The grand dividends value

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16 March 2021

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February 18, 2022

Abstract

We introduce a new value for games with transferable utility, called grand dividends value. In the payoff calculation, the grand dividends value takes into account the worths of all subcoalitions of a player set. The concept of grand dividends, representing the surplus (which can also be non-positive) of the worth of the grand coalition over the worths of all coalitions that lack one player of the player set, is the initial point here. Many of the properties known from the Shapley value are also satisfied by the grand dividends value. Along with new axioms having a similar correspondence to axioms satisfied by the Shapley value, axiomatizations arise that have an analogous equivalent for the Shapley value, including the classics by Shapley and Young. The new concept of multiple dividends provides a close connection to the Shapley value.

Keywords Cooperative game · (Harsanyi/Grand/Multiple) Dividends · Shapley value · Grand dividends value

1. Introduction

The concept of a coalition function, also called characteristic function, goes back to von Neumann and Morgenstern (1944). In Shapley (1953b), a TU-game is given by a finite subset N of the universe of all possible players and a superadditive set function (the coalition function) from the subsets of N into the real numbers with the only condition that the worth of the empty set is zero. We will follow Shapley's approach but dispense with superadditivity. The coalition function can be used, for example, to model and analyze economic, political, or other social phenomena. In general, the worth of a coalition is the reward that this coalition can guarantee its players, regardless of what the other players outside the coalition do.

In the model of Harsanyi (1959, 1963), the fundamental assumption is that each player is simultaneously a member of all possible different coalitions (Harsanyi uses the term 'syndicate') which contain it. Introducing the important concept of his (Harsanyi) dividends,

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he assumes that each coalition guarantees a certain payment, the Harsanyi dividend, which should be divided among the members of this coalition. Moreover, these dividends should be assumed in addition to any dividends that each member of the coalition may receive from other coalitions. Under these assumptions, Harsanyi can show that his solution for TU-games provides each player with an equal share of all Harsanyi dividends from coalitions containing it and coincides with the Shapley value. Thus, by Harsanyi, the coalition function inherently justifies the Shapley value, but only under the assumptions noted above.

For many scenarios, these assumptions are quite reasonable. But other situations are also conceivable. Harsanyi (1959) himself points out that in von Neumann and Morgenstern (1944), it is assumed that each player is a member of only one coalition of players from a player set. For the equal division value (see, e.g., Zou et al. (2021)), we can assume that the grand coalition (the coalition containing all players) is the only coalition that actually forms. If we assume that if the grand coalition does not form, no other coalition is formed, then in the case of cooperation only the grand coalition receives a dividend equal to its worth, which is then distributed equally among all players. Considering the equal surplus division value, introduced in Driessen and Funaki (1991) as the center-of-gravity of the imputation-set value, if the grand coalition would not form, the singletons can be assumed to be the only coalitions formed. Here, each player receives its stand-alone worth as a dividend, paid in full, plus an equal share of the surplus of the worth of the grand coalition over the worths of the singletons as a dividend of the grand coalition.

While the last two values take into account only a (small) part of the worths of all possible coalitions, this is not the case for the Shapley value and the following new value.

In our model, unlike Harsanyi's model, we first assume that any coalition formed precludes the simultaneous formation of any own proper subcoalition. We will allow overlapping coalition formations, at least hypothetically, in which each such formed coalition will then be guaranteed the full coalitional worth simultaneously. Therefore, we will consider the worth of the grand coalition minus the sum of all the worths of coalitions that are missing exactly one player of the player set as a 'grand dividend' for the grand coalition.

But then, we can take a closer look at subgames on the player sets where one player of the original player set is removed, and, accordingly, get a grand dividend for the grand coalition in each subgame. Proceeding in this way, we obtain grand dividends for all coalitions, until finally, each player receives its stand-alone worth as a grand dividend for its singleton. Of course, we can have non-positive grand dividends, just like the Harsanyi dividends. For player sets with only two players, grand dividends coincide with Harsanyi dividends.

With the concept of grand dividends, we can introduce a new TU-value, called 'grand dividends value'. The grand dividends value is given by the fact that each player receives an equal share of the grand dividend of all subgames where that player is a member of the player set as a payoff. Note, however, that, depending on the size of the player set and the number of members in a coalition, we may have to consider the same dividend several times, just as our assumption above would dictate.

The grand dividends value satisfies many axioms that are also satisfied by the Shapley value and it also satisfies a set of new axioms that are analogous to ones also satisfied by the Shapley value. Thus, we can give axiomatizations of the grand dividends value that are analogous to axiomatizations of the Shapley value in Shapley (1953b), Myerson

(1980), and Besner (2020). In particular, the new grand dividends monotonicity, which states that for a player the payoff does not decrease if the grand dividends do not decrease, has interesting economic significance in our view, similar to strong monotonicity (Young, 1985). It offers, along with efficiency and symmetry, an analogous characterization of the grand dividends value to the axiomatization of the Shapley value in Young (1985).

For the payoff calculation, the same grand dividends of a subgame are used several times, depending on the size of the initial player set and the subgame player set. In the last content section, we combine these multiple grand dividends of the same coalition into 'multiple dividends'. If these multiple dividends are interpreted as Harsanyi dividends of a new coalition function, we can show that the Shapley value for the resulting 'multiple dividends game' is equal to the grand dividends value for the initial game, which has far-reaching consequences.

The article is organized as follows. In Section 2 we give some preliminaries. Section 3 introduces the grand dividends and the grand dividends value. An example shows that in a special case the grand dividends value is preferable to the Shapley value. In Section 4, we give two axiomatizations which are analogous to axiomatizations in Besner (2020) and (Myerson, 1980). In Sections 5 and 6, respectively, we provide axiomatizations that are similar to the classical axiomatizations of the Shapley value in Shapley (1953b) and Young (1985). Next, in Section 7, we recall first some results of the potential, the reduced game, and the consistency property in Hart and Mas-Colell (1989). Afterwards, we introduce multiple dividends and an associated multiple dividends game, which is then used to establish an intrinsic connection of the grand dividends value to the Shapley value. Finally, Section 8 contains some concluding remarks and points out possible extensions of the grand dividends value. An Appendix shows the logical independence of the axioms in our characterizations.

