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Two-dimensional Geographical Position as a Factor in Determining the Growth and Decline of Retail Agglomeration

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Abstract

We investigate where retail stores agglomerate in a road network with radial roads and a ring road in a two-dimensional space. Per-distance travel cost on the radial roads can be different from that on the ring road. The transition of the two-dimensional agglomeration patterns of retail stores is investigated with decreases in the travel costs. Results show 1) a difference in improvement sequences in the radial and ring roads generates a difference in the agglomeration patterns with different welfare levels and 2) how the two-dimensional geographical position of shopping agglomerations ensuring the highest welfare level differs from that in equilibrium.

Keywords: Agglomeration, Bifurcation, Monopolistic competition, Two-dimensional road network. JEL classification: C62, L11, L13, R12.

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1. Introduction

Shopping is an indispensable daily activity in our lives. The hollowing-out of urban commercial centers has been an economic geographical progressing problem over the past several decades. One of the factors driving the hollowing-out is a decrease in travel costs caused by automobility and road improvements. Road improvements, however, provide social benefits to consumers. If the hollowing-out harms social welfare, it is an urban problem. Hence, it is essential to elucidate how road improvements affect social welfare related to the agglomeration of retail stores in a downtown and suburbs. We explore how road improvements in a two-dimensional road network affect the agglomeration pattern and social welfare.

The location of retail stores has been studied for almost a century since Hotelling (1929). One feature of Hotelling's framework is a simplified urban space: a line segment where consumers are distributed uniformly. Although several studies extend this feature to capture some unique economic mechanisms,[1] urban spaces in the real world are more complex than the spaces employed by those studies. One realistic factor increasing complexity is a road network embedded in a two-dimensional space. The road network generates geographical heterogeneity, such as a center and suburbs. In order to explore recent urban problems (e.g., the hollowing-out of the center), it is essential to differentiate the center from the suburbs.

Some studies focus on the differentiation of the center from the suburbs. For example, Braid (1993) explores price competition among retail stores on the Manhattan roadway grid. Similarly, Braid (2013) investigates the optimal locations of retail stores in a city with a central intersection and radial roadways extending from a center to

[1] For example, a circle (e.g., Mulligan, 1996), a line segment where consumers are distributed non-uniformly (Tabuchi and Thisse, 1995), a homogeneous two-dimensional space (Tabuchi, 1994), and a homogeneous n-dimensional space (Irmen and Thisse, 1998) are employed.

suburbs. Kishi, Kono and Nozoe (2015) analyze a spatial price model à la Capozza and Van Order (1977) in a similar space. Guo and Lai (2015) analyze the Cournot competition in a circle with a diameter as a main street. These studies differentiate the center from the suburbs by embedding a two-dimensional road network. On the other hand, Ushchev, Sloev and Thisse (2015) analyze competition between retail stores in a downtown (i.e., a center) and a shopping mall in a suburb in a line segment.

In contrast to these previous studies, we focus on heterogeneity in road networks observed in the real world Actually, per-distance travel cost near a center is different from those near suburbs in a real network. Moreover, roads in a city are not simultaneously improved by a local government.[2] Focusing on how an improvement sequence on a road network affects the agglomeration patterns of retail stores and social welfare, we investigate where retail stores are located in such a heterogeneous road network.

We build on the spatial price competition model proposed by Tabuchi (2009). This model comprises a homogeneous space, monopolistic competition among retail stores,[3] and a dynamical system that describes changes in the sizes of marketplaces where the retail stores are located. Tabuchi (2009) shows that the self-organization of the retail stores, which can be interpreted as the emergence of subcenters, occurs as a result of their competition in the homogeneous space.

Our paper differs from Tabuchi (2009) in a space where retail stores can be located. We employ a regular-hexagonal shape with one center and six suburbs (Figure 1), which are potential marketplaces for retail stores. In the real world, road networks in cities are constructed around the central business district. Most cities have radial roads and ring roads in the network. Hence, the regular-hexagonal shape is a simplified description of

[2]For example, Mun (1997) shows that the asymmetry of transport cost in a road network generates a difference in city size distribution.

[3]Recently, locations where firms operate under monopolistic competition have been investigated (e.g., Ago, 2008; Ushchev et al., 2015).

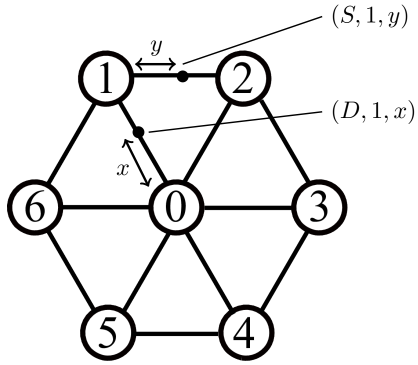

Figure 1: City shape. Black lines: the road network in the city; node 0: the center in the city; nodes $1, \ldots, 6$: the suburbs.

real road networks for our theoretical analysis.[4]

Actually, hexagonal domains have recently been employed as a two-dimensional spatial platform for New Economic Geography models. Ikeda, Murota, Akamatsu, Kono and Takayama (2014), for example, explore where and how population agglomeration takes place in a hexagonal domain by bifurcation analysis. Some theoretical properties of the location patterns on hexagonal domains have been clarified (e.g., Ikeda et al., 2017, 2018, 2019). Conducting bifurcation analysis introduced by Ikeda et al. (2014), we investigate market equilibria.

Moreover, we relax the uniform per-distance travel cost assumption employed in many spatial competition models. In our model, the per-distance travel cost on the radial roads can be different from that on the ring road. Such a relaxation captures one of the features of road networks in the real world. Combining the spatial platform and this relaxation, we investigate how improvement sequences in the road network affect the agglomeration pattern of retail stores and social welfare. In particular, we explore where retail stores should be located from the viewpoint of social welfare.

[4] Since our analysis focuses on the symmetry of the location of suburbs, a different number of suburbs would not qualitatively affect our result unless the symmetry differed.

The contribution of our paper is twofold. First, we show that a difference in improvement sequences in the road network generates a difference in agglomeration patterns in equilibrium even for the same travel costs parameters. Conducting bifurcation analysis to explore market equilibria, we demonstrate that all the retail stores agglomerate in the center if the radial roads are improved first. In contrast, the stores are located in the center as well as in several suburbs if the ring road is improved first.

Second, we show that the scale of agglomeration of retail stores in each marketplace as well as the two-dimensional location pattern of marketplaces in which stores operate at a market equilibrium differ from those at the first-best situation particularly when the travel costs are low. This implies that policymakers should guide stores to form an appropriate location pattern with policies such as land-use regulation.[5]

The rest of our paper is organized as follows. A spatial competition model is introduced in Section 2. Agglomeration patterns of retail stores are explored in Section 3. Our theoretical results are verified with numerical comparative statics analysis of the distribution of retail stores in Section 4. Section 5 concludes our paper.

2. Model

2.1. City and goods

We consider a city composed of seven potential marketplaces labeled $0, 1, \dots, 6$. Marketplace 0 and marketplaces $1, \dots, 6$ are in the center and suburbs, respectively. The center is connected to the suburbs by radial roads, whereas the suburbs are located on a ring road. We consider that the radial roads and the ring road form a regular-hexagonal road network as shown in Figure 1. For simplicity, the length of all the line

[5] Land-use regulation can be practical alternatives to superior policies that are often politically infeasible. The effect of the land-use regulation on social welfare has been theoretically investigated by many papers (e.g., Brueckner, 2009; Kono and Joshi, 2018, 2019).

segments between the marketplaces is assumed to be one.

We consider two types of goods: horizontally differentiated goods and an outside good. The differentiated goods are supplied by a large number of profit-maximizing retail stores in the marketplaces. The outside good is supplied by perfectly competitive firms and chosen as a numéraire good.

2.2. Consumers

Consumers in the city are uniformly distributed over the road network with the density normalized to 1. Let $\mathcal{L}$ denote all the positions on the road network. The utility of consumers residing at $\ell \in \mathcal{L}$ and visiting marketplace j is given by $U(\ell, j) = \ln M_j(\ell) + A(\ell)$, where $M_j(\ell) = \left(\int_0^{n_j} q(\ell, k)^{\frac{\sigma-1}{\sigma}} \,\mathrm{d}k\right)^{\frac{\sigma}{\sigma-1}}$. $q(\ell, k)$ is the consumption of the kth variety, n_j is the mass of varieties supplied in marketplace j, σ (> 1) is the elasticity of substitution between any two varieties, and $A(\ell)$ is the consumption of the outside good.

If consumers choose to visit marketplace j, then the budget constraint is given by $\int_0^{n_j} p_j(k) q(\ell, k) \,\mathrm{d}k + t(\ell, j) + A(\ell) = W$, where $p_j(k)$ is the price of the kth variety in marketplace j, W is the income, and $t(\ell, j)$ is the travel cost paid by the consumers.[6]

Solving the utility maximization problem, we obtain demand functions:

$$q(\ell, k) = p_j(k)^{-\sigma} R_j^{-1}, \tag{1}$$

$$A(\ell) = W - t(\ell, j) - 1, \tag{2}$$

where $R_j = \int_0^{n_j} p_j(k)^{1-\sigma} \,\mathrm{d}k$. We assume that income W is high so that $A(\ell)$ is positive in equilibria.

[6] We ignore commuting in order to focus on how decreases in the travel cost (i.e., road improvements) affect the equilibrium of shopping stores.

2.3. Retail stores

Retail stores are located in marketplaces. These stores share the same marginal production cost c and the same fixed cost f. We assume that retail stores in the same marketplace are under monopolistic competition. The total number of retail stores at each marketplace is determined by free entry.

Let $\pi_i(k)$ be the profit of the retail store producing the kth variety at marketplace i. $\pi_i(k)$ is given by

$$\pi_i(k) = (p_i(k) - c)Q_i(k) - f, \tag{3}$$

where $Q_i(k)$ is the total demand for the kth variety at marketplace i. Each retail store has a negligible impact on the prices of other goods in the marketplace because its supply is very small compared to the total supply of all the stores. That is, R_i does not change, as in Dixit and Stiglitz (1977). Using (1), we obtain the profit-maximizing prices, which are the same across all the varieties and marketplaces: $p_i(k) = p^*$ $(\forall i, k)$, where $p^* = c\sigma/(\sigma-1)$. We regard $\pi_i(k)$ as π_i because each firm at the same marketplace can be treated symmetrically.

2.4. Market area

Consumers are assumed to visit one marketplace where they can obtain the highest utility. Hence, total demand for a retail store $Q_i(k)$ is determined by consumers' behavior. To obtain $Q_i(k)$, we introduce 'market area', which is all the residential locations of the consumers visiting the same marketplace. We classify the two-dimensional agglomeration patterns of retail stores with the market area in Section 3.

Substituting p^* into (1), we obtain demand for the kth variety (i.e., $q(\ell, k)$) for consumers at ℓ $(\in \mathcal{L})$ visiting marketplace j: $q(\ell, k) = 1/(p^* n_j)$. Substituting this function and (2) into the utility function, we obtain the indirect utility of the consumers:

$$V(\ell, j) = \sigma_{-1} \ln n_j - t(\ell, j) + V_D, \tag{4}$$

where $\sigma_{-1} = (\sigma - 1)^{-1}, V_D = -\ln p^* + W - 1$. We define the set of the indirect utilities that the consumers can obtain by visiting a marketplace:

$$\mathcal{V}(\ell) = \{V(\ell, 0), V(\ell, 1), \ldots, V(\ell, 6)\}, \quad \ell \in \mathcal{L}.$$

Using $\mathcal{V}(\ell)$, we mathematically define the market area.

