline results can be found in the Appendix of the paper. Fourth, using these results we can show that all factor income shares must be strictly positive and time-invariant. Finally, it is shown that the remaining variables, such as wage rate and individual consumption, will grow at a constant rate.

Lemma 1 *Suppose the production function in (7) satisfies Assumptions A1 and A2. Then condition (vii) implies the existence of a positive constant κ^* such that $K_t = \kappa^* Y_t$ for all t. This means Y_t and K_t must be growing at the same rate over time.*

Before proceeding further, we first review the fundamental results in AGI, where government intervention is absent (i.e., $\mu = 0$). According to their Lemma 1 and Proposition 1, if the production function is given by

$$Y_t = B_t K_t^{\alpha} N_t^{\beta} X_t^{\nu},$$

where $\alpha > 0$, $\beta > 0$, $\nu > 0$, $\alpha + \beta + \nu = 1$, and B_t is a measure of total factor productivity (TFP) that grows exogenously at a constant rate $b > 0$, then a unique balanced growth equilibrium exists in which per-worker output, per-worker capital, individual consumption and wage rate all grow at the same rate. The common growth factor γ^* and the utilisation rate τ^* are jointly determined by

$$\frac{\gamma^* (1+n)}{(1-\tau^*)} = \frac{\alpha (1+n)(2+\theta)\gamma^*}{\beta - (2+\theta)\nu(1-\tau^*)/\tau^*} + 1 - \delta, \tag{17}$$

$$\gamma^* = (1+b)^{\frac{1}{1-\alpha}} \left(\frac{1-\tau^*}{1+n}\right)^{\frac{\nu}{1-\alpha}}. \tag{18}$$

[18] Stiglitz (1974) and Groth and Shou (2007) are among the studies that consider equilibria with a constant extraction rate. Scholz and Ziemes (1999) and Grimaud and Rougé (2003) are two examples that consider equilibria with a constant growth rate of X_t. These two conditions are equivalent given (13).

Once τ^* and γ^* are known, the value of r^* and κ^* are determined by

$$1 + r^* = \frac{\gamma^*(1+n)}{1-\tau^*} \qquad \text{and} \qquad \kappa^* = \frac{\alpha}{r^* + \delta}. \tag{19}$$

In the rest of the paper, we will refer to this as the AGI solution or the endogenous growth solution.

The main implication of the AGI solution is that both τ^* and γ^* are jointly determined by a host of factors, including the TFP growth rate (b), population growth rate (n), depreciation rate (δ), the share of factor incomes in total output (α, β and ν), and the consumers' rate of time preference (θ). If we decompose B_t according to $B_t \equiv Q_t^{\nu} A_t^{\beta}$ and define $\widehat{k}_t \equiv K_t / (A_t N_t)$ and $\widehat{x}_t \equiv (Q_t X_t) / (A_t N_t)$, then the AGI solution also implies

$$\frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \left(\frac{\widehat{x}_{t+1}}{\widehat{x}_t}\right)^{\frac{\nu}{1-\alpha}} = \left[\frac{(1+q)(1-\tau^*)}{(1+a)(1+n)}\right]^{\frac{\nu}{1-\alpha}}. \tag{20}$$

Thus, depending on the solution of (17)-(18), $\widehat{k}_t$ and $\widehat{x}_t$ can be monotonically increasing, monotonically decreasing or constant over time in the unique balanced growth equilibrium.

To comprehend the significance of these findings, consider an alternate economy with $\nu = 0$ in AGI's production function. This means natural resources are no longer needed in the production process and, as a result, $B_t \equiv A_t^{1-\alpha}$. In this case, a constant r_t immediately implies a constant $\widehat{k}_t$. This in turn implies that per-worker capital and per-worker output must be growing at the same rate as A_t, so that $\gamma^* = (1+a)$. This can also be seen by setting $\tau^* = 0$ and $\nu = 0$ in equations (18) and (20). This result is well-known in the economic growth literature: In the standard neoclassical growth model where production function exhibits CRTS in K_t and $A_t N_t$, perpetual growth in per-capita variables is entirely driven by the exogenous labour-augmenting technological factor. This result holds regardless of whether the consumers are infinitely-lived or have a finite lifetime.

When compared to this alternative economy, the AGI solution shows that introducing productive natural resources can transform an otherwise exogenous growth model into one featuring endogenous growth. If, in addition, the solution of (17)-(18) satisfies $(1+q)(1-\tau^*) > (1+a)(1+n)$, then per-capita variables will grow at a faster rate than the technological factor A_t, i.e., $\gamma^* > 1 + a$.

We now return to the question of whether the AGI solution remains valid under a more general production function. Our main theorems provide an answer to this question based on the composite function in (7). Let $G(\cdot)$ be a general function that satisfies Assumptions A1 and A2. At the core of the analysis is the elasticity of substitution between the inputs of $G(\cdot)$. This can be defined using the function $g(\widehat{x}) \equiv G(\widehat{x}, 1)$ for $\widehat{x} \geq 0$, where $\widehat{x} \equiv QX/(AN)$. By the CRTS property of $G(\cdot)$, we can write

$$G(QX, AN) = AN \cdot g(\widehat{x}).$$

Under Assumption A1, $g(\cdot)$ is twice continuously differentiable with $g'(\cdot) > 0$ and $g''(\cdot) < 0$. As shown in Arrow *et al.* (1961) and Palivos and Karagiannis (2010), the elasticity of substitution of $G(\cdot)$ can be expressed as[19]

$$\sigma_G(\widehat{x}) = -\frac{g'(\widehat{x})}{\widehat{x}g(\widehat{x})}\frac{g(\widehat{x}) - \widehat{x}g'(\widehat{x})}{g''(\widehat{x})} > 0, \qquad \text{for all } \widehat{x} > 0. \tag{21}$$

In particular, $G(\cdot)$ is Cobb-Douglas if and only if $\sigma_G(\cdot)$ is identical to one.

Given that Y_t and K_t are growing at the same rate (Lemma 1), the homogeneity of $F(\cdot)$ implies that $Z_t \equiv G(Q_t X_t, A_t N_t)$ must be growing at the same rate as well, i.e.,

$$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \frac{Z_{t+1}}{Z_t} = \gamma^*(1+n). \tag{22}$$

If $G(\cdot)$ takes a Cobb-Douglas form as in

$$Z_t = G(Q_t X_t, A_t N_t) = (Q_t X_t)^{1-\phi}(A_t N_t)^{\phi}, \qquad \text{with } \phi \in (0,1), \tag{23}$$

then the growth factor of Z_t is a weighted geometric average of the growth factor of $Q_t X_t$ and $A_t N_t$, i.e.,

$$\frac{Z_{t+1}}{Z_t} = \left(\frac{Q_{t+1}X_{t+1}}{Q_t X_t}\right)^{1-\phi}\left(\frac{A_{t+1}N_{t+1}}{A_t N_t}\right)^{\phi}$$
$$\Rightarrow \gamma^*(1+n) = [(1+q)(1-\tau^*)]^{1-\phi}[(1+a)(1+n)]^{\phi}. \tag{24}$$

Obviously, this equation alone is not enough to pin down the two endogenous variables γ^* and τ^*. The extra degree of freedom is what makes the endogenous growth solution possible. In the current model, γ^* and τ^* are jointly determined by equation (24) and the capital market clearing condition in (16). Hence, any factors that appear in these two conditions (which include preference parameters and the resource tax) will affect economic growth. These ideas are encapsulated in our Theorem 1. Note that the results in Theorem 1 are valid even if $F(\cdot)$ is not Cobb-Douglas. Thus, our Theorem 1 subsumes and generalises the endogenous growth solution in AGI. The effect of

[19] The derivation of (21) rests upon two assumptions: (i) the factor markets and goods markets are perfectly competitive and (ii) $G(\cdot)$ exhibits CRTS [see Arrow *et al.* (1961, p.228-229)]. Both assumptions are satisfied in our model.

Table 1 Summary of Baseline Results

	$F(\cdot)$	$G(\cdot)$	Findings
Theorem 1	General	Cobb-Douglas	Characterisation of the endogenous growth solution.
Proposition 1*	CES with $\sigma_F \geq 1$	Cobb-Douglas	Existence and uniqueness of endogenous growth solution.
Proposition 2*	CES with $\sigma_F \geq 1$	Cobb-Douglas	Resource taxation is growth-promoting.
Theorem 2*	General	General ($\sigma_G \neq 1$)	Characterisation of the exogenous growth solution.
Proposition 3 (i)*	CES	CES	Existence and uniqueness of exogenous growth solution.
Proposition 3 (ii)*	CES	CES	Increase in μ will raise $\widehat{x}^*$ and $\widehat{k}^*$ if $\sigma_G > 1$.
			Increase in μ will lower $\widehat{x}^*$ and $\widehat{k}^*$ if $\sigma_G < 1$.

