Macroeconomic General Constrained Dynamic models (GCD models)

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Abstract

In economics balance identities as e.g. C+K'-Y(L,K) = 0 must always apply. Therefore, they are called constraints. This means that variables C,K,L cannot change independently of each other. In the general equilibrium theory (GE) the solution for the equilibrium is obtained as an optimisation under the above or similar constraints. The standard method for modelling dynamics in macroeconomics is DSGE. Dynamics in DSGE models result from the maximisation of an intertemporal utility function that results in the Euler-Lagrange equations. The Euler-Lagrange equations are differential equations that determine the dynamics of the system. In Glötzl, Glötzl, und Richters (2019) we have introduced an alternative method to model dynamics, which is a natural extension of GE theory. It is based on the standard method in physics for modelling dynamics under constraints. We therefore call models of this type "General Constrained Dynamic (GCD)" models. In this paper we apply this method to macroeconomic models of increasing complexity. The target of this labour is primarily to show the methodology of GCD models in principle and why and how it can be useful to analyse the macroeconomy with this method. Concrete economic statements play only a subordinate role. All calculations, even for GCD models of any complexity, can be easily performed with the open-source program GCDconfigurator.

Contents

1. Introduction

Recently, there has been a renewed interest in alternative approaches to macroeconomics. In Zaman (2020) four different methodological principles are presented which lie outside the framework of the conventional approach. One of these concepts is called GCD (General Constrained Dynamics) and is based on the standard method of physics for modelling a dynamic under constraints. It can be seen as a natural extension of the GE theory for modelling dynamics in economics and can be thought of as an alternative to DSGE. The method was first introduced in Glötzl (2015) under the name Newtonian Constrained Dynamics, a name that was later changed to General Constrained Dynamics. The principles, many references and an application to the microeconomic Edgeworth box model are presented in Glötzl, Glötzl, und Richters (2019). In Richters und Glötzl (2020) it is shown that SFC models (stock flow consistent models (Godley und Lavoie 2012) can be understood as special GCD models. In (Richters 2021) a more complex macroeconomic model is used to show that GCD models converge to the classical equilibrium solution under some assumptions.

The aim of this paper is to show how macroeconomic GCD models can be built in a systematic way and how they can be used for macroeconomic analysis. In particular, we want to point out that all calculations for all GCD models (with non-intertemporal utility functions) can be performed easily and conveniently with the open-source program GCDconfigurator, which is published in GitHub (Glötzl und Binter 2022).

As intertemporal utility functions are essential in many applications and in DSGE models only intertemporal utility functions are used, it is essential to extend the GCD framework to intertemporal utility functions as well. The principles, how intertemporal utility functions can be incorporated into the framework is laid out in Glötzl (2022c).

DSGE models are typically used to analyse economic shocks. Therefore, another paper (Glötzl 2022a) describes how any type of economic shock, e.g., demand, supply or price shocks, can be modelled with GCD.

Non-intertemporal GCD models and intertemporal GCD models can be seen as an essential contribution to solving problem 8 of the 18 major problems of dynamics listed by Stephen Smale (Smale 1991; Smale Institute 2003).

In chapter 2 we give a literature review of the modelling of economic dynamics under constraints, which is essentially based on Richters (2021). Many further references can be found mainly in Glötzl, Glötzl, und Richters (2019).

In Chapter 3 we present the general structure of GCD models. A GCD model consists of agents (households, firms, banks, government...) that produce or buy goods. The behaviour of the different variables (stocks and flows) is described by a differential algebraic system of equations derived from the utility functions for each agent and the ability of each agent to influence the evolution of the variables over time. Furthermore, a set of algebraic equations, called constraints, describes the constraints of possible solutions in the same way as in general equilibrium models. In particular, equilibrium identities are always an important constraint. These guarantee that all GCD models also have the important SFC property ("Selfconsistant Stocks and Flows").

In chapter 4 we give possible utility functions for households, firms, commercial banks, central banks and governments.

In chapter 5 we discuss what GCD models can be used for and why they are also helpful for the theoretical understanding of economics. On the one hand, they are suitable for many practical applications, such as the analysis of business cycles or the influence of economic policy instruments. On the other hand, they provide the theoretical insight that different economic theories essentially only differ in terms of different assumptions about the economic power of agents, which are described by the so-called "power factors". A continuous change in the power factors results in a smooth transition from one theory to another. In this sense, GCD models can also be understood as a meta-methodology for economic models. In this chapter we also provide an overview of what further research tasks still need to be done in connection with GCD models in the future.

In Chapter 5 we describe the basic structure of utility functions and give examples of possible utility functions for the following agents: Household, Firm, Bank, Central Bank and State.

The solutions of the system can be calculated numerically e.g. with Mathematica. In chapter 6 we refer to the open-source program GCDconfigurator, which facilitates the derivation of the differential algebraic equation system for any GCD model with arbitrary agents, arbitrary (non-intertemporal) utility functions, arbitrary power factors and arbitrary constraints. GCDconfigurator is freely accessible via GitHub (Glötzl und Binter 2022) and can be downloaded under

https://github.com/lbinter/gcd

All Mathematica program codes used for calculations of the various GCD models can be downloaded under

https://www.dropbox.com/sh/npis47xjqkecggv/AAAMzCVhmhDYIIhoB5MfATFya?dl=0

In chapter 7 we present the simplest macroeconomic model A1, which consists of 1 firm, 1 good and 1 household. Even this simple model shows business cycles for a certain selection of the underlying parameters. We discuss the typical characteristics of these business cycles.