2. Preliminaries

Let $\mathfrak{U}$ be a countably infinite set, the universe of all players and let $\mathcal{N}$ be the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) with a player set $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Each subset $S \subseteq N$ is called a **coalition**, $v(S)$ is the **worth** of the coalition S and Ω^S denotes the set of all non-empty subsets of S. For each $S \in \Omega^N$, $|S|$ or s respectively denotes the cardinality of S, especially n denotes the cardinality of a player set N. $\mathbb{V}(N)$ denotes the set of all TU-games with player set N. The **restriction** of (N, v) to a player set $S \in \Omega^N$ is denoted by (S, v). A **unanimity game** (N, u_S), $S \in \Omega^N$, is defined for all $T \subseteq N$ by $u_S(T) = 1$, if $S \subseteq T$, and $u_S(T) = 0$, otherwise.

Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$. For all $S \in \Omega^N$, the **Harsanyi dividends** $\lambda_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\lambda_v(S) := \begin{cases} 0, \text{ if } S = \emptyset, \\ v(S) - \sum_{R \subsetneq S} \lambda_v(R) \text{ otherwise.} \end{cases} \tag{1}$$

The **marginal contribution** MC_i^v of a player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) =$

$v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, if we have $v(\{i\}) = 0$, i is called a **null player** in (N, v). Two players $i, j \in N$, $i \neq j$, are **symmetric** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

For all $N \in \mathcal{N}$, a **TU-value** or **solution** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$.

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$. The **equal division value** ED is given by

$$ED_i(N, v) := \frac{v(N)}{n} \text{ for all } i \in N.$$

The **equal surplus division value** ESD (Driessen and Funaki, 1991), also known as the center of imputation set (CIS-vector), is given by

$$ESD_i(N, v) := \frac{v(N) - \sum_{i \in N} v(\{i\})}{n} + v(\{i\}) \text{ for all } i \in N.$$

The **Shapley value** Sh (Shapley, 1953b), is given by

$$Sh_i(N, v) := \sum_{S \subseteq N, S \ni i} \frac{(s-1)!\,(n-s)!}{n!} \big[v(S) - v(S \backslash \{i\})\big] \text{ for all } i \in N. \tag{2}$$

We refer to the following well-known axioms for TU-values φ which hold for all $N \in \mathcal{N}$:

Efficiency, E. For all $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Efficiency means that the worth of the grand coalition is fully shared among all players. The following axiom states that a player who does not contribute anything to a (non-empty) coalition should receive only the singleton's worth.

Dummy player, N. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ such that i is a dummy player in (N, v), we have $\varphi_i(N, v) = v(\{i\})$.

A null player receives nothing.

Null player, N. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in (N, v), we have $\varphi_i(N, v) = 0$.

Additivity, A. For all $(N, v), (N, w) \in \mathbb{V}(N)$, we have $\phi(N, v) + \phi(N, w) = \phi(N, v + w)$.

Additivity requires that it is irrelevant whether one first adds the games and then applies the solution concept, or whether one first applies the solution concept to the individual games and then adds the payoffs.

Symmetry, Sym. For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are symmetric in (N, v), we have $\phi_i(N, v) = \phi_j(N, v)$.

Symmetry means that two players who contribute the same amount to each coalition should receive the same payoff.

Balanced contributions, BC (Myerson, 1980)**.** For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, we have $\varphi_i(N, v) - \varphi_i(N \backslash \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \backslash \{i\}, v)$.

By this property, for two players the amount that one player would win or lose if the other player drops out of the game is the same for both players.

Strong monotonicity, SMon (Young, 1985)**.** For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $MC_i^v(S) \leq MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v) \leq \varphi_i(N, w)$.

Strong monotonicity states that a player's payoff should not decrease if the worth of the coalitions containing that player increases or stays the same compared to the worth of the coalitions that do not contain that player.

Marginality, Mar (Young, 1985)**.** For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v) = \varphi_i(N, w)$.

By marginality, only a player's marginal contributions are relevant to the player's payoff. The following axiom states that the payoff differences of two players should be the same for different worths of the grand coalition.

Equal (aggregate) monotonicity, EMon (Béal et al., 2018)**.** For all $(N, v) \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, we have

$$\varphi_i(N, v) - \varphi_i(N, v + \alpha \cdot u_N) = \varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N) \text{ for all } i, j \in N,$$

Standardness, St (Hart and Mas-Colell, 1989)**.** For all $(N, v) \in \mathbb{V}^N$, $N = \{i, j\}$, $i \neq j$, we have

$$\phi_i(\{i, j\}, v) = v(\{i\}) + \frac{1}{2}\big[v(\{i, j\}) - v(\{i\}) - v(\{j\})\big].$$

Standardness implies that in two-player games cooperation results in the surplus being shared equally.

3. The grand dividends value

Harsanyi (1959, 1963), proposing Harsanyi dividends, assumes that all possible coalitions are formed simultaneously. The Harsanyi dividend of a singleton equals the worth of the singleton, and for all other coalitions, we have recursively that their Harsanyi dividends equal their worth minus the Harsanyi dividends of all proper subcoalitions. This means that Harsanyi dividends, and thus the worth of each coalition, are regarded as independent not only of what players outside that coalition do but also of external effects of the players in that coalition with players from outside, i.e., the worth of overlapping coalitions. Thus, in Harsanyi's model, the grand coalition has no cooperation benefit if the worth of the grand coalition is equal to the worth of the sum of all Harsanyi dividends of all proper subcoalitions of the grand coalition.

Initially, in what follows, we take a different approach to the introduction and theoretical justification of our new TU-value. The main difference is that by forming a coalition, we preclude the formation of subcoalitions of that coalition but forming overlapping coalitions is no problem. That is, the total worth of two simultaneously formed overlapping coalitions is the sum of the worths of both coalitions, while in Harsanyi's model, the total worth of both coalitions is the sum of the Harsanyi dividends of the set containing all subcoalitions of these coalitions, including their own Harsanyi dividends. In our model, it follows that we have no cooperation benefit by forming the grand coalition if the worth of the grand coalition equals the sum of the worth of all the subcoalitions of the grand coalition with one less player and these coalitions had previously formed.

