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Abstract

This paper analyzes local and global equilibrium dynamics in an optimizing endogenous growth model under expenditure-based fiscal austerity feedback policies expressed relative to the private capital stock—prescribing spending cuts in reaction to public debt accumulation. Because the present value of equilibrium primary surpluses turns to be a nonlinear function of debt, two steady state equilibria are shown to emerge, one exhibiting low debt and high growth, one exhibiting high debt and low growth. Local analysis reveals that the low-debt/high-growth steady state is saddle-path stable while the high-debt/low-growth steady state is unstable—the latter thus indicating the possibility of self-defeating austerity, characterized by off-equilibrium upward spirals in debt because of persistent policy-induced adverse effects on growth dividends and fiscal revenues. However, when global nonlinear dynamics are taken into account, it is demonstrated that the two steady states are endogenously connected. In particular, global analysis reveals that even if the high-debt/low-growth steady state is locally unstable, there exists a unique and possibly non-monotonic saddle connection making the economy converge to the low-debt/high-growth steady state. The existence of the saddle connection guarantees global determinacy of perfect foresight equilibrium should the high-debt/low-growth steady state be a node, ruling out multiple explosive paths incompatible with the government's intertemporal budget constraint and the private agents' transversality condition. The foregoing results are robust with respect to the adoption of an output-based—rather than a capital-based—policy function as long as the rule is nonlinear and sufficiently reactive to debt changes.

JEL Classification: H63; E62; O40; C62.

Keywords: Fiscal Austerity; Feedback Policy Rules; Endogenous Growth; Multiple Equilibria; Local Dynamics; Global Dynamics.

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> "There has to be a final farewell to austerity, perhaps the greatest error of the crisis."
>
> —Jean Paul Junker, then President of the European Commission (*Interview to EL PAÍS, Strasbourg, European Parliament, November 20, 2017*)

> "There's a quite good case to be made that austerity in the face of a depressed economy is, literally, a false economy—that it actually makes long-run budget problems worse. People like me have been hesitant to make this argument loudly, for fear of being cast as the left equivalent of Arthur Laffer—but the heck with it, I'm going to lay it out. [...] In short, there's a very good case to be made that austerity now isn't just a bad idea because of its impact on the economy and the unemployed; it may well fail even at the task of helping the budget balance."
>
> —Paul Krugman, (*Self-Defeating Austerity, New York Times, July 7, 2010*)

1 Introduction

This paper characterizes analytically local and global equilibrium dynamics induced by expenditure-based fiscal austerity policies in an endogenous growth model whereby productive government spending is financed by distortionary taxation and issuance of public debt. It is demonstrated that fiscal austerity may be self-defeating around an unintended balanced growth path that exhibits relatively high debt and relatively low growth, but turns to be stabilizing from a global-dynamics perspective—which takes into account non-linearities and potential connections between multiple steady-state equilibria. In particular, fiscal retrenchment typically enables the economic system to uniquely converge towards a balanced growth path that displays relatively low debt and relatively high growth.

Whether austerity in front of rapidly growing public debt-to-GDP ratios is capable of bringing about macroeconomic stability constitutes a pressing issue in recent public policy debates. The European sovereign debt crisis of 2009-2012, the subsequent critical phases of Greece's insolvency protracted up to 2018, and the alleged massive increase in public debts as a result of expansionary fiscal policies aimed to offset the corona crisis have sparked off an intense concern over the question of fiscal sustainability in the European Union. The austerity measures implied by the Fiscal Compact Treaty,[1] in force since 2013, are traditionally considered by a number of influential European policy makers as the correct "exit strategy" to avoid explosive dynamics in the debt-to-GDP ratio. Fiscal retrenchment is regarded as necessary to guarantee debt consolidation and preserve governments' solvency, and at the same time as a prerequisite to foster long-run growth.[2] With a tax burden in the vicinity of one half of GDP for several European countries (Eurostat, 2020)—around the top of the "Laffer curve" (Trabandt and Uhlig, 2011, 2012)—and a central bank independently committed to price stability, thus prevented from creating inflation taxes, expenditure cuts are periodically advocated to embark economies with high debt-to-GDP ratios on dynamic paths converging to the 60-percent Maastricht reference value.[3]

However, at the same time a number of authors have severely questioned the fiscal austerity strategy undertaken by the European Union in the aftermath of the debt crisis of 2009-2012, stressing that it may easily open the door to unintended consequences for aggregate stability. According to Krugman (2010, 2015), fiscal austerity is likely to be "self-defeating" and harmful for long-run sustainability, in the sense that the contractionary effect of a spending cut on output (Blanchard and Leigh, 2013; Auerbach and Gorodnichenko, 2017) may well offset the deficit reduction, thereby causing an undesired debt-to-GDP ratio increase. DeLong and

[1]Formally, "Treaty on Stability, Coordination and Governance in the Economic and Monetary Union".

[2]For instance, it has been argued by the then President of the European Central Bank, Jean-Claude Trichet, that "the idea that austerity measures could trigger stagnation is incorrect. [...] In fact, in these circumstances, everything that helps to increase the confidence of households, firms and investors in the sustainability of public finances is good for the consolidation of growth and job creation. [...] Confidence-inspiring policies will foster and not hamper economic recovery, because confidence is the key factor today" (*Interview to La Repubblica, June 24, 2010*).

[3]Notable academic appeals of advocates of expenditure-based austerity are the empirical works by Giavazzi and Pagano (1990), Alesina and Perotti (1995, 1997), Alesina and Ardagna (2010), and more recently Alesina, Favero and Giavazzi (2019a). For comprehensive reviews of literature on a wide range of empirical findings, see Briotti (2005) and Alesina, Favero and Giavazzi (2018a, 2018b, 2019b).

Summers (2012) provide theoretical support to Krugman's argument, showing in the context of a baseline non-optimizing framework that spending cuts to reduce the deficit as debt accumulates may effectively worsen—rather than improving—the long-run fiscal position, by means of endogenous permanently adverse effects on production and consequently on the riskiness of the outstanding bond stock.[4]

The question of whether fiscal austerity is stabilizing or self-defeating in terms of debt and growth dynamics, while being heavily debated and controversial both theoretically and empirically, is nevertheless rather under-explored by macroeconomic theory employing the discipline of optimizing general equilibrium models.[5] Importantly, consistent with the business cycle literature, the approach in the existing studies restricts attention to local dynamics around a particular calibrated steady state with a zero growth rate, thereby abstracting from endogenous growth, global nonlinearities and a potential multiplicity of steady-state equilibria.

This paper, on the other hand, aims to examine the dynamic consequences of fiscal austerity in an optimizing general equilibrium framework with endogenous growth, in which nonlinear dynamics and possible multiple stationary states are explicitly considered. Specifically, the contribution of the present study is to analyze local and global dynamics in a growth-setup of the type originally developed in the seminal work by Barro (1990), here extended for our purposes to incorporate endogenous government debt dynamics induced by expenditure-based fiscal austerity feedback policies—prescribing spending cuts in reaction to debt accumula-

[4]Empirical evidence supporting this criticism is provided by Cafiso and Cellini (2014), Fatás and Summers (2018), Fatás (2019), and House, Proebsting and Tesar (2019).

[5]Using a dynamic stochastic general equilibrium model for a currency union, Erceg and Lindé (2013) find that large expenditure-based consolidations may be counterproductive in the short-term. In a model with imperfect credibility, Lemoine and Lindé (2016) reinforce the results obtained in Erceg and Lindé (2013). Denes, Eggertsson and Gilbukh (2013) develop a thorough analysis of the limitations of austerity measures in the typical New Keynesian framework, which are found to be amplified when the nominal interest rate is zero. Bi, Leeper and Leith (2013) investigate the macroeconomic implications of fiscal consolidations with uncertain timing and composition, and find that the conditions that could render them expansionary in the medium term are unlikely to occur in the current economic environment. Working in both closed- and open-economy standard New Keynesian setups, Alesina *et al.* (2017) argue that wealth effects on aggregate demand mitigate the negative impact of persistent spending cuts.

tion.[6] Analogously to Benhabib, Schmitt-Grohé and Uribe (2001), Benassy (2007), Cochrane (2011), and Schmitt-Grohé and Uribe (2017), who however concentrate on monetary policy feedback rules in exogenous growth models, in the presence of nonlinearities we shall use the criterion of global (in)determinacy to evaluate the connection between fiscal policy rules and macroeconomic (in)stability.

Because the present value of equilibrium primary surpluses proves to be a nonlinear function of debt—even in the presence of linear expenditure-based fiscal policy rules—two steady state equilibria are shown to emerge, one exhibiting low debt and high growth, one exhibiting high debt and low growth. Local analysis reveals that the low-debt/high-growth steady state is saddle-path stable while the high-debt/low-growth steady state is unstable—the latter thus indicating the possibility of self-defeating austerity, characterized by off-equilibrium upward spirals in debt because of persistent policy-induced adverse effects on growth dividends and consequently on fiscal revenues. However, when global nonlinear dynamics are taken into account, it is demonstrated that the two steady states are endogenously connected. In particular, global analysis reveals that even if the high-debt/low-growth steady state is locally unstable, there exists a unique saddle connection making the economy converge to the low-debt/high-growth steady state. The existence of the saddle connection guarantees global determinacy of perfect foresight equilibrium should the high-debt/low-growth steady state be an unstable node, ruling out multiple explosive paths incompatible with the government's intertemporal budget constraint and the private agents' transversality condition.

[6] There is an extensive literature working in the context of endogenous growth models that incorporate public debt dynamics. See Turnovsky (2000), Irmen and Kuehnel (2009), Greiner and Fincke (2015), Maebayashi, Hori and Futagami (2017), and references therein. However, a common approach of these studies—also in this case—is to focus on local dynamics, thus overlooking global dynamics activated by expenditure-based austerity policy actions—the subject of this paper. A prominent exception accounting for global dynamics is the recent work by Cheron et al. (2019), who find, in the context a Barro (1990)-type model, potential global indeterminacy associated to a fiscal policy targeting the stock of government debt relative to GDP. The present study can be seen as complementary to this line of research, for we attempt to elucidate local and global dynamic properties associated to fiscal feedback instrument rules, reacting to debt upward changes that may arise endogenously by means of cuts in government expenditures.

Remarkably, from our analysis it emerges that the equilibrium trajectory connecting the two steady states may be non-monotonic in terms of both debt and growth dynamics. That is, along the saddle connection originating around the steady-state with high debt and low growth, initially debt may display an increase and, at the same time, the growth rate may display a decline, potentially for long periods of time. In this case, econometricians using first data generated from the saddle connection equilibrium to estimate the macroeconomic effects of fiscal retrenchment could well provide empirical support for the argument of self-defeating austerity—like the one recently detected by some authors for advanced economies in the aftermath of the Great Recession (Fatás and Summers, 2018; Fatás, 2019; House, Proebsting and Tesar, 2020)—even though the economy is instead on an equilibrium path that will approach a steady state with lower debt and higher growth.[7]

The intuition behind the global determinacy result along a non-monotonic heteroclinic orbit connecting the two steady-state equilibria is as follows. Suppose that the economy is in the neighborhood of the high-debt/low-growth balanced growth path with an initial debt-to-capital ratio larger than its steady-state value. The fiscal austerity policy rule implies a lower spending-to-capital ratio, which is associated to a lower marginal productivity of private capital and consequently to a diminished real interest rate, compared to the values prevailing in the unintended steady state. According to the Euler equation on optimal consumption-saving decisions, the expected growth rate falls, thereby triggering a negative effect on

[7]An analogous empirical implication in terms of inflation dynamics in a different environment with multiple steady states under monetary policy feedback rules, globally giving rise to unintended liquidity traps along a saddle connection, has originally been developed by Benhabib, Schmitt-Grohé and Uribe (2001). One might critically argue, however, that the time effectively spent in the neighborhood of the unintended steady-state equilibrium could be relatively short as this equilibrium is a repulsor. Estimating how much time the orbit would take in order to reverse the self-defeating dynamics, by means of a "half-life" estimator, and evaluating the implied welfare burden are thus important considerations for future research. The present framework aimed to analyze the dynamic effects of fiscal austerity and the derived results could then constitute a fruitful benchmark for a more complex investigation along this direction.

expected fiscal revenues. In equilibrium, the present value of expected primary surpluses turns to increase by less than one-to-one with respect to positive deviation of debt with respect to the steady state, hence bringing about a temporarily unstable debt path associated with decreases in the economy's growth rate. Two mechanisms induced by fiscal austerity actions in response to potentially unstable trajectories in debt help reverse dynamics. First, to the extent that austerity decreases real interest rates, debt service falls. Second, austerity spurs capital accumulation over time because it increases private saving directly, by means of spending cuts, and indirectly, by means of policy-induced contractions in households' consumption. Both effects tend to improve the government's fiscal position over time, thus inverting the global dynamics of the debt-to-capital ratio and the rate of growth along the saddle connection.

To sum up, the macroeconomic instability outcome that would arise by performing exclusively a local analysis around the steady state with high debt and low growth, which generates the possibility of self-defeating fiscal consolidations, disappears as soon as a global analysis of equilibrium dynamics is conducted. The global determinacy result, in conjunction with the (possibly non-monotonic) convergence of the unique globally stable equilibrium path towards an alternative steady state with low debt and high growth, gives by contrast analytical foundations to the view that committing governments' actions to fiscal austerity policy rules is likely to pay off in the long run.

More in general, from a methodological point of view, the analytical results demonstrated in this paper show that evaluating the dynamic effects of fiscal austerity from a global-dynamics perspective, which incorporates the possibility of multiple balanced growth paths and the potential existence of heteroclinic orbits connecting different steady states, proves to be essential for a general characterization of the macroeconomic consequences of fiscal consolidations. Our analytically tractable setup here presented could then be used as a useful benchmark for more complex investigations along the foregoing lines.

The remainder of the paper is organized in five sections. Section 2 presents the optimizing endogenous growth model and specifies the fiscal policy framework. Section 3 examines the implied steady states and characterizes local equilibrium dynamics. Section 4 analyzes the issue of global equilibrium dynamics and establishes the paper's main results. Section 5 addresses the issue of results' robustness by exploring the dynamic consequences of an output-based—rather than a capital-based—policy function. The concluding Section 6 provides summary comments and final remarks.