Definition 1. *The market area of marketplace i ($i = 0, 1, \ldots, 6$) is the following set:*

$$\mathcal{M}_i = \{\ell \in \mathcal{L} \mid \max \mathcal{V}(\ell) = V(\ell, i)\}. \tag{5}$$

We can obtain $Q_i(k)$ using the defined market area. Let $\mu(\mathcal{M}_i)$ denote the total length of market area $\mathcal{M}_i$. Using demand function $q(\ell, k)$ and $\mu(\mathcal{M}_i)$, we obtain the total demand:

$$Q_i(k) = \begin{cases} \mu(\mathcal{M}_i)/(p^* n_i) & (n_i > 0), \\ 0 & (n_i = 0). \end{cases} \tag{6}$$

2.5. Market equilibrium

We introduce the market equilibrium condition of the size of marketplaces. Let $\boldsymbol{n} = (n_0, \ldots, n_6)^\top$ denote the distribution of the retail stores across the marketplaces in the city. The market equilibrium condition for $\boldsymbol{n}$ is the following condition:

$$\begin{cases} \pi_i = 0 & \text{if} \quad n_i > 0, \\ \pi_i \leq 0 & \text{if} \quad n_i = 0, \end{cases} \qquad i = 0, 1, \ldots, 6. \tag{7}$$

Condition (7) implies that retail stores have no incentive to locate at marketplace i if the profit they obtain at marketplace i is not positive. Note that profit π_i is a function of $\boldsymbol{n}$ because $Q_i(k)$ in (6) depends on $\boldsymbol{n}$.

We employ a dynamical system to investigate the stability of equilibria. We assume that $\boldsymbol{n}$ gradually evolves in proportion to both profit $\boldsymbol{\pi}$ and state $\boldsymbol{n}$ itself as follows:[7]

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{n}), \tag{8}$$

where $\boldsymbol{F}(\boldsymbol{n}) = (F_0(\boldsymbol{n}), F_1(\boldsymbol{n}), \ldots, F_6(\boldsymbol{n}))^\top$ and $F_i(\boldsymbol{n}) = n_i \pi_i$ $(i = 0, 1, \ldots, 6)$. Since dynamics (8) implies that the growth rate of n_i per unit time is equal to profit π_i, retail stores are attracted to marketplaces where they can obtain profits. This dynamics has an advantage shown by the following lemma.[8]

Lemma 1. *$\boldsymbol{n}$ is the market equilibrium iff $\boldsymbol{n}$ is a stationary point of dynamics (8).*

Proof. See Appendix A. □

We investigate the market equilibria by finding stationary points of dynamics (8). A stationary point is linearly-stable if every eigenvalue of Jacobian matrix $\partial \boldsymbol{F}/\partial \boldsymbol{n}$ has a negative real part. We call linearly-stable stationary points stable equilibria. We also investigate transitions from unstable equilibria under dynamics (8) in Section 4.

2.6. Travel cost

Consumers have several route choices to the marketplaces. The consumers choose the route with the lowest travel cost. We mathematically define $\mathcal{L}$ to express travel costs, which are determined by the distance between consumers and a marketplace. Let D and S denote the radial roads and the ring road in the city, respectively. Since we assume that the length of each road between the marketplaces is 1, we can represent $\mathcal{L}$

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[7] Such a modeling methodology is called the Boltzmann, Lotka and Volterra method, which has been applied to statistical physics as well as regional science (e.g., Harris and Wilson (1978); Wilson (2008); Osawa et al. (2017)).

[8] The dynamics assumed by Tabuchi (2009) implies that changes in the number of retail stores per unit time depend on profits only. The dynamics assumed by Tabuchi (2009) and us are qualitatively the same. Furthermore, the dynamics in our paper can capture corner equilibria as stationary points of the dynamics.

by $\mathcal{L} = \mathcal{A} \times \mathcal{P} \times X$, where $\mathcal{A} = \{D, S\}$, $\mathcal{P} = \{1, 2, \ldots, 6\}$, and $X = (0, 1)$. $(D, i, x) \in \{D\} \times \mathcal{P} \times X$ is equal to position x distant from the center on the radial road between the center and suburb i (e.g., see $(D, 1, x)$ in Figure 1). Similarly, $(S, i, y) \in \{S\} \times \mathcal{P} \times X$ is equal to position y distant from suburb i on the ring road between suburb i and j ($\equiv i + 1 \bmod 6$) (e.g., see $(S, 1, y)$ in Figure 1). Therefore, $\{D\} \times \mathcal{P} \times X$ is all the positions on the radial roads, whereas $\{S\} \times \mathcal{P} \times X$ is that on the ring road.

For consumers residing at $\ell = (D, i, x) \in \{D\} \times \mathcal{P} \times X$ (i.e., consumers residing along the radial roads), the travel cost is given by

$$t(\ell, j) = \begin{cases} \phi x & (j = 0), \\ \min\{\phi(1 + x), \phi(1 - x) + \tau L_{ij}\} & (j \in \mathcal{P}), \end{cases} \tag{9}$$

where ϕ is the per-distance travel cost on the radial roads, τ is the per-distance travel cost on the ring road, and $L_{ij} = \min\{|i - j|, 6 - |i - j|\}$. ϕx is the travel cost when the consumers visit the center; $\phi(1+x)$ is when the consumers visit suburb j ($\in \mathcal{P}$) via the center; $\phi(1 - x) + \tau L_{ij}$ is when via the suburbs. On the other hand, for consumers residing at $\ell = (S, i, x) \in \{S\} \times \mathcal{P} \times X$ (i.e., consumers residing along the ring road), the travel cost is given by

$$t(\ell, j) = \begin{cases} \phi + \tau(1/2 - |x - 1/2|) & (j = 0), \\ \tau \times \min\{|i + x - j|, 6 - |i + x - j|\} & (j \in \mathcal{P}). \end{cases} \tag{10}$$

$\phi + \tau(\cdots)$ is the travel cost when the consumers visit the center via the nearest suburb; $\tau \times \min\{\cdots\}$ is the travel cost when the consumers visit marketplace j via the shortest route along the ring road.

2.7. Welfare

We measure the efficiency of the distribution of retail stores. Since the retail stores' profits are zero in the equilibria by condition (7), social welfare SW is total consumer utility (in monetary terms).

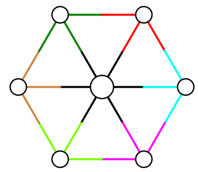


Figure 2: The dispersion. Black area: $\mathcal{M}_0$; green: $\mathcal{M}_1$; red: $\mathcal{M}_2$; sky blue: $\mathcal{M}_3$; pink: $\mathcal{M}_4$; yellowgreen: $\mathcal{M}_5$; brown: $\mathcal{M}_6$; the size of $\bigcirc$: the number of retail stores.

3. Agglomeration patterns of retail stores

We focus on some agglomeration patterns of retail stores. These patterns are possible market equilibria, which are investigated in Section 4.

3.1. A simple agglomeration pattern

We focus on the agglomeration pattern of retail stores in which every marketplace has a market area (i.e., $\mathcal{M}_i \neq \emptyset$ $(i = 0, 1, \dots, 6)$). We define this market area pattern as market pattern (D), and the equilibria that forms market pattern (D) as the dispersion (Figure 2). See Appendix B.2 for the details of these definitions.

A symmetric assumption for an equilibrium is often employed when the change of an agglomeration pattern with the change in an exogenous parameter is investigated (e.g., Ikeda et al., 2014). Assuming $n_1 = n_2 = \dots = n_6$, we investigate how decreases in per-distance travel cost ϕ and τ affect the dispersion. For $\boldsymbol{n} = (n_0, n_1, \dots, n_1)$, we can obtain dynamics (8) as follows.

Lemma 2. *For $\boldsymbol{n} = (n_0, n_1, \dots, n_1)$, dynamics (8) under market pattern (D) is*

$$F_0(\boldsymbol{n}) = \frac{3}{\sigma}\left(\frac{\ln(n_0/n_1)}{\phi(\sigma-1)} + 1\right) - fn_0, \tag{11}$$

$$F_i(\boldsymbol{n}) = \frac{1}{2\sigma}\left(-\frac{\ln(n_0/n_1)}{\phi(\sigma-1)} + 3\right) - fn_1, \quad i = 1, 2, \dots, 6. \tag{12}$$

Proof. See Appendix B.3.1. □

Let $\boldsymbol{n}_d \equiv (n_0, n_1, \ldots, n_1)$ be a symmetric equilibrium of the dispersion (i.e., $F_i(\boldsymbol{n}_d) = 0$ $(i = 0, 1, \ldots, 6)$). First, we investigate the change in $\boldsymbol{n}_d$ and the emergence of another agglomeration pattern with a decrease in ϕ. This change is summarized as follows.

Lemma 3. *If $n_0 > n_1$ holds in linearly-stable $\boldsymbol{n}_d$, then n_0 and n_1 in the equilibria change monotonously with an increase in ϕ:*

$$\frac{\mathrm{d}\, n_0}{\mathrm{d}\, \phi} < 0, \quad \frac{\mathrm{d}\, n_1}{\mathrm{d}\, \phi} > 0. \tag{13}$$

On the other hand, if $n_0 < n_1$ holds in linearly-stable $\boldsymbol{n}_d$, then

$$\frac{\mathrm{d}\, n_0}{\mathrm{d}\, \phi} > 0, \quad \frac{\mathrm{d}\, n_1}{\mathrm{d}\, \phi} < 0. \tag{14}$$

Proof. See Appendix B.3.2. □

Monotonicity (13) in Lemma 3 indicates that the full agglomeration of retail stores in the center is a possible market equilibrium.

Next, we investigate the change in $\boldsymbol{n}_d$ with a decrease in τ and the emergence of another agglomeration pattern. Since τ is not included in (11) or (12), τ does not affect the change in n_0 and n_1 of $\boldsymbol{n}_d$. On the other hand, a decrease in τ can affect the linear stability.

We briefly investigate the change in linearly-unstable $\boldsymbol{n}_d$ at a certain level of τ. Solutions starting near unstable $\boldsymbol{n}_d$ under dynamics (8) are classified into 1) the solution diverging from $\boldsymbol{n}_d$ and 2) the solution converging to $\boldsymbol{n}_d$. In particular, near $\boldsymbol{n}_d$, the motion of any solution diverging from $\boldsymbol{n}_d$ is almost equal to a linear combination of the eigenvectors for the eigenvalues of $\partial \boldsymbol{F}/\partial \boldsymbol{n}$ that has a positive real part.[9] Hence, the linear combination is the most likely change from $\boldsymbol{n}_d$ after it is not linearly-stable.

[9]Such a classification can be applied to stationary points of general dynamical systems. See, e.g., Kuznetsov (Chapter 2.2, 2004) for the theoretical details.

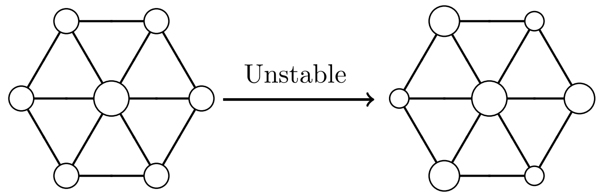


Figure 3: The change from $\boldsymbol{n}_d$ at a certain level of τ by a decrease in τ. The size of ◯: the number of retail stores.

Lemma 4. *Just after stationary point $\boldsymbol{n}_d$ is unstable at a certain level of τ by a decrease in τ, the eigenvector for the eigenvalues of $\partial \boldsymbol{F}/\partial \boldsymbol{n}$ that has a positive real part is $w(0, 1, -1, 1, -1, 1, -1)^\top$ $(w \in \mathbb{R})$.*

Proof. See Appendix B.3.3. □

Lemma 4 indicates that the dispersion changes into an agglomeration pattern where large agglomerations and small agglomerations alternately emerge on the ring road (See Figure 3).[10]

3.2. Corner equilibria

We next focus on the equilibria in which some marketplaces have no market area. We call these equilibria corner equilibria. Various symmetric corner equilibria can hold because the geometrical symmetry of the road network generates symmetric market area patterns. Among the corner equilibria, we investigate four corner equilibria (Figure 4): the full agglomeration ($\mathcal{M}_0 \neq \emptyset$), the period-doubling pattern ($\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_3, \mathcal{M}_5 \neq \emptyset$), the asymmetric pattern ($\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_3 \neq \emptyset$), and the linear pattern ($\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_4 \neq \emptyset$). These equilibria are possible agglomeration patterns into which the dispersion changes with decreases in ϕ and τ.

[10]This change is qualitatively the same result as the spatial period-doubling bifurcation, which is often observed in the New Economic Geography (e.g., Ikeda et al. (2012)).