In chapter 8 we extend the A1 model to the A2 model, which has 2 objectives. First, we show how financial assets and their counterpart, financial liabilities, can be modelled in GCD models. Second, we show how the main constraints of a given model can be derived in a systematic way from the graph of the model or from the transaction matrices for each commodity.

In chapter 9 we introduce in model B1 the banking system consisting of 1 central bank and 1 commercial bank. We model and discuss the effects of a central bank monetary policy compared to an interest rate policy and the difference between exogenous and endogenous money.

In chapter 10 we model the behaviour of the central bank in terms of the Taylor rule in model B2.

In chapter 11 we introduce the government as an agent in model C1 in order to be able to analyse, for example, various fiscal policy measures of the government.

In chapter 12 we create model C2 by describing the behaviour of the central bank in model C1 by means of the Taylor rule.

In chapter 13 we extend model C2 to the comprehensive model D2.

In chapter 14 we use appropriate GCD models to explain the theoretical insight that different economic theories differ only in terms of different power factors.

In chapter 15 we describe another simple model for describing environmental impacts.

In chapter 16 we give a summary of the main features of GCD models, their advantages and disadvantages for describing economics, and a large list of further research to be done.

2. Literature review on the modelling of dynamics under constraints

This literature review is essentially based on Richters (2021). In general, a dynamic economic model is described by agents and variables that can correspond to any stock or flow of goods, resources, financial liabilities or other variables or parameters such as prices or interest rates. The behaviour of these variables is described by equations of behaviour.

The behaviour of these variables may be restricted by economic constraints described by additional equations. In particular, all balance sheet identities are subject to such constraints. However, constraints can also be relationships "which by definition apply" (Allen 1982, 4). In material flow analysis (Brunner und Rechberger 2004) these constraints also include laws of nature such as the conservation of mass and energy ("first principles" of chemistry and thermodynamics). Input-output relationships or production functions imply certain technological constraints, while budget constraints are derived from the behavioural assumption that no one gives money away without appropriate remuneration. Respect for identities is "the beginning of wisdom" in economics, but they must not be "misused to imply causes" (Tobin 1995, 11).

In general, the introduction of additional constraints to the behavioural equations can lead to the system of equations becoming overdetermined and thus unsolvable. The schools of economic thought differ in how they make this system of equations solvable (Sen 1963; Taylor 1991) , a topic which is discussed in Chapter 14

In most general equilibrium models, each agent fully controls and voluntarily adjusts all stocks and flows directly affecting him or her (such as individual working hours or savings), resulting in various individual first-order conditions. The fulfilment of the systemic constraints of market exchange can only be ensured if prices are adjusted in such a way that all individual plans are compatible with each other (neoclassical closure). Interaction via price signals, restrictions by other agents or system characteristics can be fully anticipated by the agents (Arrow und Hahn 1971).

The core of most Dynamic Stochastic General Equilibrium (DSGE) models is based on a representative agent with rational expectations, which solves an intertemporal optimisation problem under consideration of the constraints. The properties of utility and production functions, the Euler equation describing the dynamics of the system, and the transversality condition as an infinite time boundary condition guarantee that a clear and stable equilibrium path exists. External shocks in combination with various resistances that slow down the return to equilibrium can cause deviations from this optimum (Christiano, Eichenbaum, und Trabandt 2018; Lindé 2018; Becker 2008; Colander 2009; Kamihigashi 2008). While more recent DSGE models also include some heterogeneity between households and firms (Kaplan, Moll, und Violante 2018; Christiano, Eichenbaum, und Trabandt 2018), many aspects of heterogeneity must usually be left out in order to apply this approach at all (Galí 2018, 101).

Each optimisation approach requires a single function to be optimised. Therefore, the utility functions of a society of utility maximizers need to be aggregated into a single social welfare function. Aggregation is possible if and only if demand is independent of the distribution of income between agents (Gorman 1961; Stoker 1993; Kirman und Koch 1986; Kirman 1992), which (Rizvi 1994, 363) describes as an "extremely special situation". These mathematical reasons limit the admission of broader heterogeneity and social influences into DSGE models.

Keynesian disequilibrium models deviate from the assumption that price adjustments can clean up markets sufficiently quickly. However, the deviation from equilibrium assumptions implies that ex-ante (planned) behaviour does not necessarily meet the economic constraints. The (actual) ex-post dynamics are influenced both by systemic constraints and by the actions of others. Demand and supply do not necessarily coincide, and terms such as "forced saving" or "involuntary unemployment" (Barro und Grossman 1971) imply that agents cannot have complete control over the variables that influence them. For example, in some Keynesian imbalance models, demand is limited by insufficient supply or otherwise depending on market conditions (Benassy 1975; Malinvaud 1977).

In contrast, some post-Keynesian models consider the labour market to be purely demand-driven, and employees have no influence on working hours: The constraints that guarantee the consistency of stocks are satisfied by simply omitting an equal number of behavioural equations (Godley und Lavoie 2012; Caverzasi und Godin 2015). This method, namely simply dropping behavioural equations ("drop closure") to make the system of equations solvable, is justified if and only if the stocks or flows are not determined by the agents but only by the constraints (for a criticism see Richters und Glötzl (2020).

