The grand dividends value

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Abstract

We propose a value for games with transferable utility, called the grand dividends value. This new value is an alternative to the Shapley value, especially in games where the worth of a coalition depends on goods that are more or less arbitrarily multipliable or applicable, particularly in the intellectual property domain. The concept of grand dividends, representing the surplus (which can also be non-positive) of the worth of the grand coalition over the worths of all coalitions where one player of the player set has been removed, is the initial point. All the axiomatizations presented have an analogous equivalent for the Shapley value, including the classics by Shapley and Young. A further new concept, called multiple dividends, provides a close connection to the Shapley value.

Keywords Cooperative game · (Harsanyi/Grand/Multiple) Dividends · Shapley value · Grand dividends value

1. Introduction

The concept of a coalition function, also called characteristic function, goes back to von Neumann and Morgenstern (1944). In Shapley (1953b), a TU-game is given by a finite subset N of the universe of all possible players and a superadditive set function (the coalition function) from the subsets of N into the real numbers with the only condition that the worth of the empty set is zero. We will follow Shapley's approach but dispense with superadditivity. The coalition function can be used, for example, to model and analyze economic, political, or other social phenomena. In general, the worth of a coalition is the reward that this coalition can guarantee its players, regardless of what the other players outside the coalition do.

In the model of Harsanyi (1959, 1963), the fundamental assumption is that each player is simultaneously a member of all possible different coalitions (Harsanyi uses the term 'syndicate') which contain it. Introducing the important concept of his (Harsanyi) dividends, he assumes that each coalition guarantees a certain payment, the Harsanyi dividend,

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which should be divided among the members of this coalition. Moreover, these dividends should be assumed in addition to any dividends that each member of the coalition may receive from other coalitions. Under these assumptions, Harsanyi can show that his solution for TU-games provides each player with an equal share of all Harsanyi dividends from coalitions containing this player and coincides with the Shapley value. Thus, according to Harsanyi, the coalition function inherently justifies the Shapley value, but only under the above assumptions; in particular, no externalities must have to be taken into account.

For many scenarios, these assumptions are quite reasonable. But other situations are also conceivable. Harsanyi (1959) himself points out that von Neumann and Morgenstern (1944) assume that each player is a member of only one coalition of players from a player set. For the equal division value (see, e.g., Zou et al. (2021)), we can assume that the grand coalition (the coalition containing all players) is the only coalition that forms. If we assume that if the grand coalition does not form, no other coalition is formed, then, in the case of cooperation, only the grand coalition receives a dividend equal to its worth, which is then distributed equally among all players. Considering the equal surplus division value, introduced in Driessen and Funaki (1991) as the center-of-gravity of the imputation-set value, if the grand coalition would not form, the singletons can be assumed to be the only coalitions formed. As a payoff for the equal surplus division value, each player receives an equal share of the surplus of the worth of the grand coalition over the worths of the singletons as a dividend of the grand coalition and its stand-alone worth as a dividend, paid in full, for the surplus of the singleton over the empty set.

While the last two values consider only a (small) part of the worths of all possible coalitions, this is not the case for the Shapley value and the following new value. Just as Shapley (1953b) did for the introduction of his value, we focus on games without externalities.

Unlike in the model of Harsanyi (1959, 1963), in our model, we assume that, in the process of forming coalitions, each coalition formed prevents the simultaneous formation of any proper subcoalition. As in Hart and Kurz (1983), we posit that "...interactions among players will be conducted on two levels: first, among the coalitions, and second, within each coalition." But we are not restricted to considering just partitions of the player set, called coalition structures (Aumann and Drèze, 1974; Owen, 1977). In this process, we will allow overlapping coalition formations in which each such formed coalition will then be guaranteed, at least hypothetically, the full coalitional worth at the same time. Therefore, for the coalition function, we assume it represents the worths of coalitions of players who bargain less with physical goods and more with goods that are more or less arbitrarily multipliable or applicable, particularly in the area of intellectual property. These would be, for example, patents, data, software, process engineering and production methods, film and music industry products, multi-agent systems in artificial intelligence, and the like. Also, as in Hart and Kurz (1983), "... we assume as a postulate that society as a whole acts efficiently;...", which, in our model, means that in the end, only the grand coalition should emerge.

In this context, we will consider the worth of the grand coalition minus the sum of all the worths of coalitions that are missing exactly one player of the player set as a *grand dividend* for the grand coalition. But then, we can take a closer look at subgames on the player sets where one player of the original player set is removed and, accordingly, get a grand dividend for the grand coalition in each subgame. Proceeding in this way, we obtain grand dividends for all coalitions until finally, each player receives its stand-alone worth as a grand dividend for its singleton on the one-player game. Of course, non-positive grand dividends may also occur, just as happens for Harsanyi dividends. For player sets with only two players, grand dividends coincide with Harsanyi dividends.

With the concept of grand dividends, we can introduce a new TU-value, called *grand dividends value*. As a payoff, for the grand dividends value, each player receives an equal share of the grand dividends of all subgames in which the player is a member of the player set. Note, however, that, depending on the size of the player set and the number of members in a coalition, we may have to take into account the same dividend several times, just as our assumption above would dictate.

The grand dividends value satisfies many axioms that are also satisfied by the Shapley value, and it also satisfies a set of new axioms that are analogous to ones also satisfied by the Shapley value. Therefore, we can give axiomatizations of the Grand Dividends value which are analogous to axiomatizations of the Shapley value in Shapley (1953b), Myerson (1980), and Besner (2020). Especially, the grand dividends monotonicity, which states that for a player, the payoff does not decrease if the grand dividends do not decrease, has interesting economic significance, similar to strong monotonicity (Young, 1985). It offers, along with efficiency and symmetry, an analogous characterization of the grand dividends value to the axiomatization of the Shapley value in Young (1985).

For the payoff calculation, the same grand dividends of a subgame are used several times, depending on the size of the initial player set and the subgame player set. In the last content section, we combine these multiple grand dividends of the same coalition into *multiple dividends*. If these multiple dividends are interpreted as Harsanyi dividends of a new coalition function, we can show that the Shapley value for the resulting *multiple dividends game* is equal to the grand dividends value for the original game, which has far-reaching consequences.