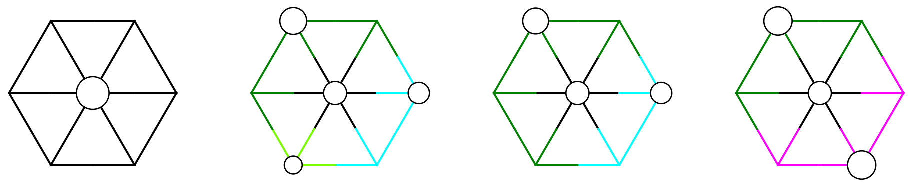


Figure 4: Corner equilibria under investigation. (a): the full agglomeration; (b) the period-doubling pattern; (c) the asymmetric pattern; (d) the linear pattern. ○: the number of retail stores. Black area: $\mathcal{M}_0$; green: $\mathcal{M}_1$; sky blue: $\mathcal{M}_3$; pink: $\mathcal{M}_4$; yellowgreen: $\mathcal{M}_5$.

Note that these corner equilibria hold under some inequality conditions. We investigate these conditions with the definitions of market area patterns. We define a market area pattern for the full agglomeration as market pattern (F), a market area pattern for the period-doubling pattern as market pattern (P), a market area pattern for the asymmetric pattern as market pattern (A), and a market area pattern for the linear pattern as market pattern (L). See Appendix C for these detailed explanations.

It is most likely that the full agglomeration and the period-doubling pattern[11] are corner equilibria into which the dispersion changes (Lemmas 3 and 4). Moreover, we can infer from Lemma 3 that the number of retail stores in marketplace 5 under the period-doubling pattern decreases with a decrease in τ. Hence, the asymmetric pattern is a possible equilibrium into which the period-doubling pattern changes.

On the other hand, the linear pattern seems to be the most efficient equilibrium

[11]Note that market areas $\mathcal{M}_1$, $\mathcal{M}_3$, and $\mathcal{M}_5$ are asymmetric in market pattern (P) (i.e., $\mu(\mathcal{M}_1) > \mu(\mathcal{M}_3) > \mu(\mathcal{M}_5)$). When $\mu(\mathcal{M}_1) = \mu(\mathcal{M}_3) = \mu(\mathcal{M}_5)$ holds at τ less than ϕ, some consumers residing along the radial roads are indifferent to choosing one of suburbs 1, 3, and 5. If a retail store enters one of the suburbs in such an agglomeration pattern, the suburb attracts the consumers. Hence, the symmetry of the market areas in the suburbs breaks. We thus focus on market pattern (P) rather than symmetric patterns where suburbs 1, 3, and 5 each have a market area.

in the equilibria in which retail stores are located in three marketplaces. However, the linear pattern is not a corner equilibrium pattern into which the period-doubling pattern or the asymmetric pattern changes under dynamics (8).

Proposition 1. *Neither any distribution* $\boldsymbol{n}$ *in market pattern (P) nor that in (A) changes into any distribution in market pattern (L) under dynamics (8).*

Proof. See Appendix D. □

Proposition 1 shows that neither the period-doubling pattern nor the asymmetric pattern changes into the linear pattern with decreases in ϕ and τ. In other words, improvements in the road network do not change the period-doubling pattern (the asymmetric pattern) into the linear pattern.

Moreover, while one may intuitively consider that the dispersion tends to change into the linear pattern, Lemma 4 indicates that such a result does not occur. This is verified in Section 4.

4. Two-dimensional geographical positions of retail stores

We explore how road improvements affect equilibria and social welfare. In our paper, we regard road improvements as decreases in travel costs (ϕ and τ). Conducting bifurcation analysis to explore market equilibria, we show how road improvement sequences affect the equilibrium.

In our model, there are two parameters that affect equilibria: elasticity of substitution σ and fixed cost f. The elasticity is investigated by various empirical studies. Referring to Bergstrand et al. (2013), we set $\sigma = 6.0$. On the other hand, we set $f = 20$.

4.1. Dependency of stable agglomeration patterns on travel costs

Since there are numerous road improvement sequence patterns in our model as well as in the real world, we focus on which road improvement sequence can generate a difference in equilibria and social welfare. Such a difference can occur with travel costs parameters for which multiple stable equilibria exist. Hence, we examine whether or not multiple stable equilibria exist with travel costs parameters.

The stability of equilibria introduced in Section 3 in the space of $(\phi, \tau) \in (0,1)\times(0,1)$ was investigated and the zones in which they were stable are enclosed by solid lines in Figure 5(a). According to the result, the dispersion tends to be stable with relatively high ϕ and τ. On the other hand, the period-doubling, the asymmetric, and the linear pattern tend to be stable with relatively higher ϕ than τ. The full agglomeration is always stable in the space.[12]

As Figure 5(a) shows, multiple stable equilibria exist in the space. Hence, an equilibrium forming an agglomeration pattern on specific travel costs is likely to change into an equilibrium forming another agglomeration pattern with decreases in travel costs. For example, the dispersion at stage α marked by $\diamond$ in Figure 5(a) is likely to change into the full agglomeration or the asymmetric pattern on stage γ. This example indicates that the road improvement sequence affects which agglomeration pattern emerges on stage γ.

We investigate which agglomeration pattern is most efficient in terms of the social welfare in the space. In Figure 5(b), the color of each zone (black, blue, orange, and red) represents the most efficient agglomeration pattern in that zone.[13] For example, the full agglomeration is the most efficient agglomeration pattern at stage γ.

The linear pattern is the most efficient agglomeration pattern in a part of the zone where multiple equilibria exist. This pattern, however, is predicted not to emerge from the dispersion, the period-doubling pattern, or the asymmetric pattern with decreases in travel costs (Lemma 4 and Proposition 1). Hence, the result indicates that roads

[12]The full agglomeration is always linearly stable. See Appendix C.1 for details.

[13]Note that the asymmetric pattern is not the most efficient in $(0, 1) \times (0, 1)$.

improvements do not generate an efficient agglomeration pattern even in terms of the locations of marketplaces where retail stores are located.

4.2. Dependency of agglomeration patterns on improvement sequences in the road network

Conducting the numerical comparative statics analysis of equilibria, we verify that road improvement sequences generate differences in the agglomeration patterns and the social welfare in equilibrium. We show two main findings through the numerical comparative statics analysis of the equilibrium for the travel costs:

- main finding 1: a difference in improvement sequences in the road network finally generates a difference in the equilibria.

- main finding 2: the welfare of the linear pattern is higher than that of the asymmetric pattern while the market system does not produce the linear pattern.

We investigate the transition of the stable dispersion from stage α to stage γ shown in Figure 5. Among various road improvement sequence patterns to stage γ, we focus on two simple improvement sequence patterns: (1) the radial roads are improved first and (2) the ring road is improved first.[14]

We investigate the following two cases of changes in the travel costs:

- The radial-roads first case:

$$(\phi, \tau) = \underbrace{(1.0, 1.0)}_{\text{Stage } \alpha} \xrightarrow{\text{Transition 1}} \underbrace{(0.35, 1.0)}_{\text{Stage } \beta_1} \xrightarrow{\text{Transition 2}} \underbrace{(0.35, 0.17)}_{\text{Stage } \gamma}.$$

[14] One may think that radial roads are improved first in the real world. Figure 5 (a) indicates that even if the road improvement sequence is initial radial roads, ring road, and additional radial roads, the equilibrium generated by the sequence is the same as that by the ring-road first case.

- The ring-road first case:

$$(\phi, \tau) = \underbrace{(1.0, 1.0)}_{\text{Stage } \alpha} \xrightarrow{\text{Transition 1}} \underbrace{(1.0, 0.17)}_{\text{Stage } \beta_2} \xrightarrow{\text{Transition 2}} \underbrace{(0.35, 0.17)}_{\text{Stage } \gamma}.$$

The radial-roads first case is that the radial roads are improved first, and the ring road is improved next.[15] On the other hand, the ring-road first case is that the ring road is improved first.

First, we focus on the result of the radial-roads first case shown in Figure 6. Figure 6(a-1) is the comparative statics analysis with a decrease in ϕ, which is equal to Stage 1. Solid lines A_1B_1 and A_2B_2 are the stable dispersion and the stable full agglomeration, respectively. Both the dispersion and the full agglomeration exist for large ϕ (> 0.37). In the dispersion, number of retail stores in the center n_0 increases and market area of the center $\mathcal{M}_0$ expands with a decrease in ϕ (Lemma 3). For small ϕ ($= 0.37$), $\mathcal{M}_0$ entirely covers the radial roads.

We investigate how a point in a neighborhood of B_1 changes under dynamics (8). Let $\widehat{B_1}$ denote the point.[16] The solution starting at $\widehat{B_1}$ under dynamics (8) is shown in Figure 6(c-1).[17] This solution converges at the square marker (□). The point marked by the square marker shows the full agglomeration. In summary, the dispersion changes into the full agglomeration when the radial roads are improved.[18]

The full agglomeration is always stable (Figure 6(a-1)). The red point in Figure 6(a-

[15] With lower travel costs, other agglomeration patterns not shown in the previous section can emerge (e.g., one downtown and a marketplace in the suburbs). To accomplish our aim, we have only to focus on the agglomeration patterns shown in the previous section.

[16] In this case, at B_1, $(\phi, n_0, n_1) = (0.366, 4.88 \times 10^{-2}, 0.853 \times 10^{-2})$. On the other hand, at $\widehat{B_1}$, $(\phi, n_0, n_1) = (0.365, 4.89 \times 10^{-2}, 0.852 \times 10^{-2})$. We obtained the solution from $\widehat{B_1}$ with dynamics (8) in market pattern (D) because the market area at $\widehat{B_1}$ have this pattern (Lemma 7 in Appendix C.1).

[17] We obtained the solution with the Runge-Kutta 4th order method.

[18] This change is called boundary equilibrium bifurcation in the dynamical systems theory (see e.g., Bernardo, Budd, Champneys and Kowalczyk, 2008).

1) is equal to the agglomeration pattern at stage β_1 (i.e., the full agglomeration).

We focus on stage γ of the radial-roads first case. Figure 6(a-2) is the comparative statics analysis with a decrease in τ, which is equal to Transition 2. The green point in Figure 6(a-2) shows the agglomeration pattern of the final stage (i.e., the full agglomeration). Hence, the radial-roads first case results in the full agglomeration emerging from the dispersion.

Next, we focus on the result of the ring-road first case shown in Figure 7. Figure 7(a-1) is the comparative statics analysis with a decrease in τ, which is equal to Stage 1. Solid line A_1B_1 is the stable dispersion; A_3B_3 is the period-doubling pattern; A_4B_4 is the linear pattern; A_5B_5 is the asymmetric pattern. The dispersion becomes unstable at point B_1, which is a bifurcation point. Three unstable equilibria emerge at this point. When the dispersion is unstable, a small perturbation to this state generates an agglomeration pattern where large agglomerations and small agglomerations alternately emerge on the ring road, as shown in Figure 3 (Lemma 4).

To investigate the change from the unstable equilibrium, we investigate how a point in a neighborhood of bifurcation point B_1 changes under dynamics (8). The solution starting at this point under dynamics (8) is shown in Figure 7(c-1).[19] The solutions shown with the blue line and the pink line in Figure 7(c-1) are obtained under market areas (D) and (P), respectively.[20] The solution converges at the point marked by the square marker (□). This result shows that the dispersion changes into the period-doubling pattern. Moreover, in the period-doubling pattern, n_1 increases with

[19] In this case, at B_1, $(\tau, n_1) = (0.306, 1.17 \times 10^{-2})$. At $\widehat{B_1}$, $(\tau, n_1) = (0.305, 1.18 \times 10^{-2})$.

[20] In this numerical analysis, the path-following of the solution stopped at the point at which the color of the line changes. This point is at the boundary between market area conditions (D) and (P) (See Lemmas 5 in Appendix B and 9 in Appendix C). We obtained the solution to the square marker by following the solution starting at a point satisfying market area condition (P) in the neighborhood of this boundary.

a decrease in τ. This pattern disappears at point B_3 (Figure 7(a-1)).

To elucidate the change from point B_3, we investigate how a point in the neighborhood of point B_3 changes under dynamics (8). The solution starting at this point is shown in Figure 7(d-1).[21] Near $\widehat{B_3}$, we obtained this solution with dynamics (8) in market pattern (P). On the other hand, for small n_5, the solution was obtained under that in market pattern (A).[22] This result shows that the period-doubling pattern changes into the asymmetric pattern. Hence, the asymmetric pattern emergin at stage β_2 is the red point in Figure 7(a-1).