Agent-based models (ABM) assume that individuals cannot solve infinitely dimensional optimisation problems, but instead make use of limited rationality, which is often modelled as a sequence of simple rules. Interactions between heterogeneous agents are important beyond market prices, and social interaction, social norms, power relations or institutions influence economic decisions. Compared to selfish utility maximizers, this corresponds to a broader version of methodological individualism (Gallegati und Richiardi 2009). ABM describe how quantities and prices can converge to a (statistical) equilibrium, but discontinuities, tipping points, lock-ins or path dependencies can also be investigated (Kirman 2010). ABM lack a common core and, depending on the economic assumptions, consistency of stocks and flows is ensured by price adjustments, auctions, matching algorithms or quantity rationing (Tesfatsion 2006; Gintis 2007; Page 2008; Gallegati und Richiardi 2009; Ballot, Mandel, und Vignes 2015; Riccetti, Russo, und Gallegati 2015; Haldane und Turrell 2018). In any case, the constraints of macroeconomic accounting must be taken into account, including in the modelling of bankruptcies or the entry and exit process of companies (Caiani u. a. 2016; Caverzasi und Russo 2018).

In order to avoid the above-mentioned problems, we have introduced in Glötzl, Glötzl, und Richters (2019) a different method for modelling dynamics as an alternative to DSGE and ABM models. This method is a natural extension of GE theory and is based on the standard method in physics for modelling dynamics under constraints. We therefore call this method "General Constrained Dynamics GCD". The GCD method is a "closure" method to solve an overdetermined system of equations (due to the additional constraints) by introducing additional Lagrange multipliers. It can also be understood as a method to transfer the concept of

Lagrange multipliers from optimisation problems under constraints to dynamic systems under constraints. This is done in analogy to how it is done in classical mechanics. In contrast, if the behaviour in GE models is referred to as "total utility maximisation", the behaviour in GCD models can best be described as "individual utility optimisation".

In comparison to DSGE, GCD models initially go back two steps and do without intertemporal optimisation and stochastic shocks. However, GCD models do not require the restriction that all utility functions can be aggregated to a social welfare function. GCD models describe the interaction of limited rational agents that under economic constraints exert economic forces to improve their individual situation (gradient increase). The processes of trade and price adjustment take place simultaneously and can converge towards equilibrium. However, they do not necessarily converge towards equilibrium in every case.

In this article we apply the GCD-Method to macroeconomic models of increasing complexity. The aim of this article is to show how GCD models are constructed in principle and why and how it can be useful to analyse macroeconomics with this method.

All calculations, even for arbitrarily complex GCD models (with non-intertemporal utility functions), can be easily performed with the open-source program GCDconfigurator (Glötzl und Binter 2022) which can be downloaded under

https://github.com/lbinter/gcd

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In further contributions we show how the above-mentioned limitations can be overcome compared to DSGE models. The basic ideas of how GCD models can be adapted to intertemporal utility functions are shown in Glötzl (2022c) and how any kind of economic shock, e.g. demand, supply or price shocks, can be modelled with GCD is shown in Glötzl (2022a).

3. The principle set up of GCD models

3.1. The model graph

It has proved to be extremely helpful to present each model in the form of a model graph. This provides an immediate overview of the agents, stock variables and flow variables. Using model A2 we also show how the constraints can be systematically determined from the model graph (see chapter 8.2.). Another possibility for the systematic representation of a model results from specifying the corresponding transaction matrices. This method is often used to describe SFC models (stock flow consistent models). Constraints can also be derived from this in a systematic way (see Chap. 8.3). However, we prefer the description of a model with model graphs, as long as the models are not so complex that the graphs become unclear.

In detail a GCD model consists of the following elements:

3.2. Agents

In principle, any number of any agents is possible, e.g:

- One or more households
- One or more companies
- One or more banks
- A central bank
- One State
- Any other agents

3.3. Goods

Agents exchange goods (flows) and/or store them (stocks) or create or destroy them. In GCD models it is useful to consider not only money but also all other goods that are usually exchanged for money at a certain price.

In principle any number of any goods is possible, e.g:

- Money
- Goods
- Services
- Labour
- debt notes (promissory notes)

  (receivables = positive stock of debt notes, liabilities = negative stock of debt notes). The immediate price of a debt note is usually 1 (e.g.: for lending 100 € you get 100 debt notes). However, debt notes usually trigger corresponding interest payments.

- Energy
- Raw materials
- etc.

3.4. Variables

All stocks, all flows and all creation and destruction processes are represented by time-dependent variables.

It is important to distinguish between 2 types of variables: Differentially defined variables and algebraically defined variables.

We first assume that only differentially defined variables occur. This means that the behavioural equations of all variables that appear in the utility functions are given by the differential equations of the general GCD model equations in the form **Fehler! Verweisquelle konnte nicht gefunden werden.**. We therefore refer to these variables as differentially defined variables. However, in the models variables are also possible for which the behavioural equations are not given by a differential equation but by an algebraic equation, e.g. by assuming a certain production function

$$Y(t) = \beta L^{\alpha} K^{(1-\alpha)}$$

or a specific rule for determining the amount of household income tax

$T^H(t) = 0.3wL$

In chapter 3.11 the algebraically defined variables are explained in more detail.