2 The Model

We set forth an analytically tractable endogenous growth model with the public sector of the type originally proposed by Barro (1990), here extended to embody debt-financed government spending, in order to show how fiscal austerity policies may easily lead to multiplicity of steady-state equilibria—exhibiting different properties in terms of local dynamics.

Consider an economy populated by a large number of identical infinitely lived households deriving utility from consumption and supplying one unit of labor inelastically. The lifetime utility function of the representative household is given by

$$\int_0^\infty e^{-\rho t} \log C(t)\, dt, \tag{1}$$

where $\rho > 0$ and $C(t)$ denote the rate of time preference and consumption at the instant of time t, respectively. The household maximizes (1) subject to the instant budget constraint

$$\dot{K}(t) + \dot{B}(t) = (1-\tau)\left[r(t)\left(K(t) + B(t)\right) + w(t)\right] - C(t) \tag{2}$$

and to the borrowing limit condition precluding Ponzi's games

$$\lim_{t\to\infty} e^{-\int_0^t (1-\tau) r(v) dv} \left(K(t) + B(t)\right) \geq 0, \tag{3}$$

where $K(t)$ denotes the stock of capital, $B(t)$ the stock of government bonds, $r(t)$ the interest rate, $w(t)$ the wage rate, and τ the income tax rate, assumed to be time-invariant for analytical convenience. Intertemporal optimization implies the Euler equation and the tranversality condition:

$$\mu(t) = (1-\tau) r(t) - \rho, \tag{4}$$

$$\lim_{t\to\infty} e^{-\rho t} \left(C(t)\right)^{-1} \left(K(t) + B(t)\right) = 0, \tag{5}$$

where $\mu(t) \equiv \dot{C}(t)/C(t)$.

Following Barro (1989; 1990), Turnovsky and Fisher (1995), Bruce and Turnovsky (1999), and Futagami, Iwaisako and Ohdoi (2008), government spending is assumed to yield productive services.[8] Specifically, perfectly competitive firms face a Cobb-Douglas production function of the form

$$Y(t) = AK(t)^a \left(G(t) L(t)\right)^{1-\alpha}, \tag{6}$$

where $A > 0$ is a technology parameter measuring the total factor productivity, $Y(t)$ denotes output, $L(t)$ the labor input, and $G(t)$ the quantity of public services

[8]Notably, there is a wide empirical literature supporting the hypothesis of productive public expenditure. See the seminal work by Aschauer (1989) and the comprehensive literature review by de Haan and Romp (2007).

provided to producers.[9] Profit maximization implies

$$r(t) = \alpha A \left(\frac{G(t) L(t)}{K(t)} \right)^{1-\alpha}, \tag{7}$$

$$w(t) = (1-\alpha) A G(t) \left(\frac{G(t) L(t)}{K(t)} \right)^{-\alpha}. \tag{8}$$

The government issues bonds to finance budget deficits. The instant budget constraint of the public sector is therefore given by

$$\dot{B}(t) = r(t) B(t) - \tau \left[r(t) \left(K(t) + B(t) \right) + w(t) \right] + G(t). \tag{9}$$

The government must obey the no-Ponzi game condition,

$$\lim_{t \to \infty} e^{-\int_0^t (1-\tau) r(v) dv} B(t) = 0. \tag{10}$$

For our purposes, we restrict attention to a fiscal policy that takes the form of an expenditure-based feedback rule whereby government spending is set as a decreasing function of the outstanding debt. Specifically, we assume a nonlinear rule of the form

$$\frac{G(t)}{K(t)} = \Phi \left(\frac{B(t)}{K(t)} \right), \tag{11}$$

where function $\Phi(\cdot)$ is continuous and satisfies $\Phi'(\cdot), \Phi''(\cdot) < 0$. Two observations are worth pointing out. The first is that the fiscal authority is assumed to implicitly target the ratio of debt to the size of the economy measured by the stock of private capital, as in Futagami, Iwaisako and Ohdoi (2008).[10] We shall address

[9] Following the influential studies of Futagami, Morita and Shibata (1993) and Turnovsky (1997), an alternative route would be to include the stock of public capital, rather than the flow of public services, in the production function. In this paper, we follow the standard approach *à la* Barro (1990), analogously to Cheron et al. (2019), because it enables the dynamic system of the model to be analytically tractable for our study of global dynamics resulting from fiscal austerity policies.

[10] See Bruce and Turnovsky (1999) and Canofari, Piergallini and Piersanti (2020) for reasons for preferring the debt-to-capital ratio to the debt-to-output ratio in the context of endogenous

the robustness of this assumption for our results in Section 5, in which we investigate the dynamic consequences of an expenditure-based feedback policy function expressed relative to output—rather than relative to capital. The second observation is that the government is assumed to react to fiscal imbalances in a non-linear way, through an increasing marginal adjustment of the fiscal instrument to upward changes in debt, in line with a fairly well-established empirical evidence.[11]

There are relevant reasons that justify our focus on an expenditure-based policy rule. In the European context characterized by tax burden larger than 40 percent of GDP for a number of countries (Eurostat, 2020)—in the vicinity of the top of the "Laffer curve" according to Trabandt and Uhlig (2011, 2012)—and a price-stability-oriented central bank precluded from opening the door to inflation taxes, expenditure cuts are often advocated to lead economies with high debt-to-GDP ratios on dynamic trajectories converging to the 60-percent Maastricht reference value. In particular, the European Commission periodically recommends implementing fiscal consolidations by recourse to spending cuts rather than tax increases (e.g., European Commission, 2020), to the extent they are argued to be more effective in lowering the outstanding debt burden according to part of empirical literature (e.g., Favero and Mei, 2019). On this ground, the Stability and Convergence Programs in the European Union in the aftermath of the debt crisis erupted in 2009 have been primary based on expenditure cuts, including decreases in public investment. From 2010 to 2019, specifically, the European Union has cut the public expenditure-to-GDP ratio by 4.3 percentage points, while has increased the tax revenue-to-GDP ratio by 1.5 percentage points (Eurostat, 2020). Remarkably, over the same period many other economies have opted mainly for reducing public expenditure as a share of output to face the increased debt levels following the global financial crisis of 2007-09—for instance the United States by 4.2 and the

growth models.

[11]See Piergallini (2019) and references therein.

United Kingdom by 6.4 percentage points, in front of rises in taxation by 0.5 and 0.8 percentage points of GDP, respectively (IMF, 2020; Eurostat, 2020).

Now, using $Y(t) = r(t) K(t) + w(t)$, the constraint (9) can be written as

$$\dot{B}(t) = (1-\tau) r(t) B(t) + G(t) - \tau Y(t) \tag{12}$$

and further rearranged as

$$\frac{\dot{b}(t)}{b(t)} = (1-\tau) r(t) + \frac{G(t)}{B(t)} - \frac{\tau Y(t)}{B(t)} - \frac{\dot{K}(t)}{K(t)}, \tag{13}$$

where $b(t) \equiv B(t)/K(t)$. Substituting (6) and (7) into (13) yields

$$\frac{\dot{b}(t)}{b(t)} = \alpha A(1-\tau) \left(\frac{G(t) L(t)}{K(t)}\right)^{1-\alpha} + \frac{G(t)}{B(t)} - \frac{\tau A K(t)^{a} (G(t) L(t))^{1-\alpha}}{B(t)} - \frac{\dot{K}(t)}{K(t)}. \tag{14}$$

Using next the goods market clearing condition

$$\dot{K}(t) = AK(t)^{a} (G(t) L(t))^{1-\alpha} - C(t) - G(t), \tag{15}$$

the labor market clearing condition $L(t) = 1$, and rule (11) into (14) results in the following equilibrium dynamic equation for government debt:

$$\frac{\dot{b}(t)}{b(t)} = -\left[1 - \alpha(1-\tau) + \frac{\tau}{b(t)}\right] A\Phi(b(t))^{1-\alpha} + \left(1 + \frac{1}{b(t)}\right) \Phi(b(t)) + c(t), \tag{16}$$

where $c(t) \equiv C(t)/K(t)$.

From the Euler equation (4), the evolution of consumption over time can be rearranged as

$$\frac{\dot{c}(t)}{c(t)} = (1-\tau) r(t) - \frac{\dot{K}(t)}{K(t)} - \rho. \tag{17}$$

Substituting (7) and (15) into (17) results in the following equilibrium dynamic equation for consumption:

$$\frac{\dot{c}(t)}{c(t)} = c(t) - [1 - \alpha(1-\tau)] A\Phi(b(t))^{1-\alpha} + \Phi(b(t)) - \rho. \tag{18}$$

3 Steady-State Equilibria and Local Dynamics

In this section, we initially develop the steady-state analysis. In this context, we show how fiscal austerity rules that react to upward changes in debt through spending cuts bring about multiplicity of balanced growth paths. Then, after investigating the implied properties in terms of equilibrium debt levels and growth rates, we explore the issue of local dynamics in a small neighborhood around each steady state that arises.

Equilibrium steady states are positive bounded values for (b, c) such that $\dot{b}(t) = 0$, $\dot{c}(t) = 0$, and the agents' transversality condition (5) together with the government's no-Ponzi game condition (10) hold.[12] Because under expenditure-based fiscal austerity policies, using (6) and (11), in equilibrium the primary surplus-to-capital ratio $(\tau Y(t) - G(t))/K(t)$ turns out to be a nonlinear function of the debt-to-capital ratio of the form

$$\Psi(b(t)) \equiv \tau A\Phi(b(t))^{1-\alpha} - \Phi(b(t)), \tag{19}$$

the next proposition applies.

Proposition 1 *(**Steady-State Analysis**) Suppose that fiscal policy follows an expenditure-based austerity policy rule* ($\Phi', \Phi'' < 0$) *expressed relative to the capital stock and satisfies*

$$\hat{b} < \frac{\Psi\left(\hat{b}\right)}{\rho}, \tag{20}$$

[12] In what follows, as usual in the growth literature, we rule out the case in which $c = 0$ in order to concentrate on steady states that have an economic sense.

where

$$\hat{b} = \arg\max_{b} \left\{ \frac{\Psi(b)}{\rho} - b \right\}, \tag{21}$$

which precludes the government from engaging in Ponzi's games, and

$$\Psi(0) < 0, \tag{22}$$

which stipulates the occurrence of a positive primary deficit in the case of a debt equal to zero. Then, from the equilibrium system (16) and (18),(a) there exist two steady states, (b^*, c^*) *and* $(\bar{b}, \bar{c})$, *obeying* $b^* < \bar{b}$ *and*

$$c^* \gtreqless \bar{c} \Longleftrightarrow (1-\alpha)(1-\tau) A \left(\Phi(b^*)^{1-\alpha} - \Phi(\bar{b})^{1-\alpha} \right) \gtreqless \rho(\bar{b} - b^*); \tag{23}$$

(b) the growth rate at low-debt steady state (b^*, c^*), μ^*, *is higher than the growth rate at high-debt steady state* $(\bar{b}, \bar{c})$, $\bar{\mu}$.

Proof. See the Appendix.

According to Proposition 1, the induced inverted U-shaped non-linearity of the present value of equilibrium primary surpluses with respect to debt—which would occur even in the limiting case of a linear expenditure-based policy rule—opens the door to multiplicity of steady-state equilibria. In particular, two steady states emerge, one intended steady state (b^*, c^*) displaying low debt and high growth and one unintended steady state $(\bar{b}, \bar{c})$ displaying high debt and low growth. As Figures 1-2 in the Appendix show, this is because under expenditure-based fiscal austerity measures, an increase in the present value of primary surpluses more than one-to-one with respect to an increase in debt cannot hold globally and, specifically, turns to be unfeasible beyond a threshold level of public debt given by $\hat{b}$. Spending cuts as debt expands bring about a contraction in the output of the economy relative to the capital stock, thus dampening fiscal revenues, in a way that the present value of primary surpluses for $\hat{b} < b < b^{\max} \equiv \Phi^{-1}\left(\tau A\left(1-\alpha\right)\right)^{1/\alpha}$ increases less than proportionally, and for $b > b^{\max}$ even decreases.

Explore now local dynamics. Linearizing equations (16) and (18) in the neighborhood of any steady-state point (b, c), one obtains the Jacobian

$$J^{(b,c)} = \begin{pmatrix} J_{11}^{(b,c)} & b \\ J_{21}^{(b,c)} & c \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned} J_{11}^{(b,c)} &= -\left[1-\alpha\left(1-\tau\right)+\frac{\tau}{b}\right] A\left(1-\alpha\right)\Phi\left(b\right)^{-\alpha}\Phi'\left(b\right)b+\frac{\tau}{b}A\Phi\left(b\right)^{1-\alpha} \\ &\quad +\left(1+b\right)\Phi'\left(b\right)-\frac{\Phi\left(b\right)}{b} \\ &= -\Phi'\left(b\right)b\left\{\left[1-\alpha\left(1-\tau\right)\right]A\left(1-\alpha\right)\Phi\left(b\right)^{-\alpha}-1\right\}+\left(\rho-\Psi'\left(b\right)\right), \end{aligned} \tag{25}$$

$$J_{21}^{(b,c)} = -\Phi'\left(b\right)c\left\{\left[1-\alpha\left(1-\tau\right)\right]A\left(1-\alpha\right)\Phi\left(b\right)^{-\alpha}-1\right\}. \tag{26}$$

The determinant and the trace of the Jacobian matrix are

$$\det\, J^{(b,c)} = \left(1-\frac{\Psi'\left(b\right)}{\rho}\right)\rho c, \tag{27}$$

$$\begin{aligned} \mathrm{tr}\; J^{(b,c)} &= -\left[\left(1-\tau\right)\left(1-\alpha\right)\right]A\left(1-\alpha\right)\Phi\left(b\right)^{-\alpha}\Phi'\left(b\right)b+\left(\rho-\Psi'\left(b\right)\right)\left(1+b\right) \\ &\quad +\left(1-\alpha\right)\left(1-\tau\right)A\Phi\left(b\right)^{1-\alpha}+\rho. \end{aligned} \tag{28}$$

Hence, the following proposition holds.