We focus on stage γ. Figure 7(a-2) is the comparative statics analysis with a decrease in ϕ, which is equal to Stage 2. In both the asymmetric pattern and the linear pattern, n_1 decreases with a decrease in ϕ. The green point in Figure 7(a-2) is equal to the agglomeration pattern of stage γ (i.e., the asymmetric pattern). This pattern is not the agglomeration pattern that emerges in the radial-roads first case. In summary, the improvement sequences in the road network finally generate the difference in the agglomeration pattern. This observation is main finding 1.

4.3. Welfare analysis

We discuss the welfare analysis results shown in Figures 6 and 7. First, we compare the welfare of stage γ of the radial-roads first case and that of the ring-road first case. The welfare is 11.7 in the radial-roads first case (Figure 6(b-2)), whereas the welfare is 10.8 in the ring-road first case (Figure 7(b-2)). These demonstrations show that the radial-roads first case is more effective than the ring-road first case.

Next, we focus on the welfare of the linear pattern shown in the ring-road first case. In the ranges of travel costs ($0.01 < \tau < 0.34$ in Figure 7(b-1) and $0.35 < \phi \leq 1.00$ in Figure 7(b-2)), the welfare in the linear pattern is higher than that in the other patterns

[21] B_3 is $(\tau, n_1) = (0.188, 3.36 \times 10^{-2})$. On the other hand, $\widehat{B_3}$ is $(\tau, n_1) = (0.187, 3.36 \times 10^{-2})$.

[22] We obtained this result by the same procedure as we did for Figure 7(c-1).

in the ring-road first case. In particular, the social welfare of the linear pattern is higher than that of the asymmetric pattern at the same travel costs. However, not the linear pattern but the asymmetric pattern emerges from the dispersion in the market system. That is, the two-dimensional shape of the location in the market system is not that of the first-best location. This result is main finding 2, which indicates that policies that change the locations of marketplaces are needed (e.g., land-use regulations).

5. Conclusion

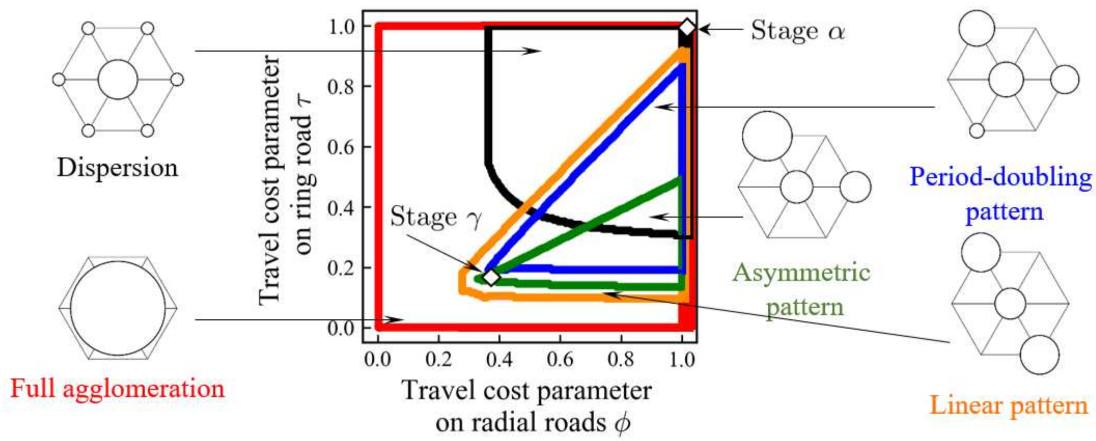
We have investigated how improvement sequences on a two-dimensional road network affect the agglomeration patterns of retail stores and social welfare. We have two main findings: (1) the improvement sequence in the road network finally generates the difference in agglomeration patterns and (2) the two-dimensional shape of the locations in the market system differs from that in the first-best location. In particular, the asymmetric pattern emerges if the ring road is improved first. This result contrasts with the main result of Tabuchi (2009), which is the emergence of the Christaller-Lösch system of hexagonal market area in a two-dimensional homogeneous space. The improvement sequence in the road network generates this contrast.

Our model is specific, but more realistic assumptions can be considered with this model. We would like to review our three assumptions one by one in the following.

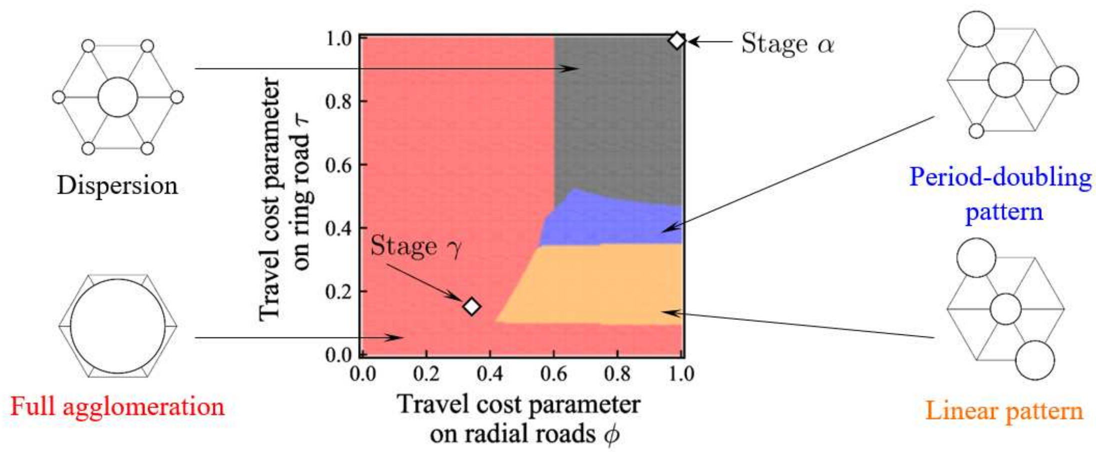
First, we assume so-called one-stop shopping, in contrast to two-stop shopping models which have been developed in recent years (Kim and Serfes, 2006; Brandão et al., 2014; Ushchev et al., 2015; Anderson et al., 2017). The assumption of two-stop shopping, however, makes the analysis more complex. Moreover, the results of the agglomeration of retail stores are similar to that of one-stopping shopping. One-stop shopping thus has a benefit to simply investigate the agglomeration patterns of retail stores, which is suitable for accomplishing our objective.

Second, we assume a uniform consumers-distribution in our model. This distribution is observed in local cities in the real world. Our model mainly targets the store-agglomeration mechanism in these cities. On the other hand, non-uniform distribution or endogenous consumer distribution has been considered in spatial competition models (e.g., Tabuchi and Thisse, 1995; Fujita and Thisse, 1986). Our analysis focuses on the symmetry of the road network. The assumption of the non-uniform distribution would not qualitatively affect our result unless the symmetry of the distribution differed. The assumption of exogenous consumers-distribution thus has a benefit to simply investigate the agglomeration mechanism of retail stores in a city. However, if we particularly investigate the interaction between consumers distribution and the location of shopping centers, it is necessary to consider endogenous consumer distribution.

Third, our model does not consider online shopping which prevails nowadays (e.g., Amazon). However, so-called brick-and-mortar retailers have at least one advantage over online retailers; consumers can identify the quality of a good (e.g., clothing) well. On the other hand, the assumption of online shopping has been introduced to the framework of spatial competition models (e.g., Guo and Lai, 2017). It is also essential to investigate the agglomeration mechanism regarding competition between online retailers and brick-and-mortar retailers (e.g., book stores) in a road network in the future.



(a)



(b)

Figure 5: (a) Zones of stable equilibria in $(\phi, \tau) \in (0,1) \times (0,1)$. The zone bounded by black lines: the dispersion; blue: the period doubling pattern; green: the asymmetric pattern; orange: the linear pattern; red: the full agglomeration. (b) The most efficient agglomeration pattern in terms of the social welfare. Black zone: the dispersion; blue zone: the period-doubling pattern; orange zone: the linear pattern; red zone: the full agglomeration.

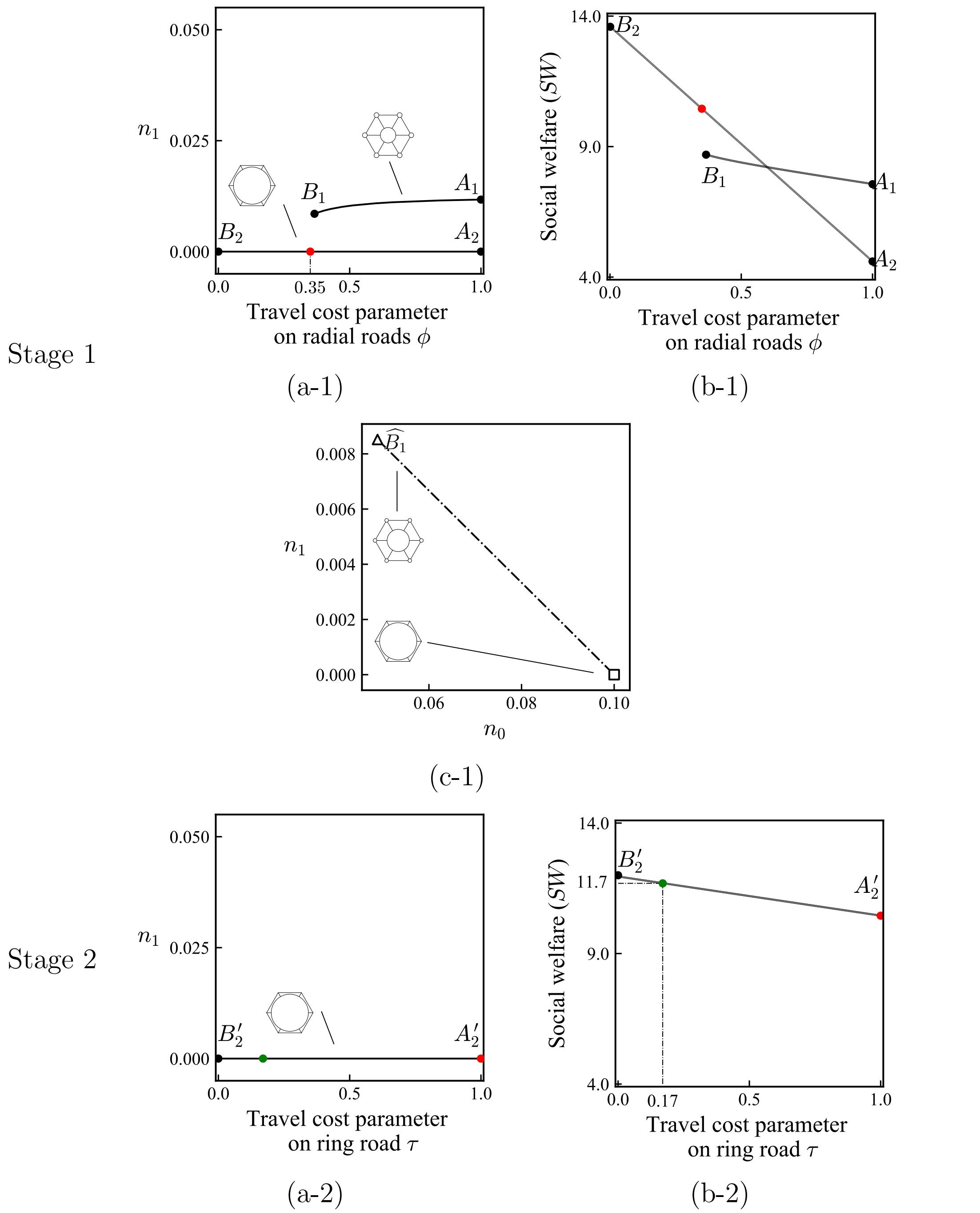


Figure 6: The radial-roads first case. (a) The market equilibria with decreases in the travel costs. Solid line: stable equilibria. (b) Social welfare of stable equilibria in (a). (c-1) The solution starting at a point in the neighborhood of point B_1. Dashed-dotted line: the solution under market pattern (F).

