# Zeta annuities, fractional calculus, and polylogarithms

Tao, Jim

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Jim Tao
jtao@alumni.caltech.edu

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## 1 Introduction

Mathematical jokes about analytic continuation often conflate values of the Riemann zeta function with divergent sums. For example, $1 + 1 + 1 + \cdots$ is jokingly claimed to be $\zeta(0) = -1/2$, and $1 + 2 + 3 + \cdots$ is jokingly claimed to be $\zeta(-1) = -1/12$. Another mathematical joke involves a lottery in which the prize is an infinite amount of money. When the winning ticket is drawn, the jubilant winner comes to claim his prize, and the mathematician who organized the lottery explains the mode of payment: "1 dollar now, 1/2 dollar next week, 1/3 dollar the week after that..." The joke here is that the harmonic series technically diverges but grows so slowly as to be insignificant in one's lifetime.

If the weekly effective interest rate $i$ is positive, the lottery prize thus described actually has finite value. The same can be said for $1 + 1 + 1 + \cdots$ and $1 + 2 + 3 + \cdots$ if the same mode of payment is used. Formulas for the present value of the latter two sums paid out as annuities-due are well known in the actuarial literature, namely as $\ddot{a}_{\overline{\infty|}} = (1 + i)/i$ and $(I\ddot{a})_{\overline{\infty|}} = (1 + i)^2/i^2$, respectively [1]. One might ask what is the present value of the harmonic series lottery prize or in general the value of 1 dollar paid now, $2^{-s}$ dollars paid next week, $3^{-s}$ dollars paid the week after that, etc. for some real number $s$. As we'll show in this paper, the present value of such a *zeta annuity* can be determined using polylogarithms after expressing annuity formulas in terms of $v = (1 + i)^{-1}$, a financial quantity known as a discount factor. More generally, in the case where the annuity is finite and ends with some final $n$th payment, the present value can be determined using fractional calculus after expressing annuity formulas in terms of $\delta = \ln(1 + i)$, a financial quantity known as the force of interest.

## 2 Background

Let $i > 0$ be the periodic effective interest rate. Let $v = (1 + i)^{-1}$ be the corresponding *discount factor*, and let $d = i/(1+i)$ be the corresponding *discount rate*. Let $\delta = \ln(1 + i)$ be the corresponding *force of interest*.

## 2.1 Level annuities

**Definition 2.1.1.** *A* level annuity-immediate *makes a constant payment at the end of each of $n$ units of time.*

**Lemma 2.1.2.** *The present value of a level annuity-immediate with each payment 1 is*

$$a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

*Proof.* Since $0 < v < 1$ we may apply the formula for a geometric series:

$$\begin{aligned}
a_{\overline{n}|} &= \frac{1}{1+i} + \left(\frac{1}{1+i}\right)^2 + \left(\frac{1}{1+i}\right)^3 + \cdots + \left(\frac{1}{1+i}\right)^n \\
&= v + v^2 + v^3 + \cdots + v^n \\
&= v \cdot \frac{1 - v^n}{1 - v} \\
&= \frac{1 - v^n}{i},
\end{aligned}$$

where

$$\frac{v}{1-v} = \frac{1}{(1+i)(1-v)} = \frac{1}{1+i-(1+i)v} = \frac{1}{1+i-1} = \frac{1}{i}.$$

□

Multiplying by $(1+i)^n$, we get the following corollary:

**Corollary 2.1.3.** *The accumulated value of a level annuity-immediate at time $n$ with each payment 1 is*

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

**Definition 2.1.4.** *A* level annuity-due *makes a constant payment at the beginning of each of $n$ units of time.*

Multiplying the formulas for level annuity-immediate by $1 + i$ and using the identity $d = i/(1+i)$, we get the corresponding formulas for level annuity-due:

**Corollary 2.1.5.** *The present value of a level annuity-due with each payment 1 is*

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d}.$$

**Corollary 2.1.6.** *The accumulated value of a level annuity-due at time $n$ with each payment 1 is*

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

**Definition 2.1.7.** *A* continuously payable level annuity *makes a uniform continuous payment throughout each of $n$ units of time.*