Proposition 2 ***(Local Analysis)*** *Suppose that fiscal policy follows an expenditure-based austerity policy rule* $(\Phi', \Phi'' < 0)$ *expressed relative to the capital stock and satisfies (20) and (22). Then, from the equilibrium system (16) and (18), locally (a) the steady state* (b^*, c^*) *is a saddle point; (b) the steady state* $\left(\bar{b}, \bar{c}\right)$ *is an unstable* *node or an unstable spiral point.*

Proof. See the Appendix.

Consider part *(a)* of Proposition 2, pertaining to the dynamic properties of the economy in a small neighborhood around the low-debt/high-growth steady state (b^*, c^*). Because $c(t)$ is a "jump" variable and $b(t)$ is a predetermined variable, around (b^*, c^*) local determinacy applies. That is, there exists a unique perfect foresight equilibrium converging asymptotically to the steady state. In particular, the only trajectory of $(b(t), c(t))$ leading to (b^*, c^*) is given by the saddle-path solution expressed by

$$c(t) = c^* + \left(\frac{J_{21}^{(b^*,c^*)}}{\lambda_1 - c^*} \right) (b(t) - b^*), \tag{29}$$

$$b(t) = b^* + (b(0) - b^*) e^{\lambda_1 t}, \tag{30}$$

where $\lambda_1 < 0$ is the negative eigenvalue associated to $J^{(b^*,c^*)}$.

The economic intuition behind the local determinacy result around (b^*, c^*) is as follows. Suppose that private agents, in response to a sunspot, develop expectations of a higher growth rate. As a result of the Euler equation, characterizing the households' consumption-saving optimal decision, this revision in expectations must be associated with a higher real interest rate. The increase in the real interest rate rises both interest payments for the government and costs of production for firms, which optimally respond by reducing the capital input. Both effects tend to deteriorate the fiscal position of the government in terms of debt-to-capital ratio. Because fiscal authorities are engaged in austerity policies in response to debt accumulation, the government is expected to cut public expenditures. As long as the debt burden is below the threshold level $\hat{b}$ (see Figures 1-2 in the Appendix), the present value of primary surpluses rises more than one-to-one with respect to increases in debt. As a consequence, both the growth rate and the debt-to-capital ratio fall. First, the decline in the growth rate does not validate the original expectations, ruling out equilibrium indeterminacy and hence the possibility of self-fulfilling expectational equilibria. Second, the decline in the debt-to-capital ratio makes the economy embark on a stable path converging to the steady state (b^*, c^*).

Nevertheless, consider now part *(b)* of Proposition 2, pertaining to the high-debt/low-growth steady state $(\bar{b}, \bar{c})$. The dynamic properties under fiscal austerity rules change dramatically. If $b(0) \neq \bar{b}$, no perfect foresight equilibria exist in which $(b(t), c(t))$ converge asymptotically to $(\bar{b}, \bar{c})$, except the steady state itself. Local instability applies, in contrast with the previous stabilizing properties of fiscal austerity policies. It thus follows that, in the case of real roots, there exists a continuum of unstable curved paths, bounded below by the unstable branch given by

$$c(t) = \bar{c} + \left(\frac{\eta_2 - J_{11}^{(\bar{b},\bar{c})}}{\bar{b}} \right) \left(b(t) - \bar{b} \right), \tag{31}$$

with $\eta_2 < \eta_1$, where η_1, η_2 are the positive eigenvalues associated to $J^{(\bar{b},\bar{c})}$, in which the debt-to-capital ratio embarks on potentially explosive trajectories.

Such a self-defeating outcome of fiscal austerity, featured by unintended off-equilibrium upward spirals in debt, occurs precisely in spite of the associated spending cuts implemented by fiscal authorities in the attempt to offset the worsened debt burden and restore fiscal discipline. The point, however, is that fiscal retrenchment around the steady state with high debt is now more pronounced if compared to the steady state with low debt, thereby causing a higher decline in the growth rate of the economy[13] and, in turn, inducing more adverse endogenous

[13]Using (7) and (11) into (4) and applying the labor market equilibrium condition yield $\mu(t) = \alpha A (1-\tau) \Phi(b(t))^{1-\alpha} - \rho$. So $d\mu(t)/db(t) = \alpha A (1-\tau)(1-\alpha) \Phi'(b(t))^{-\alpha} < 0$ because $\Phi' < 0$ and $d^2\mu(t)/db(t)^2 = -\alpha^2 A (1-\tau)(1-\alpha) \Phi''(b(t))^{-\alpha-1} > 0$ because $\Phi'' < 0$. Two considerations are thus made explicit. First, the growth rate is a decreasing function of the debt-to-capital ratio. Second, the growth rate turns to be crowded out due to fiscal austerity actions by more the higher the debt-to-capital ratio.

effects on fiscal revenues. Consequently, as long as the debt burden is now above the threshold level $\hat{b}$, in equilibrium the present value of expected primary surpluses increases by less than one-to-one with respect to increases in debt, at odds with usual debt sustainability requirements.

From the above analysis, one may be tempted to conclude that, if the economy lies in the vicinity of the high-debt/low-growth steady state, enforcing fiscal austerity is likely to persistently exacerbate—rather than improving—the debt accumulation problem, hence warning against the adoption of spending cuts in reaction to debt upward changes. A conclusion of this type, however, could be misleading, because from a global perspective the picture may be radically different. The central reason is that the steady states may endogenously be connected. Globally, trajectories of the economy diverging from one steady state can turn out to be equilibrium paths if they converge to the other steady state. To examine this possibility, we must depart from local analysis.

4 Global Dynamics

This section characterizes global equilibrium dynamics. To this end, it will prove convenient for the analysis that follows to rewrite the system (16) and (18) as

$$\dot{b}(t) = (c(t) - \Omega(b(t)))\, b(t), \tag{32}$$

$$\dot{c}(t) = (c(t) - \Gamma(b(t)))\, c(t), \tag{33}$$

where

$$\Omega(b(t)) \equiv \left[1 - \alpha(1-\tau) + \frac{\tau}{b(t)}\right] A\Phi(b(t))^{1-\alpha} - \left(1 + \frac{1}{b(t)}\right)\Phi(b(t)), \tag{34}$$

$$\Gamma(b(t)) \equiv [1 - \alpha(1-\tau)]\, A\Phi(b(t))^{1-\alpha} - \Phi(b(t)) + \rho. \tag{35}$$

The following proposition establishes the main results.

Proposition 3 ***(Global Analysis)*** *Suppose that fiscal policy follows an expenditure-based austerity policy rule* ($\Phi', \Phi'' < 0$) *expressed relative to the capital stock and satisfies (20) and (22). Then, from the equilibrium system (16) and (18), globally there exists a heteroclinic orbit connecting the steady states* $(\bar{b}, \bar{c})$ *and* (b^*, c^*)*; the unique trajectory along the orbit originates in the neighborhood of the steady state* $(\bar{b}, \bar{c})$ *and makes the economy converge asymptotically to the steady state* (b^*, c^*)*. Moreover, in the case in which* $(\bar{b}, \bar{c})$ *is an unstable node, (a) if* $c^* < \bar{c}$ *and* $\Gamma'\left(\bar{b}\right) > 0$*, the orbit is non-monotonic in terms of dynamics of the debt-to-capital ratio and the growth rate; (b) if* $c^* < \bar{c}$ *and* $\Gamma'\left(\bar{b}\right) < 0$*, the orbit is non-monotonic in terms of dynamics of either the debt-to-capital ratio and the growth rate or the consumption-to-capital ratio; (c) if* $c^* > \bar{c}$ *and* $\Gamma'(b^*) > 0$*, the orbit is non-monotonic in terms of dynamics of the consumption-to-capital ratio; (d) if* $c^* > \bar{c}$ *and* $\Gamma'(b^*) < 0$*, the orbit is monotonic.*

Proof. See the Appendix.

The central result that comes from Proposition 3 is that, under a fiscal policy commitment to expenditure-based austerity policies, the global dynamics is characterized by the existence of a unique trajectory—the heteroclininc orbit H—joining the two steady states $(\bar{b}, \bar{c})$ and (b^*, c^*), as shown in Figures 3-5 in the Appendix. In particular, the saddle connection originates in the neighborhood of the high-debt/low-growth steady state $(\bar{b}, \bar{c})$ and makes the economy converge asymptotically to the low-debt/high-growth steady state (b^*, c^*). Three important implications emerge. First, the trajectory of debt and consumption along the heteroclinic orbit must be qualified as a "perfect foresight equilibrium" path of the economy because, in spite of the fact that it diverges away from the high-debt/low-growth steady state, it is however globally stabilizing to the extent it converges to the low-debt/high-growth steady state, hence satisfying both the government's no-Ponzi game condition and the private agents' transversality condition—as we have demonstrated in the proof of Proposition 1.

Second, the existence of a unique saddle connection ensures "global determinacy" of perfect foresight equilibrium when the steady state with high debt and low growth is an unstable node. This means that all other trajectories of debt and consumption are *not* equilibrium paths because they diverge from *both* steady states and, as the analysis that follows shows, are not compatible with the government's intertemporal budget constraint and/or the private agents' intertemporal optimization.[14]

To see this point, we begin by noting that from the phase diagrams that illustrate potential global dynamics that may occur (see Figures 3-5 in the Appendix), outside the heteroclinic orbit paths diverging from both steady states can either lead to $c(t) = 0$ or be explosive in both $b(t)$ and $c(t)$. The first case cannot be an equilibrium outcome because households are not satisfying their intertemporal optimization condition for consumption dynamics, given by equation (4). The second case can be supported as an equilibrium outcome if and only if the government's no-Ponzi game condition (10) and the private agents' transversality condition (5) are verified. Using (4), condition (10) can be rewritten as

$$\lim_{t\to\infty} e^{-\rho t}\frac{B(t)}{C(t)} = \lim_{t\to\infty} e^{-\rho t}\frac{b(t)}{c(t)} = 0. \tag{36}$$

Condition (5) can be expressed as

$$\lim_{t\to\infty} e^{-\rho t}\frac{(K(t)+B(t))}{C(t)} = \lim_{t\to\infty} e^{-\rho t}\left(\frac{1}{c(t)}+\frac{b(t)}{c(t)}\right) = 0. \tag{37}$$

[14]On the other hand, if the eigenvalues of the Jacobian matrix evaluated at the steady state with high debt and low growth are complex numbers, then "global determinacy" does not hold. In this case, indeed, being that steady state an unstable spiral point, the economy fluctuates around it for a while.

Equations (36) and (37) are thus satisfied if and only if, along explosive dynamics, $b(t)/c(t)$ grows at a rate strictly lower than ρ. From (16) and (18), the growth rate of $b(t)/c(t)$ is given by

$$\begin{aligned}\frac{\dot{b}(t)}{b(t)}-\frac{\dot{c}(t)}{c(t)} &= -\left[1-\alpha(1-\tau)+\frac{\tau}{b(t)}\right]A\Phi(b(t))^{1-\alpha}+\left(1+\frac{1}{b(t)}\right)\Phi(b(t)) \\ &\quad +c(t)-\left\{c(t)-[1-\alpha(1-\tau)]A\Phi(b(t))^{1-\alpha}+\Phi(b(t))-\rho\right\} \\ &= \rho+\frac{\Phi(b(t))-\tau A\Phi(b(t))^{1-\alpha}}{b(t)}, \qquad (38)\end{aligned}$$

so that

$$\lim_{t\to\infty}\left(\frac{\dot{b}(t)}{b(t)}-\frac{\dot{c}(t)}{c(t)}\right)=\rho, \qquad (39)$$

because $\Phi(\infty)=0$. Consequently, both conditions (36) and (37) are violated. This implies that no explosive path of $(b(t),c(t))$ can be sustained as a perfect foresight equilibrium.

Third, as elucidated in parts (a)-(b) of Proposition 3 and illustrated in Figures 3-4 in the Appendix, the heteroclinic orbit connecting the two steady states may be "non-monotonic" in terms of both debt and growth dynamics. This means that, along the saddle connection originating around the steady-state with high debt and low growth, initially debt may feature an increase and, at the same time, the growth rate may feature a decline, potentially for long periods of time. Under these circumstances, using first data generated from the saddle connection equilibrium to infer the macroeconomic effects of fiscal retrenchment, econometricians could well lend empirical support to the claim of counterproductive austerity—like the one recently detected by some authors for a number of advanced economies over the post-Great Recession period (see, e.g., Fatás and Summers, 2018; Fatás, 2019; House, Proebsting and Tesar, 2020)—even though the economy is instead on an equilibrium trajectory that will reverse direction and point towards a steady state with lower debt and higher growth.

The main economic intuition behind the result of non-monotonic dynamics along the heteroclinic orbit, which gives the flavor of the proof of Proposition 3, is as follows. Suppose that the economy is around the high-debt/low-growth steady state with an initial debt-to-capital ratio larger than its steady-state value. The fiscal austerity policy rule (11) prescribes a lower spending-to-capital ratio, which is associated to a lower marginal productivity of private capital and hence to a lower real interest rate according to (7), in relation to the values that obtain in the unintended balanced growth path. From the Euler equation (4) characterizing optimal consumption-saving decisions, the expected growth rate declines, thus bringing about an adverse effect on expected fiscal revenues. In equilibrium, the present value of expected primary surpluses turns to increase by less than one-to-one with respect to positive deviation of debt with respect to the steady state, thereby triggering a temporarily unstable debt path associated with decreases in the economy's growth rate. The subsequent occurrence of an inverted dynamics involves two major effects induced globally by fiscal austerity actions in response to potentially unstable trajectories in debt. First, decreases in real interest rates driven by expenditure cuts cause debt service to fall. Second, from (15) austerity spurs capital accumulation over time because it increases private saving directly, by means of spending cuts, and indirectly, by means of policy-induced contractions in households' consumption. Both effects tend to improve the government's fiscal position over time, reversing the dynamics of the debt-to-capital ratio and the rate of growth along the saddle connection.