3.5. Constraint conditions

For every agent and every good, the following conservation equation, which is called a constraint, must necessarily apply:

Incoming goods - outgoing goods + production of goods - destruction of goods - - change in stock of goods = 0

E.g. for a company that produces a number $Y(t)$ of machines, designate

$C(t)$ the part of the machines which are sold,
$S(t)$ the stock in the warehouse,
$K(t)$ the number of machines used for production, i.e. the real capital stock and
$I(t)$ the investment, i.e the part of production used for its own further production, the following constraint holds

$$Y(t) - C(t) - S'(t) - I(t) = Y(t) - C(t) - S'(t) - K'(t) = 0$$

We avoid the formulation of this constraint by valuation at market prices p

$$pY(t) - pC(t) - pS'(t) - pI(t) = 0$$

because only the term $pC(t)$ corresponds to a real flow, namely the flow of money when machines are sold, whereas the other terms correspond to a flow of values. However, since valuations can change very easily, the conservation equation for values generally applies only to a very limited extent and must be applied with great caution.

In addition to the above-mentioned constraints, which are derived from the conservation equations for each good for each agent, there are also other constraints imposed by model assumptions, such as the assumption that all consumer goods are consumed immediately and not stored.

Model graphs in the form of flow charts and/or transaction matrices for all goods are very helpful in establishing the constraints. We show model graphs in the form of flow charts for each model. We explain the use of the corresponding transaction matrices with an example in chapter 8.3.

Note: The conservation equations for GCD models are closely related to the conservation equations of physics and chemistry, e.g:

1st law of thermodynamics (conservation of energy)
1st law of chemistry (conservation of mass)

Since debts (liabilities) and accounts (receivables) always arise simultaneously and in the same amount, it applies that in a closed system the sum of debts (liabilities) must always be the same as the sum of accounts (receivables). This analogy to the conservation laws of physics makes it reasonable to call this fundamental relationship for a monetary economy "1st law of economics" (Glötzl 1999; 2009)

3.6. Utility functions for each agent

The behaviour of an agent is described by its utility function. These utility functions are not subject to any restrictions and can basically depend on all variables (stocks and flows) and any parameters.

3.7. Power factors for each agent for each variable

An agent's interest in changing variables does not per se lead to actual change, because the agent must also have the power or opportunity to actually implement its desire for change. This is described by the so-called power factor μ_x^A, which can assume values between 0 and ∞. A high-power factor leads to a rapid temporal adjustment of the variables. The power factors in some sense can therefore also be interpreted as speed adjustment factors.

3.8. GCD model equations for the simple case (utility functions and constraints depend only on (x_1, x_2))

3.8.1. Ex-ante equations of motion

We explain the principle for 2 agents A, B and 2 variables x_1, x_2.

The utility functions of A, B are $U^A(x_1, x_2), U^B(x_1, x_2)$. The interest of A is to change x_1, x_2 so that the increase of his utility function is maximal. This is given, if the change of x_1, x_2 is done in the direction of the gradient of $U^A(x_1, x_2)$ $U^A(x_1, x_2)$, i.e.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \textit{proportional} \quad \begin{pmatrix} \dfrac{\partial U^A}{\partial x_1} \\ \dfrac{\partial U^B}{\partial x_1} \end{pmatrix}$$

The interest of A in a change of the variables does not lead alone to an actual change, because the household must have also the power and/or possibility of actually implementing its change desire. For example, a household cannot or can only partially enforce its additional consumption desire, e.g., to go to the cinema or go on vacation, because it is possibly quarantined or the borders are closed. This limitation of the possibility to enforce his consumption change requests is described by a (possibly time-dependent and endogenously determined) "power factor" μ_C^H. In general, the change request for each of the variables is described by "power factors" $\mu_{x_1}^A, \mu_{x_2}^A, \mu_{x_1}^B, \mu_{x_2}^B$. Considering the power factors, the following applies to the change of x_1, x_2 (due to the interest of A and the power of A to enforce this interest)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \textit{proportional} \quad \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix}$$

Just as A has an interest, to change x_1, x_2, also B has an interest to change these two variables. The actual change is therefore the result of the two individual efforts to change, weighted with the power factors. We therefore refer to this behaviour as "**individual utility optimisation**".

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} \qquad \text{<3.1>}$$

In case there is a "master utility function" MU such that

$$\begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} \qquad \text{<3.2>}$$

the two utility functions can be **aggregated**. Then

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} \qquad \text{<3.3>}$$

Equation <3.3> describes the temporal change of the variable along the gradient of MU . If MU is convex, (x_1, x_2) converges to the maximum value of MU , i.e.

$$\lim_{t\to\infty}(x_1(t), x_2(t)) = (x_1^{\max}, x_2^{\max}) \qquad with\ MU(x_1^{\max}, x_2^{\max}) = maximal$$

Define the overall utility function $GU = U^A + U^B$. If the overall utility function equals the master utility function, i.e. $GU = MU$, we therefore refer to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial GU}{\partial x_1} \\ \dfrac{\partial GU}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial (U^A + U^B)}{\partial x_1} \\ \dfrac{\partial (U^A + U^B)}{\partial x_2} \end{pmatrix} \qquad \text{<3.4>}$$

as **"overall utility maximisation"**.

These equations of motion <3.1> resp. <3.4> describe the dynamics of (x_1, x_2) under the condition that there are no constraints that restrict the dynamics. It is therefore referred to as the **ex-ante equation of motion**.