* Additional assumptions on parameter values are necessary.

resource taxation on this type of balanced-growth equilibria is analysed in Proposition 2.

On the other hand, if $\sigma_G(\cdot)$ is never equal to one (which means it is either uniformly bounded above or uniformly bounded below by one), then condition (22) is satisfied only if $\{Z_t, Q_t X_t, A_t N_t\}$ are all growing at the same rate, so that

$$\gamma^* (1+n) = (1+q)(1-\tau^*) = (1+a)(1+n). \tag{25}$$

These equations uniquely pin down the value of γ^* and τ^*. In particular, the growth rate of per-worker output is now solely determined by the growth rate of A_t, i.e., $\gamma^* = 1+a$. Hence, the endogenous growth solution is no longer valid. We refer to this type of balanced-growth equilibria as exogenous growth solutions and further characterise them in Theorem 2. One immediate implication is that the resource tax μ can only affect the *level* of economic variables in an exogenous growth solution but not their growth rate. The effects of μ are formally stated in Proposition 3. Table 1 summarises the main findings of our benchmark model.

Theorem 1 *Suppose $F(\cdot)$ satisfies Assumptions A1 and A2 and $G(\cdot)$ takes the Cobb-Douglas form in (23). Define $b \equiv (1+a)^{\phi}(1+q)^{1-\phi} - 1$. Then any equilibrium that satisfies conditions (vi)-(viii), if exists, must also satisfy*

$$\gamma^* = (1+b)\left(\frac{1-\tau^*}{1+n}\right)^{1-\phi}, \tag{26}$$

$$(1+r^*)(1-\tau^*) = \gamma^*(1+n), \tag{27}$$

$$\gamma^*(1+n) = \chi^* F_2(1,\chi^*)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right)\right], \tag{28}$$

$$F_1(1,\chi^*) = r^* + \delta. \tag{29}$$

In addition, all factor income shares are strictly positive and time-invariant, while wage rate and individual consumption grow at the same rate as per-worker output.

Theorem 1 describes a balanced growth equilibrium that is similar in spirit to the AGI solution. This equilibrium is characterised by four key variables, namely the growth factor of per-worker output (γ^*), the utilisation rate of natural resources (τ^*), the rate of return from physical capital (r^*) and the ratio between $(\widehat{x}_t)^{1-\phi}$ and $\widehat{k}_t$ (denoted by χ^*). All other variables can be uniquely determined using these four values. Similar to the AGI solution, the utilisation rate τ^* must be greater than a certain threshold $\overline{\tau}(\mu) \in (0,1)$ which depends on μ. To see this, note that both $\gamma^*(1+n)$ and $\chi^* F_2(1,\chi^*)$ are strictly positive, hence (28) requires

$$\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right) > 0$$

$$\Rightarrow \tau^* > \overline{\tau}(\mu) \equiv \frac{(2+\theta)(1-\phi)}{\phi(1+\mu) + (2+\theta)(1-\phi)} \in (0,1). \tag{30}$$

It is obvious from (30) that $\overline{\tau}(\mu)$ is strictly decreasing in μ.

Note that equation (26) dictates an inverse relationship between γ^* and τ^*. This can be explained as follows: A lower utilisation rate of natural resources means that the resource stock will deplete at a slower pace. Thus, a larger stock of natural resources will be available in each subsequent time period. By the complementarity between capital input and resource input in the production function, such a change will raise the marginal product of capital and the return from capital holding in all future time periods. This will in turn promote capital accumulation and economic growth. As we will see below, this inverse relationship is specific to the endogenous growth solution and it can help us understand the growth effect of resource taxation.

Since the balanced growth path in AGI also satisfies conditions (vi)-(viii), it can be recovered from the system of equations in Theorem 1. First, by setting $\mu = 0$ and $F(K_t, Z_t) = K_t^{\alpha} Z_t^{1-\alpha}$, with $\alpha \in (0,1)$, we can get

$$\chi^* = \left(\frac{r^*+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad \chi^* F_2(1,\chi^*) = \frac{1-\alpha}{\alpha}(r^*+\delta).$$

Upon substituting these into (28) and setting $\phi = \beta/(1-\alpha)$ and $(1-\phi) = \nu/(1-\alpha)$, we get

$$\gamma^*(1+n) = \frac{1}{\alpha}(r^*+\delta)\left[\frac{\beta-(2+\theta)\nu(1-\tau^*)/\tau^*}{2+\theta}\right].$$

This, together with (27), gives us

$$\frac{\alpha(2+\theta)(1+n)\gamma^*}{\beta-(2+\theta)\nu(1-\tau^*)/\tau^*} = r^*+\delta = \frac{\gamma^*(1+n)}{1-\tau^*} - (1-\delta),$$

which is the same equation as in AGI's Lemma 1 part (i). According to their Proposition 1, a unique balanced growth equilibrium exists when both $F(\cdot)$ and $G(\cdot)$ are Cobb-Douglas. We now show that a unique equilibrium can be obtained if $F(\cdot)$ is a CES function with elasticity of substitution strictly greater than one and $\mu \geq 0$.

Proposition 1 *Suppose $F(\cdot)$ takes the CES form in (8) with elasticity of substitution $\sigma_F \equiv (1-\eta)^{-1} \geq 1$ and $G(\cdot)$ takes the Cobb-Douglas form in (23). Then the economy has at least one balanced growth equilibrium that satisfies (26)-(29). If, in addition,*

$$\left\{(1+b)\left[\frac{1+n}{1-\overline{\tau}(\mu)}\right]^{\phi} - (1-\delta)\right\}^{\eta} > \alpha(1-\eta)^{1-\eta}, \tag{31}$$

where $\overline{\tau}(\mu)$ is the threshold level defined in (30), then a unique balanced growth equilibrium exists.

We now turn to the effect of resource taxation on the endogenous growth solution. Consider two economies that are otherwise identical except for the tax rate on resource input, denoted by $\mu_2 > \mu_1 \geq 0$. In both economies, $F(\cdot)$ takes the CES form in (8) with elasticity of substitution $\sigma_F \geq 1$ and $G(\cdot)$ takes the Cobb-Douglas form in (23). Suppose a unique balanced growth equilibrium exists in both economies.[20] Let τ_i^* and γ_i^* denote, respectively, the equilibrium utilisation rate and common growth factor in the economy with tax rate μ_i, for $i \in \{1,2\}$. Then the economy with a higher tax rate will also have a faster growth rate, i.e., $\gamma_2^* > \gamma_1^*$ for any $\mu_2 > \mu_1 \geq 0$. In other words, resource taxation is *growth-enhancing*. This result is formally stated in Proposition 2. The intuition behind this is straight-forward: Increasing the tax rate μ will raise the cost of resource input and discourage utilisation, i.e., $\tau_2^* < \tau_1^*$ for any $\mu_2 > \mu_1 \geq 0$. A higher growth rate then follows from the inverse relationship between τ^* and γ^* described earlier.

[20] It suffice to assume that condition (31) is satisfied under the higher tax rate, i.e., $\mu_2 > 0$. The details of this are shown in the proof of Proposition 2.