Since the present value of a payment of 1 spread out over a unit of time is

$$\int_0^1 e^{-\delta t}\,\mathrm{d}t = \left.\frac{e^{-\delta t}}{-\delta}\right|_0^1 = \frac{1-e^{-\delta}}{\delta} = \frac{1-(1+i)^{-1}}{\delta} = \frac{d}{\delta},$$

we can multiply the formulas for level annuity-due by $d/\delta$ and get the corresponding formulas for a continuously payable level annuity:

**Corollary 2.1.8.** *The present value of a continuously payable level annuity with each payment 1 is*

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}.$$

**Corollary 2.1.9.** *The accumulated value of a continuously payable level annuity at time $n$ with each payment 1 is*

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n-1}{\delta}.$$

## 2.2 Arithmetic progression annuities

**Definition 2.2.1.** *An* arithmetic progression annuity-immediate *makes a payment at the end of each of $n$ units of time following an arithmetic progression.*

**Lemma 2.2.2.** *The present value of an arithmetic progression annuity-immediate with payment $j$ at time $j$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.$$

*Proof.* Adding the present value of the individual payments, we get

$$(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1} + nv^n.$$

Multiplying by $1+i$, we get

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + (n-1)v^{n-2} + nv^{n-1}.$$

Subtracting, we get

$$i(Ia)_{\overline{n}|} = (1 + v + v^2 + \cdots + v^{n-1}) - nv^n = \ddot{a}_{\overline{n}|} - nv^n.$$

Dividing by $i$, we get

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i},$$

as desired. □

Multiplying by $(1+i)^n$, we get the following corollary:

**Corollary 2.2.3.** *The accumulated value of an arithmetic progression annuity-immediate at time $n$ with payment $j$ at time $j$ for $j \in \{1, 2, 3, \dots, n\}$ is*

$$(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}.$$

**Definition 2.2.4.** *An* arithmetic progression annuity-due *makes a payment at the beginning of each of $n$ units of time following an arithmetic progression.*

Multiplying the formulas for arithmetic progression annuity-immediate by $1+i$ and using the identity $d = i/(1+i)$, we get the corresponding formulas for arithmetic progression annuity-due:

**Corollary 2.2.5.** *The present value of an arithmetic progression annuity-due with payment $j$ at time $j-1$ for $j \in \{1, 2, 3, \dots, n\}$ is*

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}.$$

**Corollary 2.2.6.** *The accumulated value of an arithmetic progression annuity-due at time $n$ with payment $j$ at time $j-1$ for $j \in \{1, 2, 3, \dots, n\}$ is*

$$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}.$$

**Definition 2.2.7.** *An* continuously payable arithmetic progression annuity *makes a uniform continuous payment throughout each of $n$ units of time following an arithmetic progression.*

Since the present value of a payment of 1 spread out over a unit of time is

$$\int_0^1 e^{-\delta t}\,\mathrm{d}t = \left.\frac{e^{-\delta t}}{-\delta}\right|_0^1 = \frac{1-e^{-\delta}}{\delta} = \frac{1-(1+i)^{-1}}{\delta} = \frac{d}{\delta},$$

we can multiply the formulas for arithmetic progression annuity-due by $d/\delta$ and get the corresponding formulas for a continuously payable arithmetic progression annuity:

**Corollary 2.2.8.** *The present value of a continuously payable arithmetic progression annuity with payment $j$ from time $j-1$ to time $j$ for $j \in \{1, 2, 3, \dots, n\}$ is*

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}.$$

**Corollary 2.2.9.** *The accumulated value of a continuously payable arithmetic progression annuity at time $n$ with payment $j$ from time $j-1$ to time $j$ for $j \in \{1, 2, 3, \dots, n\}$ is*

$$(I\bar{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta}.$$

## 3 Zeta annuities

### 3.1 Integer values of $s$

**Definition 3.1.1.** *A* Laurent polynomial progression annuity-immediate *makes a payment at the end of each of $n$ units of time following a Laurent polynomial progression.*

**Theorem 3.1.2.** *Let $s$ be an integer. The present value of a Laurent polynomial progression annuity-immediate with payment $j^{-s}$ at time $j$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)a)_{\overline{n|}} = \left(v\frac{d}{dv}\right)^{-s} \frac{1-v^n}{v^{-1}-1}.$$