3.8.2. Ex-post equations of motion

3.8.2.1. Vertical constraint forces

If a constraint

$Z(x_1, x_2) = 0$

has to be fulfilled, an additional constraint force f^Z has to be added to the ex-ante force

$$x_i' = \sum_{j=1}^{J} \mu_i^j f_i^j + f^Z \qquad i = 1,2,\ldots,I \qquad \text{<3.5>}$$

to ensure the constraint Z to be fulfilled at all times. In physics, this constraint force f^Z is perpendicular to the constraint at all times due to the so-called d'Alembert principle, i.e.

$$f^Z(x_1,x_2) = \begin{pmatrix} f_1^Z(x_1,x_2) \\ f_2^Z(x_1,x_2) \end{pmatrix} = \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<3.6>}$$

We therefore refer to this type of constraint forces as **"vertical constraint forces"**. The time-dependent factor $\lambda = \lambda(t)$ is called Lagrange multiplier, as in the case of optimisation under constraints.

Vertical constraint forces can also be characterised by the following equivalent principles. This is because the theorem (Glötzl 2018) holds that the following principles are equivalent:

(1) **d'Alembert's principle** (constraint forces do no work)

(2) **vertical constraint forces** (constraint forces are perpendicular to the manifold of constraint conditions)

(3) **Gaussian principle of least constraint** (those constraint forces f^{Z_i} occur for which $\|f^{Z_i}\| \rightarrow minimal$)

(4) **unnamed principle**

If x is a solution of

$$x' = f(x) + f^Z(x)$$
$$0 = Z(x)$$

then f^Z satisfies the unnamed principle:$\Leftrightarrow \frac{d\|x'\|}{dt} = \frac{\langle x', f\rangle}{\|x'\|}$

Note: If one of the equivalent principles is satisfied, then the constraint force has no effect on $\|x'\|$ but only on the direction of x'. Note, however, that the inverse does not hold.

It is therefore plausible in many cases to model constraint forces in economics in an analogous way to physics in terms of d'Alembert's principle respectively as vertical constraint forces.

From <3.1> and <3.6> results the "equation of motion considering the constraint condition", called **ex-post equation of motion**:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B x_1,x_2}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<3.7>}$$
$$0 = Z(x_1,x_2)$$

If U^A, U^B can be aggregated to a master utility function MU , the equation of motion is as follows

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial MU(x_1,x_2)}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<3.8>}$$
$$0 = Z(x_1,x_2)$$

and if the master utility function MU is convex, (x_1,x_2) converge to the maximum value of MU under the constraint Z , i.e.

$\lim_{t\to\infty}(x_1(t), x_2(t)) = (x_1^{\max,Z}, x_2^{\max,Z})$ *with* $MU(x_1^{\max,Z}, x_2^{\max,Z}) = maximal\ under\ constraint\ Z$

and it holds that the dynamics at $(x_1^{\max,Z}, x_2^{\max,Z})$ is stationary, i.e.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} = 0 \qquad \text{<3.9>}$$

or equivalently

$$\begin{pmatrix} \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} = -\lambda \begin{pmatrix} \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} \qquad \text{<3.10>}$$

In general, for

J agents with the designations $j = 1, 2, ..., J$

I Variables with the designations x_i $i = 1, 2, ..., I$ $x = (x_1, x_2, ..., x_I)$

K Constraints with the designations Z^k $k = 1, 2, ..., K$ Z_k $k = 1, 2, ..., K$

the I **general GCD model equations for vertical constraint forces** are obtained analogously

$$\boxed{x_i' = \sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad i = 1, 2, ...I \qquad \text{<3.11>}}$$

If there is a "master utility function" MU such that

$$\sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} = \frac{\partial MU}{\partial x_i} \qquad i = 1, 2, ..., I \qquad \text{<3.12>}$$

the utility functions $U^j,\ j=1,2,...,J$ are called **aggregable**.

If $MU=\sum_{j=1}^{J}U^j$, the master utility function is called the total utility function. If the master utility function MU is convex, x converges to the maximum value of MU under the constraint conditions $Z^k,\ k=1,2,...,K$.

3.8.2.2. Other constraint forces

Another type of constraint force that can occur, especially in the case of a constraint force describing a limited resource, is a constraint force that is centrally directed to the origin. We therefore refer to this as a "**central constraint force**".

$$f^Z(x_1(t),x_2(t))=\begin{pmatrix} f_1^Z(x_1(t),x_2(t)) \\ f_2^Z(x_1(t),x_2(t)) \end{pmatrix}=\varphi(t)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \text{<3.13>}$$

A model for this are constraint forces such as occur in theoretical biology in the derivation of the so-called replicator equation (Glötzl 2022b). In biology, this model assumption of a central constraint force is equivalent to the assumption that in the struggle for limited resources, equally high death rates are triggered for all species.