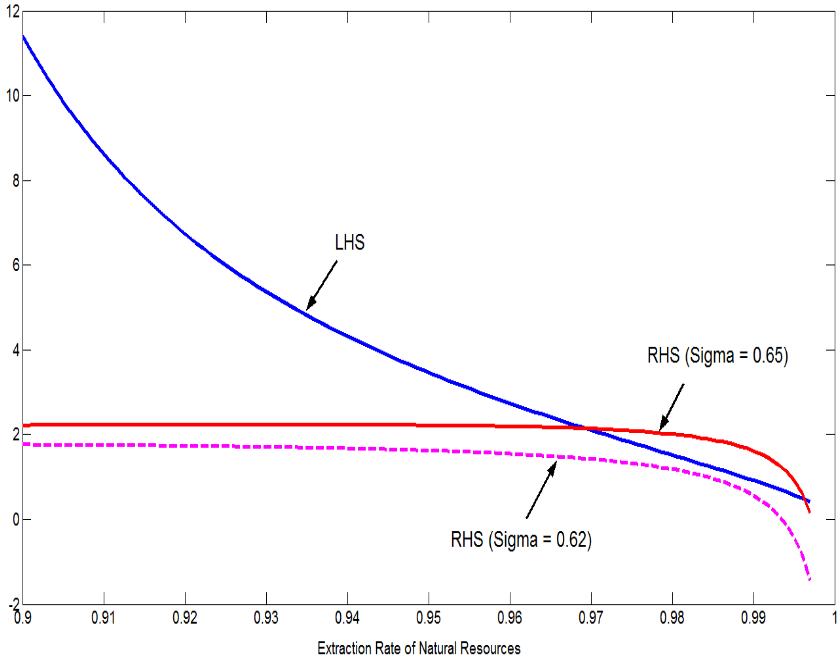

Figure 1 Numerical Example

Proposition 2 *Suppose $F(\cdot)$ takes the CES form in (8) with elasticity of substitution $\sigma_F \equiv (1-\eta)^{-1} \geq 1$ and $G(\cdot)$ takes the Cobb-Douglas form in (23). Suppose the condition in (31) is satisfied under $\mu_2 > 0$. Then $\tau_2^* < \tau_1^*$ and $\gamma_2^* > \gamma_1^*$, for any $\mu_2 > \mu_1 \geq 0$.*

Next, we turn to the case when $\sigma_F < 1$ (or equivalently, $\eta < 0$). It turns out to be more difficult to ensure the existence and uniqueness of balanced growth equilibrium in this case. This is because slight changes in σ_F within this range (strictly less than one) can potentially lead to drastic changes in equilibrium outcomes. The following numerical example is intended to demonstrate this. First, we combine equations (26)-(29) to form a single equation in τ^*, which is

$$\frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau^*)^{1-\phi}}{\phi - \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)(1-\phi)(1+\mu)^{-1}} = \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}, \qquad (32)$$

where $r(\tau^*) \equiv (1+b)(1+n)^{\phi}(1-\tau^*)^{-\phi} - 1$. We then evaluate both sides of this equation over a range of τ using the following parameterisation: Suppose one model period takes 25 years. We set $\theta = 1.775$ so that the annual subjective discount factor is 0.96. We set the annual employment growth rate to 1.6%, which matches the average annual growth rate of U.S. employment over the period 1953-2008. This implies $n = (1.0160)^{25} - 1 = 0.4871$. The annual TFP growth rate is taken to be 1.05%, which is in line with the estimates reported by Feng and Serletis (2008, p.300). The implied value of b is 0.2984 over a 25-year period. We also set $\mu = 0$, $\delta = 1$, $\phi = 0.38$ and $\alpha = 0.24$. Figure 1 plots the left-hand side (LHS) and the right-hand side (RHS) of equation (32) under two different values of σ_F, namely 0.62 and 0.65. Both fall within the range of estimates reported by Henningsen *et al.* (2019, Table 4).[21] As shown in the diagram, equation (32) has no solution when $\sigma_F = 0.62$ $(\eta = -0.613)$, which means there is no equilibrium that satisfies conditions (vi)-(viii). But when σ_F is raised to 0.65 ($\eta = -0.538$), the same equation has at least two solutions, which are $\tau^* = 0.9695$ and $\tau^* = 0.9964$. The possibility of multiple equilibria, however, does not alter the fundamental nature of the endogenous growth solution — in each of these equilibria, the common growth factor γ^* is determined by a host of factors.

When there are more than one balanced growth equilibria, the effects of resource tax may differ across equilibria. For instance, consider the case when $\sigma_F = 0.65$ in the above example. Let (τ_1^*, γ_1^*) and (τ_2^*, γ_2^*) denote the two balanced growth equilibria, with $\tau_1^* < \tau_2^*$. It follows from (26) that $\gamma_1^* > \gamma_2^*$. Note that the resource tax μ only appears on the left-hand side of (32). In particular, any increase in μ will shift the LHS curve in Figure 1 down but leave the RHS curve unaffected. It follows that a small increase in μ will lower the value of τ_1^* and raise the value of γ_1^*, but have the opposite effects on (τ_2^*, γ_2^*).

So far we have only considered the case when $\sigma_G(\cdot)$ is identical to one. In the rest of this section, we will focus on the case when $\sigma_G(\widehat{x}) \neq 1$ for all $\widehat{x} > 0$. The main results are presented in Theorem 2, which holds for any general $F(\cdot)$ and $G(\cdot)$ that satisfy Assumptions A1 and A2.[22]

Theorem 2 *Suppose the production function in (7) satisfies Assumptions A1 and A2, and the elasticity of substitution of $G(\cdot)$ is bounded above or below by one. Suppose further that $1 + q > (1 + a)(1 + n)$. Then any equilibrium that satisfies conditions (vi)-(viii), if exists, must also satisfy $\gamma^* = 1 + a$, $r^* = q$, and*

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}. \tag{33}$$

Such an equilibrium will have $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$ for all t, where $\widehat{k}^*$ and $\widehat{x}^*$ are determined by

$$F_1\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) = q + \delta, \tag{34}$$

[21] In Henningsen *et al.* (2019, Table 4), the elasticity of substitution between the inputs of $F(\cdot)$ is denoted by $\sigma_{(LE)K}$, where E stands for commerical energy consumption (as a proxy for natural resource input).

[22] The results in Theorem 2 can be obtained under a slightly weaker set of conditions. In particular, the same results will hold for any equilibrium that satisfies condition (vi) and (viii) and has a constant ratio between K_t and Y_t. The details of this are available from the authors upon request.

$$(1+a)(1+n)\,\widehat{k}^{*}=F_{2}\left(\widehat{k}^{*},G\left(\widehat{x}^{*},1\right)\right)\left[\frac{G_{2}\left(\widehat{x}^{*},1\right)}{2+\theta}-\left(\frac{1-\tau^{*}}{\tau^{*}}\right)\frac{\widehat{x}^{*}G_{1}\left(\widehat{x}^{*},1\right)}{1+\mu}\right]. \tag{35}$$

In addition, all factor income shares are strictly positive and time-invariant, while wage rate and individual consumption grow at the same rate as per-worker output.

Theorem 2 presents a balanced growth equilibrium that is in stark contrast to the AGI solution. Specifically, if $\sigma_G(\cdot)$ is bounded away from one, then either there is no equilibrium that satisfies conditions (vi)-(viii) or any such equilibrium will have a common growth rate in per-capita variables that is solely driven by the exogenous growth factor A_t. This theorem also highlights two important differences between the two technological factors A_t and Q_t. First, the growth rate of A_t determines the common growth factor (γ^*), while the growth rate of Q_t determines the rate of return from physical capital (r^*). This follows from the fact that in an exogenous growth solution, any changes in Q_t will be absorbed by the resource price p_t. This, together with the Hotelling rule, then implies that $r^* = q$. The second difference is that, holding other factors constant, a higher growth rate of A_t will suppress the utilisation rate τ^* while a higher growth rate of Q_t will promote it. This can be explained as follows: By the complementarity between Q_tX_t and A_tN_t in $G(\cdot)$, a higher growth rate of A_t will raise the marginal product of resource input in all future time periods (when other things are kept constant). This will induce an intertemporal substitution in resource utilisation by shifting the demand from the current period to the future periods. Such a shift will slow down the depletion of the resource stock, which is equivalent to lowering the value of τ^*. On the contrary, a higher growth rate of Q_t will lower the marginal product of resource input in all future time periods and have the opposite effect on τ^*.

Since τ^* must be confined between zero and one, it is necessary to impose the restriction $1+q > (1+a)(1+n)$. This means the growth rate of resource-augmenting technological factor must be strictly positive, even when there is no population growth (i.e., $n = 0$) and no labour-augmenting technological progress (i.e., $a = 0$). Intuitively, this is saying that a minimum degree of resource-augmenting technological progress is necessary in order to compensate for the decline in X_t over time and make perpetual economic growth possible.

To sharpen our understanding of the exogenous growth solution, we focus on the case when $F(\cdot)$ and $G(\cdot)$ take the CES form in (8) and (9). Define an auxiliary notation Θ according to

$$\Theta \equiv \frac{q+\delta}{\alpha(2+\theta)}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}}-\alpha\right].$$

The first part of Proposition 3 establishes the existence and uniqueness of the exogenous growth solution, while the second part summarises the effects of resource taxation. Since $\{\tau^*, \gamma^*, r^*\}$ are all independent of μ, any changes in this tax rate will only affect the level of $\widehat{x}^*$ and $\widehat{k}^*$.