Let us now hypothetically assume that all coalitions of a player set have formed at the same time, each of which is missing one player of the original player set. Now, when the **grand coalition** (the coalition consisting of all players) is forming, we have a (not necessarily positive) surplus of the worth of the grand coalition over the sum of the worths of the previously formed coalitions. Formally, for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we call this surplus as the **grand dividend** $\delta_v(N)$, given by

$$\delta_v(N) := v(N) - \sum_{j \in N} v(N \backslash \{j\}). \tag{3}$$

At this point, we can specify an algorithm for calculating a player's payoff. As a reward for forming the grand coalition N, each subcoalition $N\backslash\{j\}$, $j \in N$, receives an equal share of the grand dividend $\delta_v(N)$, which can be divided equally among the members of each coalition. Therefore, each player in the player set receives an equal share of $\delta_v(N)$. In the next step, each coalition $N\backslash\{j\}$ can play an independent game since, according to our assumption, all these coalitions have a worth independent of the worth of other coalitions.[1] In these subgames, again, the grand dividend of the new grand coalition can be distributed and so on. We obtain a recursive formula of a new TU-value.[2]

Definition 3.1. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, the* ***grand dividends value*** *Ψ is inductively given by*

$$\Psi_i(N, v) := \frac{\delta_v(N)}{n} + \sum_{j \in N,\, j \neq i} \Psi_i(N \backslash \{j\}, v) \text{ for all } i \in N.^{3} \tag{4}$$

An interpretation of the equal surplus division value could be that it distributes the surplus of the worth of the grand coalition over the sum of the worth of the singletons evenly among the singletons and thus among the individual players. Then the players play a game on the singletons where they get an efficient payoff, namely the worth of the singleton. At first glance, we can think of the grand dividends value as an extension of the equal surplus division value that step by step passes through all coalition sizes. For a two-player game the payoffs match, both TU-values satisfy standardness.

The Shapley value also satisfies standardness. By Harsanyi (1959), equivalent to (2), for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, the Shapley value Sh is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\lambda_v(S)}{s} \quad \text{for all} \quad i \in N. \tag{5}$$

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[1] For bargaining with overlapping coalitions, see Ray (2007).

[2] Already in Hart and Mas-Colell (1989) and Sprumont (1990), a related recursive formula for the Shapley value can be found, which is later proposed in Kongo and Funaki (2016), given by

$$Sh_i(N, v) := \frac{1}{n}\big(v(N) - v(N \backslash \{i\}\big) + \frac{1}{n} \sum_{j \in N,\, j \neq i} Sh_i(N \backslash \{j\}, v) \text{ for all } i \in N.$$

[3] If $n = 1$, we use the convention that an empty sum evaluates to zero.

The Shapley value assigns to each player an equal share of the Harsanyi dividends of all coalitions in which that player is a member. In the following proposition, we can find a related formula for the grand dividends value. For all subgames in which a particular player is part of the player set, that player receives an equal share of the grand dividend. However, since we successively consider all subgames when assigning dividends, depending on the size of the set of players, the respective coalitions are thus considered multiple times, so that we multiply each grand dividend by the number of times it occurs.

Proposition 3.2. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, the grand dividend value Ψ is given by*

$$\Psi_i(N, v) = \sum_{S \subseteq N,\, S \ni i} \frac{(n-s)!}{s} \delta_v(S) \text{ for all } i \in N. \tag{6}$$

Proof. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and φ be a TU-value, given by

$$\varphi_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{(n-s)!}{s} \delta_v(S) \text{ for all } i \in N. \tag{7}$$

Each coalition $S \subsetneq N$, $S \ni i$, is a subset of $(n - s)$ different coalitions $T \subsetneq N$, $|T| = n - 1$, $T \ni i$. Therefore, we have

$$\sum_{j \in N,\, j \neq i} \left[\sum_{S \subseteq N \setminus \{j\},\, S \ni i} \frac{(n-1-s)!}{s} \delta_v(S) \right] = \sum_{S \subsetneq N,\, S \ni i} \frac{(n-s)!}{s} \delta_v(S) \text{ for all } i \in N. \tag{8}$$

It follows, for all $i \in N$,

$$\begin{aligned}
\varphi_i(N, v) &\underset{(7)}{=} \frac{\delta_v(N)}{n} + \sum_{S \subsetneq N,\, S \ni i} \frac{(n-s)!}{s} \delta_v(S) \\
&\underset{(8)}{=} \frac{\delta_v(N)}{n} + \sum_{j \in N,\, j \neq i} \left[\sum_{S \subseteq N \setminus \{j\},\, S \ni i} \frac{(n-1-s)!}{s} \delta_v(S) \right] \\
&\underset{(7)}{=} \frac{\delta_v(N)}{n} + \sum_{j \in N,\, j \neq i} \varphi_i(N \setminus \{j\}, v) \underset{(4)}{=} \Psi_i(N, v).
\end{aligned}$$

□

In the following example, we show that, at least in some special cases, the grand dividend value is preferable to the Shapley value.

Example 3.3. We consider a three-player game in which the players are companies which hold a certain number of patents that are necessary to produce electronic devices such as smartphones, tablet-PCs, notebooks, radios, e-readers, navigation devices, or the like. While players 2 and 3 are able to produce some (few) devices only with their own patents, player 1 cannot produce any device based on its own patents. When two-player coalitions form, they can produce more electronics items more cheaply, with better quality, or both, because of the greater number of patents, which prevents them from continuing to produce as a single company. In our example, we assume that all two-player coalitions produce different goods. Furthermore, since the market is assumed to be large enough so that the purchase of products from one two-player coalition does not affect the purchase of products from another two-player coalition, and since these coalitions can also borrow any missing production capital at extremely low interest rates, the worth of a two-player coalition is assumed to be independent of the other two, and all three two-player coalitions can exist simultaneously. When the three-player coalition forms, we have even more goods that can be produced even cheaper, which should prevent the players from continuing to produce as (proper) subcoalitions. Hence, we have exactly the hypothetical situation mentioned above.