From Proposition 3, notice that a necessary condition for such a non-monotonic scenario to take place is $c^* < \bar{c}$. Therefore, from (23), a prediction of the present model is that the above non-monotonic outcome is more likely to occur when the difference between the high-debt and the low-debt steady-state levels is relatively high, the income tax rate is relatively high, the elasticity of output with respect to government spending in the aggregate production function is relatively low, and the total factor productivity is relatively low—features that arguably characterize many European countries for which austerity has been found to be self-defeating in the aftermath of the Great Recession, such as GIIPS economies[15] (see House, Proebsting and Tesar, 2020).

In synthesis, to summarize the main implications of our theoretical findings, the macroeconomic instability outcome that would emerge by focusing solely on a local analysis around the steady state with high debt and low growth—incorporating the possibility of self-defeating fiscal consolidations—disappears as soon as one moves to a global analysis of equilibrium dynamics. The global determinacy result, in conjunction with the (possibly non-monotonic) convergence of the unique globally stable equilibrium path towards an alternative steady state with low debt and high growth, provides by contrast sound analytical foundations to the view that committing governments' actions to fiscal austerity policy rules is not in vain and is likely to pay off in the long run.

5 Dynamic Analysis under a GDP-Based Rule

Our investigation so far has been conducted by employing the private capital stock as the measure of the size of the economy in the specification of the fiscal austerity rule, along the lines of Futagami, Iwaisako and Ohdoi (2008). This assumption has enabled us to express both debt and consumption dynamics in terms of ratios to capital, which typically simplifies the analysis in endogenous growth models. However, in the context of the Futagami-Iwaisako-Ohdoi framework with a debt targeting rule, Minea and Villieu (2013) demonstrate that defining instead the target in terms of debt-to-GDP ratio yields steady-state uniqueness and equilibrium determinacy, in contrast with the results of multiplicity of balanced growth paths

[15]Greece, Ireland, Italy, Portugal and Spain.

and possible indeterminacy that apply when the debt target is expressed as a ratio to private capital. Therefore, the purpose of this section is to explore whether by adopting output as the measure of the size of the economy in the policy function, our main findings so far obtained significantly modify.

To directly compare results with Minea and Villieu (2013), we first concentrate on the case a *linear* fiscal rule, as assumed by Futagami, Iwaisako and Ohdoi (2008). Specifically, the policy function now takes the form

$$\gamma(t) = \gamma^* - \xi(\theta(t) - \theta^*), \tag{40}$$

where $\gamma(t) \equiv G(t)/Y(t)$, $\theta(t) \equiv B(t)/Y(t)$, $\xi > 0$ is a policy parameter measuring the "strenght" of fiscal austerity actions, and γ^*, θ^* are the target steady state levels of the spending-to-GDP ratio and the debt-to-GDP ratio, respectively. Using the relationships

$$g(t) = (A\gamma(t))^{\frac{1}{\alpha}}, \tag{41}$$

$$b(t) = \theta(t) A^{\frac{1}{\alpha}} \gamma(t)^{\frac{1-\alpha}{\alpha}}, \tag{42}$$

where $g(t) \equiv G(t)/K(t)$, together with rule (40), yields the dynamic system given by

$$\begin{aligned}\frac{\dot{\theta}(t)}{\theta(t)} &= \left[1 - \frac{(1-\alpha)}{a}\xi\frac{\theta(t)}{\gamma(t)}\right]^{-1} \times \\ &\left\{\begin{array}{c} -[1-\alpha(1-\tau)] A^{\frac{1}{\alpha}} [\gamma^* - \xi(\theta(t) - \theta^*)]^{\frac{1-\alpha}{\alpha}} + \\ \frac{[\gamma^* - \xi(\theta(t) - \theta^*)] - \tau}{\theta(t)} + A^{\frac{1}{\alpha}} [\gamma^* - \xi(\theta(t) - \theta^*)]^{\frac{1}{\alpha}} + c(t) \end{array}\right\},\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= c(t) - [1-\alpha(1-\tau)] A^{\frac{1}{\alpha}} [\gamma^* - \xi(\theta(t) - \theta^*)]^{\frac{1-\alpha}{\alpha}} + \\ &\quad A^{\frac{1}{\alpha}} [\gamma^* - \xi(\theta(t) - \theta^*)]^{\frac{1}{\alpha}} - \rho.\end{aligned} \tag{44}$$

Hence, we can state the following proposition.

Proposition 4 ***(Steady-State and Dynamic Analysis under a Linear GDP-Based Rule)*** *Suppose that fiscal policy follows a linear expenditure-based austerity policy rule ($\xi > 0$) expressed relative to output. Then, from the equilibrium system (43) and (44),(a) there exists one steady state (θ^*, c^*); (b) if $[a/(1-\alpha)](\gamma^*/\theta^*) > \rho$, the steady state (θ^*, c^*) is a saddle point if and only if*

$$\rho < \xi < \frac{\alpha}{(1-\alpha)} \frac{\gamma^*}{\theta^*}; \tag{45}$$

if $[a/(1-\alpha)](\gamma^*/\theta^*) < \rho$, the steady state (θ^*, c^*) is a saddle point if and only if

$$\frac{\alpha}{(1-\alpha)} \frac{\gamma^*}{\theta^*} < \xi < \rho; \tag{46}$$

(c) if either conditions (45) or (46) are not satisfied, the steady state (θ^*, c^*) can be either a sink or an unstable node/spiral point.

Proof. See the Appendix.

From Proposition 4, as in Minea and Vilneu (2013), steady-state multiplicity is removed. Intuitively, when the GDP is taken as a measure of the size of the economy within a linear fiscal austerity rule, in the steady state the present value of the primary surplus-to-GDP ratio turns out to be globally a linear increasing function of the debt-to-GDP ratio, thereby ensuring steady-state uniqueness in correspondence of the target level θ^*. Differently from Minea and Vilneu (2013), however, under fiscal feedback instrument rules—as opposed of targeting rules—saddle-path stability and thus equilibrium determinacy[16] is not always verified.

[16] The reason why saddle-path stability ensures equilibrium determinacy is that the system (43)-(44) consists of one jumping variable with a free initial condition, $c(t)$, and one non-jumping variable with a predetermined intial condition, $\theta(t)$, because the ratio $b(0) \equiv B(0)/K(0) = (B(0)/Y(0))(Y(0/K(0))) = \theta(0) A\frac{1}{\alpha}[\gamma^* - \xi(\theta(0) - \theta^*)]^{\frac{1-\alpha}{\alpha}}$ cannot jump—being the initial stocks of public debt $B(0)$ and capital $K(0)$ both predetermined.

According to Proposition 4, in fact, determinacy requires that austerity policies are designed to be sufficiently reactive to debt increases but at the same time not overly aggressive. The reason is that excessively pronounced expenditure cuts in response to the accumulation of the debt-to-GDP ratio are self-defeating for they generate an excessive crowding-out effect of growth dividends and fiscal revenues, bringing about as a consequence either instability or indeterminacy of equilibrium. Nevertheless, it is worth emphasizing that, should the feedback policy parameter ξ respect the range defined by either condition (45) or condition (46) in part (*b*) of Proposition 4, the general result of global determinacy under expenditure-based austerity policies obtained in Section 4 still applies—although the economy in this case does not display the presence of a self-defeating temporarily unstable path around an alternative balanced growth path exhibiting relatively high debt and relatively low growth.

Explore now the case of a *nonlinear* fiscal rule, compatible with the well-documented empirical evidence showing that marginal fiscal adjustments increase in the level of debt (see Piergallini, 2019, and references therein). Specifically, consider a policy function of the form

$$\gamma(t) = \Xi(\theta(t)), \tag{47}$$

where function $\Xi(\cdot)$ is continuous and obeys $\Xi'(\cdot), \Xi''(\cdot) < 0$. Using (41), (42) and (47), the equilibrium system is described by

$$\begin{aligned} \frac{\dot{\theta}(t)}{\theta(t)} &= \left[1 + \frac{(1-\alpha)}{a}\Xi'(\theta(t))\frac{\theta(t)}{\Xi(\theta(t))}\right]^{-1} \times \\ &\left\{ \begin{array}{c} -\left[1-\alpha(1-\tau)\right] A^{\frac{1}{\alpha}}\Xi(\theta(t))^{\frac{1-\alpha}{\alpha}} + \frac{\Xi(\theta(t))-\tau}{\theta(t)} + \\ A^{\frac{1}{\alpha}}\Xi(\theta(t))^{\frac{1}{\alpha}} + c(t) \end{array} \right\}, \end{aligned} \tag{48}$$

$$\frac{\dot{c}(t)}{c(t)} = c(t) - [1-\alpha(1-\tau)] A^{\frac{1}{\alpha}} \Xi(\theta(t))^{\frac{1-\alpha}{\alpha}} + A^{\frac{1}{\alpha}} \Xi(\theta(t))^{\frac{1}{\alpha}} - \rho. \tag{49}$$

Thus, the following proposition applies.

Proposition 5 ***(Steady-State and Dynamic Analysis under a Nonlinear GDP-Based Rule)*** *Suppose that fiscal policy follows a nonlinear expenditure-based austerity policy rule* ($\Xi', \Xi'' < 0$) *expressed relative to output and satisfies* $\tau - \rho\hat{\theta} < \Xi\left(\hat{\theta}\right)$, *where* $\hat{\theta} = \arg\max_{\theta} \left\{\Xi\left(\hat{\theta}\right) - \left(\tau - \rho\hat{\theta}\right)\right\}$, *and* $\Xi(0) < \tau$. *Then, from the equilibrium system (48) and (49),(a) there exist two steady states,* (θ^*, c^*) *and* $\left(\bar{\theta}, \bar{c}\right)$, *obeying* $\theta^* < \bar{\theta}$ *and* $c^* \gtreqless \bar{c} \Longleftrightarrow [1-\alpha(1-\tau)] \left(\Xi(\theta^*)^{\frac{1-\alpha}{\alpha}} - \Xi\left(\bar{\theta}\right)^{\frac{1-\alpha}{\alpha}}\right) \gtreqless \Xi(\theta^*)^{\frac{1}{\alpha}} - \Xi\left(\bar{\theta}\right)^{\frac{1}{\alpha}}$; *(b) the growth rate at low-debt steady state* (θ^*, c^*), μ^*, *is higher than the growth rate at high-debt steady state* $\left(\bar{\theta}, \bar{c}\right)$, $\bar{\mu}$; *(c) if* $[a/(1-\alpha)](\Xi(\theta^*)/\theta^*) < \rho$ *and* $\left(\partial\dot{\theta}/\partial\theta\right)\Big|_{\left(\bar{\theta},\bar{c}\right)} + \bar{c} > 0$, *globally there exists a heteroclinic orbit connecting the steady states* $\left(\bar{\theta}, \bar{c}\right)$ *and* (θ^*, c^*), *analogously to Proposition 3, should fiscal policy be sufficiently reactive to debt changes, such that*

$$\left|\Xi'(\theta^*) \frac{\theta^*}{\Xi(\theta^*)}\right| > \frac{\alpha}{1-\alpha}. \tag{50}$$

Proof. See the Appendix.

Two key results emerge from Proposition 5. First, nonlinearity in expenditure-based fiscal policy making justifiable on the basis of large empirical evidence makes steady-state multiplicity reappear. This is because the present value of primary surplus-to-GDP ratios turns to increase nonlinearly with respect to upward changes in the debt-to-GDP ratio, thereby leading again to two steady state equilibria, (θ^*, c^*) and $\left(\bar{\theta}, \bar{c}\right)$, the first exhibiting low debt and high growth and the second exhibiting high debt and low growth. In addition, if fiscal policy is sufficiently reactive to debt changes in a way to induce saddle-path stability around (θ^*, c^*) but to cause instability around $\left(\bar{\theta}, \bar{c}\right)$, then the global dynamics is featured again by the existence of a unique and possibly non-monotonic trajectory—the heteroclininc orbit H—connecting the two steady states, as shown in Figure 6 in the Appendix. Clearly, in this case the essence of the results obtained in the foregoing section is not altered in any fundamental dimension.

6 Conclusions

The possibility of self-defeating fiscal austerity is a popular argument in academic and public policy debates, but widely uninvestigated by macroeconomic theory in the context of optimizing general equilibrium models whereby potential nonlinear global dynamics and steady-state multiplicity are taken into account. The present paper uses the discipline of an optimizing endogenous growth model with productive government spending, financed by distortionary taxes and government debt, in order to scrutinize the dynamic effects of expenditure-based fiscal austerity feedback policies evaluated relative to the stock of private capital—stipulating spending cuts in response to debt upward changes.

The study has five main conclusions. First, because the present value of equilibrium primary surpluses proves to be a nonlinear function of debt—even in the presence of a linear expenditure-based fiscal policy rule—multiplicity of balanced growth paths stands out. In particular, in the present framework two steady states arise, one displaying low debt and high growth, the other displaying high debt and low growth.

Second, from local analysis, unintended consequences of austerity in terms of macroeconomic stability may take place. This is because, while the low-debt/high-growth steady state is saddle-path stable, the high-debt/low-growth steady state is unstable, confirming the potential occurrence of self-defeating austerity, featured by off-equilibrium upward trajectories in debt driven by endogenous negative effects of fiscal consolidations on growth dividends and fiscal revenues, which dominate the benefits induced by decreases in government expenditures.

Third, from global analysis, the two steady states turn out to be endogenously connected. Specifically, although the high-debt/low-growth steady state is locally unstable, there exists a unique and potentially non-monotonic saddle connection originating in the neighborhood of the foregoing steady state and converging to the low-debt/high-growth steady state. Because of existence of the saddle connection, global determinacy of perfect foresight equilibrium prevails when the high-debt/low-growth steady state is a node, ruling out the multiple explosive trajectories incompatible with the government's intertemporal budget constraint and the private agents' tranversality condition.

Fourth, the equilibrium trajectory connecting the two steady states may be non-monotonic in terms of both debt and growth dynamics. That is, along the saddle connection originating in the vicinity of the steady-state with high debt and low growth, initially debt may exhibit a continuous increase and the rate of growth rate may exhibit a continuous decline, even for protracted periods of time. In these circumstances, econometricians using early data generated from a saddle connection to estimate the macroeconomic effects of fiscal consolidations could therefore support the case for self-defeating austerity—as some authors argue with reference to a number of advanced economies in the aftermath of the Great Recession—even though the economy is instead embarking on an equilibrium path leading to a steady state with lower debt and higher growth.