The article is organized as follows. In Section 2 we give some preliminaries. Section 3 introduces the grand dividends and the grand dividends value. An example shows that in a special case, which corresponds to our assumptions above, the grand dividends value is preferable to the Shapley value. In Section 4, we give two axiomatizations which are analogous to axiomatizations in Besner (2020) and Myerson (1980). In Sections 5 and 6, respectively, we provide axiomatizations that are similar to the classical axiomatizations of the Shapley value in Shapley (1953b) and Young (1985). Next, in Section 7, we recall some results of the potential, the reduced game, and the consistency property in Hart and Mas-Colell (1989). Afterward, we introduce multiple dividends and an associated multiple dividends game, which is then used to establish a strong connection of the grand dividends value to the Shapley value. Finally, Section 8 contains some concluding remarks and points out possible extensions of the grand dividends value. The Appendix shows the logical independence of the axioms in our characterizations.

2. Preliminaries

Let $\mathfrak{U}$ be a countably infinite set, the universe of all players and let $\mathcal{N}$ be the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) with a player set $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$.

Each subset $S \subseteq N$ is called a **coalition**, $v(S)$ is the **worth** of the coalition S and Ω^S denotes the set of all non-empty subsets of S. For each $S \in \Omega^N$, $|S|$ or s respectively denotes the cardinality of S, in particular, n denotes the cardinality of a player set N. $\mathbb{V}(N)$ denotes the set of all TU-games with the player set N. The **restriction** of (N, v) to a player set $S \in \Omega^N$ is denoted by (S, v). A **unanimity game** (N, u_S), $S \in \Omega^N$, is defined for all $T \subseteq N$ by $u_S(T) = 1$, if $S \subseteq T$, and $u_S(T) = 0$, otherwise.

Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$. For all $S \in \Omega^N$, the **Harsanyi dividends** $\lambda_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\lambda_v(S) := \begin{cases} 0, \text{ if } S = \emptyset, \\ v(S) - \sum_{R \subsetneq S} \lambda_v(R) \text{ otherwise.} \end{cases} \tag{1}$$

The **marginal contribution** MC_i^v of a player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, if we have $v(\{i\}) = 0$, i is called a **null player** in (N, v). Two players $i, j \in N$, $i \neq j$, are **symmetric** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

For all $N \in \mathcal{N}$, a **TU-value** or **solution** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$.

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. The **equal division value** ED is given by

$$ED_i(N, v) := \frac{v(N)}{n} \text{ for all } i \in N.$$

The **equal surplus division value** ESD (Driessen and Funaki, 1991), also known as the center of imputation set (CIS-vector), is given by

$$ESD_i(N, v) := \frac{v(N) - \sum_{i \in N} v(\{i\})}{n} + v(\{i\}) \text{ for all } i \in N.$$

The **Shapley value** Sh (Shapley, 1953b), is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{(s-1)!\,(n-s)!}{n!} \big[v(S) - v(S \backslash \{i\})\big] \text{ for all } i \in N. \tag{2}$$

We refer to the following well-known axioms for TU-values φ which hold for all $N \in \mathcal{N}$:

Efficiency, E. For all $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Efficiency means that the worth of the grand coalition is fully shared among all players. The following axiom states that a player who does not contribute anything to a (non-empty) coalition should receive only the singleton's worth.

Dummy player, N. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ such that i is a dummy player in (N, v), we have $\varphi_i(N, v) = v(\{i\})$.

A null player receives nothing.

Null player, N. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in (N, v), we have $\varphi_i(N, v) = 0$.

Additivity, A. For all $(N, v), (N, w) \in \mathbb{V}(N)$, we have $\varphi(N, v) + \varphi(N, w) = \varphi(N, v + w)$.

Additivity requires that it is irrelevant whether one first adds the games and then applies the solution concept, or whether one first applies the solution concept to the individual games and then adds the payoffs.

Symmetry, Sym. For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are symmetric in (N, v), we have $\varphi_i(N, v) = \varphi_j(N, v)$.

Symmetry means that two players who contribute the same amount to each coalition should receive the same payoff.

Balanced contributions, BC (Myerson, 1980)**.** For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, we have $\varphi_i(N, v) - \varphi_i(N \backslash \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \backslash \{i\}, v)$.

By this property, for two players the amount that one player would win or lose if the other player drops out of the game is the same for both players.

Strong monotonicity, SMon (Young, 1985)**.** For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $MC_i^v(S) \leq MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v) \leq \varphi_i(N, w)$.

Strong monotonicity states that a player's payoff should not decrease if the worth of the coalitions containing that player increases or stays the same compared to the worth of the coalitions that do not contain that player.

Marginality, Mar (Young, 1985)**.** For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v) = \varphi_i(N, w)$.

By marginality, only a player's marginal contributions are relevant to the player's payoff. The following axiom states that the payoff differences of two players should be the same for different worths of the grand coalition.

Equal (aggregate) monotonicity[1], EMon (Béal et al., 2018)**.** For all $(N, v) \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, we have

$$\varphi_i(N, v) - \varphi_i(N, v + \alpha \cdot u_N) = \varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N) \text{ for all } i, j \in N,$$

Standardness, St (Hart and Mas-Colell, 1989)**.** For all $(N, v) \in \mathbb{V}(N)$, $N = \{i, j\}$, $i \neq j$, we have

$$\varphi_i(\{i, j\}, v) = v(\{i\}) + \frac{1}{2}\big[v(\{i, j\}) - v(\{i\}) - v(\{j\})\big].$$

Standardness implies that in two-player games cooperation results in the surplus being shared equally.

3. The grand dividends value

Harsanyi (1959, 1963), proposing Harsanyi dividends, assumes that all possible coalitions are formed simultaneously. The Harsanyi dividend of a singleton equals the worth of the

[1] As mentioned in Béal et al. (2018) for proportional monotonicity, also equal (aggregate) monotonicity is not in itself related to monotonicity (Meggido, 1974), called aggregate monotonicity in Young (1985), but, along with efficiency, it implies monotonicity.

singleton, and for all other coalitions, we have recursively that their Harsanyi dividends equal their worth minus the Harsanyi dividends of all proper subcoalitions. This means that Harsanyi dividends, and thus the worth of each coalition, are regarded as independent not only of what the outside players do but also of external effects of the inside players with players from outside, i.e., the worth of overlapping coalitions. Thus, in Harsanyi's model, the grand coalition has no cooperation benefit if the worth of the grand coalition is equal to the worth of the sum of all Harsanyi dividends of all proper subcoalitions of the grand coalition.

In what follows, we take a different approach to the introduction and theoretical justification of our new TU-value. The fundamental difference is that, by forming a coalition S, we prevent the formation of proper subcoalitions of S, but forming overlapping coalitions is no problem. That is, the total worth of two simultaneously formed overlapping coalitions is the sum of the worths of both coalitions, while in Harsanyi's model, the total worth of both coalitions is the sum of the Harsanyi dividends of the set containing all subcoalitions of these coalitions, including their own Harsanyi dividends.