*(Note that $(\zeta(0)a)_{\overline{n|}} = a_{\overline{n|}}$ and $(\zeta(-1)a)_{\overline{n|}} = (Ia)_{\overline{n|}}$.)*

*Proof.* For the base case $s = 0$,

$$(\zeta(0)a)_{\overline{n|}} = a_{\overline{n|}} = \frac{1-v^n}{i} = \frac{1-v^n}{v^{-1}-1}.$$

Suppose our formula holds for $s = -k \leq 0$ where $k$ is nonnegative. Then

$$\left(v\frac{d}{dv}\right)^{k} \frac{1-v^n}{v^{-1}-1} = (\zeta(-k)a)_{\overline{n|}} = v + 2^k v^2 + 3^k v^3 + \cdots + n^k v^n.$$

Differentiating with respect to $v$ and then multiplying by $v$ gives us

$$\left(v\frac{d}{dv}\right)^{k+1} \frac{1-v^n}{v^{-1}-1} = v + 2^{k+1}v^2 + 3^{k+1}v^3 + \cdots + n^{k+1}v^n = (\zeta(-k-1)a)_{\overline{n|}},$$

so induction is complete for nonpositive $s$.

Suppose our formula holds for $s = k \geq 0$ where $k$ is nonnegative. Then

$$\left(w\frac{d}{dw}\right)^{-k} \frac{1-w^n}{w^{-1}-1} = (\zeta(k)a)_{\overline{n|}} = w + 2^{-k}w^2 + 3^{-k}w^3 + \cdots + n^{-k}w^n.$$

Dividing by $w$ and integrating with respect to $w$ from $w = 0$ to $w = v$, we get

$$\left(v\frac{d}{dv}\right)^{-k-1} \frac{1-v^n}{v^{-1}-1} = v + 2^{-k-1}v^2 + 3^{-k-1}v^3 + \cdots + n^{-k-1}v^n = (\zeta(k+1)a)_{\overline{n|}},$$

so induction is complete for nonnegative $s$. $\square$

Multiplying by $(1+i)^n = v^{-n}$, we get the following corollary:

**Corollary 3.1.3.** *Let $s$ be an integer. The accumulated value of a Laurent polynomial progression annuity-immediate at time $n$ with payment $j^{-s}$ at time $j$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)S)_{\overline{n|}} = v^{-n} \left(v\frac{d}{dv}\right)^{-s} \frac{1-v^n}{v^{-1}-1}.$$

*(Note that $(\zeta(0)S)_{\overline{n|}} = s_{\overline{n|}}$ and $(\zeta(-1)S)_{\overline{n|}} = (Is)_{\overline{n|}}$.)*

**Definition 3.1.4.** *A* Laurent polynomial progression annuity-due *makes a payment at the beginning of each of $n$ units of time following a Laurent polynomial progression.*

Multiplying the formulas for Laurent polynomial progression annuity-immediate by $1 + i = v^{-1}$, we get the corresponding formulas for Laurent polynomial progression annuity-due:

**Corollary 3.1.5.** *Let $s$ be an integer. The present value of a Laurent polynomial progression annuity-due with payment $j^{-s}$ at time $j-1$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)\ddot{a})_{\overline{n|}} = v^{-1}\left(v\frac{d}{dv}\right)^{-s}\frac{1-v^n}{v^{-1}-1}.$$

*(Note that $(\zeta(0)\ddot{a})_{\overline{n|}} = \ddot{a}_{\overline{n|}}$ and $(\zeta(-1)\ddot{a})_{\overline{n|}} = (I\ddot{a})_{\overline{n|}}$.)*

**Corollary 3.1.6.** *Let $s$ be an integer. The accumulated value of a Laurent polynomial progression annuity-due at time $n$ with payment $j^{-s}$ at time $j-1$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)\ddot{s})_{\overline{n|}} = v^{-n-1}\left(v\frac{d}{dv}\right)^{-s}\frac{1-v^n}{v^{-1}-1}.$$

*(Note that $(\zeta(0)\ddot{s})_{\overline{n|}} = \ddot{s}_{\overline{n|}}$ and $(\zeta(-1)\ddot{s})_{\overline{n|}} = (I\ddot{s})_{\overline{n|}}$.)*