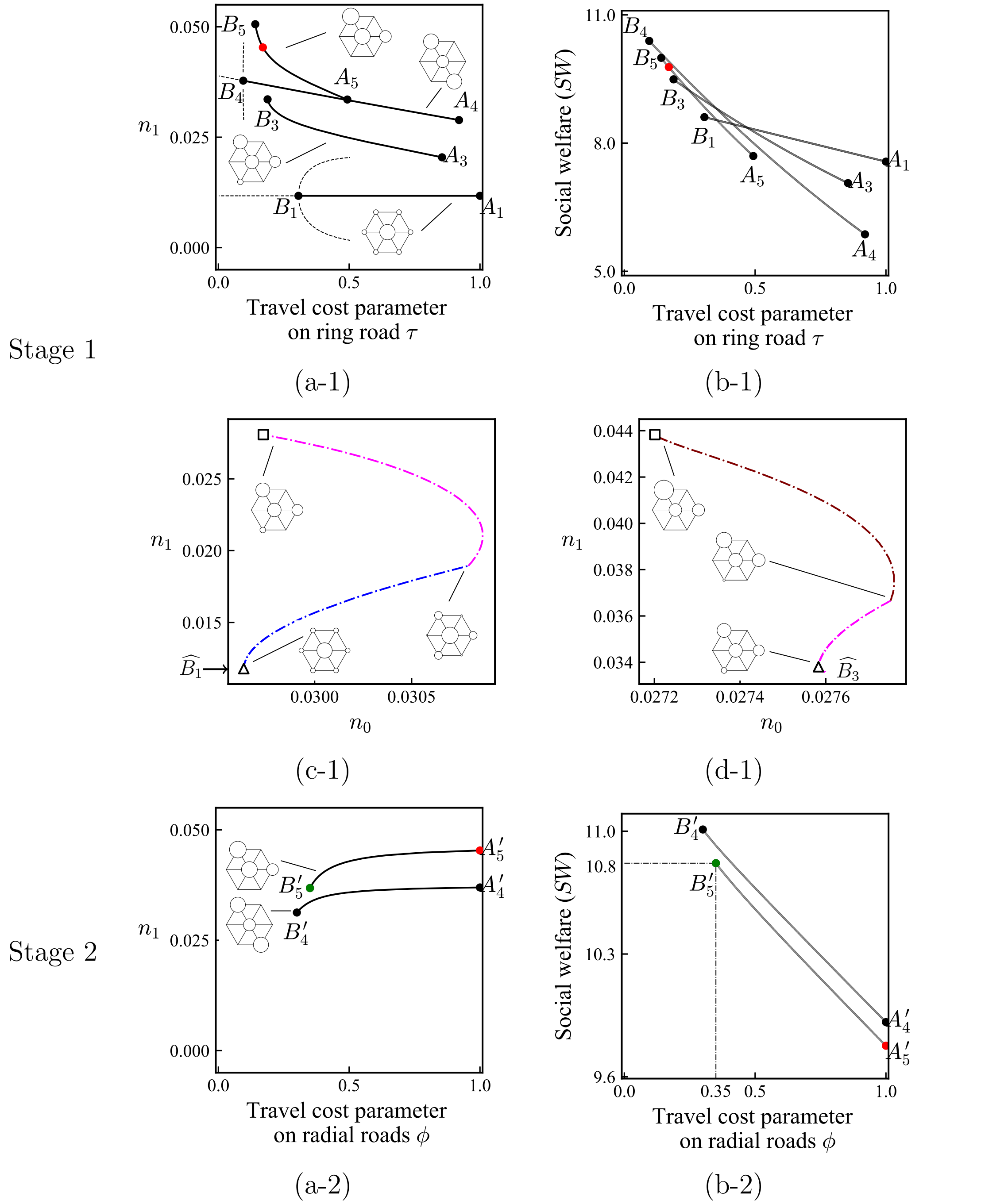


Figure 7: The ring-road first case. (a) The market equilibria with decreases in the travel costs. Solid line: stable equilibria; dashed line: unstable equilibria (b) Social welfare of stable equilibria in (a). (c-1) The solution of the dynamics starting at a point in a neighborhood of B_1. Blue dashed-dotted line: the solution under market pattern (D); pink: market pattern (P). (d-1) The solution starting at a point in a neighborhood of B_3. Pink: market pattern (P); brown: market pattern (A).

A. Proof of Lemma 1

($\Rightarrow$) It is obvious. ($\Leftarrow$) Let $\boldsymbol{n}^* = (n_0^*, n_1^*, \ldots, n_6^*)$ be the stationary point of dynamics (8). Using $\boldsymbol{n}^*$ and dynamics (8), we obtain $n_i^* \pi_i(\boldsymbol{n}^*) = 0$ $(i = 0, 1, \ldots, 6)$. $n_i^* \pi_i(\boldsymbol{n}^*) = 0$ holds if and only if

$$n_i^* = 0 \text{ or } \pi_i(\boldsymbol{n}^*) = 0. \tag{A1}$$

We check that market equilibria condition (7) holds at $\boldsymbol{n}^*$. If $n_i^* > 0$, then we obtain $\pi_i(\boldsymbol{n}^*) = 0$ by condition (A1). On the other hand, if $n_i^* = 0$, $Q_i = 0$ holds by Eq. (6). Therefore, $\pi_i(\boldsymbol{n}^*) = -f < 0$ holds.

B. Theoretical properties of the dispersion

B.1. Market boundary

We focus on market boundaries. A market boundary is a position at which consumers obtain the same indirect utility across multiple marketplaces.[23] Let t_i denote the market boundary between the center and suburb i and T_i denote the market boundary between suburb i and j $(\equiv i + 1 \bmod 6)$. Since the length of all the roads between the marketplaces is one, $t_i, T_i \in (0, 1)$ $(i = 0, 1, \ldots, 6)$ hold.

Using the market boundaries, we express the market areas. To express the market areas as subsets of all the positions on the road network $\mathcal{L}$, we define the following sets: $\mathcal{D}_i(Y) = \{(D, i, x) \in \mathcal{L} \mid x \in Y\}$, $\mathcal{S}_i(Y) = \{(S, i, x) \in \mathcal{L} \mid x \in Y\}$ $(i \in \mathcal{P})$. $\mathcal{D}_i(Y)$ denotes an area on the radial road between the center and suburb i, whereas $\mathcal{S}_i(Y)$ denotes an area on the ring road between suburb i and j $(\equiv i+1 \bmod 6)$. These subsets are employed in Appendix B.2 and C.

[23] Market boundaries between marketplaces i and j $(i \neq j)$, for example, are $\mathcal{M}_i \cap \mathcal{M}_j$.

B.2. The definitions of market pattern (D) and the dispersion

We define market pattern (D) with the market boundaries.

Definition 2. *Market pattern (D) is market areas given by*

$$\mathcal{M}_0 = \cup_{m\in\mathcal{P}}\ \mathcal{D}_m(\,(0,t_m]\,), \tag{B1}$$

$$\mathcal{M}_i = \mathcal{D}_i(\,[t_i,1)\,)\cup\mathcal{S}_i(\,(0,T_i]\,)\cup\mathcal{S}_j(\,[T_j,1)\,), \quad i,j\in\mathcal{P},\ j\equiv i-1 \bmod 6. \tag{B2}$$

The definition of market pattern (D) implies that every marketplace has a market area nearby. We can obtain the market boundaries as follows.

Lemma 5. *In market pattern (D), market boundaries t_i, T_i ($i\in\mathcal{P}$) are given by*

$$t_i = \frac{1}{2}\left(\frac{\ln\left(n_0/n_i\right)}{\phi(\sigma-1)}+1\right), \tag{B3}$$

$$T_i = \frac{1}{2}\left(\frac{\ln\left(n_i/n_j\right)}{\tau(\sigma-1)}+1\right), \quad j\equiv i+1 \text{ mod } 6. \tag{B4}$$

Proof. See Supplement SA.1. □

We can obtain dynamics (8) with t_i and T_i in market pattern (D).

Lemma 6. *Dynamics (8) in market pattern (D) is given by*

$$F_0(\boldsymbol{n}) = n_0\left(\frac{1}{\sigma n_0}\sum_{m=1}^{6} t_m - f\right), \tag{B5}$$

$$F_i(\boldsymbol{n}) = n_i\left(\frac{1}{\sigma n_i}((1-t_i)+T_i+(1-T_j))-f\right), \quad i,j\in\mathcal{P}, \quad j\equiv i-1 \bmod 6. \tag{B6}$$

Proof. See Supplement SA.2. □

The dispersion is the stationary points of dynamics (8) given by (B5) and (B6).

B.3. Proofs of Lemmas in Section 3.1

B.3.1. Proof of Lemma 2

Substituting $\boldsymbol{n} = (n_0, n_1, \ldots, n_1)$ into (B5) and (B6), we can obtain (11) and (12).

B.3.2. Brief proof of Lemma 3

We use the implicit function theorem. Using the Jacobian matrix of $\widetilde{\boldsymbol{F}} = (F_0, F_1)^\top$ with respect to $\widetilde{\boldsymbol{n}} = (n_0, n_1)$ and the linearly-stable condition of the dispersion, we can prove the Lemma. See Supplement SA.4 for details.

B.3.3. Brief proof of Lemma 4

We can analytically obtain the eigenvalues of $\partial \boldsymbol{F}/\partial \boldsymbol{n}$ at $\boldsymbol{n}_d$. Obtaining the eigenvector for the largest of these eigenvalues, we can prove the Lemma. See Supplement SA.5 for details.

C. Mathematical explanation of the corner equilibria in Section 3.2

In this appendix, we show the definitions of market area patterns and corner equilibria shown in Section 3.2. To express $\boldsymbol{F}(\boldsymbol{n})$ as a matrix, we appropriately permute the components of $\boldsymbol{F}(\boldsymbol{n})$ as follows:

$$\widehat{\boldsymbol{F}}(\boldsymbol{n}) = \begin{pmatrix} \boldsymbol{F}^+(\boldsymbol{n}) \\ \boldsymbol{F}^0(\boldsymbol{n}) \end{pmatrix}, \tag{C1}$$

where

$$\boldsymbol{F}^+(\boldsymbol{n}) = (F_{i_1}(\boldsymbol{n}), \dots, F_{i_m}(\boldsymbol{n}))^\top, \quad (i_1 < \cdots < i_m, \quad \mathcal{M}_{i_1}, \dots, \mathcal{M}_{i_m} \neq \phi),$$
$$\boldsymbol{F}^0(\boldsymbol{n}) = (F_{j_1}(\boldsymbol{n}), \dots, F_{j_k}(\boldsymbol{n}))^\top, \quad (j_1 < \cdots < j_k, \quad \mathcal{M}_{j_1} = \cdots = \mathcal{M}_{j_k} = \phi).$$

i_r $(r = 1, \dots, m)$ denotes an index assigned to a marketplace with a market area, whereas j_r $(r = 1, \dots, k)$ denotes an index assigned to one with no market area.

C.1. The full agglomeration

We focus on market pattern (F) and the full agglomeration.

Definition 3. *Market pattern (F) is market areas given by $\mathcal{M}_0 = \mathcal{L}, \mathcal{M}_i = \emptyset$ $(i \in \mathcal{P})$.*

The definition of market pattern (F) implies that the center has the market area entirely covering the entire city. We can obtain inequality conditions in this market pattern.

Lemma 7. *Market pattern (F) holds if and only if the following inequality holds.*

$$\phi \leq \sigma_{-1} \ln\left(n_0/n_i\right), \quad i \in \mathcal{P}, \tag{C2}$$

where $\sigma_{-1} = (\sigma - 1)^{-1}$.

Proof. See Supplement SB.1. □

In market pattern (F), $i_1 = 0$ and $(j_1, \ldots, j_6) = (1, \ldots, 6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this market pattern as follows.

Lemma 8. *In market pattern (F),* $\boldsymbol{F}^{+}(\boldsymbol{n})$ *and* $\boldsymbol{F}^{0}(\boldsymbol{n})$ *are given by*

$$\boldsymbol{F}^{+}(\boldsymbol{n}) = \frac{12}{\sigma} - f n_0, \quad \boldsymbol{F}^{0}(\boldsymbol{n}) = -f \boldsymbol{n}^{0}, \tag{C3}$$

where $\boldsymbol{n}^{0} = (n_1, \ldots, n_6)^{\top}$.