In our model, we have no cooperation benefit by forming the grand coalition if the worth of the grand coalition equals the sum of the worth of all the subcoalitions of the grand coalition with one less player and these coalitions had previously formed.

Let us now hypothetically assume that all coalitions of a player set have formed at the same time, each of which is missing one player of the original player set. Now, when the **grand coalition** (the coalition comprising all players) is forming, we have a (not necessarily positive) surplus of the worth of the grand coalition over the sum of the worths of the previously formed coalitions. Formally, for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we call this surplus as the **grand dividend** $\delta_v(N)$, given by

$$\delta_v(N) := v(N) - \sum_{j \in N} v(N \backslash \{j\}). \tag{3}$$

At this point, we can specify an algorithm for calculating a player's payoff. As a reward for forming the grand coalition N, each subcoalition $N \backslash \{j\}$, $j \in N$, receives an equal share of the grand dividend $\delta_v(N)$, which can be divided equally among the members of each coalition. Therefore, each player in the player set receives an equal share of $\delta_v(N)$. In the next step, each coalition $N \backslash \{j\}$ can play an independent game since, according to our assumption, all these coalitions have a worth independent of the worth of other coalitions.[2] In these subgames, again, the grand dividend of the new grand coalition can be distributed and so on. We obtain a recursive formula of a new TU-value.

Definition 3.1. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, the **grand dividends value** Ψ is inductively given by*

$$\Psi_i(N, v) := \frac{\delta_v(N)}{n} + \sum_{j \in N, j \neq i} \Psi_i(N \backslash \{j\}, v) \text{ for all } i \in N.^{3} \tag{4}$$

[2]For bargaining with overlapping coalitions, see Ray (2007).

[3]If $n = 1$, we use the convention that an empty sum evaluates to zero.

Already in Hart and Mas-Colell (1989) and Sprumont (1990), a related recursive formula for the Shapley value can be found, which is later proposed, e.g., in Pérez-Castrillo and Wettstein (2001) or Kongo and Funaki (2016), given by

$$Sh_i(N,v) := \frac{1}{n}\big(v(N) - v(N\backslash\{i\}\big) + \frac{1}{n}\sum_{j\in N,\, j\neq i} Sh_i(N\backslash\{j\},v) \text{ for all } i\in N.$$

For each player i, the Shapley value is the average of i's marginal contribution to the grand coalition and the average of i's Shapley values in the games in which one player other than i is removed in each case. The grand dividends value of the player i is the average of the contribution of all coalitions missing one player to the grand coalition, and all grand dividends values of the player i in the games, missing another player.

An interpretation of the equal surplus division value could be that it distributes the surplus of the worth of the grand coalition over the sum of the worth of the singletons evenly among the singletons and thus among the individual players. Then the players play a game on the singletons where they get an efficient payoff, namely the worth of the singleton. At first glance, we can think of the grand dividends value as an extension of the equal surplus division value that, step-by-step, passes through all coalition sizes. For a two-player game, the payoffs match, both TU-values satisfy standardness **St**.

The Shapley value also satisfies standardness. By Harsanyi (1959), equivalent to (2), for all $N\in\mathcal{N}$, $(N,v)\in\mathbb{V}(N)$, the Shapley value Sh is given by

$$Sh_i(N,v) := \sum_{S\subseteq N,\, S\ni i}\frac{\lambda_v(S)}{s} \text{ for all } i\in N. \tag{5}$$

The Shapley value assigns to each player an equal share of the Harsanyi dividends of all coalitions of which that player is a member. In the following proposition, we can find a related formula for the grand dividends value. For all subgames in which a player i is part of the player set, the player i receives an equal share of the grand dividend. However, since we successively consider all subgames when assigning dividends, depending on the size of the set of players, the respective coalitions are thus considered multiple times. Therefore, we multiply each grand dividend by the number of times it occurs.

Proposition 3.2. *For all $N\in\mathcal{N}$, $(N,v)\in\mathbb{V}(N)$, the grand dividends value Ψ is given by*

$$\Psi_i(N,v) = \sum_{S\subseteq N,\, S\ni i}\frac{(n-s)!}{s}\delta_v(S) \textit{ for all } i\in N. \tag{6}$$

Proof. Let $N\in\mathcal{N}$, $(N,v)\in\mathbb{V}(N)$, and φ be a TU-value, given by

$$\varphi_i(N,v) := \sum_{S\subseteq N,\, S\ni i}\frac{(n-s)!}{s}\delta_v(S) \text{ for all } i\in N. \tag{7}$$

Each coalition $S\subsetneq N$, $S\ni i$, is a subset of $(n-s)$ different coalitions $T\subsetneq N$, $|T| = n-1$, $T\ni i$. Therefore, we have

$$\sum_{j\in N,\, j\neq i}\left[\sum_{S\subseteq N\backslash\{j\},\, S\ni i}\frac{(n-1-s)!}{s}\delta_v(S)\right] = \sum_{S\subsetneq N,\, S\ni i}\frac{(n-s)!}{s}\delta_v(S) \text{ for all } i\in N. \tag{8}$$

It follows, for all $i \in N$,

$$\begin{aligned}
\varphi_i(N,v) &\underset{(7)}{=} \frac{\delta_v(N)}{n} + \sum_{S \subsetneq N,\, S \ni i} \frac{(n-s)!}{s} \delta_v(S) \\
&\underset{(8)}{=} \frac{\delta_v(N)}{n} + \sum_{j \in N,\, j \neq i} \left[\sum_{S \subseteq N \setminus \{j\},\, S \ni i} \frac{(n-1-s)!}{s} \delta_v(S) \right] \\
&\underset{(7)}{=} \frac{\delta_v(N)}{n} + \sum_{j \in N,\, j \neq i} \varphi_i(N \setminus \{j\}, v) \underset{(4)}{=} \Psi_i(N,v).
\end{aligned}$$

□

Remark 3.3. *For all $(N,v) \in \mathbb{V}(N)$, $N \in \mathcal{N}$, the grand dividends value Ψ coincides with the Shapley value Sh if $v(S) = 0$ for all $S \subseteq N$, $|S| \leq |N| - 2$. In particular, this is the case if $|N| = 2$.*

In the following example, we show that, at least sometimes, the grand dividends value is preferable to the Shapley value.