Formally, let $(N, v) \in \mathbb{V}(N)$, $N = \{1, 2, 3\}$, be a TU-game, given by

$$\begin{aligned} &v(\{1\}) = 0, && v(\{2\}) = 1, && v(\{3\}) = 4, && v(\{1,2\}) = 2, \\ &v(\{1,3\}) = 6, && v(\{2,3\}) = 9, && v(\{1,2,3\}) = 18. && \end{aligned}$$

The crucial issue now is how the cooperation benefits of the grand coalition will be distributed. We have

$$Sh(N, v) = (7/2, 11/2, 9) \text{ and } \Psi(N, v) = (11/6, 29/6, 34/3).$$

If player 3 has a choice, he will not participate in a three-player coalition if the payoff is to be made with the Shapley value. Since the worths of the two-player coalitions are independent of each other, it is better for player 3 to play a separate two-player game for each of these two coalitions. We have $Sh_3(\{1,3\}, v)+Sh_3(\{2,3\}, v) = 11 > 9 = Sh_3(N, v)$. Since the other two players, player 1 and player 2, improve their payoffs in the singleton or two-player game on the set of players {1,2}, respectively, if they also play the other possible two-player games, they will agree to those games as well. However, since they will then further improve their payoff using the grand dividends value in the three-player game, all players will finally prefer the grand dividends value over the Shapley value in the absence of any external constraints.

4. Inessential grand dividends and balanced summarized contributions

We call a TU-game $(N, v) \in \mathbb{V}(N)$ an **inessential grand dividend game** if $v(N) = \sum_{j\in N} v(N\backslash\{j\})$ which is, by (3), equivalent to $\delta_v(N) = 0$. The following property states that in an inessential grand dividend game, the payoff to a player is completely determined by the sum of the player's payoffs in all subgames in which one player of the player set is removed at a time.

Inessential grand dividend, IGD.[4] For all $N \in \mathcal{N}$ and all inessential grand dividend games $(N, v) \in \mathbb{V}(N)$, we have $\varphi_i(N, v) = \sum_{j\in N} \varphi_i(N\backslash\{j\}, v)$ for all $i \in N$.

It follows a first axiomatization of the grand dividends value.

[4]This axiom is related to the inessential grand coalition property in Besner (2020).

Theorem 4.1. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **IGD**, and **EMon**.*[5]

Proof. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$.

I. Existence: **IGD** and **EMon** follow immediately by (4). We show **E** by induction on the size n.

Initialization: Let $n = 1$. Then, **E** is satisfied by (3) and (4).

Induction step: Let $n \geq 2$. Assume that Ψ satisfies **E** for all n', $n' < n$, (IH). We have

$$\sum_{i\in N} \Psi_i(N, v) \underset{(4)}{=} \sum_{i\in N} \left[\frac{\delta_v(N)}{n} + \sum_{j\in N,\, j\neq i} \Psi_i(N\backslash\{j\}, v)\right] \underset{(IH)}{=} \delta_v(N) + \sum_{i\in N} v(N\backslash\{i\}) \underset{(3)}{=} v(N),$$

and **E** is shown.

II. Uniqueness: Let φ be a TU-value which satisfies all axioms from Theorem 4.1. We show uniqueness by induction on the size n.

Initialization: Let $n = 1$. Then, uniqueness is satisfied by **E**.

Induction step: Let $n \geq 2$. Assume that φ is unique for all n', $n' < n$, (IH). Then, by (IH) and **IGD**, φ is unique on the inessential grand dividend game $\big(N, v - \delta_v(N)u_N\big)$. By **EMon**, we have, for all $i, j \in N$,

$$\begin{aligned} &\varphi_i(N, v) = \varphi_i\big(N, v - \delta_v(N) \cdot u_N\big) + \varphi_j(N, v) - \varphi_j\big(N, v - \delta_v(N) \cdot u_N\big) \\ \Leftrightarrow\ &\sum_{k\in N} \varphi_k(N, v) = \sum_{k\in N} \varphi_k\big(N, v - \delta_v(N) \cdot u_N\big) + n \cdot \big[\varphi_j(N, v) - \varphi_j\big(N, v - \delta_v(N) \cdot u_N\big)\big] \end{aligned}$$

and, by **E** and (IH), φ is unique for the player j. Since j is arbitrary, uniqueness and, therefore, also Theorem 4.1 is shown. □

For game situations like our example, this axiomatization seems quite convincing. If the worth of the grand dividend is equal to the sum of the worths of the two-player coalitions, it should not matter whether the grand coalition forms or not, and if only the grand dividend changes, then for fairness the payoff for all players should change by the same amount.

The balanced contributions property **BC** states that for any two players, the amount that one player would win or lose if the other player drops out of the game is the same for both players. By the following property, the gain or loss for two players of a player set is the same if they would play the game with the entire player set instead of playing games with player sets, each missing one of their original players.

Balanced summarized contributions, BSC. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, we have

$$\varphi_i(N, v) - \sum_{k\in N,\, k\neq i} \varphi_i(N\backslash\{k\}, v) = \varphi_j(N, v) - \sum_{k\in N,\, k\neq j} \varphi_j(N\backslash\{k\}, v).$$

The balanced summarized contributions property has a strong connection to the grand dividends value. Similar as the Shapley value can be characterized by **E** and **BC** (Myerson, 1980), the grand dividends value can be characterized by **E** and **BSC**.

[5] A related axiomatization of the Shapley value can be found in Besner (2020) where the inessential grand dividend property is replaced by the inessential grand coalition property.

Theorem 4.2. *The grand dividends value Ψ is the unique TU-value that satisfies **E** and **BSC**.*

Proof. Since **E** is already shown in the proof of Theorem 4.1 and **BSC** follows immediately from (4), we only need to show uniqueness.

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and φ be a TU-value which satisfies **E** and **BSC**. We show uniqueness by induction on the size n.

Initialization: Let $n = 1$. Then, uniqueness is satisfied by **E**.

Induction step: Let $n \geq 2$. Assume that φ is unique for all n', $n' < n$, (IH). By **BSC**, we have

$$\varphi_i(N, v) - \sum_{k \in N,\, k \neq i} \varphi_i(N \backslash \{k\}, v) = \varphi_j(N, v) - \sum_{k \in N,\, k \neq j} \varphi_j(N \backslash \{k\}, v)$$

$$\Leftrightarrow \quad n \cdot \varphi_i(N, v) - n \cdot \sum_{k \in N,\, k \neq i} \varphi_i(N \backslash \{k\}, v) = \sum_{k \in N} \varphi_k(N, v) - \sum_{j \in N} \sum_{k \in N,\, k \neq j} \varphi_j(N \backslash \{k\}, v)$$

and, by **E** and (IH), φ is unique for the player i. Since i is arbitrary, uniqueness and, therefore, Theorem 4.2 is shown. $\square$

5. An axiomatization in the spirit of Shapley

We pick the original axiomatization of the Shapley value as the starting point of this section.