Fifth, the foregoing results prove to be robust with respect to the adoption of policy function in which output—rather than the stock of private capital—is employed as the measure of the size of the economy, as long as the fiscal feedback rule is nonlinear and sufficiently reactive to debt changes.

In sum, the possible self-defeating outcome, obtained conducting exclusively a local analysis around the steady state with high debt and low growth, vanishes as soon as a global-dynamics perspective is adopted. The convergence of the economy along a unique heteroclinic orbit leading to a different steady state exhibiting low debt and high growth reveals to be a sound analytical argument in favor of the view that fiscal policy commitment to austerity rules is likely to pay off in the long run.

Finally, from a methodological perspective, the theoretical results derived in this paper imply that making use of a global analysis of macroeconomic dynamics may be essential to obtain a comprehensive characterization of the dynamic properties of fiscal consolidations. The analytically tractable framework we have studied here to convey our arguments in a direct and transparent way could then be employed as a useful benchmark for more complex investigations along the foregoing lines.

Appendix: Proofs

Proof of Proposition 1. *(a)* From equation (16), $\dot{b}=0$ if and only if

$$c=\left[1-\alpha\left(1-\tau\right)+\frac{\tau}{b}\right]A\Phi\left(b\right)^{1-\alpha}-\left(1+\frac{1}{b}\right)\Phi\left(b\right). \tag{51}$$

From equation (18), $\dot{c}=0$ if and only if

$$c=\left[1-\alpha\left(1-\tau\right)\right]A\Phi\left(b\right)^{1-\alpha}-\Phi\left(b\right)+\rho. \tag{52}$$

Combining (51) with (52) and using (19) yield

$$b=\frac{\Psi\left(b\right)}{\rho}. \tag{53}$$

Equation (53) is the government's intertemporal budget constraint at the steady states, according to which the steady-state ratio of government debt to capital must equal the steady-state present discounted value of the ratio of primary surpluses to

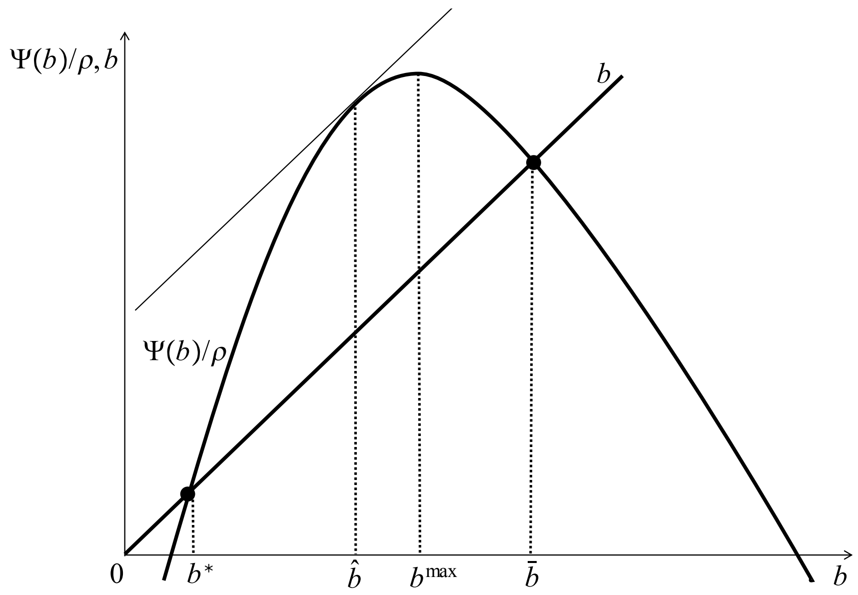

Figure 1: *Multiple steady states with* $\bar{b} > b^{\max}$

capital. In the steady state, which features a balanced growth path $\dot{C}(t)/C(t) = \dot{K}(t)/K(t) = \dot{B}(t)/B(t) = \mu < (1-\tau)r$, not only the government's no-Ponzi game condition (10) but also the agents' transversality condition (5) is verified, because $(C(t))^{-1}(K(t)+B(t)) = (c(t))^{-1}(1+b(t))$ is constant over time. Now, because

$$\frac{\Psi'(b)}{\rho} = -\frac{\Phi'(b)}{\rho}\left[1-\tau A(1-\alpha)\Phi(b)^{-\alpha}\right] \gtreqless 0 \Longleftrightarrow b \lesseqgtr b^{\max} \equiv \Phi^{-1}(\tau A(1-\alpha))^{\frac{1}{\alpha}},$$

then, if conditions (20) and (22) hold, the steady-state relation (53) has two solutions, $b^*, \bar{b} > 0$. Figures 1-2 show the two steady-state equilibria for the government debt-to-capital ratio, with $b^* < \bar{b}$, which occur at the intersections of functions $\Psi(b)/\rho$ and b. Notice that multiple steady states due to the non-linearity of $\Psi(b)/\rho$ would take place even in the presence of a linear fiscal policy rule ($\Phi'' = 0$). In this limiting case, $\Psi''(b)/\rho = -\left\{[\Phi'(b)]^2/\rho\right\}\tau A\alpha(1-\alpha)\Phi(b)^{-\alpha-1} < 0$. More-

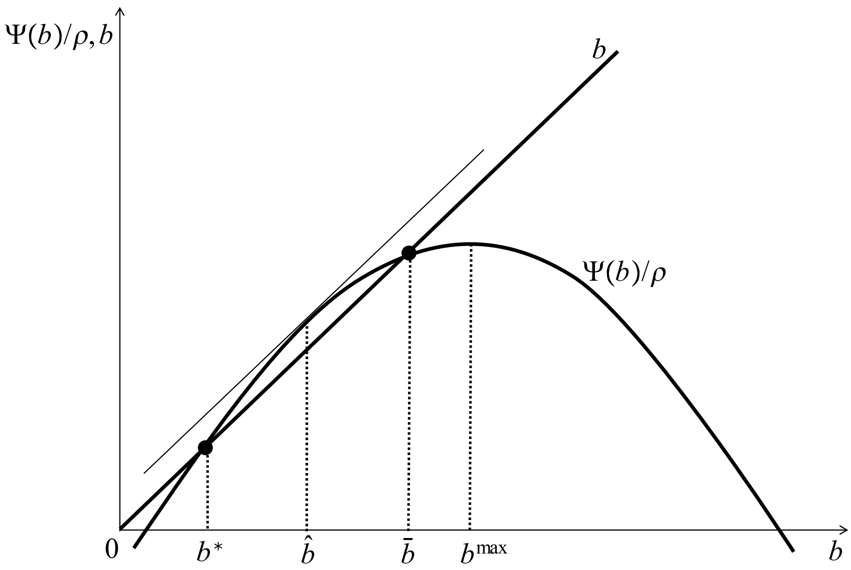

Figure 2: *Multiple steady states with* $\bar{b} < b^{\max}$

over, from (19), (52) and (53),

$$c = (1-\alpha)(1-\tau) A\Phi(b)^{1-\alpha} + \rho(1+b).$$

It follows that $c^* \gtreqless \bar{c}$ if and only if

$$(1-\alpha)(1-\tau) A\left(\Phi(b^*)^{1-\alpha} - \Phi(\bar{b})^{1-\alpha}\right) \gtreqless \rho(\bar{b} - b^*).$$

(b) Substituting (7) and (11) into (4), and applying the labor market equilibrium condition, one obtains

$$\mu = \alpha A(1-\tau)\Phi(b)^{1-\alpha} - \rho.$$

Because $\Phi' < 0$ and $b^* < \bar{b}$, it follows that $\mu^* > \bar{\mu}$. ■

Proof of Proposition 2. *(a)* Let $J^{(b^*,c^*)}$ be the Jacobian of (16) and (18) evaluated at (b^*, c^*). We have

$$\det\, J^{(b^*,c^*)} = \left(1 - \frac{\Psi'(b^*)}{\rho}\right)\rho c^* < 0,$$

because $\Psi'(b^*)/\rho > 1$. Therefore, (b^*, c^*) is a saddle point, with the stable branch given by

$$c(t) = c^* + \left(\frac{J_{21}^{(b^*,c^*)}}{\lambda_1 - c^*}\right)(b(t) - b^*),$$

with

$$b(t) = b^* + (b(0) - b^*)\, e^{\lambda_1 t},$$

where $\lambda_1 < 0$ is the negative eigenvalue associated to $J^{(b^*,c^*)}$. *(b)* Let $J^{(\bar{b},\bar{c})}$ be the Jacobian of (16) and (18) evaluated at $(\bar{b}, \bar{c})$. We have

$$\det\, J^{(\bar{b},\bar{c})} = \left(1 - \frac{\Psi'(\bar{b})}{\rho}\right)\rho\bar{c} > 0,$$

$$\begin{aligned} \mathrm{tr}\; J^{(\bar{b},\bar{c})} &= -[(1-\tau)(1-\alpha)]\, A(1-\alpha)\,\Phi(\bar{b})^{-\alpha}\,\Phi'(\bar{b})\,\bar{b} + \left(\rho - \Psi'(\bar{b})\right)\left(1+\bar{b}\right) \\ &\quad + (1-\alpha)(1-\tau)\, A\Phi(\bar{b})^{1-\alpha} + \rho \\ &> 0, \end{aligned}$$

because $\Psi'(\bar{b})/\rho < 1$. This implies that, under real eigenvalues, $(\bar{b}, \bar{c})$ is an unstable node, with the two unstable eigenspaces given by

$$c(t) = \bar{c} + \left(\frac{\eta_1 - J_{11}^{(\bar{b},\bar{c})}}{\bar{b}}\right)\left(b(t) - \bar{b}\right),$$

$$c(t) = \bar{c} + \left(\frac{\eta_2 - J_{11}^{(\bar{b},\bar{c})}}{\bar{b}}\right)\left(b(t) - \bar{b}\right),$$

with

$$b\,(t) = \bar{b} + X_1 e^{\eta_1 t} + X_2 e^{\eta_2 t},$$

where X_1 and X_2 are constant and η_1, η_2 are the positive eigenvalues associated to $J^{(\bar{b},\bar{c})}$, satisfying $\eta_1 > \eta_2$. On the other hand, under complex eigenvalues, $(\bar{b}, \bar{c})$ is an unstable spiral point. ■

Proof of Proposition 3. As the system (16) and (18) does not have an explicit solution, we must employ qualitative methods in order to study global dynamics. One traditional method is to find a first integral of system (16) and (18), that is, a Lyapunov function $V\,(\cdot)$ such that $V(b, c) =$ constant. We could not find this function. Then we employ a qualitative method widely adopted in dynamic analysis that consists in determining a "trapping region" for the heteroclinic orbit. The rationale is as follows. Because the steady state $(\bar{b}, \bar{c})$ is a source, the unstable manifold is the set $\mathbb{R}_+/(\bar{b}, \bar{c})$; because the steady state (b^*, c^*) is a saddle point, the stable manifold is, locally, composed by a single trajectory belonging to $\mathbb{R}_+$; therefore there is an intersection of the unstable manifold of the first point and of the stable manifold of the second point that is non-empty and that qualifies the existence of a heteroclinic orbit connecting the steady states $(\bar{b}, \bar{c})$ and (b^*, c^*). In order to prove that it exists, and to characterize it, we build a trapping region enclosing the heteroclinic, and demonstrate that all the trajectories starting inside the trapping area escape from it, with the exception of those starting at any point along the heteroclinic orbit. First notice that setting $\dot{b}\,(t) = 0$ in equation (32) yields the locus given by

$$c\,(t) = \Omega\,(b\,(t))\,. \tag{54}$$

Because

$$\left.\frac{dc(t)}{db(t)}\right|_{\dot{b}(t)=0} = \Omega'(b(t)) = \left[1-\alpha(1-\tau)+\frac{\tau}{b(t)}\right] A(1-\alpha)\,\Phi(b(t))^{-\alpha}\,\Phi'(b(t))$$
$$-\frac{\tau}{b(t)^2}A\Phi(b(t))^{1-\alpha} - \left(1+\frac{1}{b(t)}\right)\Phi'(b(t)) - \frac{\Phi(b(t))}{b(t)^2}$$
$$= -\Phi'(b(t))\left\{1-[1-\alpha(1-\tau)]\,A(1-\alpha)\,\Phi(b(t))^{-\alpha}\right\}$$
$$+\frac{1}{b(t)}\left(\Psi'(b(t)) - \frac{\Psi(b(t))}{b(t)}\right)$$
$$\gtreqless 0 \Longleftrightarrow b(t) \lesseqgtr \tilde{b}_{\Omega},$$

where

$$\tilde{b}_{\Omega} = \arg\max_{b(t)} \Omega(b(t)),$$

in the phase plane $(b(t), c(t))$ the $\dot{b}(t)=0$–locus (54) is inverted U-shaped. Setting $\dot{c}(t)=0$ in equation (33) yields the locus given by

$$c(t) = \Gamma(b(t)). \tag{55}$$

Because

$$\left.\frac{dc(t)}{db(t)}\right|_{\dot{c}(t)=0} = \Gamma'(b(t)) = -\Phi'(b(t))\left\{1-[1-\alpha(1-\tau)]\,A(1-\alpha)\,\Phi(b(t))^{-\alpha}\right\}$$
$$\gtreqless 0 \Longleftrightarrow b(t) \lesseqgtr \tilde{b}_{\Gamma} \equiv \Phi^{-1}\left\{[1-\alpha(1-\tau)]\,A(1-\alpha)\right\}^{\frac{1}{\alpha}},$$

in the phase plane $(b(t), c(t))$ the $\dot{c}(t)=0$–locus (55) is also inverted U-shaped. From the properties of the Jacobian matrix evaluated at the steady states (b^*, c^*) and $(\bar{b}, \bar{c})$ (see Proposition 2), we have

$$\det J^{(b^*,c^*)} = J_{11}^{(b^*,c^*)}c^* - J_{21}^{(b^*,c^*)}b^*$$
$$= -\Omega'(b^*)\,c^*b^* + \Gamma'(b^*)\,c^*b^* < 0,$$