**Definition 3.1.7.** *An* continuously payable Laurent polynomial progression annuity *makes a uniform continuous payment throughout each of $n$ units of time following a Laurent polynomial progression.*

Since the present value of a payment of 1 spread out over a unit of time is

$$\int_0^1 e^{-\delta t}\,\mathrm{d}t = \left.\frac{e^{-\delta t}}{-\delta}\right|_0^1 = \frac{1-e^{-\delta}}{\delta} = \frac{1-(1+i)^{-1}}{\delta} = \frac{1-v}{\ln v^{-1}} = \frac{v-1}{\ln v},$$

we can multiply the formulas for a Laurent polynomial progression annuity-due by $(v-1)/\ln v$ and get the corresponding formulas for a continuously payable Laurent polynomial progression annuity:

**Corollary 3.1.8.** *Let $s$ be an integer. The present value of a continuously payable Laurent polynomial progression annuity with payment $j^{-s}$ from time $j-1$ to time $j$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)\bar{a})_{\overline{n|}} = \frac{1-v^{-1}}{\ln v}\left(v\frac{d}{dv}\right)^{-s}\frac{1-v^n}{v^{-1}-1}.$$

*(Note that $(\zeta(0)\bar{a})_{\overline{n|}} = \bar{a}_{\overline{n|}}$ and $(\zeta(-1)\bar{a})_{\overline{n|}} = (I\bar{a})_{\overline{n|}}$.)*

**Corollary 3.1.9.** *Let $s$ be an integer. The accumulated value of a continuously payable Laurent polynomial progression annuity at time $n$ with payment $j^{-s}$ from time $j-1$ to time $j$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)\bar{s})_{\overline{n|}} = v^{-n}\frac{1-v^{-1}}{\ln v}\left(v\frac{d}{dv}\right)^{-s}\frac{1-v^n}{v^{-1}-1}.$$

*(Note that $(\zeta(0)\bar{s})_{\overline{n|}} = \bar{s}_{\overline{n|}}$ and $(\zeta(-1)\bar{s})_{\overline{n|}} = (I\bar{s})_{\overline{n|}}$.)*

## 3.2 Arbitrary real values of $s$ with $n$ finite

To be able to apply the formulas derived in the previous section in the case where $s$ is real but not an integer, it looks like we'll need to do fractional calculus with powers of the operator $v\frac{d}{dv}$. However, as we'll see from a simple change of variables as done in [2], it's not necessary to look at fractional powers of a product of two noncommuting operators. If we make the substitutions $\sigma_j = -\ln t_j$ and $\delta = -\ln v$, we find that for any polynomial function $g : \mathbb{R} \to \mathbb{R}$ passing through the origin,

$$\int_0^v \frac{1}{t_1} \int_0^{t_1} \cdots \int_0^{t_{s-1}} \frac{1}{t_s} g(t_s)\, \mathrm{d}t_s \cdots \mathrm{d}t_2\, \mathrm{d}t_1 = \int_\delta^\infty \int_{\sigma_1}^\infty \cdots \int_{\sigma_{s-1}}^\infty g(e^{-\sigma_s})\, \mathrm{d}\sigma_s \cdots \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1,$$

so if we express the annuity formulas in terms of $\delta$, we just have to do fractional derivatives and integrals, which are much easier and more well-known. To get started, we prove an analog of a Cauchy formula for repeated integration:

**Lemma 3.2.1.** *Let $s$ be a positive integer and $f : [a, \infty) \to \mathbb{R}$ be a function for some $a > 0$ such that, for all $x \in [a, \infty)$, $f(x) < c\exp(-\lambda x)$ for some $c, \lambda > 0$. Then for $\delta \geq a$, the integral*

$$\left(-\frac{d}{d\delta}\right)^{-s} f(\delta) = \int_\delta^\infty \int_{\sigma_1}^\infty \cdots \int_{\sigma_{s-1}}^\infty f(\sigma_s)\, \mathrm{d}\sigma_s \cdots \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1$$

*is given by*

$$\left(-\frac{d}{d\delta}\right)^{-s} f(\delta) = \frac{1}{(s-1)!} \int_\delta^\infty (t-\delta)^{s-1} f(t)\, \mathrm{d}t.$$