Theorem 5.1 (Shapley, 1953b)**.** *The Shapley value Sh is the unique TU-value that satisfies **E**, **A**, **N**, and **Sym**.*

We would like to point out that Nowak and Radzik (1994) also introduced their solidarity value with an axiomatization similar to this one. Their axiomatization differs from Shapley's by replacing the null player axiom **N** with their A-null player axiom. Further axiomatizations which differ only in the null player axiom from Shapley's axiomatization are the axiomatization of the equal division value in van den Brink (2007), using the nullifying player property, the axiomatization of the equal surplus division value by Casajus and Huettner (2014), using the dummifying player property, and, as a recent result, the axiomatization of the average surplus value in Li et al. (2021), using the A-null surplus player property.

Our next axiomatization of the grand dividends value also differs from Shapley's only in the null player axiom. A null player is a dummy player with a stand-alone worth of zero. Since the dummy player property **D** implies the null player property **N** and the Shapley value also satisfies **D**, the Shapley value can also be axiomatized by replacing **N** with the stronger **D**.

It is well-known and easy to prove that $i \in N$ is a dummy player in (N, v) if $\lambda_v(S) = 0$ for all $S \subseteq N$, $S \ni i, S \neq \{i\}$. Analogously, we call a player $i \in N$ a **multiplier** in (N, v) if $\delta_v(S) = 0$ for all $S \subseteq N$, $S \ni i, S \neq \{i\}$. The following axiom states that, depending on the size of the player set N, a multiplier receives as a payoff $(n-1)!$ times its stand-alone-worth.

Multiplier, Mul. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a multiplier in (N, v), we have $\varphi_i(N, v) = (n-1)!\, \delta_v(\{i\})$.

Remark 5.2. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. If a multiplier $i, i \notin N$, joins the player set, the multiplier ensures that the worth of any coalition $S \cup \{i\}$, $S \in \Omega^N$, has the worth of the sum of its subcoalitions with one less player. It is easy to show, by induction on s, that in this case we have

$$v(S \cup \{i\}) = s!\, v(\{i\}) + \sum_{R \subseteq S} (s-r)!\, v(R). \tag{9}$$

Thus, the name multiplier seems more than justified and the multiplier is content with the multiple that its stand-alone worth contributes to the worth of the new grand coalition.

By (6), it is obvious that the grand dividends value satisfies **Mul**. If the worth of the singleton of a multiplier equals zero, we call the multiplier a **null multiplier**. This yields the following property which is implied by **Mul**.

Null multiplier, NMul. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a null multiplier in (N, v), we have $\varphi_i(N, v) = 0$.

It may not be entirely fair, but it is not that far-fetched for a null multiplier i to receive a payoff of zero. Each coalition containing player i has as its worth the sum of worths of coalitions that all do not contain player i. In this sense, player i does not contribute to the worth of any coalition, thus the other players split the full payoff among themselves. A null multiplier can be seen in this sense as a kind of catalyst that enables the other players to multiply, but does not change itself. We give a new axiomatization.

Theorem 5.3. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **A**, **NMul**, and **Sym**.*

Proof. In unanimity games (N, u_S), $S \in \Omega^N$, which form a basis for $\mathbb{V}(N)$ (see Shapley (1953b)), we have $\lambda_{u_S}(S) = 1$ and $\lambda_{u_S}(T) = 0$, $T \in \Omega^N$, $T \neq S$. Analogosly, we introduce another basis. For each coalition $S \in \Omega^N$, we use a TU-game $(N, z_S) \in \mathbb{V}(N)$ such that

$$\delta_{z_S}(T) := \begin{cases} 1, & \text{if } T = S, \\ 0, & \text{if } , T \in \Omega^N,\ T \neq S. \end{cases} \tag{10}$$

Due to (3), we have $z_S(S) = 1$ and all coalitions which are no supersets of S have a worth of zero. Each coalition T containing S as a proper subset, contains $\binom{t-s}{t-s-1} = t - s$ coalitions of the size $t - s - 1$ containing S and all other coalitions which are subsets of the same size have a worth of zero. Thus, each TU-game (N, z_S), $S \in \Omega^N$, is given, by

$$z_S(T) := \begin{cases} (t-s)!, & \text{if } S \subseteq T, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

Since a $(2^n - 1) \times (2^n - 1)$ matrix A of the $2^n - 1$ entries of the $2^n - 1$ coalition functions z_S, $S \in \Omega^N$, correspondingly ordered, is a triangular matrix with $det\, A = 1 \neq 0$, we have found a basis for $\mathbb{V}(N)$.

Let now $N \in \mathcal{N}$, $(N,v),(N,w) \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$.

I. Existence: **E** is shown in the proof of Theorem 4.1. By (6), Ψ obviously satisfies **NMul** and **Sym**. Since we have, by (3), $\delta_{v+w} = \delta_v + \delta_w$, **A** is satisfied by (6).

II. Uniqueness: Let φ be a TU-value which satisfies all axioms from Theorem 5.3. For all $S \in \Omega^N$, $i \in N$, we have $\varphi_i(N, \alpha z_S) = 0$, by **Sym** and **E**, if $\alpha = 0$, and, by **NMul**, if $i \in N \backslash S$. By **E**, **Sym**, and (11), it follows $\varphi_i(N, \alpha z_S) = \alpha \dfrac{(n-s)!}{s}$ for all $i \in S$. Therefore, φ is unique on all games $(N, \alpha z_S)$ for all $\alpha \in \mathbb{R}$ and all $S \in \Omega^N$. But then, by **A**, uniqueness is shown and the proof is complete. $\square$

For all multipliers which are no null multipliers, the payoff according to the multiplier property seems to be much fairer than for a null multiplier. Therefore, the next axiomatization that also follows from the results of this section is probably even more compelling.

Corollary 5.4. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **A**, **Mul**, and **Sym**.*

6. An axiomatization in the spirit of Young

Certainly, the following theorem is one of the most beautiful axiomatizations of the Shapley value.