Example 3.4. We consider a three-player game in which the players are companies that hold a certain number of patents that are necessary to produce electronic devices such as smartphones, tablet-PCs, notebooks, radios, e-readers, navigation devices, or the like. While players 2 and 3 can produce some (few) devices only with their own patents, player 1 cannot produce any device based on its own patents. When two-player coalitions form, they can produce more electronics items more cheaply, with better quality, or both, because of the greater number of patents, which prevents them from continuing to produce as a single company. In our example, we assume that all two-player coalitions produce different goods. Furthermore, since the market is assumed to be large enough so that the purchase of products from one two-player coalition does not affect the purchase of products from another two-player coalition, and since these coalitions can also borrow any missing production capital at extremely low-interest rates, the worth of a two-player coalition is assumed to be independent of the other two, and all three two-player coalitions can exist at the same time. When the three-player coalition forms, we have even more goods that can be produced even cheaper, which should prevent the players from continuing to produce as (proper) subcoalitions. Hence, we have exactly the hypothetical situation mentioned above.

Formally, let $(N,v) \in \mathbb{V}(N)$, $N = \{1,2,3\}$, be a TU-game, given by

$$\begin{array}{llll}
v(\{1\}) = 0, & v(\{2\}) = 1, & v(\{3\}) = 4, & v(\{1,2\}) = 2, \\
v(\{1,3\}) = 6, & v(\{2,3\}) = 9, & v(\{1,2,3\}) = 18. &
\end{array}$$

The crucial question now is how to distribute the benefits of working together in the grand coalition. We have

$$Sh(N,v) = \left(\frac{7}{2}, \frac{11}{2}, 9 \right) \text{ and } \Psi(N,v) = \left(\frac{11}{6}, \frac{29}{6}, \frac{34}{3} \right).$$

If player 3 has a choice, that player will not take part in a three-player coalition if the payoff is to be made with the Shapley value. Since the worths of the two-player coalitions are independent of each other, player 3 should play a separate two-player game for each of these two coalitions. We have $Sh_3(\{1,3\},v) + Sh_3(\{2,3\},v) = 11 > 9 = Sh_3(N,v)$. Since the other two players, player 1 and player 2, improve their payoffs in the singleton or two-player game on the set of players {1,2}, respectively, if they also play the other possible two-player games, they will agree to those games as well. However, since they will then further improve their payoff using the grand dividends value in the three-player game, all players will finally prefer the grand dividends value over the Shapley value in the absence of any external constraints. Replacing the Shapley value by the equal division or equal surplus division value would yield analogous results.

4. Inessential grand dividends and balanced summarized contributions

We call a TU-game $(N,v) \in \mathbb{V}(N)$ an **inessential grand dividend game** if $v(N) = \sum_{j\in N} v(N\backslash\{j\})$ which is, by (3), equivalent to $\delta_v(N) = 0$. The following property states that in an inessential grand dividend game, the payoff to a player is completely determined by the sum of the player?s payoffs in all subgames in which one player of the player set is removed at a time.

Inessential grand dividend, IGD.[4] For all $N \in \mathcal{N}$ and all inessential grand dividend games $(N,v) \in \mathbb{V}(N)$, we have $\varphi_i(N,v) = \sum_{j\in N} \varphi_i(N\backslash\{j\},v)$ for all $i \in N$.

It follows a first axiomatization of the grand dividends value.

Theorem 4.1. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **IGD**, and **EMon**.*[5]

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$.

I. Existence: **IGD** and **EMon** follow immediately by (4). We show **E** by induction on the size n.

Initialization: Let $n = 1$. Then, **E** is satisfied by (3) and (4).

Induction step: Let $n \geq 2$. Assume that Ψ satisfies **E** for all n', $n' < n$, (IH). We have

$$\sum_{i\in N} \Psi_i(N,v) \underset{(4)}{=} \sum_{i\in N} \left[\frac{\delta_v(N)}{n} + \sum_{j\in N,\, j\neq i} \Psi_i(N\backslash\{j\},v)\right] \underset{(IH)}{=} \delta_v(N) + \sum_{i\in N} v(N\backslash\{i\}) \underset{(3)}{=} v(N),$$

and **E** is shown.

II. Uniqueness: Let φ be a TU-value which satisfies all axioms from Theorem 4.1. We show uniqueness by induction on the size n.

Initialization: Let $n = 1$. Then, uniqueness is satisfied by **E**.

Induction step: Let $n \geq 2$. Assume that φ is unique for all n', $n' < n$, (IH). Then, by (IH) and **IGD**, φ is unique on the inessential grand dividend game $\big(N, v - \delta_v(N)u_N\big)$.

[4]This axiom is related to the inessential grand coalition property in Besner (2020).

[5]A related axiomatization of the Shapley value can be found in Besner (2020) where the inessential grand dividend property is replaced by the inessential grand coalition property.

By **EMon**, we have, for all $i, j \in N$,

$$\begin{aligned} &\varphi_i(N,v) = \varphi_i\big(N, v - \delta_v(N)\cdot u_N\big) + \varphi_j(N,v) - \varphi_j\big(N, v-\delta_v(N)\cdot u_N\big) \\ \Leftrightarrow \quad &\sum_{k\in N}\varphi_k(N,v) = \sum_{k\in N}\varphi_k\big(N, v-\delta_v(N)\cdot u_N\big) + n\cdot\big[\varphi_j(N,v) - \varphi_j\big(N, v-\delta_v(N)\cdot u_N\big)\big] \end{aligned}$$

and, by **E** and (IH), φ is unique for the player j. Since j is arbitrary, uniqueness and, therefore, also Theorem 4.1 is shown. □

For game situations like in Example 3.4, this axiomatization seems quite convincing. If the worth of the grand dividend is equal to the sum of the worths of the two-player coalitions, it should not matter whether or not the grand coalition forms, and if only the grand dividend changes, then, for fairness, the payoff for all players should change by the same amount.

The balanced contributions property **BC** states that for any two players, the amount that one player would win or lose if the other player drops out of the game is the same for both players. By the following property, the gain or loss for two players of a player set is the same if they would play the game with the entire player set instead of playing games with player sets, each missing one of their original players.

Balanced summarized contributions, BSC. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, we have

$$\varphi_i(N,v) - \sum_{k\in N,\, k\neq i}\varphi_i(N\backslash\{k\}, v) = \varphi_j(N,v) - \sum_{k\in N,\, k\neq j}\varphi_j(N\backslash\{k\}, v).$$

The balanced summarized contributions property has a strong connection to the grand dividends value. Similar as the Shapley value can be characterized by **E** and **BC** (Myerson, 1980), the grand dividends value can be characterized by **E** and **BSC**.

Theorem 4.2. *The grand dividends value Ψ is the unique TU-value that satisfies **E** and **BSC**.*

Proof. Since **E** is already shown in the proof of Theorem 4.1 and **BSC** follows immediately from (4), we only need to show uniqueness.

Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and φ be a TU-value which satisfies **E** and **BSC**. We show uniqueness by induction on the size n.