Proof. The proof is similar to that of Lemma 6. □

The full agglomeration is the stationary points of dynamics (8) given by (C3). Note that the full agglomeration is always linearly-stable because the eigenvalues are $-f$, which is negative.

C.2. The period-doubling pattern

We focus on market pattern (P) and the period-doubling pattern.

Definition 4. *Market pattern (P) is market areas given by*

$$\mathcal{M}_0 = \cup_{m \in \mathcal{P}} \ \mathcal{D}_m(\,(0, t_m]\,), \tag{C4}$$

$$\mathcal{M}_1 = \left(\cup_{m \in \{1,2,6\}} \ \mathcal{D}_m(\,[t_m, 1)\,)\right) \cup \left(\cup_{m \in \{1,6\}} \ \mathcal{S}_m(X)\right) \cup \mathcal{S}_2(\,(0, T_2]\,) \cup \mathcal{S}_5(\,[T_5, 1)\,), \tag{C5}$$

$$\mathcal{M}_3 = (\cup_{m\in\{3,4\}}\ \mathcal{D}_m(\,[t_m,1)\,)) \cup \mathcal{S}_3(X) \cup \mathcal{S}_2(\,[T_2,1)\,) \cup \mathcal{S}_4(\,(0,T_4]\,), \tag{C6}$$

$$\mathcal{M}_5 = \mathcal{D}_5(\,[t_5,1)\,) \cup \mathcal{S}_4(\,[T_4,1)\,) \cup \mathcal{S}_5(\,(0,T_5]\,), \tag{C7}$$

$$\mathcal{M}_2 = \mathcal{M}_4 = \mathcal{M}_6 = \emptyset. \tag{C8}$$

The definition of market pattern (P) has three features: (1) the center has a market area only on the radial roads, (2) suburbs 1, 3, and 5 have a market area on both the radial roads and the ring road, and (3) $\mu(\mathcal{M}_5) < \mu(\mathcal{M}_3) < \mu(\mathcal{M}_1)$ always holds. We can obtain inequality conditions in this market pattern.

Lemma 9. *Market pattern (P) holds if and only if the following inequalities hold.*

$$-\phi < \sigma_{-1}\ln(n_i/n_j) < \phi - \tau, \quad (i,j) = (0,1),(0,3), \tag{C9}$$

$$-\phi < \sigma_{-1}\ln(n_0/n_5) < \phi, \tag{C10}$$

$$0 < \sigma_{-1}\ln(n_i/n_j) < 2\tau, \quad (i,j) = (1,3),(3,5),(1,5), \tag{C11}$$

$$\tau < \sigma_{-1}\ln(n_i/n_j)\,, \quad (i,j) = (1,2),(3,4),(1,6). \tag{C12}$$

Proof. See Supplement SB.2. □

In market pattern (P), $(i_1,i_2,i_3,i_4) = (0,1,3,5)$ and $(j_1,j_2,j_3) = (2,4,6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this market pattern as follows.

Lemma 10. *In market pattern (P), $\boldsymbol{F}^+(\boldsymbol{n})$ and $\boldsymbol{F}^0(\boldsymbol{n})$ are given by*

$$\boldsymbol{F}^+(\boldsymbol{n}) = \frac{1}{2\sigma}(A_P\boldsymbol{z}_P^+ + \boldsymbol{b}_P) - f\boldsymbol{n}_P^+, \quad \boldsymbol{F}^0(\boldsymbol{n}) = -f\boldsymbol{n}_P^0, \tag{C13}$$

where

$$A_P = \frac{1}{\sigma-1}\left(\begin{array}{c|c} 6\phi^{-1} & -\phi^{-1}\boldsymbol{c}_P^\top \\ \hline -\phi^{-1}\boldsymbol{c}_P & \phi^{-1}B_P + \tau^{-1}C_P \end{array}\right),$$

$$\boldsymbol{z}_P^+ = (\ln n_0, \ln n_1, \ln n_3, \ln n_5)^\top, \quad \boldsymbol{c}_P = (3,2,1)^\top, \quad B_P = \mathrm{diag}\,(3,2,1),$$

$$C_P = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \ \boldsymbol{b}_P = \begin{pmatrix} 3\tau\phi^{-1}+6 \\ \boldsymbol{b}_{P1} \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{b}_{P1} = (-2\tau\phi^{-1}+7, -\tau\phi^{-1}+6, 5)^\top, \\ \boldsymbol{n}_P^0 = (n_2,n_4,n_6)^\top. \end{array}$$

Proof. The proof is similar to that of Lemma 6. □

The period-doubling pattern is the stationary points of dynamics (8) given by (C13).

C.3. The asymmetric pattern

We focus on market pattern (A) and the asymmetric pattern.

Definition 5. *Market pattern (A) is market areas given by*

$$\mathcal{M}_0 = \cup_{m\in\mathcal{P}}\ \mathcal{D}_m(\,(0, t_m]\,), \tag{C14}$$

$$\mathcal{M}_1 = (\cup_{m\in\{1,2,5,6\}}\mathcal{D}_m([t_m, 1))) \cup (\cup_{m\in\{1,5,6\}}\mathcal{S}_m(X)) \cup \mathcal{S}_2((0, T_2]) \cup \mathcal{S}_4(\,[T_4, 1)\,), \tag{C15}$$

$$\mathcal{M}_3 = (\cup_{m\in\{3,4\}}\ \mathcal{D}_m(\,[t_m, 1)\,)) \cup \mathcal{S}_3(X)\ \cup\ \mathcal{S}_2(\,[T_2, 1)\,) \cup \mathcal{S}_4(\,(0, T_4]\,), \tag{C16}$$

$$\mathcal{M}_2 = \mathcal{M}_4 = \mathcal{M}_5 = \mathcal{M}_6 = \emptyset. \tag{C17}$$

The definition of market pattern (A) has three features: (1) the center has a market area only on the radial roads, (2) suburbs 1 and 3 each have a market area on both the radial roads and the ring road, and (3) $\mu(\mathcal{M}_3) < \mu(\mathcal{M}_1)$ always holds. We can obtain inequality conditions as follows.

Lemma 11. *Market pattern (A) holds if and only if the following inequalities hold.*

$$-\phi < \sigma_{-1}\ln\left(n_0/n_1\right) < \phi - 2\tau, \tag{C18}$$

$$-\phi < \sigma_{-1}\ln\left(n_0/n_3\right) < \phi - \tau, \tag{C19}$$

$$0 < \sigma_{-1}\ln\left(n_1/n_3\right) < 2\tau, \tag{C20}$$

$$\tau < \sigma_{-1}\ln\left(n_i/n_j\right), \quad (i, j) = (1, 2), (1, 6), (3, 4), \tag{C21}$$

$$2\tau < \sigma_{-1}\ln\left(n_1/n_5\right). \tag{C22}$$

Proof. The proof is similar to that of Lemma 9. □

In market pattern (A), $(i_1, i_2, i_3) = (0, 1, 3)$ and $(j_1, j_2, j_3, j_4) = (2, 4, 5, 6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this market pattern as follows.

Lemma 12. *In market pattern (A), $\boldsymbol{F}^{+}(\boldsymbol{n})$ and $\boldsymbol{F}^{0}(\boldsymbol{n})$ are given by*

$$\boldsymbol{F}^{+}(\boldsymbol{n}) = \frac{1}{2\sigma}(A_A \boldsymbol{z}_A^{+} + \boldsymbol{b}_A) - f\boldsymbol{n}_A^{+}, \quad \boldsymbol{F}^{0}(\boldsymbol{n}) = -f\boldsymbol{n}_A^{0}, \tag{C23}$$

where

$$A_A = \frac{1}{\sigma-1}\left(\begin{array}{c|c} 6\phi^{-1} & -\phi^{-1}\boldsymbol{c}_A^{\top} \\ \hline -\phi^{-1}\boldsymbol{c}_A & \phi^{-1}B_A + \tau^{-1}C_A \end{array}\right),$$

$$\boldsymbol{z}_A^{+} = (\ln n_0, \ln n_1, \ln n_3)^{\top}, \quad \boldsymbol{c}_A = (4,2)^{\top}, \quad B_A = \mathrm{diag}\,(4,2),$$

$$C_A = 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \boldsymbol{b}_A = \begin{pmatrix} 5\tau\phi^{-1}+6 \\ \boldsymbol{b}_{A1} \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{b}_{A1} = (-4\tau\phi^{-1}+10, -\tau\phi^{-1}+8)^{\top}, \\ \boldsymbol{n}_A^{0} = (n_2, n_4, n_5, n_6)^{\top}. \end{array}$$

Proof. The proof is similar to that of Lemma 6. □

The asymmetric pattern is the stationary points of dynamics (8) given by (C23).

C.4. The linear pattern

We focus on market pattern (L) and the linear pattern.

Definition 6. *Market pattern (L) is market areas given by*

$$\mathcal{M}_0 = \cup_{i=1}^{6}\ \mathcal{D}_i(\,(0,t_i]\,), \tag{C24}$$

$$\mathcal{M}_1 = (\cup_{i\in\{1,2,6\}}\ \mathcal{D}_i(\,[t_i,1)\,)) \cup (\cup_{i\in\{1,6\}}\ \mathcal{S}_i(X)) \cup \mathcal{S}_2(\,(0,T_2]\,) \cup \mathcal{S}_5(\,[T_5,1)\,), \tag{C25}$$

$$\mathcal{M}_4 = (\cup_{i\in\{3,4,5\}}\ \mathcal{D}_i(\,[t_i,1)\,)) \cup (\cup_{i\in\{3,4\}}\ \mathcal{S}_i(X)) \cup \mathcal{S}_2([T_2,1)) \cup \mathcal{S}_5(\,(0,T_5]\,), \tag{C26}$$

$$\mathcal{M}_2 = \mathcal{M}_3 = \mathcal{M}_5 = \mathcal{M}_6 = \emptyset. \tag{C27}$$

The definition of market pattern (L) has two features: (1) the center has a market area only on the radial roads, (2) suburbs 1 and 4 each have a market area on both the radial roads and the ring road. We can obtain inequality conditions as follows.

Lemma 13. *Market pattern (L) holds if and only if the following inequalities hold.*

$$-\phi < \sigma_{-1}\ln\,(n_0/n_j) < \phi - \tau, \quad j = 1,4, \tag{C28}$$

$$-\tau < \sigma_{-1}\ln\,(n_1/n_4) < \tau, \tag{C29}$$

$$\tau < \sigma_{-1}\ln\,(n_i/n_j)\,, \quad (i,j) = (1,2),(1,6),(4,3),(4,5). \tag{C30}$$

Proof. The proof is similar to that of Lemma 9. □

In market pattern (L), $(i_1, i_2, i_3) = (0, 1, 4)$ and $(j_1, j_2, j_3, j_4) = (2, 3, 5, 6)$. We can obtain $\widehat{\boldsymbol{F}}(\boldsymbol{n})$ in this pattern as follows.

Lemma 14. *In market pattern (L), $\boldsymbol{F}^{+}(\boldsymbol{n})$ and $\boldsymbol{F}^{0}(\boldsymbol{n})$ are given by*

$$\boldsymbol{F}^{+}(\boldsymbol{n}) = \frac{1}{2\sigma}(A_L \boldsymbol{z}_L^{+} + \boldsymbol{b}_L) - f\boldsymbol{n}_L^{+}, \quad \boldsymbol{F}^{0}(\boldsymbol{n}) = -f\boldsymbol{n}_L^{0}, \tag{C31}$$

where

$$A_L = \frac{1}{\sigma - 1}\left(\begin{array}{c|c} 6\phi^{-1} & -\phi^{-1}\boldsymbol{3}_2^{\top} \\ \hline -\phi^{-1}\boldsymbol{3}_2 & 3\phi^{-1}I_2 + \tau^{-1}C_L \end{array}\right), \quad \boldsymbol{z}_L^{+} = (\ln n_0, \ln n_1, \ln n_4)^{\top},$$

$$C_L = 2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \boldsymbol{b}_L = \begin{pmatrix} 4\tau\phi^{-1} + 6 \\ \boldsymbol{b}_{L1} \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{b}_{L1} = (-2\tau\phi^{-1} + 9, -2\tau\phi^{-1} + 9)^{\top}, \\ \boldsymbol{n}_L^{0} = (n_2, n_3, n_5, n_6)^{\top}. \end{array}$$

Proof. The proof is similar to that of Lemma 6. □

The linear pattern is the stationary points of dynamics (8) given by (C31).