Proposition 3 *Suppose* $F(\cdot)$ and $G(\cdot)$ *take the CES form in (8) and (9), respectively. Suppose further that* $\min\{\Theta, 1+q\} > (1+a)(1+n)$. *Then the following results hold.*

(i) *There exists a unique balanced growth equilibrium that satisfies* $\gamma^* = 1+a$, $r^* = q$, *and (33)-(35).*

(ii) *An increase in* μ *will raise the value of* $\widehat{x}^*$ *and* $\widehat{k}^*$ *if* $\sigma_G \equiv (1-\psi)^{-1} > 1$ *and lower their value if* $\sigma_G \equiv (1-\psi)^{-1} < 1$.

The results in the second part of Proposition 3 can be explained as follows: In the unique balanced-growth equilibrium with $r^* = q$, the first-order condition in (10) can be rewritten as

$$F_1\left[1, \frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right] = q + \delta.$$

This shows that the ratio $G(\widehat{x}^*, 1)/\widehat{k}^*$ is independent of μ, which in turn implies that any changes in μ will affect $\widehat{x}^*$ and $\widehat{k}^*$ in the same direction. Hence, it suffice to focus on the effect on $\widehat{x}^*$. Holding other things constant, an increase in the resource tax rate will raise the cost of resource input and discourage the utilisation of natural resources. But as shown in (33), τ^* in the exogenous growth solution is independent of μ . This means the before-tax input price p_t must decrease in order to promote utilisation and neutralise the effect of μ. The overall effect on resource input $\widehat{x}^*$ then depends on how responsive p_t is to μ. This is determined by the elasticity of substitution σ_G. In order to gain further insight into this, first rewrite (11) as

$$(1+\mu)p_t = Q_t F_2\left[1, \frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right] G_1(\widehat{x}^*, 1)$$

$$\Rightarrow \frac{d}{d\mu}[(1+\mu)p_t] = Q_t F_2\left[1, \frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right] G_{11}(\widehat{x}^*, 1)\frac{d\widehat{x}^*}{d\mu}, \tag{36}$$

where $G_{11}(\widehat{x}^*, 1) < 0$ by the concavity of $G(\cdot)$. Next, using both (11) and (12), we can get

$$\frac{(1+\mu)p_t}{w_t} = \frac{Q_t}{A_t}\frac{G_1(\widehat{x}^*, 1)}{G_2(\widehat{x}^*, 1)} = \frac{\phi}{1-\phi}\frac{Q_t}{A_t}(\widehat{x}^*)^{\psi-1}.$$

The second equality uses the CES production function in (9). It follows that

$$\frac{d}{d\mu}\left[\frac{(1+\mu)\,p_t}{w_t}\right] = \left[\frac{\phi}{1-\phi}\frac{Q_t}{A_t}\right]\underbrace{(\psi-1)}_{(-)}(\widehat{x}^*)^{\psi-2}\frac{d\widehat{x}^*}{d\mu}. \tag{37}$$

The results in part (ii) of Proposition 3 can now be interpreted as follows: If the elasticity of substitution σ_G is sufficiently high (i.e., $\sigma_G > 1$), then an increase in μ will induce a large reduction in p_t so that both the after-tax input price $(1+\mu)\,p_t$ and the relative input price $(1+\mu)\,p_t/w_t$ will fall, and more resource input will be used (i.e., $\widehat{x}^*$ will increase). On the contrary, if σ_G is less than one, then an increase in μ will raise both $(1+\mu)\,p_t$ and $(1+\mu)\,p_t/w_t$, and lower $\widehat{x}^*$.

Two final remarks are in order. First, Proposition 3 covers the special case in which $F(\cdot)$ and $G(\cdot)$ share the same constant elasticity of substitution, i.e., $\eta = \psi$. In this case, the production function in (7) becomes

$$Y_t = \left[\alpha K_t^{\eta} + (1-\alpha)\,\phi\,(Q_t X_t)^{\eta} + (1-\alpha)\,(1-\phi)\,(A_t N_t)^{\eta}\right]^{\frac{1}{\eta}},$$

which is the familiar Dixit–Stiglitz aggregator function. Second, the main results in Theorem 1 and Theorem 2 can be readily extended to an environment with infinitely-lived consumers.[23]

4 Further Results and Discussions

4.1 Alternative Use of Tax Revenues

Most of the theoretical results in Section 3 will remain valid if all the revenues collected from the resource tax are redistributed evenly among the young consumers through a lump-sum transfer.[24] Under this alternative arrangement, a young consumer at time t faces the following budget constraint:

$$c_{1,t} + s_t + p_t m_t = w_t + \xi_t,$$

where ξ_t is the transfer at time t. The consumer's optimal choices are now given by

$$c_{1,t} = \left(\frac{1+\theta}{2+\theta}\right)(w_t + \xi_t), \qquad c_{2,t+1} = \left(\frac{1+r_{t+1}}{2+\theta}\right)(w_t + \xi_t)$$

$$s_t = \frac{w_t + \xi_t}{2+\theta} - p_t m_t. \tag{38}$$

[23] The details are shown in Section C of the online Mathematical Appendix.

[24] Due to page limitations, we only highlight the key points here. Further details are available from the authors upon request.

The government's budget is balanced in every time period, so that

$$\mu p_t X_t = N_t \xi_t, \qquad \text{for all } t \geq 0. \tag{39}$$

The rest of the economy is the same as in the benchmark model.

Since the policy variables μ and ξ_t do not affect the production technology directly, most of the results in Theorem 1 and Theorem 2 will remain valid. Specifically, it remains the case that if the elasticity of substitution of $G(\cdot)$ is constant and equal to one, then the endogenous growth solution will prevail; but if this elasticity is bounded away from one, then γ^* and τ^* are again determined by (25). The proof of these statements are essentially the same as the proof of Theorem 1 and Theorem 2, hence they are not repeated here. The only parts that need to be modified are (28) and (35), which are derived from the capital-market-clearing condition. In particular, equation (28) in Theorem 1 will be replaced by

$$\gamma^* (1+n) = \chi^* F_2 (1, \chi^*) \left[\frac{\phi}{2+\theta} + \left(\frac{\mu}{2+\theta} - \frac{1-\tau^*}{\tau^*} \right) \frac{1-\phi}{1+\mu} \right]. \tag{40}$$

This equation also implies that τ^* must be greater than the threshold

$$\widetilde{\tau}(\mu) \equiv \frac{(1-\phi)(2+\theta)}{\phi + \mu + (1-\phi)(2+\theta)}, \tag{41}$$

which is strictly decreasing in μ. Similarly, equation (35) in Theorem 2 will be replaced by

$$(1+a)(1+n)\widehat{k}^* = F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) \left[\frac{G_2(\widehat{x}^*, 1)}{2+\theta} + \left(\frac{\mu}{2+\theta} - \frac{1-\tau^*}{\tau^*} \right) \frac{\widehat{x}^* G_1(\widehat{x}^*, 1)}{1+\mu} \right].$$

The results of Propositions 1-3 also remain valid, except for some minor changes. First, consider the case when $G(\cdot)$ is Cobb-Douglas and $F(\cdot)$ is a CES function with $\sigma_F \equiv (1-\eta)^{-1} \geq 1$. It can be shown that (i) a balanced growth equilibrium that satisfies (26), (27), (29) and (40) always exists; (ii) a unique balanced growth equilibrium exists if the condition below is satisfied,

$$\left\{ (1+b) \left[\frac{1+n}{1-\widetilde{\tau}(\mu)} \right]^{\phi} - (1-\delta) \right\}^{\eta} > \alpha (1-\eta)^{1-\eta},$$

where $\widetilde{\tau}(\mu)$ is the threshold defined in (41); and (iii) an increase in μ will lower the value of τ^* but increase the common growth factor γ^*.

Finally, consider the case when both $F(\cdot)$ and $G(\cdot)$ take the CES form as in Proposition 3. In the benchmark model, the value of $\widehat{x}^*$ is uniquely determined by

$$(\widehat{x}^*)^{\psi} = \frac{1-\phi}{\phi} \frac{\Theta - (1+a)(1+n)}{(1+a)(1+n) + \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\Theta}. \tag{42}$$

The derivation of (42) is shown in the proof of Proposition 3. When the tax revenues are refunded to the consumers, the value of $\widehat{x}^*$ is determined by

$$(\widehat{x}^*)^{\psi} = \frac{1-\phi}{\phi} \frac{\Theta - (1+a)(1+n)}{(1+a)(1+n) + \left[\left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right) - \frac{\mu}{1+\mu}\right]\Theta}. \tag{43}$$

In both settings, the utilisation rate τ^* is determined by (33). Note that the right-hand side of both (42) and (43) are strictly increasing in μ. Thus, an increase in μ will raise (or lower) the value of $\widehat{x}^*$ if $\psi > 0$ (or $\psi < 0$).