Initialization: Let $n = 1$. Then, uniqueness is satisfied by **E**.

Induction step: Let $n \geq 2$. Assume that φ is unique for all n', $n' < n$, (IH). By **BSC**, we have

$$\begin{aligned} &\varphi_i(N,v) - \sum_{k\in N,\, k\neq i}\varphi_i(N\backslash\{k\}, v) = \varphi_j(N,v) - \sum_{k\in N,\, k\neq j}\varphi_j(N\backslash\{k\}, v) \\ \Leftrightarrow \quad & n\cdot\varphi_i(N,v) - n\cdot\sum_{k\in N,\, k\neq i}\varphi_i(N\backslash\{k\}, v) = \sum_{k\in N}\varphi_k(N,v) - \sum_{j\in N}\sum_{k\in N,\, k\neq j}\varphi_j(N\backslash\{k\}, v) \end{aligned}$$

and, by **E** and (IH), φ is unique for the player i. Since i is arbitrary, uniqueness and, therefore, Theorem 4.2 is shown. □

This axiomatization has a direct relationship to the statement in Hart and Kurz (1983), cited in the Introduction: In the first step, all coalitions that have one player less than the grand coalition consider whether to merge; if they should, in the second step, the players see that it is fair to all of them; if the chosen TU-value also satisfies efficiency, it is better to do so if the worth of the grand coalition is higher than the sum of the worths of all coalitions with one player less.

5. An axiomatization in the spirit of Shapley

We pick the original axiomatization of the Shapley value as the starting point of this section.

Theorem 5.1 (Shapley, 1953b)**.** *The Shapley value Sh is the unique TU-value that satisfies* ***E****,* ***A****,* ***N****, and* ***Sym****.*

We would like to point out that Nowak and Radzik (1994) also introduced their solidarity value with an axiomatization similar to this one. Their axiomatization differs from Shapley's by replacing the null player axiom **N** with their A-null player axiom. Further axiomatizations which differ only in the null player axiom from Shapley's axiomatization are the axiomatization of the equal division value in van den Brink (2007), using the nullifying player property, the axiomatization of the equal surplus division value by Casajus and Huettner (2014), using the dummifying player property, and, as a recent result, the axiomatization of the average surplus value in Li et al. (2021), using the A-null surplus player property.

Our next axiomatization of the grand dividends value also differs from Shapley's only in the null player axiom. A null player is a dummy player with a stand-alone worth of zero. Since the dummy player property **D** implies the null player property **N** and the Shapley value also satisfies **D**, the Shapley value can also be axiomatized by replacing **N** with the stronger **D**.

It is well-known and easy to prove that $i \in N$ is a dummy player in (N, v) if $\lambda_v(S) = 0$ for all $S \subseteq N$, $S \ni i, S \neq \{i\}$. Analogously, we call a player $i \in N$ a **multiplier** in (N, v) if $\delta_v(S) = 0$ for all $S \subseteq N$, $S \ni i, S \neq \{i\}$. The following axiom states that, depending on the size of the player set N, a multiplier receives as a payoff $(n-1)!$ times its stand-alone-worth.

Multiplier, Mul. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a multiplier in (N, v), we have $\varphi_i(N, v) = (n-1)!\, v(\{i\})$.

Remark 5.2. Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. If a multiplier $i, i \notin N$, joins the player set, the multiplier ensures that the worth of any coalition $S \cup \{i\}$, $S \in \Omega^N$, has the worth of the sum of its subcoalitions with one less player. It is easy to show, by induction on s, that in this case we have

$$v(S \cup \{i\}) = s!\, v(\{i\}) + \sum_{R \subseteq S} (s-r)!\, v(R). \tag{9}$$

Thus, the name multiplier seems more than justified. Moreover, the direct contribution of a multiplier to the grand coalition corresponds exactly to the associated payoff by the multiplier property **Mul**.

If the worth of the singleton of a multiplier equals zero, we call the multiplier a **null multiplier**. This yields the following property which is implied by **Mul**.

Null multiplier, NMul. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and $i \in N$ such that i is a null multiplier in (N, v), we have $\varphi_i(N, v) = 0$.

It may not be entirely fair, but, by (9), each coalition containing player i has as its worth a (multiplied) sum of the worths of coalitions that all do not contain player i. In this sense, player i does not contribute to the worth of any coalition. Therefore, the other players split the full payoff among themselves. We can see a null multiplier as a kind of catalyst that enables the other players to multiply but does not change itself. We give a new axiomatization.

Theorem 5.3. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **A**, **NMul**, and **Sym**.*

Proof. In unanimity games (N, u_S), $S \in \Omega^N$, which form a basis for $\mathbb{V}(N)$ (see Shapley (1953b)), we have $\lambda_{u_S}(S) = 1$ and $\lambda_{u_S}(T) = 0$, $T \in \Omega^N$, $T \neq S$. Analogously, we introduce another basis. For each coalition $S \in \Omega^N$, we use a TU-game $(N, z_S) \in \mathbb{V}(N)$ such that

$$\delta_{z_S}(T) := \begin{cases} 1, & \text{if } T = S, \\ 0, & \text{if } , T \in \Omega^N,\ T \neq S. \end{cases} \tag{10}$$

Due to (3), we have $z_S(S) = 1$ and all coalitions which are no supersets of S have a worth of zero. Each coalition T containing S as a proper subset, contains $\binom{t-s}{t-s-1} = t - s$ coalitions of the size $t - s - 1$ containing S and all other coalitions which are subsets of the same size have a worth of zero. Thus, each TU-game (N, z_S), $S \in \Omega^N$, is given, by

$$z_S(T) := \begin{cases} (t-s)!, & \text{if } S \subseteq T, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Since a $(2^n - 1) \times (2^n - 1)$ matrix A of the $2^n - 1$ entries of the $2^n - 1$ coalition functions z_S, $S \in \Omega^N$, correspondingly ordered, is a triangular matrix with $det\, A = 1 \neq 0$, we have found a basis for $\mathbb{V}(N)$.

Let now $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$.

I. Existence: **E** is shown in the proof of Theorem 4.1. By (6), Ψ obviously satisfies **NMul** and **Sym**. Since we have, by (3), $\delta_{v+w} = \delta_v + \delta_w$, **A** is satisfied by (6).