*Proof.* The base case $s = 1$ follows from the fundamental theorem of calculus:

$$-\frac{d}{d\delta} \int_\delta^N f(t)\, \mathrm{d}t = f(\delta),$$

after taking the limit $N \to \infty$. Suppose the lemma holds for $s = k$. By the Leibniz integral rule, which holds by dominated convergence,

$$-\frac{d}{d\delta}\left[\frac{1}{k!}\int_\delta^\infty (t-\delta)^k f(t)\, \mathrm{d}t\right] = \frac{1}{(k-1)!}\int_\delta^\infty (t-\delta)^{k-1} f(t)\, \mathrm{d}t.$$

Applying the inductive hypothesis, we get

$$\begin{aligned}
\left(-\frac{d}{d\delta}\right)^{-k-1} f(\delta) &= \int_\delta^\infty \int_{\sigma_1}^\infty \cdots \int_{\sigma_k}^\infty f(\sigma_{k+1})\, \mathrm{d}\sigma_{k+1} \cdots \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 \\
&= \int_\delta^\infty \frac{1}{(k-1)!} \int_\delta^\infty (t-\delta)^{k-1} f(t)\, \mathrm{d}t\, \mathrm{d}\sigma_1 \\
&= \int_\delta^\infty \left(-\frac{d}{d\delta}\right) \left[\frac{1}{k!}\int_\delta^\infty (t-\delta)^k f(t)\, \mathrm{d}t\right] \mathrm{d}\sigma_1 \\
&= \frac{1}{k!}\int_\delta^\infty (t-\delta)^k f(t)\, \mathrm{d}t,
\end{aligned}$$

as desired. $\square$

Using the Gamma function $\Gamma(s) = (s-1)!$, we can extend the above formula to real values of $s$:

$$\left(-\frac{d}{d\delta}\right)^{-s} f(\delta) = \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} (t-\delta)^{s-1} f(t)\, \mathrm{d}t.$$

For non-integer values of $s$, we can calculate the present and accumulated value of zeta annuities:

**Corollary 3.2.2.** *Let $1-k \leq s < 2-k$ where $k$ is an integer. The present value of a zeta annuity-immediate with payment $j^{-s}$ at time $j$ for $j \in \{1, \ldots, n\}$ is*

$$(\zeta(s)a)_{\overline{n|}} = \left(-\frac{d}{d\delta}\right)^{-s-k} \left[\left(-\frac{d}{d\delta}\right)^{k} \left(\frac{1-e^{-\delta n}}{e^{\delta}-1}\right)\right].$$

*(Note that $(\zeta(0)a)_{\overline{n|}} = a_{\overline{n|}}$ and $(\zeta(-1)a)_{\overline{n|}} = (Ia)_{\overline{n|}}$.)*

**Example 3.2.3.** *Let $\delta = .05$ and $n = 5$. The present value of a $\zeta(-1/2)$ (square root) annuity-immediate is*

$$\begin{aligned}
(\zeta(-1/2)a)_{\overline{5|}} &= \left(-\frac{d}{d\delta}\right)^{1/2-2} \left[\left(-\frac{d}{d\delta}\right)^{2} \left(\frac{1-e^{-5\delta}}{e^{\delta}-1}\right)\right] \\
&= \frac{1}{\Gamma(3/2)} \int_{.05}^{\infty} (t-.05)^{1/2} \left(-\frac{d}{dt}\right)^{2} \left(\frac{1-e^{-5t}}{e^{t}-1}\right) \mathrm{d}t \\
&= 7.10057.
\end{aligned}$$

*The result of the above numerical integration agrees with the sum*

$$e^{-.05}\sqrt{1} + e^{-.1}\sqrt{2} + e^{-.15}\sqrt{3} + e^{-.2}\sqrt{4} + e^{-.25}\sqrt{5}$$

*when input into a calculator.*

Multiplying by $(1+i)^n = e^{\delta n}$, we get the following corollary:

**Corollary 3.2.4.** *Let $1-k \leq s < 2-k$ where $k$ is an integer. The accumulated value of a zeta annuity-immediate at time $n$ with payment $j^{-s}$ at time $j$ for $j \in \{1, 2, 3, \ldots, n\}$ is*

$$(\zeta(s)S)_{\overline{n|}} = e^{\delta n} \left(-\frac{d}{d\delta}\right)^{-s-k} \left[\left(-\frac{d}{d\delta}\right)^{k} \left(\frac{1-e^{-\delta n}}{e^{\delta}-1}\right)\right].$$