4.2 Alternative Specifications of Production Function

In this subsection, we will consider two alternative specifications of the production function. These are given by

$$Y_t = F\left(A_t N_t, G\left(K_t, Q_t X_t\right)\right), \tag{44}$$

$$Y_t = F\left(Q_t X_t, G\left(K_t, A_t N_t\right)\right). \tag{45}$$

To maintain consistency across all three specifications, we use $G(\cdot)$ to represent the "inner" aggregator function and $F(\cdot)$ to represent the "outer" aggregator function in (7), (44) and (45). All three specifications will coincide with AGI's production function if both $G(\cdot)$ and $F(\cdot)$ take the Cobb-Douglas form. Our main interest here is to examine the properties of balanced growth equilibrium when *one* of the aggregator functions in (44) and (45) *does not* take the Cobb-Douglas form. To this end, we consider four different parametric production functions based on (44) and (45). In the first two specifications, the inner aggregator function is Cobb-Douglas but the outer one has a CES form, so that

$$Y_t = \left\{\phi\left(A_t N_t\right)^{\psi} + (1-\phi)\left[K_t^{\alpha}\left(Q_t X_t\right)^{1-\alpha}\right]^{\psi}\right\}^{\frac{1}{\psi}}, \tag{46}$$

$$Y_t = \left\{\phi\left(Q_t X_t\right)^{\psi} + (1-\phi)\left[K_t^{\alpha}\left(A_t N_t\right)^{1-\alpha}\right]^{\psi}\right\}^{\frac{1}{\psi}}, \tag{47}$$

with $\alpha \in (0,1)$, $\phi \in (0,1)$ and $\psi < 1$. In the second group, the inner aggregator function is a CES function and the outer one is Cobb-Douglas, so that

$$Y_t = \left[\phi K_t^{\psi} + (1-\phi)(Q_t X_t)^{\psi}\right]^{\frac{1-\beta}{\psi}} (A_t N_t)^{\beta} \tag{48}$$

$$Y_t = (Q_t X_t)^{\nu} \left[\phi K_t^{\psi} + (1-\phi)(A_t N_t)^{\psi}\right]^{\frac{1-\nu}{\psi}}, \tag{49}$$

with $\beta \in (0,1)$, $\nu \in (0,1)$, $\phi \in (0,1)$ and $\psi < 1$. The parameters β and ν have the same economic meaning as in AGI. Specifically, they represent the share of total output distributed as labour income and expenses on natural resource input, respectively. The rest of the economy is the same as in the benchmark model. The main result of this subsection is summarised in Theorem 3.[25]

Theorem 3 *Suppose the production function takes one of the forms in (46)-(49). Then any equilibrium that satisfies conditions (vi)-(viii), if exists, must satisfy* $\gamma^* = 1 + a$, $r^* = q$, *and*

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}.$$

The main message of Theorem 3 is clear: despite the differences in appearance, all the production functions in (46)-(49) will deliver the same type of balanced growth equilibria in which the common growth factor γ^* is pinned down by the exogenous growth rate of A_t. Hence, there is no room for endogenous growth.

4.3 Comparison with Uzawa Growth Theorem

Our findings suggest that the AGI solution is valid only under the "knife-edge" condition of a unitary elasticity of substitution between effective labour input and effective resource input. If we rewrite (23) as

$$G(Q_t X_t, A_t N_t) = \left[A_t (Q_t X_t)^{\frac{1-\phi}{\phi}} N_t\right]^{\phi},$$

then the expression $\widetilde{X}_t \equiv A_t (Q_t X_t)^{\frac{1-\phi}{\phi}}$ can be viewed as a labour-augmenting factor and serves as the engine of growth. When viewed through this lens, our results suggest that the AGI solution

[25] This result is established by "brute force." Specifically, we show one by one that endogenous growth is not possible for all the specifications in (46)-(49). The proof is mostly technical in nature, and does not add much to the understanding of our main results. For this reason, the full proof is omitted here but it is shown in the online Mathematical Appendix.

is valid only when effective resource input is labour-augmenting in the production function, i.e.,

$$Y_t = F\left(K_t, \left(\widetilde{X}_t N_t\right)^{\phi}\right).$$

This result may remind one of the celebrated Uzawa Growth Theorem [Uzawa (1961)]. But there are at least two important differences between the two. First, the Uzawa Growth Theorem and its variants are typically derived from a CRTS production function with only two inputs, namely physical capital and labour [see, for instance, Uzawa (1961), Schlicht (2006), Jones and Scrimgeour (2008) and Grossman *et al.* (2017)]. It is not immediately clear how the Uzawa Growth Theorem can be extended to a general CRTS production function with more than two inputs, such as the one considered here. Second, and more importantly, the Uzawa Growth Theorem states the conditions under which balanced growth equilibria can emerge, without explicitly mentioning whether the "engine of growth" is exogenous or endogenous. The distinction between exogenous and endogenous growth, however, is the main focus of our analysis.

5 Conclusions

The primary objective of this study is to showcase the importance of σ_G [i.e., the elasticity of substitution between effective resource input (QX) and effective labour input (AN)] in generating endogenous growth and in policy analysis within the framework of Agnani, Gutiérrez and Iza (2005). Specifically, we show that for a number of empirically plausible specifications of production function, endogenous economic growth will emerge only if σ_G is constant and equal to one. This condition, however, has found little support in the empirical literature. For all other specifications that we have considered (some of which have been tested in empirical studies), the common growth rate in balanced growth equilibria is pinned down by the exogenous growth rate of labour-augmenting technological factor. This is not just a robustness check of AGI's result. We also show that the assumption on σ_G will affect the policy implications of the model. If we assume that σ_G is constant and equal to one, then an increase in resource tax rate will raise the economic growth rate in a balanced growth equilibrium. But if σ_G is strictly less than one (as some empirical studies suggested), then an increase in resource tax rate will suppress capital formation and lower aggregate output.

Appendix: Proofs

Proof of Lemma 1

Suppose there exists a real number $r^* > -\delta$ such that

$$F_1 \left[K_t, G\left(Q_t X_t, A_t N_t\right) \right] = F_1 \left[1, \frac{G\left(Q_t X_t, A_t N_t\right)}{K_t} \right] = r^* + \delta > 0.$$

The first equality follows from the homogeneity property of $F_1\left(\cdot\right).$ Since $F_1\left(1, \cdot\right)$ is continuous and strictly decreasing under Assumption A1, there exists a non-negative real number χ^* such that

$$\frac{G\left(Q_t X_t, A_t N_t\right)}{K_t} = \chi^* \geq 0.$$

Note that there are two possible cases: In this first one, $\chi^* = 0$ which can happen if $\lim_{k \to \infty} \sigma_F\left(k\right) > 1.$ A formal proof of this can be found in Palivos and Karagiannis (2010). Under this scenario, K_t is persistently growing at a higher rate than $G\left(Q_t X_t, A_t N_t\right).$ so that

$$\frac{K_t}{G\left(Q_t X_t, A_t N_t\right)} \to \infty, \qquad \text{as } t \to \infty.$$

But this also means that total labour income $\left(w_t N_t\right)$ will converge to zero, which violates the necessary condition for capital accumulation, i.e., $w_t N_t > \left(2 + \theta\right) p_t M_{t+1} \geq 0$. Hence, this type of "unbalanced" growth paths are not sustainable in equilibrium. In this second case, we have $\chi^* > 0$. By the homogeneity property of $F\left(\cdot\right),$ we can write

$$Y_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) = K_t F\left(1, \chi^*\right), \qquad \text{for all } t,$$

$$\Rightarrow K_t = \left[F\left(1, \chi^*\right)\right]^{-1} Y_t.$$

The desired results follow by setting $\kappa^* \equiv \left[F\left(1, \chi^*\right)\right]^{-1} > 0$. This completes the proof of Lemma 1.