II. Uniqueness: Let φ be a TU-value which satisfies all axioms from Theorem 5.3. For all $S \in \Omega^N$, $i \in N$, we have $\varphi_i(N, \alpha z_S) = 0$, by **Sym** and **E**, if $\alpha = 0$, and, by **NMul**, if $i \in N \backslash S$. By **E**, **Sym**, and (11), it follows $\varphi_i(N, \alpha z_S) = \alpha \dfrac{(n-s)!}{s}$ for all $i \in S$. Therefore, φ is unique on all games $(N, \alpha z_S)$ for all $\alpha \in \mathbb{R}$ and all $S \in \Omega^N$. But then, by **A**, uniqueness is shown and the proof is complete. □

For all multipliers which are no null multipliers, the payoff, according to the multiplier property, seems to be much fairer than for a null multiplier. Therefore, since, by (6), the grand dividends value obviously satisfies **Mul**, the next axiomatization that also follows from the results of this section should be even more convincing.

Corollary 5.4. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **A**, **Mul**, and **Sym**.*

6. An axiomatization in the spirit of Young

Certainly, the following theorem is one of the most beautiful axiomatizations of the Shapley value.

Theorem 6.1 (Young, 1985)**.** *The Shapley value Sh is the unique TU-value that satisfies **E**, **SMon**, and **Sym**.*

Thereby **SMon** can also be replaced by the weaker **Mar**. By (1), the condition '$MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N\backslash\{i\}$' in **Mar** can be equivalently replaced by '$\lambda_v(S) = \lambda_w(S)$ for all $S \subseteq N$, $S \ni i$', analogously in **SMon**. We replace marginal contributions or Harsanyi dividends respectively by grand dividends in both axioms and obtain two new properties.

Grand dividends independency, GDInd. For all $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$, and $i \in N$ such that $\delta_v(S) = \delta_w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(N, v) = \varphi_i(N, w)$.

Grand dividends monotonicity, GDMon. For all $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$, and $i \in N$ such that $\delta_v(S) \leq \delta_w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(N, v) \leq \varphi_i(N, w)$.

The grand dividends monotonicity states that the payoff to a player should not decrease if the grand dividends of all coalitions containing that player increase or stay the same. It is easy to show that **GDMon** implies **GDInd**. By this property, the payoffs remain the same if the grand dividends of all coalitions containing that player stay the same. Therefore, a player's payoff depends only on the grand dividends of coalitions containing the player. Young (1985) used **SMon** instead of **Mar** to axiomatize the Shapley value where the proof only used **Mar**. The same approach is used in the proof of our following axiomatization. We introduce **GDMon** only because it might seem even more compelling for applications than **GDInd**. We formulate an axiomatization in the spirit of the characterization of the Shapley value just mentioned.

Theorem 6.2. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **GDInd/GDMon**, and **Sym**.*

Proof. The proof is similar to the proof in Young (1985).

Since **E** is shown in the proof of Theorem 4.1 and **Sym** and **GDInd/GDMon** follow immediately from (6), we only need to show uniqueness.

The games (N, z_S), $S \in \Omega^N$, defined by (11), form a basis of $\mathbb{V}(N)$. This means, we have for any $(N, v) \in \mathbb{V}(N)$ a unique representation of the coalition function v, given by

$$v = \sum_{S \in \Omega^N} \alpha_S z_S,\ \alpha_S \in \mathbb{R}. \tag{12}$$

Note, due to (10), that for all $S \in \Omega^N$, $c \in \mathbb{R}$, and two games $(N, v), (N, w) \in \mathbb{V}(N)$, $w := v + cz_S$, we have

$$\delta_v(T) = \delta_w(T) \text{ for all } T \subseteq N,\ T \neq S. \tag{13}$$

Therefore, **GDInd** implies

$$\varphi_i(N,v) = \varphi_i(N,w) \text{ for all } i \in N \backslash S. \tag{14}$$

Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and φ be a TU-value which satisfies **E**, **Sym**, and **GDInd**. We use an induction on the size $r_v := |\{R \in \Omega^N : \delta_v(R) \neq 0\}|$.

Initialization: Let $r = 0$. We have $v(N) = 0$ and uniqueness is satisfied by **E** and **Sym**.

Induction step: Let $r \geq 1$. Assume that φ is unique for all TU-games (N,v'), $r_{v'} \leq r-1$, (IH). Let Q be the intersection of all coalitions $Q_k \in \Omega^N$, $\delta_v(Q_k) \neq 0$,

$$Q := \bigcap_{1 \leq k \leq r} Q_k.$$

Two cases can be distinguished: (a) $i \in N \backslash Q$ and (b) $i \in Q$.

(a) Each $i \in N \backslash Q$ is a member of at most $r-1$ coalitions Q_k, $\delta_v(Q_k) \neq 0$ and we have at least one coalition $Q_i \in \Omega^N$, $\delta_v(Q_i) \neq 0$. Then, by (12), exists a coalition function v_i such that

$$v_i = \sum_{S \in \Omega^N,\, S \neq Q_i} \alpha_S z_S,$$

and, by (13), we have $\delta_v(S) = \delta_{v_i}(S)$ for all $S \subseteq N$, $S \ni i$. Therefore, by **GDInd**, (14), and (IH), φ is unique on (N,v) for all $i \in N \backslash Q$.

(b) Each $i \in Q$ is a member of all coalitions Q_k, $\delta_v(Q_k) \neq 0$. Thus, all coalitions $S \in \Omega^N$, $Q \nsubseteq S$, have a grand dividend $\delta_v(S) = 0$. It follows, $v(S) = 0$ for all $S \in \Omega^N$, $Q \nsubseteq S$. Therefore, if $|Q| = 1$, by **E**, and (a), φ is unique for $i \in Q$. If $|Q| \geq 2$, we have $v(T \cup \{j\}) = v(T \cup \{k\})$ for all $j,k \in Q$ and $T \subseteq N \backslash \{j,k\}$. Hence, all players $i \in Q$ are symmetric in (N,v). By **Sym**, **E**, and (a), φ also is unique for all $i \in Q$ and the proof is complete. $\square$

7. Multiple dividends and a strong relationship to the Shapley value

In this section, we first recall the reduced game of a *TU*-game, introduced in Hart and Mas-Colell (1989).

Definition 7.1. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $R \in \Omega^N$, and a TU-value φ, the **reduced TU-game** $(R, v_R^\varphi) \in \mathbb{V}(R)$ is defined, for all $S \in \Omega^R$, by*

$$v_R^\varphi(S) := v\big(S \cup R^c\big) - \sum_{j \in R^c} \varphi_j\big(S \cup R^c, v\big),$$

where $R^c := N \backslash R$.

We can interpret this reduced game like this: if a coalition of players R^c exits the game, then in the reduced game, each coalition S which is a subset of the coalition R of the players remaining receives the worth of the coalition $S \cup R^c$ in the original game minus the payoff of the players left in the restricted game on $S \cup R^c$.