Theorem 6.1 (Young, 1985)**.** *The Shapley value Sh is the unique TU-value that satisfies **E**, **SMon**, and **Sym**.*

Thereby **SMon** can also be replaced by the weaker **Mar**. By (1), the condition '$MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$' in **Mar** can be equivalently replaced by '$\lambda_v(S) = \lambda_w(S)$ for all $S \subseteq N$, $S \ni i$', analogously in **SMon**. We replace marginal contributions or Harsanyi dividends respectively by grand dividends in both axioms and obtain two new properties.

Grand dividends independency, GDInd. For all $N \in \mathcal{N}$, $(N,v),(N,w) \in \mathbb{V}(N)$, and $i \in N$ such that $\delta_v(S) = \delta_w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(N,v) = \varphi_i(N,w)$.

Grand dividends monotonicity, GDMon. For all $N \in \mathcal{N}$, $(N,v),(N,w) \in \mathbb{V}(N)$, and $i \in N$ such that $\delta_v(S) \leq \delta_w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(N,v) \leq \varphi_i(N,w)$.

The grand dividends monotonicity states that the payoff to a player should not decrease if the grand dividends of all coalitions containing that player increase or stay the same. It is easy to show that **GDMon** implies **GDInd**. By this property, the payoffs remain the same if the grand dividends of all coalitions containing that player stay the same. Therefore, a player's payoff depends only on the grand dividends of coalitions containing the player. Young (1985) used **SMon** instead of **Mar** to axiomatize the Shapley value where the proof only used **Mar**. The same approach is used in the proof of our following axiomatization. We introduce **GDMon** only because it might seem even more compelling for applications than **GDInd**. We formulate an axiomatization in the spirit of the characterization of the Shapley value just mentioned.

Theorem 6.2. *The grand dividends value Ψ is the unique TU-value that satisfies* ***E****,* ***GDInd/GDMon****, and* ***Sym****.*

Proof. The proof is similar to the proof in Young (1985).

Since **E** is shown in the proof of Theorem 4.1 and **Sym** and **GDInd/GDMon** follow immediately from (6), we only need to show uniqueness.

The games (N, z_S), $S \in \Omega^N$, defined by (11), form a basis of $\mathbb{V}(N)$. This means, we have for any $(N, v) \in \mathbb{V}(N)$ a unique representation of the coalition function v, given by

$$v = \sum_{S \in \Omega^N} \alpha_S z_S, \; \alpha_S \in \mathbb{R}. \tag{12}$$

Note, due to (10), that for all $S \in \Omega^N$, $c \in \mathbb{R}$, and two games $(N, v), (N, w) \in \mathbb{V}(N)$, $w := v + cz_S$, we have

$$\delta_v(T) = \delta_w(T) \text{ for all } T \subseteq N, \, T \neq S. \tag{13}$$

Therefore, **GDInd** implies

$$\varphi_i(N, v) = \varphi_i(N, w) \text{ for all } i \in N \backslash S. \tag{14}$$

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and φ be a TU-value which satisfies **E**, **Sym**, and **GDInd**. We use an induction on the size $r_v := |\{R \in \Omega^N : \delta_v(R) \neq 0\}|$.

Initialization: Let $r = 0$. We have $v(N) = 0$ and uniqueness is satisfied by **E** and **Sym**.

Induction step: Let $r \geq 1$. Assume that φ is unique for all TU-games (N, v'), $r_{v'} \leq r-1$, (IH). Let Q be the intersection of all coalitions $Q_k \in \Omega^N$, $\delta_v(Q_k) \neq 0$,

$$Q := \bigcap_{1 \leq k \leq r} Q_k.$$

Two cases can be distinguished: (a) $i \in N \backslash Q$ and (b) $i \in Q$.

(a) Each $i \in N \backslash Q$ is a member of at most $r - 1$ coalitions Q_k, $\delta_v(Q_k) \neq 0$ and we have at least one coalition $Q_i \in \Omega^N$, $\delta_v(Q_i) \neq 0$. Then, by (12), exists a coalition function v_i such that

$$v_i = \sum_{S \in \Omega^N, \, S \neq Q_i} \alpha_S z_S,$$

and, by (13), we have $\delta_v(S) = \delta_{v_i}(S)$ for all $S \subseteq N$, $S \ni i$. Therefore, by **GDInd**, (14), and (IH), φ is unique on (N, v) for all $i \in N \backslash Q$.

(b) Each $i \in Q$ is a member of all coalitions Q_k, $\delta_v(Q_k) \neq 0$. Thus, all coalitions $S \in \Omega^N$, $Q \nsubseteq S$, have a grand dividend $\delta_v(S) = 0$. It follows, $v(S) = 0$ for all $S \in \Omega^N$, $Q \nsubseteq S$. Therefore, if $|Q| = 1$, by **E**, and (a), φ is unique for $i \in Q$. If $|Q| \geq 2$, we have $v(T \cup \{j\}) = v(T \cup \{k\})$ for all $j, k \in Q$ and $T \subseteq N \backslash \{j, k\}$. Hence, all players $i \in Q$ are symmetric in (N, v). By **Sym**, **E**, and (a), φ also is unique for all $i \in Q$ and the proof is complete. □

7. Multiple dividends and a strong relationship to the Shapley value

In this section, we first recall the reduced game of a TU-game, introduced in Hart and Mas-Colell (1989).

Definition 7.1. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, $R \in \Omega^N$, and a TU-value φ, the **reduced TU-game** $(R, v_R^{\varphi}) \in \mathbb{V}(R)$ is defined, for all $S \in \Omega^R$, by*

$$v_R^{\varphi}(S) := v\big(S \cup R^c\big) - \sum_{j \in R^c} \varphi_j\big(S \cup R^c, v\big), \tag{15}$$

where $R^c := N \backslash R$.

We can interpret this reduced game like this: if a coalition of players R^c exits the game, then in the reduced game each coalition S which is a subset of the coalition R of the players remaining receives the worth of the coalition $S \cup R^c$ in the original game minus the payoff of the players left in the restricted game on $S \cup R^c$.

A TU-value is called consistent if each player of the coalition R receives the same payoff in the reduced game and in the original game.