D. Proof of Proposition 1

Let $\mathcal{A}_p$ be the closure of the set of $\boldsymbol{n}$ satisfying the inequality conditions for market pattern (P), $\mathcal{A}_a$ be the closure for market pattern (A), and $\mathcal{A}_l$ be the closure for market pattern (L). $\mathcal{A}_p, \mathcal{A}_a$, and $\mathcal{A}_l$ are given by

$$\mathcal{A}_p = \mathrm{cl}\,\{\boldsymbol{n} \in \mathbb{R}_{+}^{7} \mid (C9)\text{–}(C12)\}, \tag{D1}$$

$$\mathcal{A}_a = \mathrm{cl}\,\{\boldsymbol{n} \in \mathbb{R}_{+}^{7} \mid (C18)\text{–}(C22)\}, \tag{D2}$$

$$\mathcal{A}_l = \mathrm{cl}\,\{\boldsymbol{n} \in \mathbb{R}_{+}^{7} \mid (C28)\text{–}(C30)\}, \tag{D3}$$

where $\mathrm{cl}\,\{\cdot\}$ is the closure of $\{\cdot\}$. Since the closure of (C12) and the closure of (C30) are disjoint sets. $\mathcal{A}_p \cap \mathcal{A}_l = \emptyset$ thus holds. Similarly, $\mathcal{A}_a \cap \mathcal{A}_l = \emptyset$ holds. Therefore, the solution starting at any point in market pattern (P) (or market pattern (A)) under dynamics (8) does not go to any state in market pattern (L).

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SA. Proofs in Appendix B

SA.1. Proof of Lemma 5

For any $i \in \mathcal{P}$, the following conditions hold in market pattern (D):

$$\ell = (D, i, t_i) \ \Rightarrow \ V(\ell, 0) = V(\ell, i), \tag{SA1}$$

$$\ell = (S, i, T_i), \ j \equiv i + 1 \bmod 6 \ \Rightarrow \ V(\ell, i) = V(\ell, j). \tag{SA2}$$

Using (4) and (SA1), we can obtain (B3). On the other hand, using (4) and (SA2), we can obtain (B4).

SA.2. Proof of Lemma 6

Using market boundaries t_i and T_i ($i \in \mathcal{P}$), we obtain $\mu(\mathcal{M}_0)$ and $\mu(\mathcal{M}_i)$ in market pattern (D):

$$\mu(\mathcal{M}_0) = \sum_{m=1}^{6} t_m, \tag{SA3}$$

$$\mu(\mathcal{M}_i) = (1 - t_i) + T_i + (1 - T_j), \quad i \in \mathcal{P}, \quad j \equiv i - 1 \bmod 6. \tag{SA4}$$

Substituting (SA3) (or (SA4)) into (6), we obtain Q_j ($j = 0, 1, \ldots, 6$) in market pattern (F). Since π_j in (3) is determined by Q_j, we can obtain $n_j \pi_j$ in dynamics (8), which is equal to (B5) (or (B6)).

SA.3. The eigenvalues of the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{n})$ for $\boldsymbol{n}_d$

To obtain the eigenvalues of the Jacobian matrix $\partial \boldsymbol{F}/\partial \boldsymbol{n}$ in market pattern (D), we rewrite $\boldsymbol{F}(\boldsymbol{n})$ as a matrix:

$$\boldsymbol{F}(\boldsymbol{n}) = \frac{1}{2\sigma}(A\boldsymbol{z} + \boldsymbol{b}) - f\boldsymbol{n}, \tag{SA5}$$

where

$$A = \frac{1}{\sigma - 1}\left(\begin{array}{c|c} 6a_1 & -a_1\mathbf{1}_6^\top \\ \hline -a_1\mathbf{1}_6 & a_1 I_6 + a_2 B \end{array}\right), \quad a_1 = 1/\phi, \quad a_2 = 1/\tau,$$

$$B = \begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{z} = (\ln(n_0), \ln(n_1), \ldots, \ln(n_6))^\top, \\ \boldsymbol{b} = (6,3,3,3,3,3,3)^\top. \end{array}$$

$\boldsymbol{k}_6$ is all-k 6-dimensional column vector whereas I_k is $k \times k$ identity matrix.

The following Lemma is employed for proofs of Lemmas 3 and 4.

Lemma 15. *For $\boldsymbol{n} = \boldsymbol{n}_d$, the eigenvalues of Jacobian matrix $\partial \boldsymbol{F}/\partial \boldsymbol{n}$ are given by*

$$\begin{cases} \lambda_1 = -f, \\ \lambda_2 = \frac{1}{2\phi\sigma(\sigma-1)}\left(\frac{6}{n_0} + \frac{1}{n_1}\right) - f, \\ \lambda_3 = \frac{1}{2n_1\sigma(\sigma-1)}\left(\frac{1}{\phi} + \frac{4}{\tau}\right) - f, \\ \lambda_4 = \frac{1}{2n_1\sigma(\sigma-1)}\left(\frac{1}{\phi} + \frac{3}{\tau}\right) - f \qquad \text{(repeated twice)}, \\ \lambda_5 = \frac{1}{2n_1\sigma(\sigma-1)}\left(\frac{1}{\phi} + \frac{1}{\tau}\right) - f \qquad \text{(repeated twice)}. \end{cases} \tag{SA6}$$

Proof. Let $J(\boldsymbol{n})$ denote Jacobian matrix $\partial \boldsymbol{F}/\partial \boldsymbol{n}$. Using (SA5), we obtain $J(\boldsymbol{n}_d)$ as follows:

$$J(\boldsymbol{n}_d) = \frac{1}{2\sigma}\left(A\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{n}}(\boldsymbol{n}_d)\right) - fI_7 = \left(\begin{array}{c|c} 6d_1 - f & -d_2\mathbf{1}_6^\top \\ \hline -d_1\mathbf{1}_6 & (d_2 - f)I_6 + d_3 B \end{array}\right), \tag{SA7}$$

where $d_1 = d_0/\phi n_0,\ d_2 = d_0/\phi n_1,\ d_3 = d_0/\tau n_1,\ d_0 = 1/2\sigma(\sigma-1)$. Let λ be an eigenvalue of $J(\boldsymbol{n}_d)$. Using elementary transformation of matrices, we obtain the determinant of $J - \lambda I_7$:

$$\det(J(\boldsymbol{n}_d) - \lambda I_7) = \det\left(\begin{array}{c|c} 6d_1 - f - \lambda & -d_2\mathbf{1}_6^\top \\ \hline -d_1\mathbf{1}_6 & (d_2 - f - \lambda)I_6 + d_3 B \end{array}\right)$$

$$
= \det \left(\begin{array}{c|c} -f-\lambda & \mathbf{0}_6^\top \\ \hline -d_1 \mathbf{1}_6 & J_1 \end{array} \right)
$$

$$
= (-f-\lambda) \det J_1,
$$

where $J_1 = d_1 \mathbf{1}_6 \mathbf{1}_6^\top + (d_2 - f - \lambda) I_6 + d_3 B$.

Next, we calculate $\det J_1$ with the property of orthogonal matrices. We consider the following orthogonal matrix:

$$
Q = \begin{pmatrix}
1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{12} & 1/\sqrt{4} & 1/\sqrt{12} & 1/\sqrt{4} \\
1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{12} & -1/\sqrt{4} & 2/\sqrt{12} & 0 \\
1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{12} & 0 & 1/\sqrt{12} & -1/\sqrt{4} \\
1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{12} & 1/\sqrt{4} & -1/\sqrt{12} & -1/\sqrt{4} \\
1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{12} & -1/\sqrt{4} & -2/\sqrt{12} & 0 \\
1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{12} & 0 & -1/\sqrt{12} & 1/\sqrt{4}
\end{pmatrix}. \tag{SA8}
$$

Since Q is an orthogonal matrix, $\det(J_1) = \det(Q^\top J_1 Q)$ holds. We carry out orthogonal transformation to matrices in J_1: $Q^\top \mathbf{1}_6 \mathbf{1}_6^\top Q = \mathrm{diag}(6,0,0,0,0,0), Q^\top I_6 Q = I_6, Q^\top B Q = \mathrm{diag}(0,4,3,3,1,1)$. Using these matrices, we obtain $\det(J(\boldsymbol{n}_d) - \lambda I_7) = (-f-\lambda) e_1 e_2 e_3^2 e_4^2$, where $e_1 = 6d_1 + d_2 - f - \lambda, e_2 = d_2 + 4d_3 - f - \lambda, e_3 = d_2 + 3d_3 - f - \lambda, e_4 = d_2 + d_3 - f - \lambda$. Therefore, the eigenvalues of Jacobian matrix $J(\boldsymbol{n}_d)$ are given by (SA6). □

SA.4. Detailed proof of Lemma 3

Let $\widetilde{J}$ denote $\partial \widetilde{\boldsymbol{F}} / \partial \widetilde{\boldsymbol{n}}$. Using (11) and (12), we can obtain $\widetilde{J}$:

$$
\widetilde{J} = \begin{pmatrix} 6\alpha_1 - f & -6\alpha_2 \\ -\alpha_1 & \alpha_2 - f \end{pmatrix}, \tag{SA9}
$$

where $\alpha_1 = \alpha/n_0,\ \alpha_2 = \alpha/n_1,\ \alpha = 1/2\phi\sigma(\sigma-1)$. We can obtain $\det \widetilde{J}$ with one of eigenvalues (SA6):

$$
\det \widetilde{J} = f(f - 6\alpha_1 - \alpha_2) = -f\lambda_2.
$$

Since we assume that $\boldsymbol{n}_d$ is linearly-stable, $\lambda_2 < 0$ (i.e., $\lambda_2 \neq 0$) holds. Hence, we can apply the implicit function theorem:

$$\left(\partial n_0/\partial\phi \quad \partial n_1/\partial\phi\right)^\top = \lambda_2^{-1}\gamma\left(6 \quad -1\right)^\top. \tag{SA10}$$

where $\gamma = (2\phi^2\sigma(\sigma-1))^{-1}\ln(n_0/n_1)$. Since the sign of $\partial n_0/\partial\phi$ and $\partial n_1/\partial\phi$ are determined by that of $\ln(n_0/n_1)$, we can obtain (13) and (14).

SA.5. Detailed proof of Lemma 4

We focus on the signs of the eigenvalues in (SA6). λ_1 is negative because f is positive. Moreover, if an equilibrium is stable, λ_2 is always negative regardless of τ. It is obvious that $\lambda_4, \lambda_5 < \lambda_3$. Therefore, if the equilibrium becomes unstable with a decrease in τ, λ_3 becomes positive.

We define $\xi = \frac{1}{\sqrt{6}}(0, 1, -1, 1, -1, 1, -1)^\top$. Since $J(\boldsymbol{n}_d)\xi = \lambda_3\xi$ holds, ξ is the eigenvector for λ_3.