Let us illustrate this with an example. A typical dynamic in biology is the initially independent exponential growth of 2 species A and B with birth rates b_A,b_B .

$$\begin{aligned} n_A' &= b_A n_A \qquad b_A \text{ "growth rate"} \\ n_B' &= b_B n_B \qquad b_B \text{ "growth rate"} \end{aligned} \qquad \text{<3.14>}$$

A constraint typical for biology is, for example, the assumption of limited resources. This can be given, for example, by a limitation of the food supply or also by a limitation of the habitat. This results in the sum of the number of absolute frequencies of the different species remaining constant. This is formally described by the constraint condition

$$Z(n_1,n_2,...)=\sum_i n_i - constant = 0$$

Assuming that the constraint condition triggers equally high death rates in both species, the differential algebraic equation system is obtained

$$\begin{aligned} n_A' &= b_A n_A - \varphi n_A \\ n_B' &= b_B n_B - \varphi n_B \\ Z(n_A, n_B) &= n_A + n_B - n = 0 \qquad n \text{ constant} \end{aligned} \qquad \text{<3.15>}$$

Assuming that A is twice as successful ("powerful") in the struggle for resources, the death rate for A would be half as high and thus the system of equations would be

$$\begin{aligned} n_A' &= b_A n_A - \varphi \frac{1}{2} n_A \\ n_B' &= b_B n_B - \varphi n_B \\ Z(n_A, n_B) &= n_A + n_B - n = 0 \qquad n \text{ constant} \end{aligned}$$

When applied to economic constraints, this can be interpreted as follows. Agents can have different powers to oppose constraints. For example, if raw materials are limited in total, it may be easier for some countries to obtain the necessary raw materials than for others.

In the most general case, different types of constraint forces can occur. Essential for the modeling is only that the constraint forces used must be linearly independent and multiplied by the respective Lagrange multiplier.

Note: In the case where not all constraint forces are vertical, x typically does not converge to the maximum value of MU under the constraints $Z^k, k = 1,2,...,K$, even if the master utility function is convex.

As a rule, it is sufficient to use purely vertical constraint forces. In the following, we will therefore always restrict ourselves to vertical constraint forces.

3.9. GCD model equations for the general case (utility functions and constraints also depend on antiderivatives and/or derivatives of (x_1, x_2))

3.9.1. Constraints depend on antiderivatives and/or derivatives

So far, we have assumed that the constraints depend only on x. However, the constraints can also depend on the antiderivatives $X=(X_1,X_2,...,X_I)$. This means, X_i is antiderivative of x_i, iff $X_i'=x_i$. The constraints can depend in principle, however, also on the time derivatives $x'=(x_1',x_2',...,x_I')$. In physics it is valid (Flannery 2011), that the constraint force always results from derivative with respect to the highest time derivative of x, i.e.

If $Z(...,X_i,...)$ then $f_i^Z=\frac{\partial Z}{\partial X_i}$ and

$$x_i'=\sum_{j=0}^{J}\mu_{x_i}^j\frac{\partial U^j}{\partial x_i}+\sum_{k=1}^{K}\lambda^k\frac{\partial Z^k(...,X_i,...)}{\partial X_i} \qquad i=1,2,...I \qquad \text{<3.16>}$$

If $Z(...,X_i,x_{i,}...)$ then $f_i^Z=\frac{\partial Z}{\partial x_i}$ and

$$x_i'=\sum_{j=0}^{J}\mu_{x_i}^j\frac{\partial U^j}{\partial x_i}+\sum_{k=1}^{K}\lambda^k\frac{\partial Z^k(...,X_i,x_i,...)}{\partial x_i} \qquad i=1,2,...I \qquad \text{<3.17>}$$

If $Z(...,X_i,x_i,x_i',...)$ then $f_i^Z=\frac{\partial Z}{\partial x_i'}$ and

$$x_i'=\sum_{j=0}^{J}\mu_{x_i}^j\frac{\partial U^j}{\partial x_i}+\sum_{k=1}^{K}\lambda^k\frac{\partial Z^k(...,X_i,x_i,x_i',...)}{\partial x_i'} \qquad i=1,2,...I \qquad \text{<3.18>}$$

We assume that this approach is also plausible in economics in the case of vertical constraints.

3.9.2. Utility functions depend on antiderivatives and/or derivatives

So far, we have assumed that utility functions only depend on x. But also, the utility functions can additionally depend on antiderivatives and derivatives of x. In these cases, both the antiderivatives $X=(X_1,X_2,...,X_I)$ and the derivatives $x'=(x_1',x_2',...,x_I')$ are to be considered as additional variables in their own right, i.e.

$$X = (X_1, X_2, ..., X_I) = (x_{I+1}, x_{I+2}, ..., x_{2I})$$
$$x' = (x'_1, x'_2, ..., x'_I) = (x'_{2I+1}, x'_{2I+2}, ..., x'_{3I})$$

In that case, the following additional constraints must be used

$$x'_{I+i} - x_i = 0 \qquad i = 1, 2, ..., I$$
$$x'_i - x_{2I+i} = 0 \qquad i = 1, 2, ..., I$$

3.10. Market forces

The behaviour of x_i is given by the general GCD model equation (for vertical constraint forces) for x_i <3.11>

$$x'_i = \sum_{j=0}^{J} \mu^j_{x_i} \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad i = 1, 2, ...I \qquad \text{<3.19>}$$

The right-hand side of <3.19>

$$\sum_{j=0}^{J} \mu^j_{x_i} \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i}$$

thus describes the **market forces** that lead to a change in x_i and is composed of 2 parts. The market forces that agents exert on x_i

$$\sum_{j=0}^{J} \mu^j_{x_i} \frac{\partial U^j}{\partial x_i}$$

and the market forces that the constraints Z^k exert on x_i. These are just the constraint forces

$$f^{Z^k}(x) = \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad k = 1, 2, ..., K$$