A TU-value is called consistent if each player of the coalition R receives the same payoff in the reduced game and in the original game.

Consistency, C (Hart and Mas-Colell, 1989)**.** For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, $R \in \Omega^N$, we have $\varphi_i(N,v) = \varphi_i(R, v_R^{\varphi})$ for all $i \in R$.

The Shapley value is closely connected to this axiom.

Theorem 7.2 (Hart and Mas-Colell, 1989)**.** *Sh is the unique TU-value that satisfies **C** and **St**.*

In the proof of this theorem, a function called potential in Hart and Mas-Colell (1989) is essential.

Definition 7.3. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, define $P : \mathbb{V}(N) \to \mathbb{R}$ and $P(\emptyset, v) = 0$ such that*

$$v(N) = \sum_{i \in N} D_i P(N,v), \tag{15}$$

where

$$D_i P(N,v) = P(N,v) - P(N \backslash \{i\}) \text{ for all } i \in N.$$

*Then P is called a **potential**.*

It follows a strong connection between the potential P and the Shapley value.

Theorem 7.4 (Hart and Mas-Colell, 1989)**.** *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, there exists a unique potential function P, the resulting payoff vector $(D_i P(N,v))_{i \in N}$ coincides with the Shapley value Sh of the game (N,v) and the potential $P(N,v)$ is uniquely given by (15) applied only to (N,v) and its subgames.*

At first glance, the definition of the grand dividends value, which is close to the definition of the Shapley value, should also allow a potential approach similar to this potential and a related reduced game consistency. However, certain difficulties arise in this regard. In this respect, we point to a simple expression for the worth of a coalition, given by the grand dividends of the subgames.

Proposition 7.5. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, we have*

$$v(N) = \sum_{S \in \Omega^N} (n-s)!\, \delta_v(S).$$

Proof. Let $N \in \mathcal{N}$ and $(N,v) \in \mathbb{V}(N)$. We have

$$v(N) \underset{\mathbf{E}}{=} \sum_{i \in N} \Psi_i(N,v) \underset{(6)}{=} \sum_{i \in N} \sum_{S \subseteq N, S \ni i} \frac{(n-s)!}{s} \delta_v(S) = \sum_{S \in \Omega^N} (n-s)!\, \delta_v(S).$$

□

This means that each grand dividend of a subgame (S,v) is included several times in the worth of the grand coalition N, depending on the size of N and S. Thus, analogous to the Harsanyi dividends, we can introduce new dividends that combine the multiple grand dividends of a coalition into a single dividend.

Definition 7.6. *For each $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and all $S \subseteq N$, the* ***multiple dividends*** μ_v^N *are defined by*

$$\mu_v^N(S) := \begin{cases} 0, & \text{if } S = \emptyset, \\ (n-s)!\,\delta_v(S), & \text{otherwise.} \end{cases}$$

By Proposition 7.5, the following remark is immediate.

Remark 7.7. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, we have

$$v(N) = \sum_{S \subseteq N} \mu_v^N(S).$$

The potential for the Shapley value (see Hart and Mas-Colell (1989, Formula (2.3))) is just the sum of the Harsanyi dividends divided by the size of the coalitions. And here, one of the fundamental differences of the grand dividends value compared to the Shapley value comes into play: when we consider subgames, the values of the multiple dividends for the same coalitions change, or respectively, we have a different number of grand dividends of the same coalitions to consider, depending on the size of the player set.

For this reason, it cannot be assumed that in a simple manner straight forward a modified potential and a corresponding reduced game consistency can be found since different player sets have to be considered. In the following, we choose a different path where the multiple dividends defined above are extremely useful.

We define a new coalition function v_μ^N which has the multiple dividends μ_v^N as Harsanyi dividends.

Definition 7.8. *For each $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$, the* ***multiple dividends game*** *$(N, v_\mu^N) \in \mathbb{V}(N)$ is given by*

$$v_\mu^N(S) := \sum_{T \subseteq S} \mu_v^N(T) \text{ for all } S \subseteq N.$$

Remark 7.9. By (1), for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and a corresponding multiple dividends game (N, v_μ^N), we have

$$\mu_v^N(S) = \lambda_{v_\mu^N}(S) \text{ for all } S \subseteq N.$$

Finally, by Remark 7.9, (5), and (6) we get an interesting corollary.

Corollary 7.10. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and a corresponding multiple dividends game (N, v_μ^N), we have*

$$\Psi(N, v) = Sh(N, v_\mu^N).$$

This corollary has far-reaching consequences.

Remarks 7.11. If it follows from an axiom, satisfied by the grand dividends value for all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, that a different or the same axiom is satisfied for the associated games (N, v_μ^N) and the Shapley value can be axiomatized by such axioms, then the grand dividends value can also be axiomatized by the initial axioms. I.e., for example, we could also have indirectly derived a proof of Theorem 5.3 from the axiomatization of the Shapley value in Theorem 5.1 by showing that for all games (N, v) the axioms **E**, **A**, **NMul**, and **Sym** are satisfied by the grand dividends value and, from the satisfaction of these axioms for all (N, v), the axioms **E**, **A**, **N**, and **Sym** are also satisfied for all corresponding (N, v_μ^N).

Of course, the relationship also exists in the opposite direction.

Remark 7.12. By Remark 7.11, we can interpret each game $(N, v) \in \mathbb{V}(N)$ as an multiple dividends game to a corresponding game (N, v_λ^N) which is recursively given by

$$v_\lambda^N(S) := \mu_{v_\lambda^N}(S) + \sum_{j \in S} v_\lambda^N(S \backslash \{j\}) \text{ for all } S \in \Omega^N,$$

where

$$\mu_{v_\lambda^N}(S) := \frac{\lambda_v(S)}{(n-s)!}.$$

Then, by Corollary 7.10, we have

$$Sh(N, v) = \Psi(N, v_\lambda^N).$$

Based on Theorem 7.4 und Corollary 7.10, we can use the potential P also for the grand dividends value.

Corollary 7.13. *For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and corresponding multiple dividends games $(N, v_\mu^N) \in \mathbb{V}(N)$, there exists a unique potential function P such that*

$$D_i P(N, v_\mu^N) = \Psi_i(N, v) \textit{ for all } i \in N.$$

We adapt consistency for multiple dividends games.

Multiple dividends consistency, MDC. For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, corresponding multiple dividends games $(N, v_\mu^N) \in \mathbb{V}(N)$, and $R \in \Omega^N$, we have

$$\varphi_i(N, v) = \varphi_i\Big(R, \big((v_\mu^N)_R^\varphi\big)_\lambda^R\Big) \text{ for all } i \in R,$$

where $\big(R, (v_\mu^N)_R^\varphi\big)$ is the reduced TU-game according to Definition 7.1 for (N, v_μ^N) and R, which is interpreted as an multiple dividends game to the corresponding game $\Big(R, \big((v_\mu^N)_R^\varphi\big)_\lambda^R\Big)$ according to Remark 7.12.