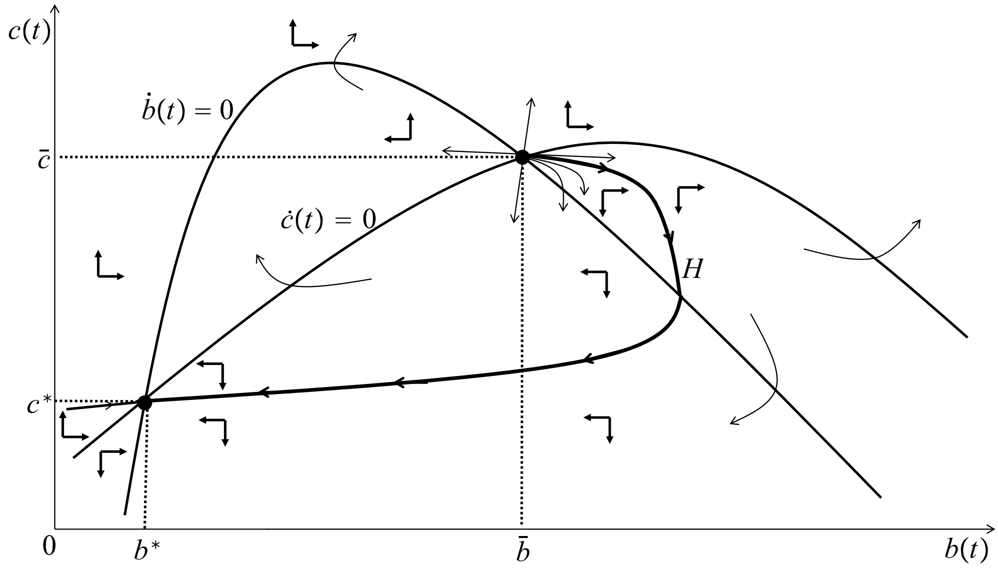

Figure 3: *Dynamic behavior of* $(b(t), c(t))$ *for Case I:* $c^* < \bar{c}$ *and* $\Gamma'(\bar{b}) > 0$

implying

$$\Omega'(b^*) > \Gamma'(b^*),$$

and

$$\begin{aligned}\det J^{(\bar{b},\bar{c})} &= J_{11}^{(\bar{b},\bar{c})}\bar{c} - J_{21}^{(\bar{b},\bar{c})}\bar{b} \\ &= -\Omega'(\bar{b})\,\bar{c}\bar{b} + \Gamma'(\bar{b})\,\bar{c}\bar{b} > 0,\end{aligned}$$

implying

$$\Omega'(\bar{b}) < \Gamma'(\bar{b}).$$

Next, observe that from (32), $\dot{b}(t) > (<) 0$ if $c(t) > (<) \Omega(b(t))$, and from (33) $\dot{c}(t) > (<) 0$ if $c(t) > (<) \Gamma(b(t))$. Then, we have four different cases. *(a)* (Case I: $c^* < \bar{c}$ and $\Gamma'(\bar{b}) > 0$) Figure 3 shows the global dynamics for Case I. The stable arm of the saddle point passing through (b^*, c^*) has locally a positive slope, given by $-\Gamma'(b^*)\,c^*/(\lambda_1 - c^*)$, which is lower than the slope of the isocline $\dot{c}(t) = 0$

evaluated at (b^*, c^*), given by $\Gamma'(b^*)$. Since the steady state $(\bar{b}, \bar{c})$ is an unstable node under real roots, there must exist one trajectory—the heteroclinic orbit H—originating in the neighborhood of the steady state $(\bar{b}, \bar{c})$, negatively slopped around $(\bar{b}, \bar{c})$ because tangent to the negatively slopped eigenspace related to the non-dominant eigenvalue $\eta_2 < \eta_1$, given by $c(t) = \bar{c} + \left[\left(\eta_2 + \Omega'\left(\bar{b}\right)\bar{b}\right)/\bar{b}\right]\left(b(t) - \bar{b}\right)$, where $\eta_2 < -\Omega'\left(\bar{b}\right)\bar{b}$, and converging asymptotically to the steady state (b^*, c^*)—locally along the associated saddle path whose stable arm is given by $c(t) = c^* - [\Gamma'(b^*)c^*/(\lambda_1 - c^*)](b(t) - b^*)$. As the heteroclinic H is tangent to the eigenspace associated to η_2 in the neighborhood $(\bar{b}, \bar{c})$, it will lie between the $\dot{b}(t) = 0$–locus and that eigenspace and will never cross this line. Hence in this case the trapping region enclosing the heteroclinic orbit is defined by the vertices $A \equiv (b^*, c^*)$, $B \equiv \left(\bar{b}, \bar{c}\right)$, and $C \equiv (b_C, c_C)$, where b_C and c_C solve the system

$$\begin{cases} c = c^* - \frac{\Gamma'(b^*)c^*}{\lambda_1 - c^*}(b - b^*) \\ c = \bar{c} + \frac{\eta_2 + \Omega'\left(\bar{b}\right)\bar{b}}{\bar{b}}\left(b - \bar{b}\right) \end{cases}$$

so that

$$b_C = \frac{\frac{\Gamma'(b^*)b^*c^*}{\lambda_1 - c^*} + \eta_2 + \Omega'\left(\bar{b}\right)\bar{b} - (\bar{c} - c^*)}{\frac{\Gamma'(b^*)c^*}{\lambda_1 - c^*} + \frac{\eta_2 + \Omega'\left(\bar{b}\right)\bar{b}}{\bar{b}}},$$

$$c_C = c^* - \frac{\Gamma'(b^*)c^*}{\lambda_1 - c^*}\left[\frac{\frac{\Gamma'(b^*)b^*c^*}{\lambda_1 - c^*} + \eta_2 + \Omega'\left(\bar{b}\right)\bar{b} - (\bar{c} - c^*)}{\frac{\Gamma'(b^*)c^*}{\lambda_1 - c^*} + \frac{\eta_2 + \Omega'\left(\bar{b}\right)\bar{b}}{\bar{b}}} - b^*\right],$$

and the sides

$$(AB) = \left\{(b, c) \in \left(b^*, \bar{b}\right) \times (c^*, \bar{c}) : \dot{c} = 0\right\},$$

$$(AC) = \left\{(b, c) \in (b^*, b_C) \times (c^*, c_C) : c = c^* - \frac{\Gamma'(b^*)c^*}{\lambda_1 - c^*}(b - b^*)\right\},$$

$$(BC) = \left\{(b, c) \in \left(\bar{b}, b_C\right) \times (\bar{c}, c_C) : c = \bar{c} + \frac{\eta_2 + \Omega'\left(\bar{b}\right)\bar{b}}{\bar{b}}\left(b - \bar{b}\right)\right\}.$$

That is, the trapping area $[A, B, C]$ has its sides given by the segment of the

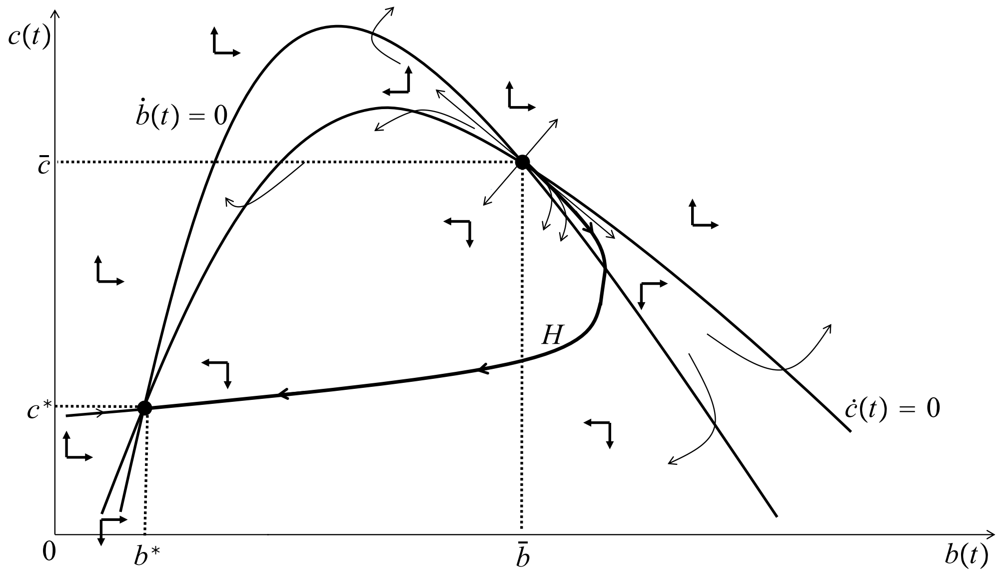

Figure 4: *Dynamic behavior of* $(b(t), c(t))$ *for Case II:* $c^* < \bar{c}$ *and* $\Gamma'(\bar{b}) < 0$*; sub-case I*

$\dot{c}(t) = 0$–locus between the two steady-state equilibria, by a line passing through the steady state $(\bar{b}, \bar{c})$ whose slope is given by the eigenvector associated to the non-dominant eigenvalue η_2, and by a line passing through the steady state (b^*, c^*) whose slope is given by the slope of the associated saddle path, between (b^*, c^*) and the previous eigenvector-line. All the trajectories starting inside the trapping area escape from it, with the exception of those starting at any point along the heteroclinic orbit. Thus only the orbit does not hit the boundaries of the trapping area, and therefore lies in the positive quadrant and converges to the point (b^*, c^*).[17] Such a global saddle connection consequently follows a non-monotonous path—in terms of dynamic behavior of $b(t)$ and $\mu(t)$—that changes direction when $\dot{b}(t) = 0$.

(b) (Case II: $c^* < \bar{c}$ and $\Gamma'(\bar{b}) < 0$) Figures 4-5 show the two possible global dynamics for Case II. In general, also in this case the stable arm of the saddle point passing through (b^*, c^*) has locally a positive slope, lower than the slope

[17]Notice that there exist two more steady states at the intersection points between the horizontal axis, along which $\dot{c}(t) = 0$, and the $\dot{b}(t) = 0$–locus.

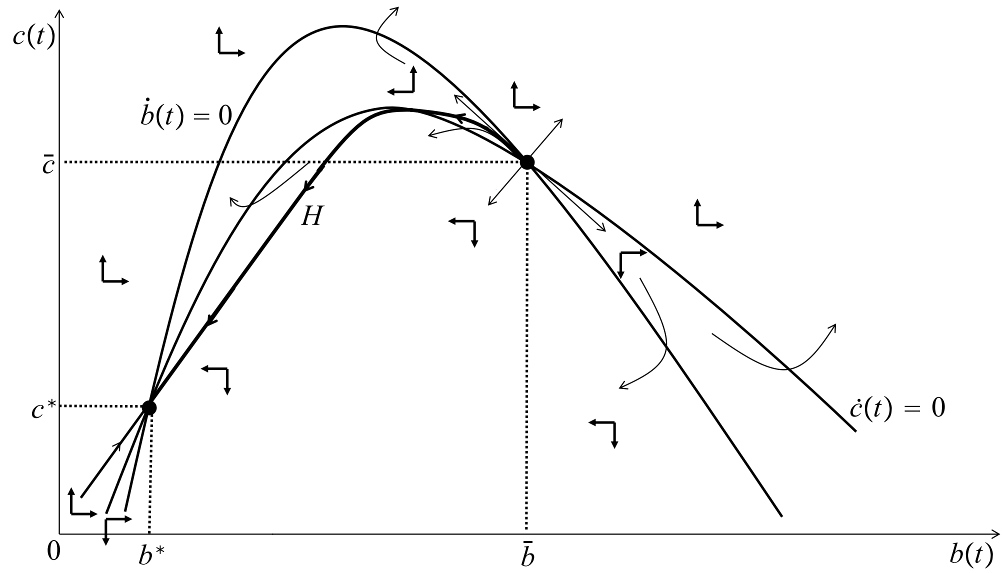

Figure 5: *Dynamic behavior of* $(b\,(t)\,, c\,(t))$ *for Case II:* $c^* < \bar{c}$ *and* $\Gamma'\left(\bar{b}\right) < 0$*; sub-case II*

of the isocline $\dot{c}\,(t) = 0$ evaluated at (b^*, c^*). The unstable eigenspace related to the non-dominant eigenvalue has a negative slope, higher in absolute value than the slope of the $\dot{c}(t) = 0$–locus evaluated at $\left(\bar{b}, \bar{c}\right)$, and lower in absolute value than the slope of the $\dot{b}(t) = 0$–locus evaluated at $\left(\bar{b}, \bar{c}\right)$. As a consequence, the heteroclinic orbit H originating in the neighborhood of the steady state $(\bar{b}, \bar{c})$ is negatively slopped around $(\bar{b}, \bar{c})$, positively slopped around (b^*, c^*), and follows a non-monotonous path—now in terms of dynamic behavior of either $b\,(t)$ and $\mu\,(t)$ (Figure 4) or $c\,(t)$ (Figure 5)—that changes direction when $\dot{b}\,(t) = 0$ in the first sub-case, and when $\dot{c}\,(t) = 0$ in the second sub-case. *(c)* (Case III: $c^* > \bar{c}$ and $\Gamma'\,(b^*) > 0$) The general characteristics of the heteroclinic orbit H are analogous to the second sub-case of Case II illustrated in Figure 5. In particular, also in this case, the global saddle connection exhibits a non-monotonous path—in terms of dynamic behavior of $c\,(t)$—that changes direction when $\dot{c}\,(t) = 0$. *(d)* (Case IV: $c^* > \bar{c}$ and $\Gamma'\,(b^*) < 0$) The stable arm of the saddle point passing through