SB. Proofs in Appendix C

SB.1. Proof of Lemma 7

$V(\ell, 0) > V(\ell, i)$ ($\forall \ell \in \mathcal{L}$, $\forall i \in \mathcal{P}$) holds if and only if market pattern (F) holds. By Eq. (4), this inequality is equivalent to the following inequalities:

$$\begin{cases} \sigma_{-1}\ln(n_0/n_j) > t((D,0,x),0) - t((D,i,x),j) \\ \sigma_{-1}\ln(n_0/n_j) > t((S,0,x),0) - t((S,i,x),j) \end{cases} \quad \forall i, j \in \mathcal{P},\ \forall x \in X. \tag{SB1}$$

By Eq. (9), one of the inequalities in (SB1) is equivalent to the following inequalities:

$$\begin{aligned} &\sigma_{-1}\ln(n_0/n_j) > t((D,0,x),0) - t((D,i,x),j) \quad \forall i,j \in \mathcal{P}, \forall x \in X \\ &\quad \Leftrightarrow \sigma_{-1}\ln(n_0/n_j) > \phi(2x-1) \quad \forall j \in \mathcal{P}, \forall x \in X \\ &\quad \Leftrightarrow \sigma_{-1}\ln(n_0/n_j) \geq \phi \quad \forall j \in \mathcal{P}. \end{aligned} \tag{SB2}$$

On the other hand, by Eq. (10), the other inequality is equivalent to the followings:

$$
\begin{aligned}
&\sigma_{-1}\ln\left(n_0/n_j\right) > t((S,0,x),0) - t((S,i,x),j) \quad \forall i,j \in \mathcal{P},\, \forall x \in X \\
&\quad \Leftrightarrow \sigma_{-1}\ln\left(n_0/n_j\right) > \phi + \tau\left(\frac{1}{2} - \left|x - \frac{1}{2}\right| - \min\left\{x, 1-x\right\}\right) \quad \forall j \in \mathcal{P},\, \forall x \in X \\
&\quad \Leftrightarrow \sigma_{-1}\ln\left(n_0/n_j\right) > \phi \quad \forall j \in \mathcal{P}. && \text{(SB3)}
\end{aligned}
$$

By (SB2) and (SB3), (SB1) is equivalent to (C2).

SB.2. Proof of Lemma 9

We rewrite inequalities (C9)–(C12) to concisely show our proof:

$$-\phi < \sigma_{-1}\ln\left(n_0/n_1\right) < \phi - \tau, \tag{SB4}$$

$$-\phi < \sigma_{-1}\ln\left(n_0/n_3\right) < \phi - \tau, \tag{SB5}$$

$$-\phi < \sigma_{-1}\ln\left(n_0/n_5\right) < \phi, \tag{SB6}$$

$$0 < \sigma_{-1}\ln\left(n_1/n_3\right) < 2\tau, \tag{SB7}$$

$$0 < \sigma_{-1}\ln\left(n_3/n_5\right) < 2\tau, \tag{SB8}$$

$$0 < \sigma_{-1}\ln\left(n_1/n_5\right) < 2\tau, \tag{SB9}$$

$$\tau < \sigma_{-1}\ln\left(n_1/n_2\right), \tag{SB10}$$

$$\tau < \sigma_{-1}\ln\left(n_3/n_4\right), \tag{SB11}$$

$$\tau < \sigma_{-1}\ln\left(n_1/n_6\right). \tag{SB12}$$

($\Rightarrow$) We check that (SB4)–(SB12) hold when market pattern (P) holds. In other words, using (C4)–(C8), we prove that inequalities (SB4)–(SB12) hold. First, since $\mathcal{M}_0 \cap \mathcal{M}_1 = \{(D,1,t_1),(D,2,t_2),(D,6,t_6)\}$ holds by (C4) and (C5), we can obtain t_1, t_2 and t_6:

$$t_1 = \frac{1}{2}\left[\frac{\ln\left(n_0/n_1\right)}{\phi(\sigma-1)} + 1\right],\; t_2 = \frac{1}{2}\left[\frac{\ln\left(n_0/n_1\right)}{\phi(\sigma-1)} + 1 + \frac{\tau}{\phi}\right],\; t_6 = \frac{1}{2}\left[\frac{\ln\left(n_0/n_1\right)}{\phi(\sigma-1)} + 1 + \frac{\tau}{\phi}\right].$$

Since $0 < t_1, t_2, t_6 < 1$, we obtain (SB4). Similarly, using (C4), (C6) and (C7), we obtain (SB5) and (SB6).

On the other hand, since $\mathcal{M}_1 \cap \mathcal{M}_3 = \{(S, 2, T_2)\}$ holds by (C5) and (C6), we can obtain $T_2 = \sigma_{-1}(2\tau)^{-1}\ln(n_1/n_3)$. Since $0 < T_2 < 1$, we obtain (SB7). Similarly, using (C5)–(C7), we obtain (SB8) and (SB9).

Next, since $\mathcal{S}_2(\,(0, T_2]\,) \subset \mathcal{M}_1$ holds by (C5), we obtain the following inequality:

$$V((S, 2, x), 1) > V((S, 2, x), 2) \quad \forall x \in (0, T_2]. \tag{SB13}$$

Using (SB13), we can obtain (SB10):

$$\begin{aligned}(SB13) &\Leftrightarrow \sigma_{-1}\ln n_2 - \tau x < \sigma_{-1}\ln n_1 - \tau(x+1) \quad \forall x \in (0, T_2] \\ &\Rightarrow \tau < \sigma_{-1}\ln(n_1/n_2)\,.\end{aligned}$$

Similarly, using (C5)–(C7), we can obtain (SB11) and (SB12).

($\Leftarrow$) We check that market pattern (P) holds when (SB4)–(SB12) hold. We first prove $\mathcal{M}_2 = \mathcal{M}_4 = \mathcal{M}_6 = \emptyset$ (i.e., (C8)). We prove $\mathcal{M}_2 = \emptyset$. The following holds by Eq. (10):

$$t(\ell, 1) - t(\ell, 2) \leq \tau \quad \forall \ell \in \cup_{m \in \mathcal{P}} S_m(X). \tag{SB14}$$

Using (4), (SB10) and (SB14), we obtain

$$V(\ell, 1) - V(\ell, 2) \geq \sigma_{-1}\ln(n_1/n_2) - \tau > \tau - \tau = 0 \quad \forall \ell \in \cup_{m \in \mathcal{P}} S_m(X). \tag{SB15}$$

On the other hand, for any $i \in \mathcal{P}$, the following holds by Eqs. (4) and (9):

$$V(\ell, 2) \in \{v_{a1}, v_{a2}\} \quad \forall \ell \in D_i(X), \tag{SB16}$$

where $v_{a1} = \sigma_{-1}\ln n_2 - \phi(1+x) + V_D, v_{a2} = \sigma_{-1}\ln n_2 - \phi(1-x) - \tau L_{i2} + V_D$. By (SB4) and (SB10), the following holds:

$$V(\ell, 0) - v_{a1} = \sigma_{-1}\ln(n_0/n_2) + \phi > \tau > 0 \quad \forall \ell \in D_i(X). \tag{SB17}$$

Moreover, since $V(\ell, 1) \geq \sigma_{-1}\ln n_1 - \phi(1-x) - \tau L_{i1} + V_D$ ($\forall \ell \in D_i(X)$) holds by (9), the following holds by (SB10):

$$V(\ell, 1) - v_{a2} \geq \sigma_{-1}\ln(n_1/n_2) + \tau L_{i2} - \tau L_{i1} > 0 \quad \forall \ell \in D_i(X). \tag{SB18}$$

Table SB1: The list of the indirect utilities for $\ell \in \cup_{m\in\mathcal{P}} S_m(X)$.

	$V(\ell,1)$	$V(\ell,3)$	$V(\ell,5)$
$\ell \in S_1(X)$	$\beta_1 - \tau x$	$\beta_2 - \tau(2-x)$	$\beta_3 - \tau(2+x)$
$\ell \in S_2(X)$	$\beta_1 - \tau(1+x)$	$\beta_2 - \tau(1-x)$	$\beta_3 - \tau(3-x)$
$\ell \in S_3(X)$	$\beta_1 - \tau(2+x)$	$\beta_2 - \tau x$	$\beta_3 - \tau(2-x)$
$\ell \in S_4(X)$	$\beta_1 - \tau(3-x)$	$\beta_2 - \tau(1+x)$	$\beta_3 - \tau(1-x)$
$\ell \in S_5(X)$	$\beta_1 - \tau(2-x)$	$\beta_2 - \tau(2+x)$	$\beta_3 - \tau x$
$\ell \in S_6(X)$	$\beta_1 - \tau(1-x)$	$\beta_2 - \tau(3-x)$	$\beta_3 - \tau(1+x)$

Notes: $\beta_1 = \sigma_{-1}\ln n_1 + V_D$; $\beta_2 = \sigma_{-1}\ln n_3 + V_D$; $\beta_3 = \sigma_{-1}\ln n_5 + V_D$.

By (SB15)–(SB18), either $V(\ell,2) < V(\ell,0)$ or $V(\ell,2) < V(\ell,1)$ holds ($\forall \ell \in \mathcal{L}$), which implies $\mathcal{M}_2 = \emptyset$. Similarly, by (SB4), (SB5), (SB11) and (SB12), $\mathcal{M}_4 = \mathcal{M}_6 = \emptyset$ holds.

Next, we prove (C4)–(C7). Note that $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_3, \mathcal{M}_5$ are determined by the relationship only among $V(\ell,0), V(\ell,1), V(\ell,3)$ and $V(\ell,5)$ because $\mathcal{M}_2 = \mathcal{M}_4 = \mathcal{M}_6 = \emptyset$.

We focus on the market areas on the ring road (i.e., $\ell \in \cup_{m\in\mathcal{P}} S_m(X)$). Using (4), (10), and (SB4), we can obtain the following inequality:

$$V(\ell,0) < V(\ell,1) \quad \forall\, \ell \in \mathcal{S}_1(X) \;\cup\; \mathcal{S}_2(\,(0,1/2]\,) \;\cup\; \mathcal{S}_5(\,[1/2,1)\,) \;\cup\; \mathcal{S}_6(X). \tag{SB19}$$

Similarly, by (SB5) and (SB6), the followings hold:

$$V(\ell,0) < V(\ell,3) \quad \forall\, \ell \in \mathcal{S}_2(\,[1/2,1)\,) \;\cup\; \mathcal{S}_3(X) \;\cup\; \mathcal{S}_4(\,(0,1/2]\,), \tag{SB20}$$

$$V(\ell,0) < V(\ell,5) \quad \forall\, \ell \in \mathcal{S}_4(\,[1/2,1)\,) \;\cup\; \mathcal{S}_5(\,(0,1/2]\,). \tag{SB21}$$

By (SB19)–(SB21), $\mathcal{M}_0 \cap (\cup_{m\in\mathcal{P}} S_m(X)) = \emptyset$ and $\cup_{m\in\mathcal{P}} S_m(X) \subset \mathcal{M}_1 \cup \mathcal{M}_3 \cup \mathcal{M}_5$ hold.

We compare $V(\ell,1), V(\ell,3)$ and $V(\ell,5)$ for $\ell \in \cup_{m\in\mathcal{P}} S_m(X)$. These functions are shown in Table SB1. By the results in Table SB1 and (SB7)–(SB9), there exist $T_2, T_4, T_5 \in X$ such that

$$\mathcal{S}_1(X) \;\cup\; \mathcal{S}_2(\,(0,T_2]\,) \;\cup\; \mathcal{S}_5(\,(T_5,1)\,) \;\cup\; \mathcal{S}_6(X) \subset \mathcal{M}_1, \tag{SB22}$$

$$\mathcal{S}_2(\,(T_2,1)\,)\ \cup\ \mathcal{S}_3(X)\ \cup\ \mathcal{S}_4(\,(0,T_4)\,) \subset \mathcal{M}_3, \tag{SB23}$$

$$\mathcal{S}_4 \times (\,(T_4,1)\,)\ \cup\ \mathcal{S}_5(\,(0,T_5)\,) \subset \mathcal{M}_5. \tag{SB24}$$

Next, we focus on the market areas on the radial roads (i.e., $\ell \in \cup_{m\in\mathcal{P}} D_m(X)$). By (SB4)–(SB6) and (SB22)–(SB24), a similar argument to the derivation of (SB16)–(SB18) shows that there exist $t_i \in X$ ($\forall i \in \mathcal{P}$) such that

$$\cup_{m=1}^{6}\ \mathcal{D}_m(\,(0,t_i]\,) \subset \mathcal{M}_0, \tag{SB25}$$

$$\mathcal{D}_1(\,[t_1,1)\,)\ \cup\ \mathcal{D}_2(\,[t_2,1)\,)\ \cup\ \mathcal{D}_6(\,[t_6,1)\,) \subset \mathcal{M}_1, \tag{SB26}$$

$$\mathcal{D}_3(\,[t_3,1)\,)\ \cup\ \mathcal{D}_4(\,[t_4,1)\,) \subset \mathcal{M}_3, \tag{SB27}$$

$$\mathcal{D}_5(\,[t_5,1)\,) \subset \mathcal{M}_5. \tag{SB28}$$

(SB22)–(SB28) are equal to (C4)–(C7) .