By Corollary 7.10, Remark 7.12, and Theorem 7.2, we present our last corollary.

Corollary 7.14. *The grand dividends value Ψ is the unique TU-value that satisfies* ***MDC*** *and* ***St****.*

8. Conclusion and extensions

Despite Corollary 7.10 and Remark 7.12, for the chicken or the egg causality dilemma, in purely chronological terms, the Shapley value came first. However, if we look more closely at our explanations in Section 7, we can also conclude that the Shapley value and the grand dividends value are two sides of the same coin. The particular side depends on what we consider to be the dividends, the Harsanyi dividends or the grand dividends. The multiple dividends are the vehicle to pass from one side to the other.

Of course, we can apply the grand dividends value to all coalition functions, just like the Shapley value. However, the corresponding axioms and, hence, the associated TU-values are most convincing when our assumptions in the Introduction or in Section 3, respectively, for the grand dividends value and those of Harsanyi (1959, 1963) for the Shapley value are satisfied. The same applies to the assumptions made in the Introduction regarding the equal division value and the equal surplus division value. For example, it may not always be appropriate to give a nullifying or zero player (see van den Brink (2007) and Deegan and Packel (1978)), who causes any coalition containing that player to receive a worth of zero, a payoff of zero with no further penalty when the cooperation of the other players is actually present.

Therefore, when selecting a TU-value for a payoff calculation, each user should pay attention not only to the desired properties the value should have, i.e., the satisfied axioms, but also to the process of coalition formation. The grand dividends value fits best when, in the process of forming a stable final state, no proper subcoalitions of a formed coalition appear and we end up with only the formed grand coalition. If it is not possible or not desired to form the grand coalition, the formulas (4) or (6) can be adjusted accordingly. It would be desirable for the worth of a coalition to be as independent as possible of the worth of other overlapping coalitions, in the sense that two overlapping coalitions could each guarantee the entire worth of their coalition simultaneously to their members. Harsanyi dividends cannot properly capture such situations because the dividends from smaller coalitions are simultaneously included in the worth of different larger coalitions.

Definition 3.1 or Proposition 3.2 immediately reveal various extensions of the grand dividends value. First, analogous to the weighted Shapley values (Shapley, 1953a), each player could be assigned a personal weight, and the summands in (6) would no longer be distributed equally among the members of the coalitions S but in proportion to these members' weights (see (16)). As in the case of the proportional Shapley value (Béal et al., 2018; Besner, 2019a), these weights could also be replaced by the stand-alone worths of the individual members.

An extension in the sense of the Harsanyi solutions (Hammer et al., 1977; Vasil'ev, 1978)[6] would also be conceivable where the weights of two players for different coalitions could be in different proportions. Moreover, a further extension is possible in which, similar to the weighted values for level structures in Besner (2021), the coalitions are given their own weights, and the grand dividends in (4) can first be distributed according to these weights before a final distribution is made according to players' weights. Or, we have a successive splitting by the weights of all the subcoalitions, again removing another player, and so on, so that, finally, we split the share of the two-player coalitions according to the weights of the two singletons that are subsets of the two-player coalition.

[6]Detailed information can be found in Derks et al. (2000) and Vasil'ev and van der Laan (2002).

Of particular interest here is that the coalitions no longer need to form partitions of the player set or its subsets. This would be in line with the idea of cooperative game theory that here, besides the members of the player set, the coalitions are also actors.

Of course, extensions to values with hierarchical structures of the player set are also conceivable, such as for the Owen value (Owen, 1977) or the Shapley levels value (Winter, 1989).

Josten(1996) implemented, as a convex combination of the Shapley value and the equal division value, a new class of TU-values, called α-egalitarian Shapley values. Analogously, such convex combinations of the grand dividends value with TU-values such as the Shapley value, the equal division value, or other TU-values would also be suitable here. In this context, we also refer to (Yokote and Funaki, 2017), where it might be interesting to see if there are connections to the convex combinations of TU-values given there and the grand dividends value, as well as connections between our TU-value and the listed relationships between monotonicity properties and linearity.

Finally, it would be a necessary step to extend the grand dividends value to games with externalities.

The investigation and axiomatizations of these extensions are left to further research.

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Appendix

We show the logical independence of the axioms in the theorems. The locical independence of the two axioms in Theorem 4.2 is obvious.

Remark 8.1. *The axioms in Theorems 4.1 and 6.2 are logically independent:*

- ***E****: The null value ϕ^0, defined by $\phi_i^0(N,v) = 0$ for all $i \in N$, satisfies* ***IGD/GDInd*** *and* ***EMon/Sym*** *but not* ***E****.*
- ***IGD/GDInd****: The Shapley value Sh satisfies* ***E*** *and* ***EMon/Sym*** *but not* ***IGD/GDInd****.*
- ***EMon/Sym****: Let $W := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$, $w_i := w(i)$ for all $w \in W$, $i \in \mathfrak{U}$, be the collection of all positive weight systems on $\mathfrak{U}$ and $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. For each $w \in W$, the* ***weighted grand dividends value*** *Ψ^w, given by*

$$\Psi_i^w(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{w_i(n-s)!}{\sum_{j \in S} w_j} \delta_v(S) \text{ for all } i \in N, \tag{16}$$

such that $w_j \neq w_k$ for at least two different players $j,k \in N$, satisfies ***E*** *and* ***IGD/GDInd*** *but not* ***EMon/Sym****.*

Remark 8.2. *The axioms in Theorem 5.3 are logically independent:*

- ***E****: The null value ϕ^0 satisfies* ***A****,* ***NMul****, and* ***Sym*** *but not* ***E****.*

- ***A***: *Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. The TU-value φ, given by*

$$\varphi_i(N,v) := \begin{cases} 0, & \text{if } i \text{ is a null multiplier,} \\ \dfrac{v(N)}{|\{j \in N : j \text{ is no null multiplier in } (N,v)\}|}, & \text{otherwise,} \end{cases}$$

*for all $i \in N$, satisfies **E**, **NMul**, and **Sym** but not **A**.*

- ***NMul***: *The Shapley value Sh satisfies **E**, **A**, and **Sym** but not **NMul**.*
- ***Sym***: *The TU-values Ψ^w, as defined in Remark 8.1, satisfy **E**, **A**, and **NMul** but not **Sym**.*

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