(b^*, c^*) has now locally a negative slope, lower in absolute value than the slope of the $\dot{c}(t) = 0$–locus evaluated at (b^*, c^*). The unstable eigenspace associated to the non-dominant eigenvalue has a negative slope, higher in absolute value than the slope of the $\dot{c}(t) = 0$–locus evaluated at $\left(\bar{b}, \bar{c}\right)$, and lower in absolute value than the slope of the $\dot{b}(t) = 0$–locus evaluated at $\left(\bar{b}, \bar{c}\right)$. As a consequence, the heteroclinic orbit H originating in the neighborhood of the steady state $(\bar{b}, \bar{c})$ is negatively slopped both around $(\bar{b}, \bar{c})$ and around (b^*, c^*), thus exhibiting a monotonous path in the dynamic behavior of $(b\,(t)\,, c\,(t))$. ■

Proof of Proposition 4. *(a)* From equation (48), $\dot{\theta} = 0$ if and only if

$$\begin{aligned} c \;=\; & \left[1-\alpha\left(1-\tau\right)\right] A^{\frac{1}{\alpha}} \left[\gamma^* - \xi\left(\theta-\theta^*\right)\right]^{\frac{1-\alpha}{\alpha}} \\ & -\frac{\left[\gamma^* - \xi\left(\theta-\theta^*\right)\right]-\tau}{\theta} - A^{\frac{1}{\alpha}} \left[\gamma^* - \xi\left(\theta-\theta^*\right)\right]^{\frac{1}{\alpha}}. \end{aligned} \tag{56}$$

From equation (44), $\dot{c} = 0$ if and only if

$$c = \left[1-\alpha\left(1-\tau\right)\right] A^{\frac{1}{\alpha}} \left[\gamma^* - \xi\left(\theta-\theta^*\right)\right]^{\frac{1-\alpha}{\alpha}} - A^{\frac{1}{\alpha}} \left[\gamma^* - \xi\left(\theta-\theta^*\right)\right]^{\frac{1}{\alpha}} + \rho. \tag{57}$$

Combining (56) with (57) yields

$$\frac{\tau - \left[\gamma^* - \xi\left(\theta-\theta^*\right)\right]}{\theta} = \rho,$$

whose unique solution is

$$\theta^* = \frac{\tau - \gamma^*}{\rho}.$$

It follows $c^* = \left[1-\alpha\left(1-\tau\right)\right] A^{\frac{1}{\alpha}} \left[\gamma^*\right]^{\frac{1-\alpha}{\alpha}} - A^{\frac{1}{\alpha}} \left[\gamma^*\right]^{\frac{1}{\alpha}} + \rho$, $\mu^* = \alpha A^{\frac{1}{\alpha}} \left(1-\tau\right) \gamma^{*\frac{1-\alpha}{\alpha}} - \rho$, $g^* = \left(A\gamma^*\right)^{\frac{1}{\alpha}}$, and $b^* = \theta^* A^{\frac{1}{\alpha}} \gamma^{*\frac{1-\alpha}{a}}$. *(b)* Linearizing equations (43) and (44) in the neighborhood of the steady-state point (θ^*, c^*), one obtains the Jacobian

$$K = \begin{pmatrix} K_{11} & \theta^* \left[1 - \frac{(1-\alpha)}{a}\xi\frac{\theta^*}{\gamma^*}\right]^{-1} \\ K_{21} & c^* \end{pmatrix},$$

where

$$\begin{aligned} K_{11} &= \left[1 - \frac{(1-\alpha)}{a}\xi\frac{\theta^*}{\gamma^*}\right]^{-1} \times \\ &\quad \left\{\frac{(1-\alpha)}{\alpha}\xi\left[1-\alpha\left(1-\tau\right)\right]A^{\frac{1}{\alpha}}\gamma^{*\frac{1-2\alpha}{\alpha}}\theta^* + (\rho-\xi) - \frac{1}{\alpha}\xi A^{\frac{1}{\alpha}}\gamma^{*\frac{1-\alpha}{\alpha}}\theta^*\right\} \\ &= \left[1 - \frac{(1-\alpha)}{a}\xi\frac{\theta^*}{\gamma^*}\right]^{-1} \times \\ &\quad \left\{\frac{1}{\alpha}\xi A^{\frac{1}{\alpha}}\gamma^{*\frac{1-\alpha}{\alpha}}\theta^*\left[(1-\alpha)\left[1-\alpha\left(1-\tau\right)\right]\gamma^{*-1}-1\right] + (\rho-\xi)\right\}, \end{aligned} \tag{58}$$

$$\begin{aligned} K_{21} &= \frac{(1-\alpha)}{\alpha}\xi\left[1-\alpha\left(1-\tau\right)\right]A^{\frac{1}{\alpha}}\gamma^{*\frac{1-\alpha}{\alpha}-1}c^* - \frac{1}{\alpha}\xi A^{\frac{1}{\alpha}}\gamma^{*\frac{1-\alpha}{\alpha}}c^* \\ &= \frac{1}{\alpha}\xi A^{\frac{1}{\alpha}}\gamma^{*\frac{1-\alpha}{\alpha}}c^*\left[\left[1-\alpha\left(1-\tau\right)\right]A^{\frac{1}{\alpha}}\gamma^{*-1}-1\right]. \end{aligned} \tag{59}$$

The determinant and the trace of the Jacobian matrix are

$$\det\ K = \left[1 - \frac{(1-\alpha)}{a}\xi\frac{\theta^*}{\gamma^*}\right]^{-1}(\rho-\xi)\,c^*, \tag{60}$$

$$\begin{aligned} \mathrm{tr}\ K &= \left[1 - \frac{(1-\alpha)}{a}\xi\frac{\theta^*}{\gamma^*}\right]^{-1} \times \\ &\quad \left\{\frac{1}{\alpha}\xi A^{\frac{1}{\alpha}}\gamma^{*\frac{1-\alpha}{\alpha}}\theta^*\left[(1-\alpha)\left[1-\alpha\left(1-\tau\right)\right]\gamma^{*-1}-1\right] + (\rho-\xi)\right\} + c^*. \end{aligned}$$

From (60), it follows that if $[a/(1-\alpha)](\gamma^*/\theta^*) > \rho$, $\det\ K < 0$ so that (θ^*, c^*) is a saddle point if and only if

$$\rho < \xi < \frac{\alpha}{(1-\alpha)} \frac{\gamma^*}{\theta^*}; \tag{61}$$

if $[a/(1-\alpha)](\gamma^*/\theta^*) < \rho$, on the other hand, det $K < 0$ so that (θ^*, c^*) is a saddle point if and only if

$$\frac{\alpha}{(1-\alpha)} \frac{\gamma^*}{\theta^*} < \xi < \rho. \tag{62}$$

(c) If either conditions (61) or (62) are not satisfied, the steady state (θ^*, c^*) is a sink if tr $K < 0$ and an unstable node/spiral point if tr $K > 0$. ■

Proof of Proposition 5. *(a)* From equation (48), $\dot{\theta} = 0$ if and only if

$$c = [1 - \alpha(1-\tau)] A^{\frac{1}{\alpha}} \Xi(\theta)^{\frac{1-\alpha}{\alpha}} - \frac{\Xi(\theta) - \tau}{\theta} - A^{\frac{1}{\alpha}} \Xi(\theta)^{\frac{1}{\alpha}}. \tag{63}$$

From equation (49), $\dot{c} = 0$ if and only if

$$c = [1 - \alpha(1-\tau)] A^{\frac{1}{\alpha}} \Xi(\theta)^{\frac{1-\alpha}{\alpha}} - A^{\frac{1}{\alpha}} \Xi(\theta)^{\frac{1}{\alpha}} + \rho. \tag{64}$$

Combining (63) with (64) gives

$$\Xi(\theta) = \tau - \rho\theta, \tag{65}$$

Now, because $\Xi', \Xi'' < 0$, if $\tau - \rho\hat{\theta} < \Xi\left(\hat{\theta}\right)$, where $\hat{\theta} = \arg\max_{\theta} \left\{\Xi\left(\hat{\theta}\right) - \left(\tau - \rho\hat{\theta}\right)\right\}$, and $\Xi(0) < \tau$, then the steady-state relation (65) has two solutions, $\theta^*, \bar{\theta} > 0$. In addition, from (64), we have that $c^* \gtreqless \bar{c}$ if and only if

$$[1 - \alpha(1-\tau)] \left(\Xi(\theta^*)^{\frac{1-\alpha}{\alpha}} - \Xi\left(\bar{\theta}\right)^{\frac{1-\alpha}{\alpha}}\right) \gtreqless \Xi(\theta^*)^{\frac{1}{\alpha}} - \Xi\left(\bar{\theta}\right)^{\frac{1}{\alpha}}.$$

(b) Substituting (7) into (4), applying the labor market equilibrium condition, and

using (41) and (47), one obtains

$$\mu = \alpha \left(1-\tau\right) A^{\frac{1}{\alpha}} \Xi\left(\theta\right)^{\frac{1-\alpha}{\alpha}} - \rho.$$

Since $\Xi' < 0$ and $\theta^* < \bar{\theta}$, it follows that $\mu^* > \bar{\mu}$. *(c)* Linearizing equations (48) and (49) in the neighborhood of any steady-state point (θ, c), one obtains the Jacobian

$$M^{(\theta,c)} = \begin{pmatrix} M_{11}^{(\theta,c)} & \theta \left[1 + \frac{(1-\alpha)}{a} \Xi'\left(\theta\right) \frac{\theta}{\Xi(\theta)}\right]^{-1} \\ M_{21}^{(\theta,c)} & c \end{pmatrix},$$

where

$$\begin{aligned} M_{11}^{(\theta,c)} &= \left[1 + \frac{(1-\alpha)}{a} \Xi'\left(\theta\right) \frac{\theta}{\Xi\left(\theta\right)}\right]^{-1} \times \\ &\quad \left\{ \begin{array}{c} -\frac{(1-\alpha)}{\alpha} \Xi'\left(\theta\right) \left[1 - \alpha\left(1-\tau\right)\right] A^{\frac{1}{\alpha}} \Xi\left(\theta\right)^{\frac{1-2\alpha}{\alpha}} \theta \\ + \left(\rho + \Xi'\left(\theta\right)\right) + \frac{1}{\alpha} \Xi'\left(\theta\right) A^{\frac{1}{\alpha}} \Xi\left(\theta\right)^{\frac{1-\alpha}{\alpha}} \theta \end{array} \right\} \\ &= \left[1 + \frac{(1-\alpha)}{a} \Xi'\left(\theta\right) \frac{\theta}{\Xi\left(\theta\right)}\right]^{-1} \times \\ &\quad \left\{ \begin{array}{c} -\frac{1}{\alpha} \xi A^{\frac{1}{\alpha}} \Xi'\left(\theta\right)^{\frac{1-\alpha}{\alpha}} \theta \left[\left(1-\alpha\right)\left[1 - \alpha\left(1-\tau\right)\right] \Xi\left(\theta\right)^{-1} - 1\right] \\ + \left(\rho + \Xi'\left(\theta\right)\right) \end{array} \right\}, \end{aligned}$$

$$\begin{aligned} M_{21}^{(\theta,c)} &= -\frac{(1-\alpha)}{\alpha} \Xi'\left(\theta\right) \left[1 - \alpha\left(1-\tau\right)\right] A^{\frac{1}{\alpha}} \Xi\left(\theta\right)^{\frac{1-\alpha}{\alpha}-1} c + \frac{1}{\alpha} \xi A^{\frac{1}{\alpha}} \Xi'\left(\theta\right)^{\frac{1-\alpha}{\alpha}} c \\ &= -\frac{1}{\alpha} \Xi'\left(\theta\right) A^{\frac{1}{\alpha}} \Xi\left(\theta\right)^{\frac{1-\alpha}{\alpha}} c \left[\left(1-\alpha\right)\left[1 - \alpha\left(1-\tau\right)\right] A^{\frac{1}{\alpha}} \Xi\left(\theta\right)^{-1} - 1\right]. \end{aligned}$$

The determinant and the trace of the Jacobian matrix are

$$\det M^{(\theta,c)} = \left[1 + \frac{(1-\alpha)}{a} \Xi'\left(\theta\right) \frac{\theta}{\Xi\left(\theta\right)}\right]^{-1} \left(\rho + \Xi'\left(\theta\right)\right) c,$$

$$\text{tr } M^{(\theta,c)} = \left[1+\frac{(1-\alpha)}{a}\Xi'(\theta)\frac{\theta}{\Xi(\theta)}\right]^{-1} \times \left\{ \begin{array}{c} -\frac{1}{\alpha}\xi A^{\frac{1}{\alpha}}\Xi'(\theta)^{\frac{1-\alpha}{\alpha}}\theta\left[(1-\alpha)\left[1-\alpha(1-\tau)\right]\Xi(\theta)^{-1}-1\right] \\ +(\rho+\Xi'(\theta)) \end{array} \right\} + c.$$