Proof of Theorem 1

The proof is divided into a number of steps:

Step 1 This part of the proof uses the same line of argument as in Schlicht (2006) and Jones and Scrimgeour (2008). First, condition (vi) implies that aggregate output Y_t grows at a constant rate $\widehat{\gamma} \equiv \gamma^* \left(1 + n\right)$ in every period, i.e., $Y_{t+1} = \widehat{\gamma} Y_t$, for all t. Rearranging terms and applying the

CRTS property of $F(\cdot)$ gives

$$\begin{aligned} Y_t &= F\left(\widehat{\gamma}^{-1} K_{t+1}, \widehat{\gamma}^{-1} G\left(Q_{t+1} X_{t+1}, A_{t+1} N_{t+1}\right)\right) \\ &= F\left(K_t, \widehat{\gamma}^{-1} G\left(Q_{t+1} X_{t+1}, A_{t+1} N_{t+1}\right)\right). \end{aligned}$$

The second line uses the fact that K_t and Y_t must grow at the same rate, as per Lemma 1. For any given $K_t > 0$, $F(K_t, Z_t)$ is strictly increasing in Z_t. Hence, the following equality must hold in any equilibrium that satisfies condition (vi),

$$G\left(Q_t X_t, A_t N_t\right) = \widehat{\gamma}^{-1} G\left(Q_{t+1} X_{t+1}, A_{t+1} N_{t+1}\right). \tag{50}$$

Note that (50) holds regardless of whether $G(\cdot)$ is Cobb-Douglas.

Suppose now $G(\cdot)$ is given by

$$G\left(Q_t X_t, A_t N_t\right) = \left(Q_t X_t\right)^{1-\phi} \left(A_t N_t\right)^{\phi}, \qquad \text{for some } \phi \in (0,1).$$

Using this, together with $A_{t+1} = (1+a) A_t$, $Q_{t+1} = (1+q) Q_t$, $X_{t+1} = (1-\tau^*) X_t$ and $N_{t+1} = (1+n) N_t$, we can rewrite (50) as

$$\left(Q_t X_t\right)^{1-\phi} \left(A_t N_t\right)^{\phi} = \widehat{\gamma}^{-1} \left[(1+q)(1-\tau^*)\right]^{1-\phi} \left[(1+a)(1+n)\right]^{\phi} \left(Q_t X_t\right)^{1-\phi} \left(A_t N_t\right)^{\phi}. \tag{51}$$

Since $\left(Q_t X_t\right)^{1-\phi} \left(A_t N_t\right)^{\phi} > 0$, (51) is valid if and only if

$$\left[(1+q)(1-\tau^*)\right]^{1-\phi} \left[(1+a)(1+n)\right]^{\phi} = \widehat{\gamma} \equiv \gamma^* (1+n)$$

$$\Rightarrow \gamma^* = (1+a)^{\phi} \left[\frac{(1+q)(1-\tau^*)}{1+n}\right]^{1-\phi}.$$

This is equation (26) in the theorem.

Step 2 Next, we will show that given condition (vii), the ratio $p_t X_t / Y_t$ must be time-invariant and strictly positive. This can then be used to derive equation (27). Suppose $r_t = r^* > -\delta$. Then by (10), we have

$$F_1\left(1, \frac{G\left(Q_t X_t, A_t N_t\right)}{K_t}\right) = F_1\left(1, \frac{\widehat{x}_t^{1-\phi}}{\widehat{k}_t}\right) = r^* + \delta > 0.$$

Since $F_1(1,\cdot)$ is strictly decreasing, it follows that the ratio between $\widehat{x}_t^{1-\phi}$ and $\widehat{k}_t$ must be constant in any equilibrium that satisfies condition (vii). Hence, we can write

$$\frac{G(Q_tX_t, A_tN_t)}{K_t} = \frac{\widehat{x}_t^{1-\phi}}{\widehat{k}_t} = \chi^* > 0. \tag{52}$$

By the homogeneity properties of $F(\cdot)$ and $F_2(\cdot)$, we can write

$$F_2(K_t, G(Q_tX_t, A_tN_t)) = F_2(1, \chi^*),$$

$$F(K_t, G(Q_tX_t, A_tN_t)) = K_tF(1, \chi^*).$$

Using these and (11), we can get

$$\begin{aligned}
\frac{p_tX_t}{Y_t} &= \frac{1}{1+\mu}\frac{Q_tX_t}{K_t}\frac{F_2(1,\chi^*)G_1(Q_tX_t, A_tN_t)}{F(1,\chi^*)} \\
&= \frac{1}{1+\mu}\frac{F_2(1,\chi^*)}{F(1,\chi^*)}\frac{G(Q_tX_t, A_tN_t)}{K_t}\frac{Q_tX_tG_1(Q_tX_t, A_tN_t)}{G(Q_tX_t, A_tN_t)} \\
&= \left(\frac{1-\phi}{1+\mu}\right)\frac{\chi^*F_2(1,\chi^*)}{F(1,\chi^*)}.
\end{aligned}$$

The last equality follows from the Cobb-Douglas specification of $G(\cdot)$. Hence, p_tX_t/Y_t must be strictly positive and time-invariant. This in turn implies

$$\frac{p_{t+1}}{p_t}\frac{X_{t+1}}{X_t} = (1+r^*)(1-\tau^*) = \frac{Y_{t+1}}{Y_t} = \gamma^*(1+n).$$

Step 3 By Lemma 1, a constant r^* means that the ratio K_t/Y_t is also constant. It follows immediately that the capital income share $(r^*+\delta)K_t/Y_t$ must be time-invariant under condition (vi). Since the production function in (7) exhibits CRTS in all three inputs,

$$\frac{p_tX_t}{Y_t} + \frac{(r^*+\delta)K_t}{Y_t} + \frac{w_tN_t}{Y_t} = 1.$$

Hence, the labour income share must be constant over time as well.

Step 4 We now derive equation (28), which is based on the capital market clearing condition in (16). As shown in Step 2, we can rewrite $F_2(K_t, G(Q_tX_t, A_tN_t))$ as $F_2(1,\chi^*)$. Substituting this and $\tau_t = \tau^*$ into (16) gives

$$K_{t+1} = F_2(1,\chi^*)\left[\frac{1}{2+\theta}A_tN_tG_2(Q_tX_t, A_tN_t) - \frac{1}{1+\mu}\left(\frac{1-\tau^*}{\tau^*}\right)Q_tX_tG_1(Q_tX_t, A_tN_t)\right].$$

Using the Cobb-Douglas specification for $G(\cdot)$, we can further simplify this to become

$$K_{t+1} = F_2(1,\chi^*)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right)\right]G(Q_tX_t, A_tN_t).$$

Dividing both sides by K_t and using (52) gives

$$\frac{K_{t+1}}{K_t} = \gamma^*(1+n) = \chi^*F_2(1,\chi^*)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right)\right].$$

Step 5 Equation (5) implies that both $c_{1,t}$ and $c_{2,t}$ will grow at the same rate as w_t when r_t is time-invariant. Since labour income share is constant over time, w_t must be growing at the same rate as per-worker output. This completes the proof of Theorem 1.

Proof of Proposition 1

Using (26) and (27), we can get

$$\gamma^*(1+n) = (1+b)(1+n)^{\phi}(1-\tau^*)^{1-\phi},$$

$$r^* = (1+b)(1+n)^{\phi}(1-\tau^*)^{-\phi} - 1 \equiv r(\tau^*).$$

The CES function in (8) implies

$$F_1(1,\chi^*) = \alpha\left[\alpha + (1-\alpha)(\chi^*)^{\eta}\right]^{\frac{1-\eta}{\eta}}, \tag{53}$$

$$F_2(1,\chi^*) = (1-\alpha)(\chi^*)^{\eta-1}\left[\alpha + (1-\alpha)(\chi^*)^{\eta}\right]^{\frac{1-\eta}{\eta}}. \tag{54}$$

Combining (29) and (53) gives

$$(1-\alpha)(\chi^*)^{\eta} = \left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha. \tag{55}$$

Substituting (55) into (54) gives

$$\begin{aligned}\chi^* F_2(1,\chi^*) &= (1-\alpha)(\chi^*)^{\eta}\left[\alpha+(1-\alpha)(\chi^*)^{\eta}\right]^{\frac{1-\eta}{\eta}} \\ &= \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}}-\alpha\right\}.\end{aligned}$$

Using these expressions, we can rewrite (28) as

$$\frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau^*)^{1-\phi}}{\phi-\left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)(1-\phi)(1+\mu)^{-1}} = \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}}-\alpha\right\}.$$