Consistency, C (Hart and Mas-Colell, 1989). For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, $R \in \Omega^N$, we have $\varphi_i(N, v) = \varphi_i(R, v_R^{\varphi})$ for all $i \in R$.

The Shapley value is closely connected to this axiom.

Theorem 7.2 (Hart and Mas-Colell, 1989). *Sh is the unique TU-value that satisfies **C** and **St**.*

In the proof of this theorem, a function called potential in Hart and Mas-Colell (1989) is essential.

Definition 7.3. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, define $P : \mathbb{V}(N) \to \mathbb{R}$ and $P(\emptyset, v) = 0$ such that*

$$v(N) = \sum_{i \in N} D_i P(N, v), \tag{16}$$

where

$$D_i P(N, v) = P(N, v) - P(N \backslash \{i\}) \text{ for all } i \in N.$$

*Then P is called a **potential**.*

It follows a strong connection between the potential P and the Shapley value.

Theorem 7.4 (Hart and Mas-Colell, 1989). *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, there exists a unique potential function P, the resulting payoff vector $(D_i P(N, v))_{i \in N}$ coincides with the Shapley value Sh of the game (N, v) and the potential $P(N, v)$ is uniquely given by (16) applied only to (N, v) and its subgames.*

At first glance, the definition of the grand dividends value, which is close to the definition of the Shapley value, should also allow a potential approach similar to this potential and a related reduced game consistency. However, certain difficulties arise in this regard. In this respect, we point to a simple expression for the worth of a coalition, given by the grand dividends of the subgames.

Proposition 7.5. *For all* $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, *we have*

$$v(N) = \sum_{S \in \Omega^N} (n-s)!\, \delta_v(S). \tag{17}$$

Proof. Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$. We have

$$v(N) \underset{\mathbf{E}}{=} \sum_{i \in N} \Psi_i(N, v) \underset{(6)}{=} \sum_{i \in N} \sum_{S \subseteq N,\, S \ni i} \frac{(n-s)!}{s} \delta_v(S) = \sum_{S \in \Omega^N} (n-s)!\, \delta_v(S).$$

□

This means that each grand dividend of a subgame (S, v) is included several times in the worth of the grand coalition N, depending on the size of N and S. Thus, analogous to the Harsanyi dividends, we can introduce new dividends that combine the multiple grand dividends of a coalition into a single dividend.

Definition 7.6. *For each* $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, *and all* $S \subseteq N$, *the* ***multiple dividends*** μ_v^N *are defined by*

$$\mu_v^N(S) := \begin{cases} 0, & \text{if } S = \emptyset, \\ (n-s)!\, \delta_v(S), & \text{otherwise.} \end{cases} \tag{18}$$

By Proposition 7.5, the following remark is immediate.

Remark 7.7. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we have

$$v(N) = \sum_{S \subseteq N} \mu_v^N(S). \tag{19}$$

The potential for the Shapley value (see Hart and Mas-Colell (1989, Formula (2.3))) is just the sum of the Harsanyi dividends divided by the size of the coalitions. And here, unfortunately, one of the fundamental differences of the grand dividends value compared to the Shapley value comes into play: when we consider subgames, the values of the multiple dividends for the same coalitions change, or respectively, we have a different number of grand dividends of the same coalitions to consider, depending on the size of the player set.

For this reason, it cannot be assumed that in a simple manner straight forward a modified potential and a corresponding reduced game consistency can be found since different sets of players have to be considered. In the following, we choose a different path where the multiple dividends defined above are extremely useful.

We define a new coalition function v_μ^N which has the multiple dividends μ_v^N as Harsanyi dividends.

Definition 7.8. *For each $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}^N$, the* ***multiple dividends game*** *$(N, v_\mu^N) \in \mathbb{V}^N$ is given by*

$$v_\mu^N(S) := \sum_{T \subseteq S} \mu_v^N(T) \text{ for all } S \subseteq N. \tag{20}$$

Remark 7.9. By (1), for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, and a corresponding multiple dividends game (N, v_μ^N), we have

$$\mu_v^N(S) = \lambda_{v_\mu^N}(S) \text{ for all } S \subseteq N. \tag{21}$$

Finally, by Remark 7.9, (5), and (6) we get an interesting corollary.

Corollary 7.10. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, and a corresponding multiple dividends game (N, v_μ^N), we have*

$$\Psi(N, v) = Sh(N, v_\mu^N). \tag{22}$$

This corollary has far-reaching consequences.

Remarks 7.11. If it follows from an axiom satisfied by the grand dividends value for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, that a different or the same axiom is satisfied for the associated games (N, v_μ^N) and the Shapley value can be axiomatized by such axioms, then the grand dividends value can also be axiomatized by the initial axioms. I.e., for example, we could also have indirectly derived a proof of Theorem 5.3 from the axiomatization of the Shapley value in Theorem 5.1 by showing that for all games (N, v) the axioms **E**, **A**, **NMul**, and **Sym** are satisfied by the grand dividends value and from the satisfaction of these axioms for all (N, v), the axioms **E**, **A**, **N**, and **Sym** are also satisfied for all corresponding (N, v_μ^N).

Of course, the relationship also exists in the opposite direction.

Remark 7.12. By Remark 7.11, we can interpret each game $(N, v) \in \mathbb{V}^N$ as an multiple dividends game to a corresponding game (N, v_λ^N) which is recursively given by

$$v_\lambda^N(S) := \mu_{v_\lambda^N}(S) + \sum_{j \in S} v_\lambda^N(S \backslash \{j\}) \text{ for all } S \in \Omega^N,$$

where

$$\mu_{v_\lambda^N}(S) := \frac{\lambda_v(S)}{(n-s)!}.$$

Then, by Corollary 7.10, we have

$$Sh(N, v) = \Psi(N, v_\lambda^N).$$

Based on Theorem 7.4 und Corollary 7.10, we can use the potential P also for the grand dividends value.

Corollary 7.13. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and corresponding multiple dividends games $(N, v_\mu^N) \in \mathbb{V}(N)$, there exists a unique potential function P such that*

$$D_i P(N, v_\mu^N) = \Psi_i(N, v) \text{ for all } i \in N. \tag{23}$$

We adapt consistency for multiple dividends games.