*(Note that $(\zeta(0)S)_{\overline{n|}} = s_{\overline{n|}}$ and $(\zeta(-1)S)_{\overline{n|}} = (Is)_{\overline{n|}}$.)*

Multiplying the formulas for zeta annuity-immediate by $1+i = e^{\delta}$, we get the corresponding formulas for zeta annuity-due:

**Corollary 3.2.5.** *Let $1-k \le s < 2-k$ where $k$ is an integer. The present value of a zeta annuity-due with payment $j^{-s}$ at time $j-1$ for $j \in \{1,\dots,n\}$ is*

$$(\zeta(s)\ddot{a})_{\overline{n}|} = e^{\delta}\left(-\frac{d}{d\delta}\right)^{-s-k}\left[\left(-\frac{d}{d\delta}\right)^{k}\left(\frac{1-e^{-\delta n}}{e^{\delta}-1}\right)\right].$$

*(Note that $(\zeta(0)\ddot{a})_{\overline{n}|} = \ddot{a}_{\overline{n}|}$ and $(\zeta(-1)\ddot{a})_{\overline{n}|} = (I\ddot{a})_{\overline{n}|}$.)*

**Corollary 3.2.6.** *Let $1-k \le s < 2-k$ where $k$ is an integer. The accumulated value of a zeta annuity-due at time $n$ with payment $j^{-s}$ at time $j-1$ for $j \in \{1,2,3,\dots,n\}$ is*

$$(\zeta(s)\ddot{s})_{\overline{n}|} = e^{\delta(n+1)}\left(-\frac{d}{d\delta}\right)^{-s-k}\left[\left(-\frac{d}{d\delta}\right)^{k}\left(\frac{1-e^{-\delta n}}{e^{\delta}-1}\right)\right].$$

*(Note that $(\zeta(0)\ddot{s})_{\overline{n}|} = \ddot{s}_{\overline{n}|}$ and $(\zeta(-1)\ddot{s})_{\overline{n}|} = (I\ddot{s})_{\overline{n}|}$.)*

Since the present value of a payment of 1 spread out over a unit of time is

$$\int_0^1 e^{-\delta t}\,\mathrm{d}t = \left.\frac{e^{-\delta t}}{-\delta}\right|_0^1 = \frac{1-e^{-\delta}}{\delta},$$

we can multiply the formulas for a zeta annuity-due by $(1-e^{-\delta})/\delta$ and get the corresponding formulas for a continuously payable zeta annuity:

**Corollary 3.2.7.** *Let $1-k \le s < 2-k$ where $k$ is an integer. The present value of a continuously payable zeta annuity with payment $j^{-s}$ from time $j-1$ to time $j$ for $j \in \{1,\dots,n\}$ is*

$$(\zeta(s)\bar{a})_{\overline{n}|} = \frac{e^{\delta}-1}{\delta}\left(-\frac{d}{d\delta}\right)^{-s-k}\left[\left(-\frac{d}{d\delta}\right)^{k}\left(\frac{1-e^{-\delta n}}{e^{\delta}-1}\right)\right].$$

*(Note that $(\zeta(0)\bar{a})_{\overline{n}|} = \bar{a}_{\overline{n}|}$ and $(\zeta(-1)\bar{a})_{\overline{n}|} = (I\bar{a})_{\overline{n}|}$.)*

**Corollary 3.2.8.** *Let $1-k \le s < 2-k$ where $k$ is an integer. The accumulated value of a continuously payable zeta annuity at time $n$ with payment $j^{-s}$ from time $j-1$ to time $j$ for $j \in \{1,\dots,n\}$ is*

$$(\zeta(s)\bar{s})_{\overline{n}|} = \frac{e^{\delta(n+1)}-e^{\delta n}}{\delta}\left(-\frac{d}{d\delta}\right)^{-s-k}\left[\left(-\frac{d}{d\delta}\right)^{k}\left(\frac{1-e^{-\delta n}}{e^{\delta}-1}\right)\right].$$