Then, under the assumptions stated in part (c) of Proposition 5, that is, $[a/(1-\alpha)] \times (\Xi(\theta^*)/\theta^*) < \rho$ and $\left(\partial\dot{\theta}/\partial\theta\right)\Big|_{(\bar{\theta},\bar{c})} + \bar{c} = M_{11}^{(\bar{\theta},\bar{c})} + \bar{c} > 0$, we have det $M^{(\theta^*,c^*)} < 0$ if

$$\left|\Xi'(\theta^*)\frac{\theta^*}{\Xi(\theta^*)}\right| > \frac{\alpha}{1-\alpha}, \tag{66}$$

because $1+[(1-\alpha)/a]\,\Xi'(\theta^*)\,\theta^*/\Xi(\theta^*) < 0$ and $-\Xi'(\theta^*) < \rho$, det $M^{(\bar{\theta},\bar{c})} > 0$, because $1+[(1-\alpha)/a]\,\Xi'\left(\bar{\theta}\right)\bar{\theta}/\Xi\left(\bar{\theta}\right) < 0$ and $-\Xi'\left(\bar{\theta}\right) > \rho$, and tr $M^{(\bar{\theta},\bar{c})} > 0$. Therefore, (θ^*, c^*) is a saddle point, with the stable branch given by

$$c(t) = c^* + \left(\frac{M_{21}^{(\theta^*,c^*)}}{\delta_1 - c^*}\right)(\theta(t)-\theta^*),$$

with

$$\theta(t) = \theta^* + (\theta(0)-\theta^*)\,e^{\delta_1 t},$$

where $\delta_1 < 0$ is the negative eigenvalue associated to $M^{(\theta^*,c^*)}$, while $\left(\bar{\theta},\bar{c}\right)$ is an unstable node under real eigenvalues, with the two unstable eigenspaces given by

$$c(t) = \bar{c} + \left\{\frac{\left(\nu_1 - M_{11}^{(\bar{\theta},\bar{c})}\right)\left[1+\frac{(1-\alpha)}{a}\Xi'\left(\bar{\theta}\right)\frac{\bar{\theta}}{\Xi(\bar{\theta})}\right]}{\bar{\theta}}\right\}\left(\theta(t)-\bar{\theta}\right),$$

$$c(t) = \bar{c} + \left\{\frac{\left(\nu_2 - M_{11}^{(\bar{\theta},\bar{c})}\right)\left[1+\frac{(1-\alpha)}{a}\Xi'\left(\bar{\theta}\right)\frac{\bar{\theta}}{\Xi(\bar{\theta})}\right]}{\bar{\theta}}\right\}\left(\theta(t)-\bar{\theta}\right),$$

with

$$\theta(t) = \bar{\theta} + Z_1 e^{\nu_1 t} + Z_2 e^{\nu_2 t},$$

where Z_1 and Z_2 are constant and ν_1, ν_2 are the positive eigenvalues associated to $M^{(\bar{\theta},\bar{c})}$ satisfying $\nu_1 > \nu_2$, and an unstable spiral point under complex eigenvalues. Hence, we can at this point characterize global dynamics using the same methods as for the proof of Proposition 3, since the nonlinear system (48) and (49) does not have an explicit solution and a Lyapunov function cannot be found. That is, we determine a trapping region enclosing the heteroclinic orbit connecting the steady states $(\bar{\theta}, \bar{c})$ and (θ^*, c^*), and demonstrate that all the trajectories originating inside the trapping area escape from it, with the exception of those originating at any point along the heteroclinic orbit. To this end, rewrite first the system (48) and (49) as

$$\dot{\theta}(t) = (c(t) - \Lambda(\theta(t))) \left[1 + \frac{(1-\alpha)}{a} \Xi'(\theta(t)) \frac{\theta(t)}{\Xi(\theta(t))}\right]^{-1} \theta(t), \tag{67}$$

$$\dot{c}(t) = (c(t) - \Upsilon(\theta(t)))\, c(t), \tag{68}$$

where

$$\Lambda(\theta(t)) = [1 - \alpha(1-\tau)]\, A^{\frac{1}{\alpha}} \Xi(\theta(t))^{\frac{1-\alpha}{\alpha}} + \frac{\tau - \Xi(\theta(t))}{\theta(t)} - A^{\frac{1}{\alpha}} \Xi(\theta(t))^{\frac{1}{\alpha}},$$

$$\Upsilon(\theta(t)) = [1 - \alpha(1-\tau)]\, A^{\frac{1}{\alpha}} \Xi(\theta(t))^{\frac{1-\alpha}{\alpha}} - A^{\frac{1}{\alpha}} \Xi(\theta(t))^{\frac{1}{\alpha}} + \rho.$$

Then notice that, setting $\dot{\theta}(t) = 0$ in equation (67), one obtains the $\dot{\theta}(t) = 0$–locus given by

$$c(t) = \Lambda(\theta(t)). \tag{69}$$

We have

$$\begin{aligned}
\left.\frac{dc(t)}{d\theta(t)}\right|_{\dot{\theta}(t)=0} &= \Lambda'(\theta(t)) = -\frac{(1-\alpha)}{\alpha}\Xi'(\theta(t))\left[1-\alpha(1-\tau)\right]A^{\frac{1}{\alpha}}\Xi(\theta(t))^{\frac{1-2\alpha}{\alpha}} \\
&\quad + \frac{(\Xi(\theta(t))-\tau)-\Xi'(\theta)\theta(t)}{\theta(t)^2} - \frac{1}{\alpha}\Xi'(\theta)A^{\frac{1}{\alpha}}\Xi(\theta)^{\frac{1-\alpha}{\alpha}} \\
&= -\frac{1}{\alpha}\Xi'(\theta)A^{\frac{1}{\alpha}}\Xi(\theta)^{\frac{1-\alpha}{\alpha}}\left\{1-(1-\alpha)\left[1-\alpha(1-\tau)\right]\Xi(\theta)^{-1}\right\} \\
&\quad + \frac{1}{\theta(t)}\left[-\Xi'(\theta) - \frac{(\tau-\Xi(\theta(t)))}{\theta(t)}\right],
\end{aligned}$$

implying that in the phase plane $(\theta(t), c(t))$ the $\dot{\theta}(t) = 0$–locus (69) can be either U-shaped or inverted U-shaped. Setting $\dot{c}(t) = 0$ in equation (68), one obtains the $\dot{c}(t) = 0$–locus given by

$$c(t) = \Upsilon(\theta(t)). \tag{70}$$

Because

$$\begin{aligned}
\left.\frac{dc(t)}{d\theta(t)}\right|_{\dot{c}(t)=0} &= \Upsilon'(\theta(t)) = -\frac{1}{\alpha}\Xi'(\theta(t))A^{\frac{1}{\alpha}}\Xi(\theta(t))^{\frac{1-\alpha}{\alpha}} \times \\
&\quad \left[1-(1-\alpha)\left[1-\alpha(1-\tau)\right]A^{\frac{1}{\alpha}}\Xi(\theta(t))^{-1}\right] \\
&\gtreqless 0 \Longleftrightarrow \theta(t) \lesseqgtr \tilde{\theta}_{\Upsilon} \equiv \Xi^{-1}\left\{(1-\alpha)\left[1-\alpha(1-\tau)\right]A^{\frac{1}{\alpha}}\right\},
\end{aligned}$$

in the phase plane $(\theta(t), c(t))$ the $\dot{c}(t) = 0$–locus (70) is inverted U-shaped. From the properties of the Jacobian matrix evaluated at the steady states (θ^*, c^*) and $(\bar{\theta}, \bar{c})$, we have

$$\begin{aligned}
\det M^{(\theta^*,c^*)} &= M_{11}^{(\theta^*,c^*)}c^* - M_{21}^{(\theta^*,c^*)}\theta^*\left[1+\frac{(1-\alpha)}{a}\Xi'(\theta^*)\frac{\theta^*}{\Xi(\theta^*)}\right]^{-1} \\
&= \left(-\Lambda'(\theta^*)+\Upsilon'(\theta^*)\right)c^*\theta^*\left[1+\frac{(1-\alpha)}{a}\Xi'(\theta^*)\frac{\theta^*}{\Xi(\theta^*)}\right]^{-1} < 0,
\end{aligned}$$

which implies, applying condition (66),

$$\Lambda'(\theta^*) < \Upsilon'(\theta^*),$$

and

$$\begin{aligned}
\det M^{(\bar{\theta},\bar{c})} &= M_{11}^{(\bar{\theta},\bar{c})}\bar{c} - M_{21}^{(\bar{\theta},\bar{c})}\bar{\theta}\left[1+\frac{(1-\alpha)}{a}\Xi'\left(\bar{\theta}\right)\frac{\bar{\theta}}{\Xi\left(\bar{\theta}\right)}\right]^{-1} \\
&= \left(-\Lambda'\left(\bar{\theta}\right)+\Upsilon'\left(\bar{\theta}\right)\right)\bar{c}\bar{\theta}\left[1+\frac{(1-\alpha)}{a}\Xi'\left(\bar{\theta}\right)\frac{\bar{\theta}}{\Xi\left(\bar{\theta}\right)}\right]^{-1} > 0,
\end{aligned}$$

which implies, using the fact that necessarily $\left|\Xi'\left(\bar{\theta}\right)\bar{\theta}/\Xi\left(\bar{\theta}\right)\right| > \alpha/(1-\alpha)$ since $\Xi''\left(\bar{\theta}\right) < 0$,

$$\Lambda'\left(\bar{\theta}\right) > \Upsilon'\left(\bar{\theta}\right).$$

Next, notice that, from equation (67), we have $\dot{\theta}(t) > (<)\, 0$ if either $c(t) > (<)\, \Lambda(\theta(t))$ and $|\Xi'(\theta(t))\,\theta(t)/\Xi(\theta(t))| < (>)\, \frac{\alpha}{1-\alpha}$ or $c(t) < (>)\, \Lambda(\theta(t))$ and $|\Xi'(\theta(t))\,\theta(t)/\,\Xi(\theta(t))| > (<)\, \frac{\alpha}{1-\alpha}$. From equation (68), on the other hand, we have $\dot{c}(t) > (<)\, 0$ if $c(t) > (<)\, \Upsilon(\theta(t))$. Let now focus on the case in which the $\dot{\theta}(t) = 0$–locus (69) is U-shaped, $c^* > \bar{c}$, and $\Upsilon'(\theta^*) > 0$. Figure 6 shows the related global dynamics. The stable arm of the saddle point passing through (θ^*, c^*) has locally a positive slope, given by $-\Upsilon'(\theta^*)\,c^*/\,(\delta_1 - c^*)$, which is lower than the slope of the isocline $\dot{c}(t) = 0$ evaluated at (θ^*, c^*), given by $\Upsilon'(\theta^*)$. Since the steady state $(\bar{\theta}, \bar{c})$ is an unstable node under real roots, there must exist one trajectory—the heteroclinic orbit H—originating in the neighborhood of the steady state $(\bar{\theta}, \bar{c})$, positively slopped around $(\bar{\theta}, \bar{c})$ because tangent to the positively

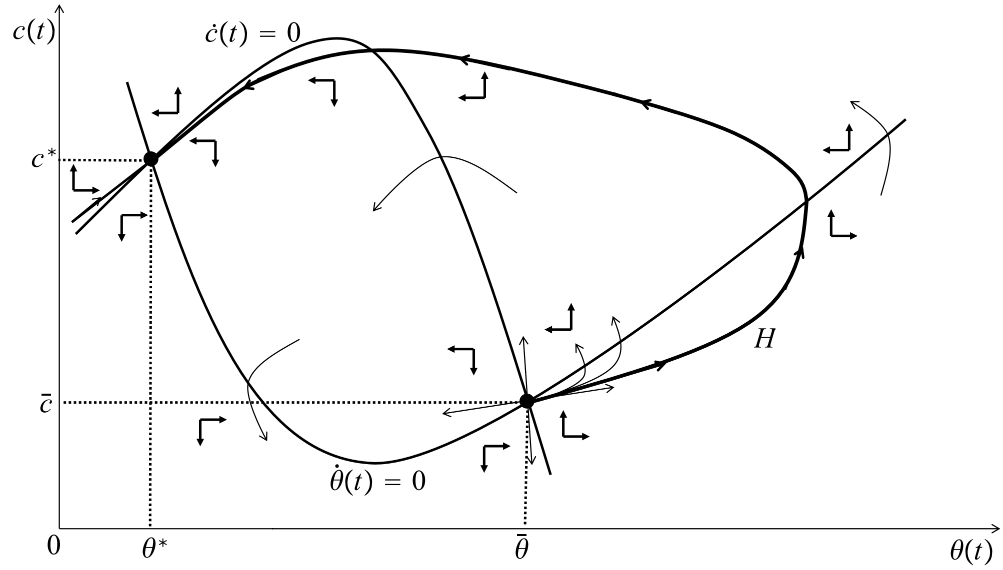

Figure 6: *Dynamic behavior of* $(\theta(t), c(t))$ *with U-shaped* $\Lambda(\theta(t))$, $c^* > \bar{c}$ *and* $\Upsilon'(\theta^*) > 0$.

slopped eigenspace related to the non-dominant eigenvalue $\nu_2 < \nu_1$, given by

$$c(t) = \bar{c} + \left\{ \frac{\left(\nu_2 - \frac{\left(-\Lambda'\left(\bar{\theta}\right)\bar{\theta}\right)}{\left[1+\frac{(1-\alpha)}{a}\Xi'\left(\bar{\theta}\right)\frac{\bar{\theta}}{\Xi\left(\bar{\theta}\right)}\right]} \right) \left[1 + \frac{(1-\alpha)}{a}\Xi'\left(\bar{\theta}\right)\frac{\bar{\theta}}{\Xi\left(\bar{\theta}\right)}\right]}{\bar{\theta}} \right\} \left(\theta(t) - \bar{\theta}\right),$$

where $\nu_2 < -\Lambda'\left(\bar{\theta}\right)\bar{\theta}\left\{1 + ((1-\alpha)/\alpha)\,\Xi'\left(\bar{\theta}\right)\bar{\theta}/\Xi\left(\bar{\theta}\right)\right\}$, and converging asymptotically to the steady state (θ^*, c^*)—locally along the associated saddle path whose stable arm is given by $c(t) = c^* - \left[\Upsilon'(\theta^*)\, c^*/\left(\delta_1 - c^*\right)\right]\left(\theta(t) - \theta^*\right)$. As the heteroclinic H is tangent to the eigenspace associated to η_2 in the neighborhood $(\bar{\theta}, \bar{c})$, it will lie between the $\dot{\theta}(t) = 0$–locus and that eigenspace and will never cross this line. Hence, analogously to the proof of Proposition 3, the trapping region $[A, B, C]$ enclosing the heteroclinic orbit is defined by the sides given by the segment of the $\dot{c}(t) = 0$–locus between the two steady-state equilibria, by a line passing through the steady state $(\bar{\theta}, \bar{c})$ whose slope is given by the eigenvector associated to the non-dominant eigenvalue ν_2, and by a line passing through the steady state (θ^*, c^*) whose slope is given by the slope of the associated saddle path, between (θ^*, c^*) and the previous eigenvector-line. All the trajectories starting inside the trapping region escape from it, with the exception of those starting at any point along the heteroclinic orbit. Thus only the orbit does not hits the boundaries of the trapping area, and therefore lies in the positive quadrant and converges to the point (b^*, c^*). Such a global saddle connection consequently follows a non-monotonous path—in terms of dynamic behavior of $b(t)$, $\mu(t)$, and $c(t)$—that changes direction both when $\dot{\theta}(t) = 0$ and when $\dot{c}(t) = 0$. ■

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