A unique balanced growth equilibrium exists if there is a unique solution for this equation. Fix $\mu \geq 0$ and define two auxiliary functions $\Lambda(\cdot)$ and $\Gamma(\cdot)$ according to

$$\Lambda(\tau;\mu) \equiv \frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau)^{1-\phi}}{\phi-\left(\frac{1-\tau}{\tau}\right)(2+\theta)(1-\phi)(1+\mu)^{-1}}, \tag{56}$$

$$\Gamma(\tau) \equiv \frac{r(\tau)+\delta}{\alpha}\left\{\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}}-\alpha\right\}. \tag{57}$$

The following properties of $\Lambda(\cdot)$ can be easily verified: $\Lambda(1;\mu)=0$; $\Lambda(\tau;\mu)\to\infty$ as τ approaches $\overline{\tau}(\mu)$ from the right, where $\overline{\tau}(\mu)\in(0,1)$ is the threshold value defined in (30); $\Lambda(\tau;\mu)<0$ for all $\tau<\overline{\tau}(\mu)$; and $\Lambda(\tau;\mu)$ is strictly decreasing in τ over the range $(\overline{\tau}(\mu),1]$. Similarly, one can show that $\Gamma[\overline{\tau}(\mu)]<\infty$ and $\Gamma(\tau)\to\infty$ as $\tau\to 1$ if $\eta\in(0,1)$. Since both $\Lambda(\cdot;\mu)$ and $\Gamma(\cdot)$ are continuous functions in τ over the range between $\overline{\tau}(\mu)$ and one, these properties ensure the existence of at least one value $\tau^*\in(\overline{\tau}(\mu),1)$ such that $\Lambda(\tau^*;\mu)=\Gamma(\tau^*)$.

If, in addition, $\Gamma(\cdot)$ is strictly increasing between $\overline{\tau}(\mu)$ and one, then a unique solution exists. Straightforward differentiation gives

$$\Gamma'(\tau)=\frac{1}{\alpha}\left\{\frac{1}{1-\eta}\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}}-\alpha\right\}(1+b)(1+n)^{\phi}\phi(1-\tau)^{-(1+\phi)}.$$

Hence, $\Gamma'(\tau)\gtrless 0$ if and only if

$$\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}}\gtrless\alpha(1-\eta)\Leftrightarrow r(\tau)+\delta\gtrless\alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1}.$$

Since $r(\tau)$ is a strictly increasing function, it follows that $\Gamma(\cdot)$ is strictly increasing between $\overline{\tau}(\mu)$ and one if and only if $r[\overline{\tau}(\mu)]+\delta>\alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1}$. This condition can be rewritten as (31). A graphical illustration of the existence and uniqueness result is shown in Figure A1. This completes the proof of Proposition 1.

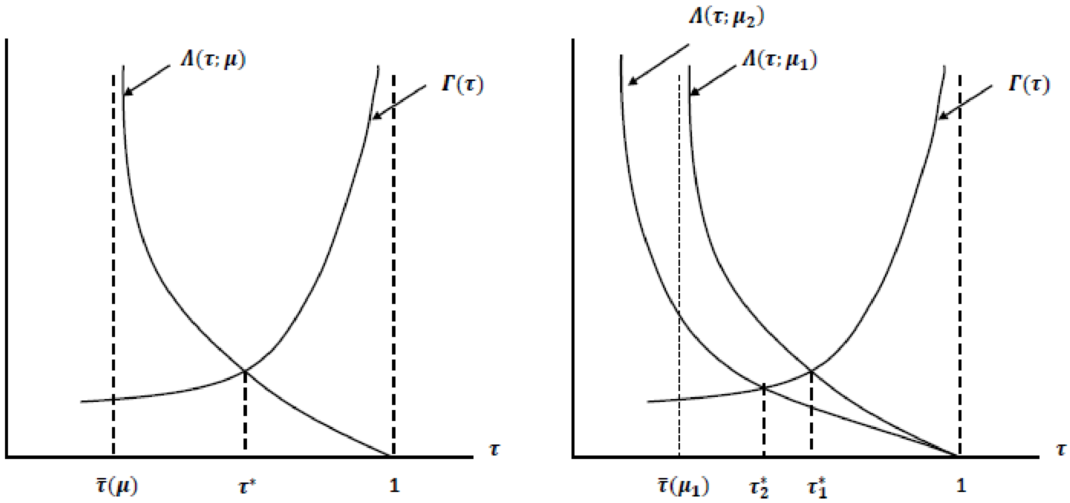

Figure A1: Existence and Uniqueness of Balanced Growth Equilibrium.

Proof of Proposition 2

Suppose condition (31) is satisfied under $\mu_2 > 0$, i.e.,

$$\left\{(1+b)\left[\frac{1+n}{1-\overline{\tau}(\mu_2)}\right]^{\phi} - (1-\delta)\right\}^{\eta} > \alpha(1-\eta)^{1-\eta}.$$

As shown in the proof of Proposition 1, this condition is sufficient to ensure the existence of a unique balanced growth equilibrium in the economy with tax rate μ_2. Rewrite the above condition as

$$r[\overline{\tau}(\mu_2)] + \delta > \alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1},$$

where $r(\tau) \equiv (1+b)(1+n)^{\phi}(1-\tau^*)^{-\phi} - 1$. Since $r(\cdot)$ is a strictly increasing function and $\overline{\tau}(\cdot)$ is strictly decreasing, it follows that

$$r[\overline{\tau}(\mu_1)] + \delta > r[\overline{\tau}(\mu_2)] + \delta > \alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1},$$

for any $\mu_2 > \mu_1 \geq 0$. Hence, (31) is also satisfied under μ_1, which ensures the existence of a unique balanced growth equilibrium in the economy with μ_1.

To establish the comparative statics result, first recall the auxiliary function $\Lambda(\tau;\mu)$ defined in (56). It is straightforward to verify that $\Lambda(\tau;\mu_2) < \Lambda(\tau;\mu_1)$ over the range $\overline{\tau}(\mu_1) \leq \tau < 1$. In addition, $\Lambda(1;\mu_2) = \Lambda(1;\mu_1) = 0$ and $\Lambda(\tau;\mu_1) \to \infty$ as $\tau \to \overline{\tau}(\mu_1)$. These conditions ensure that $\tau_2^* < \tau_1^*$ [see Figure A1]. Finally using (26), we can write

$$\gamma_2^* = (1+b)\left(\frac{1-\tau_2^*}{1+n}\right)^{1-\phi} > (1+b)\left(\frac{1-\tau_1^*}{1+n}\right)^{1-\phi} = \gamma_1^*.$$

This concludes the proof of Proposition 2.

Proof of Theorem 2

Step 1 First, we will show that $\gamma^* = 1+a$ if the elasticity of substitution of $G(\cdot)$ is never equal to one. Recall that equation (50) in the proof of Theorem 1 is valid even if $G(\cdot)$ is not Cobb-Douglas. Define $\widehat{x}_t \equiv Q_t X_t / (A_t N_t)$. Then by the CRTS property of $G(\cdot)$, equation (50) can be equivalently stated as

$$G(Q_t X_t, A_t N_t) = G\left[\frac{(1+q)(1-\tau^*)}{\widehat{\gamma}} Q_t X_t, \frac{(1+a)(1+n)}{\widehat{\gamma}} A_t N_t,\right]. \tag{58}$$

Define the following notations

$$\varsigma \equiv \frac{(1+a)(1+n)}{\widehat{\gamma}} \qquad \text{and} \qquad \varpi \equiv \frac{(1+q)(1-\tau^*)}{\widehat{\gamma}}.$$

Equation (58) is trivially satisfied if $\varsigma = \varpi = 1$, which immediately implies

$$\gamma^* = 1+a \qquad \text{and} \qquad 1-\tau^* = \frac{(1+a)(1+n)}{1+q}.$$

We now show that if $\sigma_G(\cdot) \neq 1$, then equation (58) holds if and only if $\varsigma = \varpi = 1$.