*(Note that $(\zeta(0)\bar{s})_{\overline{n}|} = \bar{s}_{\overline{n}|}$ and $(\zeta(-1)\bar{s})_{\overline{n}|} = (I\bar{s})_{\overline{n}|}$.)*

## 3.3 The case $n \to \infty$

If we take the limit $n \to \infty$, the annuity becomes a perpetuity, and the formula for the present value of a Laurent polynomial progression perpetuity-immediate coincides with the polylogarithm $\mathrm{Li}_s(v)$, a well-known special function [3] defined for arbitrary real (and complex) values of $s$:

**Corollary 3.3.1.** *Let $s$ be a real number. The present value of a zeta perpetuity-immediate with payment $j^{-s}$ at time $j$ for $j \in \mathbb{N}$ is*

$$(\zeta(s)a)_{\overline{\infty}|} = \mathrm{Li}_s(v).$$

*(Note that $(\zeta(0)a)_{\overline{\infty}|} = a_{\overline{\infty}|}$ and $(\zeta(-1)a)_{\overline{\infty}|} = (Ia)_{\overline{\infty}|}$.)*

Multiplying the formula for zeta perpetuity-immediate by $1 + i = v^{-1}$, we get the corresponding formula for zeta perpetuity-due:

**Corollary 3.3.2.** *Let $s$ be a real number. The present value of a zeta perpetuity-due with payment $j^{-s}$ at time $j - 1$ for $j \in \mathbb{N}$ is*

$$(\zeta(s)\ddot{a})_{\overline{\infty}|} = v^{-1}\mathrm{Li}_s(v).$$

*(Note that $(\zeta(0)\ddot{a})_{\overline{\infty}|} = \ddot{a}_{\overline{\infty}|}$ and $(\zeta(-1)\ddot{a})_{\overline{\infty}|} = (I\ddot{a})_{\overline{\infty}|}$.)*

**Remark 3.3.3.** *The present value of the harmonic progression perpetuity-due described in the introduction is*

$$\begin{aligned}
(\zeta(1)\ddot{a})_{\overline{\infty}|} &= v^{-1}\mathrm{Li}_1(v) \\
&= -v^{-1}\ln(1 - v) \\
&= (1 + i)(\ln(1 + i) - \ln i).
\end{aligned}$$

Since the present value of a payment of 1 spread out over a unit of time is

$$\int_0^1 e^{-\delta t}\,\mathrm{d}t = \left.\frac{e^{-\delta t}}{-\delta}\right|_0^1 = \frac{1 - e^{-\delta}}{\delta} = \frac{1 - (1 + i)^{-1}}{\delta} = \frac{1 - v}{\ln v^{-1}} = \frac{v - 1}{\ln v},$$

we can multiply the formula for a zeta perpetuity-due by $(v - 1)/\ln v$ and get the corresponding formula for a continuously payable zeta perpetuity:

**Corollary 3.3.4.** *Let $s$ be a real number. The present value of a continuously payable zeta perpetuity with payment $j^{-s}$ from time $j - 1$ to time $j$ for $j \in \mathbb{N}$ is*

$$(\zeta(s)\bar{a})_{\overline{\infty}|} = \frac{1 - v^{-1}}{\ln v}\mathrm{Li}_s(v).$$

*(Note that $(\zeta(0)\bar{a})_{\overline{\infty}|} = \bar{a}_{\overline{\infty}|}$ and $(\zeta(-1)\bar{a})_{\overline{\infty}|} = (I\bar{a})_{\overline{\infty}|}$.)*

# 4 Conclusion

We have derived the present value and accumulated value formulas for zeta annuities-immediate, due, and continuously payable for all real values of $s$. Taking the limit $n \to \infty$, the annuities become perpetuities, and the present value formula for a zeta perpetuity-immediate coincides with the polylogarithm.

# References

[1] Leslie Jane Federer Vaaler and James W. Daniel. *Mathematical interest theory*. Mathematical Association of America, Washington, DC, second edition, 2009.

[2] Roudy El Haddad. Repeated Integration and Explicit Formula for the $n$-th Integral of $x^m(\ln x)^{m'}$. *arXiv e-prints*, page arXiv:2102.11723, February 2021.

[3] Leonard Lewin, editor. *Structural properties of polylogarithms*, volume 37 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1991.