If for a particular i it holds that $\frac{\partial U^j(x)}{\partial x_i} = 0$, i.e. that the utility functions do not depend on $\mu^j_{x_1}$, or that the power factors $\mu^j_{x_1} = 0$, the general GCD model equation (for vertical constraint forces) reduces for x_i , to

$$x'_i = \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad i = 1, 2, ...I$$

In this case, the behaviour of x_i is determined exclusively by the constraint forces. Therefore, the constraint forces can also be called **"pure" market forces**,

3.11. Algebraically defined variables

So far we have assumed that the behavioural equations for all variables are given by differential equations in the form <3.11> to <3.18>. We therefore call these variables differentially determined variables. In the models, however, also variables are possible, with which the behavioural equations are not determined by a differential equation, but by an algebraic equation, e.g. by the assumption of a certain production function

$$Y(t) = \beta L^{\alpha} K^{(1-\alpha)}$$

or a specific rule for determining the amount of household income tax.

$$T^{H}(t) = 0.3 wL$$

We call these variables algebraically defined variables. These algebraic behavioural equations can often be seen as limit values of differential equations with infinitely large power factors. For example, the behaviour of the government in collecting income tax could be described by the following behaviour. It aims to collect 30% of the wage income of the household as a tax. If the tax paid is less than this, e.g. through tax evasion, the government will try to increase the collection of the tax. This behaviour can be modeled in the following way, for example:

Let $U^{G}(T^{H}) = -\frac{1}{2}(0.3 - T^{H})^{2}$ be the utility function of the Government G and Z any constraint, then results the behavioural equation

$$T^{H}\,' = \mu^{H}_{T^{H}} \frac{\partial U^{G}}{\partial T^{H}} + \lambda \frac{\partial Z}{\partial T^{H}} = \mu^{H}_{T^{H}} (0.3 wL - T^{H}) + \lambda \frac{\partial Z}{\partial T^{H}}$$

If the government has infinite power to prevent tax evasion, this results in

$$T^{H}{}' = \mu^{H}_{T^H}(0.3wL - T^H) + \lambda \frac{\partial Z}{\partial T^H} \qquad \Leftrightarrow$$

$$\Leftrightarrow \qquad \frac{T^{H}{}'}{\mu^{H}_{T^H}} = (0.3wL - T^H) + \frac{\lambda}{\mu^{H}_{T^H}} \frac{\partial Z}{\partial T^H}$$

$$\mathit{for}\ \mu^{H}_{T^H} \to \infty\ \mathit{results}$$

$$0 = (0.3wL - T^H) \qquad \Leftrightarrow$$

$$T^H = 0.3wL$$

The algebraic behavioural equation $T^H = 0.3wL$ can thus be interpreted as a differential behavioural equation with infinite power of the government.

In case of occurrence of algebraically defined variables, when forming partial derivatives of utility functions and constraints with respect to the differentially defined variables, it must be taken into account that the algebraic variables occurring in utility functions and constraints may also depend on differentially defined variables. It is best to insert the algebraically defined variables into the utility functions and constraints before the differential equations are formed.

3.12. Numerical solutions

In most cases, the differential algebraic systems of equations cannot be solved analytically, but only numerically.

3.12.1. Initial values

In ordinary differential equation systems of the 1st order, the initial values for all variables are freely selectable. In contrast to ordinary differential equation systems, not all initial values of the variables are freely selectable in differential algebraic equation systems. The reason for this is that the initial values must satisfy the differential equations and also the constraints.

If there are no time derivatives in the constraints and there are K linearly independent constraints, only $I - K$ initial values can be chosen freely. The other initial values result from the solution of the system of equations of the constraints. However, if the constraints are nonlinear an analytical solution is often not possible. In many practical applications, however, the situation is much more complex, especially if time derivatives of variables also occur in the constraints.

In the usual numerical programs for solving differential-algebraic equations, an algorithm is therefore built in, which calculates from a sufficiently large number of initial values, other possible initial values, which approximately fulfill the system of equations up to a certain tolerance. One therefore needs an understanding of the model and a certain amount of experience to determine suitable initial values.

3.12.2. Parameter selection

The parameters of a GCD model cannot be chosen arbitrarily either. For the system of equations, a solution does not have to exist for every combination of parameters or be stable over a longer period of time. Therefore, one also needs an understanding of the model and a certain experience for the selection of the values for the individual parameters.

3.12.3. Numerical solution methods

We make use of two solution methods within the framework of MATHEMTICA, namely NDSolve and Modelica. Since differential algebraic systems of equations have a much higher overall complexity than ordinary differential systems of equations, many different methods of numerical procedures are available in NDSolve.

By default, it is usually sufficient to use:

Method→Automatic

Sometimes you need:

Method→{"EquationSimplification"->"Residual"}

Sometimes one needs:

Method→{IndexReduction→ Automatic }

Sometimes one needs:

Method→{IndexReduction→{True, ConstraintMethod→Projection}}

May be in special cases also other methods must be used

For the stability of the solutions, one has to distinguish 2 cases:

- The model itself may become unstable after a certain time because, for example, certain variables become 0.
- The model is basically stable, but the numerical errors can lead to instabilities after a longer runtime.

4. Examples of possible utility functions

4.1. Household

For example, a household may have the following targets:

- **Consumption target**: he would like to consume. His desire to consume more is greater the less he is currently consuming or can consume, and his desire to consume even more is smaller the more he is already consuming.

- **Labour target**: he would like to work, but not too much and not too little.