Dividing both sides of (58) by $\varsigma A_t N_t$ and using $g(\widehat{x}) \equiv G(\widehat{x}, 1)$ give

$$g(\widehat{x}_t) = \varsigma g\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right), \qquad \text{for all } \widehat{x}_t > 0. \tag{59}$$

We first establish an intermediate result: For any $\widehat{x} > 0$,

$$\frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless 0 \qquad \text{if and only if} \qquad \sigma_G(\widehat{x}) \gtrless 1.$$

To start, straightforward differentiation gives

$$\frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] = \frac{g'(\widehat{x})}{g(\widehat{x})} - \frac{\widehat{x}\left[g'(\widehat{x})\right]^2}{\left[g(\widehat{x})\right]^2} + \frac{\widehat{x}g''(\widehat{x})}{g(\widehat{x})}. \tag{60}$$

Next, using the expression in (21), $\sigma_G(\widehat{x}) \gtrless 1$ if and only if

$$\frac{g'(\widehat{x})\left[g(\widehat{x}) - \widehat{x}g'(\widehat{x})\right]}{g(\widehat{x})} \gtrless -\widehat{x}g''(\widehat{x})$$

$$\Leftrightarrow \frac{g'(\widehat{x})}{g(\widehat{x})}\left[1 - \frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless \frac{-\widehat{x}g''(\widehat{x})}{g(\widehat{x})}$$

$$\Leftrightarrow \frac{g'(\widehat{x})}{g(\widehat{x})} - \frac{\widehat{x}\left[g'(\widehat{x})\right]^2}{\left[g(\widehat{x})\right]^2} - \frac{\widehat{x}g''(\widehat{x})}{g(\widehat{x})} = \frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless 0. \tag{61}$$

This intermediate result says that if $\sigma_G(\cdot)$ is never equal to one, then $\widehat{x}g'(\widehat{x})/g(\widehat{x})$ must be either strictly increasing or strictly decreasing for all $\widehat{x} > 0$. We will now apply this result on (59).

Since $g(\cdot)$ is continuously differentiable and (59) holds for all $\widehat{x}_t > 0$, we can differentiate both sides of (59) with respect to $\widehat{x}_t$ and get

$$g'(\widehat{x}_t) = \varpi g'\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right).$$

Combining this and (59) gives

$$\frac{\widehat{x}_t g'(\widehat{x}_t)}{g(\widehat{x}_t)} = \frac{\frac{\varpi}{\varsigma}\widehat{x}_t g'\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right)}{g\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right)}. \tag{62}$$

As mentioned above, if $\sigma_G(\cdot)$ is never equal to one, then $\widehat{x}g'(\widehat{x})/g(\widehat{x})$ must be either strictly increasing or strictly decreasing for all $\widehat{x} > 0$. Hence, the equality in (62) holds if and only if $\varpi = \varsigma$. Using this, we can rewrite (59) as $g'(\widehat{x}_t) = \varpi g'(\widehat{x}_t)$, which implies that $\varpi = 1$.

Step 2 The equalities $\varsigma = \varpi = 1$ imply that $\widehat{k}_t$ and $\widehat{x}_t$ are time-invariant in any balanced growth equilibrium, i.e., $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$. Using these, we can rewrite (10) and (11) as

$$r^* + \delta = F_1\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)$$

$$(1+\mu)\, p_t = Q_t F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) G_1(\widehat{x}^*, 1).$$

Equation (4) can now be used to obtain $r^* = q$. Equation (34) then follows.

Step 3 By Lemma 1, a constant r^* means that the ratio K_t/Y_t is also constant. It follows immediately that the capital income share $(r^* + \delta) K_t/Y_t$ must be time-invariant under condition (vi). The share $p_t X_t / Y_t$ can be expressed as

$$\frac{p_t X_t}{Y_t} = \frac{1}{1+\mu} \frac{Q_t X_t}{A_t N_t} \frac{F_2\left(\widehat{k}^*, G\left(\widehat{x}^*, 1\right)\right) G_1\left(\widehat{x}^*, 1\right)}{F\left(\widehat{k}^*, G\left(\widehat{x}^*, 1\right)\right)} = \frac{\widehat{x}^*}{1+\mu} \frac{F_2\left(\widehat{k}^*, G\left(\widehat{x}^*, 1\right)\right) G_1\left(\widehat{x}^*, 1\right)}{F\left(\widehat{k}^*, G\left(\widehat{x}^*, 1\right)\right)},$$

which is constant over time. Finally, the CRTS property of (7) implies that labour income share must be constant over time as well.

Step 4 Dividing both sides of (16) by $A_t N_t$ gives

$$(1+a)(1+n)\widehat{k}_{t+1} = F_2\left(\widehat{k}_t, G\left(\widehat{x}_t, 1\right)\right)\left[\frac{1}{2+\theta} G_2\left(\widehat{x}_t, 1\right) - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\widehat{x}_t G_1\left(\widehat{x}_t, 1\right)}{1+\mu}\right].$$

Equation (35) can be obtained by setting $\widehat{k}_{t+1} = \widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$.

Step 5 Equation (5) implies that both $c_{1,t}$ and $c_{2,t}$ will grow at the same rate as w_t when r_t is time-invariant. Since labour income share is constant over time, w_t must be growing at the same rate as per-worker output. This completes the proof of Theorem 2.

Proof of Proposition 3

Part (i) Fix $\mu \geq 0$. Suppose $F(\cdot)$ takes the CES form in (8), with $\alpha \in (0,1)$ and $\eta < 1$. Then (34) can be rewritten as

$$\alpha\left\{\alpha + (1-\alpha)\left[\frac{G\left(\widehat{x}^*, 1\right)}{\widehat{k}^*}\right]^{\eta}\right\}^{\frac{1-\eta}{\eta}} = q+\delta$$

$$\Rightarrow (1-\alpha)\left[\frac{G\left(\widehat{x}^*, 1\right)}{\widehat{k}^*}\right]^{\eta} = \left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha \tag{63}$$

Using these, we can write

$$\frac{G\left(\widehat{x}^*, 1\right)}{\widehat{k}^*} F_2\left(\widehat{k}^*, G\left(\widehat{x}^*, 1\right)\right) = \frac{q+\delta}{\alpha}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right] \equiv (2+\theta)\,\Theta,$$

where Θ is the notation defined in the text, i.e.,

$$\Theta \equiv \frac{q+\delta}{(2+\theta)\,\alpha}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right].$$

Similarly, if $G(\cdot)$ takes the CES form in (9), then we can get

$$G_2(\widehat{x}^*, 1) = (1-\phi)\left[\phi(\widehat{x}^*)^{\psi} + 1 - \phi\right]^{\frac{1}{\psi}-1} = \frac{(1-\phi)\,G(\widehat{x}^*, 1)}{\phi(\widehat{x}^*)^{\psi} + 1 - \phi},$$

$$G_1(\widehat{x}^*, 1) = \frac{\phi(\widehat{x}^*)^{\psi-1}\,G(\widehat{x}^*, 1)}{\phi(\widehat{x}^*)^{\psi} + 1 - \phi}.$$

Based on these observations, we can rewrite (35) as

$$\begin{aligned}(1+a)(1+n)\left[\phi(\widehat{x}^*)^{\psi} + 1 - \phi\right] &= \frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)\left[\frac{1-\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\phi}{1+\mu}(\widehat{x}^*)^{\psi}\right] \\ &= \Theta\left[1 - \phi - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\phi(\widehat{x}^*)^{\psi}\right],\end{aligned}$$

which can be simplified to become

$$(\widehat{x}^*)^{\psi} = \frac{1-\phi}{\phi}\frac{\Theta - (1+a)(1+n)}{(1+a)(1+n) + \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\Theta}. \tag{64}$$

The purpose of the additional condition $\min\{\Theta, 1+q\} > (1+a)(1+n)$ is twofold: First, it ensures that a unique, strictly positive value of $\widehat{x}^*$ can be obtained from the above equation. Second, it ensures that $\tau^* \in (0,1)$.

Part (ii) Differentiating both sides of (64) with respect to $\widehat{x}^*$ and μ gives

$$\psi(\widehat{x}^*)^{\psi-1}\frac{d\widehat{x}^*}{d\mu} = \frac{1-\phi}{\phi}\frac{\Theta - (1+a)(1+n)}{\left[(1+a)(1+n) + \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\Theta\right]^2}\left(\frac{1-\tau^*}{\tau^*}\right)\frac{2+\theta}{(1+\mu)^2}\Theta.$$

Since the right-hand side of the above equation is always strictly positive, it follows that

$$\frac{d\widehat{x}^*}{d\mu} \gtrless 0 \qquad \text{iff} \qquad \psi \gtrless 0.$$

Using (63), we can get

$$G_1(\widehat{x}^*, 1)\frac{d\widehat{x}^*}{d\mu} = \left\{\frac{1}{1-\alpha}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right]\right\}^{\frac{1}{\eta}}\frac{d\widehat{k}^*}{d\mu}.$$

This equation shows that $\widehat{x}^*$ and $\widehat{k}^*$ will move in the same direction whenever there is a change in μ. This completes the proof of Proposition 3.

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