Multiple dividends consistency, MDC. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}^N$, corresponding multiple dividends games $(N, v_\mu^N) \in \mathbb{V}(N)$, and $R \in \Omega^N$, we have

$$\varphi_i(N, v) = \varphi_i\Big(R, \big((v_\mu^N)_R^\varphi\big)_\lambda^R\Big) \text{ for all } i \in R,$$

where $\big(R, (v_\mu^N)_R^\varphi\big)$ is the reduced TU-game according to Definition 7.1 for (N, v_μ^N) and R, which is interpreted as an multiple dividends game to the corresponding game $\Big(R, \big((v_\mu^N)_R^\varphi\big)_\lambda^R\Big)$ according to Remark 7.12.

By Corollary 7.10, Remark 7.12, and Theorem 7.2, we present our last corollary.

Corollary 7.14. *The grand dividends value Ψ is the unique TU-value that satisfies **MDC** and **St**.*

8. Conclusion and extensions

Of course, the grand dividends value can be applied to all coalition functions, just like the Shapley value. In our view, however, the corresponding axioms and hence the associated TU-values are most convincing when our assumptions in Section 3 for the grand divends value and those of Harsanyi (1959, 1963) for the Shapley value are satisfied, respectively. The same applies to the assumptions made in the introduction regarding the equal division value and the equal surplus division value. For example, it may not always be appropriate to give a nullifying or zero player (see van den Brink (2007) and Deegan and Packel (1978)), who causes any coalition containing that player to receive a worth of zero, a payoff of zero with no further penalty when the cooperation of the other players is actually present.

Therefore, when selecting a TU-value for a payoff calculation, each user should pay attention not only to the desired properties the value should have, i.e., the satisfied axioms, but also to which coalition formations actually occur. The grand dividends value fits best when, in the process of forming a stable final state, no proper sub-coalitions of a formed coalition appear and we end up with only the formed grand coalition. If it is not possible or desired to form coalitions other than the Grand Coalition, the formulas (4) or (6) can be adjusted accordingly. It would be desirable for the worth of a coalition to be as independent as possible of the worth of other overlapping coalitions, in the sense that two overlapping coalitions could each guarantee the entire worth of their coalition simultaneously to their members. Harsanyi dividends cannot properly capture such situations because dividends from smaller coalitions are used several times for different larger coalitions.

Definition 3.1 or Proposition 3.2 immediately reveal various extensions of the grand dividends value. First, analogous to the weighted Shapley values (Shapley, 1953a), each player could be assigned a personal weight and the summands in (6) would no longer be distributed equally among the members of the coalitions S but in proportion to these members' weights (see (24)). As in the case of the proportional Shapley value (Béal et al., 2018; Besner, 2019a), these weights could also be replaced by the stand-alone worths of the individual members.

An extension in the sense of the Harsanyi solutions (Hammer et al., 1977; Vasil'ev, 1978)[6] would also be conceivable where the weights of two players for different coalitions could be in different proportions. Moreover, a further extension is possible in which, similar to the weighted values for level structures in Besner (2019b, 2021), the coalitions are given their own weights and the grand dividends in (4) can first be distributed according to these weights before a final distribution is made according to players' weights. Of particular interest here is that the coalitions no longer need to form partitions of the player set or its subsets. This would be in line with the idea of cooperative game theory that here, in addition to the members of the player set, the coalitions are also actors.

Of course, extensions to values with hierarchical structures of the player set are also conceivable, such as for the Owen value (Owen, 1977) or the Shapley levels value (Winter, 1989).

Josten(1996) implemented, as a convex combination of the Shapley value and the equal division value, a new class of TU-values, called α-egalitarian Shapley values. Analogously, such convex combinations of the grand dividends value with TU-values such as the Shapley value, the equal division value, or other TU-values would also be suitable here. In this context, we also refer to (Yokote and Funaki, 2017), where it might be interesting to see if there are connections to the convex combinations of TU-values given there and the grand dividends value, as well as connections between our TU-value and the listed relationships between monotonicity properties and linearity.

The investigation and axiomatizations of these extensions are left to further research.

Appendix

We show the logical independence of the axioms in the theorems. The locical independence of the two axioms in Theorem 4.2 is obvious.

Remark 8.1. *The axioms in Theorems 4.1 and 6.2 are logically independent:*

- ***E****: The null value ϕ^0, defined by $\phi^0_i(N,v) = 0$ for all $i \in N$, satisfies* ***IGD/GDInd*** *and* ***EMon/Sym*** *but not* ***E****.*
- ***IGD/GDInd****: The Shapley value Sh satisfies* ***E*** *and* ***EMon/Sym*** *but not* ***IGD/GDInd****.*
- ***EMon/Sym****: Let $W := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$, $w_i := w(i)$ for all $w \in W$, $i \in \mathfrak{U}$, be the collection of all positive weight systems on $\mathfrak{U}$ and $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. For each $w \in W$, the* ***weighted grand dividends value*** *Ψ^w, given by*

$$\Psi^w_i(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{w_i(n-s)!}{\sum_{j \in S} w_j} \delta_v(S) \text{ for all } i \in N, \tag{24}$$

[6]Detailed information can be found in Derks et al. (2000) and Vasil'ev and van der Laan (2002).

such that $w_j \neq w_k$ for at least two different players $j, k \in N$, satisfies ***E*** *and* ***IGD/GDInd*** *but not* ***EMon/Sym***.

Remark 8.2. *The axioms in Theorem 5.3 are logically independent:*

- ***E***: *The null value ϕ^0 satisfies* ***A***, ***GDNull***, *and* ***Sym*** *but not* ***E***.
- ***A***: *Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. The TU-value φ, given by*

$$\varphi_i(N,v) := \begin{cases} 0, & \text{if } i \text{ is a grand dividends null player,} \\ \dfrac{v(N)}{|\{j \in N : j \text{ is no grand dividends null player in } (N,v)\}|}, & \text{otherwise,} \end{cases}$$

  for all $i \in N$, satisfies ***E***, ***GDNull***, *and* ***Sym*** *but not* ***A***.
- ***GDNull***: *The Shapley value Sh satisfies* ***E***, ***A***, *and* ***Sym*** *but not* ***GDNull***.
- ***Sym***: *The TU-values Ψ^w, as defined in Remark 8.1, satisfy* ***E***, ***A***, *and* ***GDNull*** *but not* ***Sym***.

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