- **Money management target** (cash management target): he always wants to have liquid funds, not too little, so that he can buy everything he wants to buy at the moment and not too much, because he does not get any interest for it and it would be more advantageous to lend the money to the bank against interest on savings. Therefore, the higher the interest on savings, the lower his money-holding target.

- **Receivables holding target** (savings target): he would like to hold assets in the form of receivables from the bank, the more the higher the savings interest.

The stated targets of the household can be expressed, for example, in the following utility function:

$$U^H(C^H, L^H, M^H, A^H) = \left(C^H\right)^{\gamma} - \left(\hat{L}^H - L^H\right)^2 - (\hat{M}^H - M^H)^2 + A^H$$

Variable:	C^H	*consumption*
	L^H	*labour*
	M^H	*money holding (liquid assets)*
	A^H	*claims on bank (savings)*
Parameter:	γ	$0 \le \gamma \le 1$
	$\hat{L}^H$	*targeted labour*
	$\hat{M}^H$	*targeted money holding, possibly depending on the interest rate*

4.2. Firm

A firm can have the following targets, for example:

- **Profit target**: The greater the profit, the greater the utility.

- **Warehousing target:** Warehousing causes costs and should therefore be as low as possible; on the other hand, it must not be too low, otherwise fluctuations in demand cannot be compensated.

- **Investment target**: The interest in investing depends (also!) on the level of interest rates on loans. If lending rates are 0 (or even negative due to possible investment incentives), as much is invested as is organisationally feasible. If lending rates rise, correspondingly less is invested.

The stated targets of the firm can be expressed, for example, in the following utility function.

$$U^F = profit^F - (\hat{S} - S)^2 - \left(invmax\left(1 - \theta(r + r_D)\right) - inv\right)^2$$

whereby the following "algebraically" defined variables are used *:*

$$Y := \beta L^a K^{1-a}$$

$$profit^F := pY - wL - (r + r_D)(-D^F) - DP = p\beta L^a K^{1-a} - wL - (r + r_D)(-D^F) - DP$$

$$invmax := inv\ K$$

this gives the dependence of the utility function on the "differentially" defined variable,

$$U^F(p,L,K,w,D^F,DP,S,inv) = profit^F - (\hat{S} - S)^2 - \left(invmax\left(1 - \theta(r_{leit} + r_D)\right) - inv\right)^2 =$$

$$= p\beta L^a K^{1-a} - wL - (r_{leit} + r_D)(-D^F) - DP - (\hat{S} - S)^2 - \left(inv\ K\left(1 - \theta(r_{leit} + r_D)\right) - inv\right)^2$$

"differentially" defined variable *:*

	p	price
	L	*labour*
	K	*capital*
	w	*wages*
	D^F	loans payable
	DP	depreciation
	S	inventories
	inv	Net investment

"algebraically" defined variable:	Y	total output, Cobb-Douglas function
	$profit^F$	$profit$
	$invmax$	maximum net investment, when credit interest rates=0

Parameter:	α	*Cobb – Douglas parameter*
	β	technology *factor*
	r_{leit}	central bank prime rate
	r_D	ending rate premium on central bank base rate
	$\hat{S}$	stock-keeping target
	$\overline{inv}$	maximal net investment factor
	θ	factor for the interest rate dependency of the investments

Note: Note that the constraint $0 = K' - inv$ must apply to the variables K, inv in the sense of chapter 3.9.2.

4.3. Bank

For example, a bank may have the following target:

- **Profit target**: The greater the profit, the greater the utility.

The stated target of the bank can be expressed, for example, in the following utility function.

$$U^B = profit^B$$

algebraicly defined variable

$$profit^B = +(r_{leit} + r_D).(-D^F) + (r_{leit} + r_D).(-D^G) - r_{leit} A^{ZB} - (r_{leit} + r_A) A^H$$

insert in U^B

$$U^B(D^F, D^G, A^{ZB}, A^H) = +(r_{leit} + r_D).(-D^F) + (r_{leit} + r_D).(-D^G) - r_{leit} A^{ZB} - (r_{leit} + r_A) A^H$$

differentially defined variable	D^F	loans payable of firm
	D^G	loans payable of government
	A^{ZB}	loans receivable of central bank
	A^H	loans receivable of household (Savings deposits)

Parameter:	r_{leit}	central bank prime rate
	r_D	lending rate premium on central bank interest rates
	r_A	Savings interest surcharge on central bank interest rates

4.4. Central bank

The FED (Federal Reserve) has 3 targets:

- **Inflation target**: Inflation should be as close as possible to 2%.

- **Full employment target**: i.e., there should be neither unemployment nor overemployment due to overheating of the economy.

- **Target for the long-term interest rate**: moderate long-term interest rate. For the sake of simplicity, we will not consider this target any further in the following.

The first two targets can be modelled within the framework of the GCD models in the following two ways: by means of corresponding utility functions or by prescribing the setting of the prime interest rate by means of the so-called Taylor rule.

4.4.1. Utility function of a central bank

The full employment target can be expressed analogously to the utility function of the household by the term

$$-\left(\hat{L}-L\right)^2$$

in the utility function of the central bank. In contrast to the household, however, the central bank has no direct influence on employment, but only an indirect influence through its interest rate policy or its money supply policy. This means

$\mu_L^{ZB}=0$	in contrast to $\mu_L^H \neq 0$
$\mu_{r_{leit}}^{ZB}>0$	Influence on the central bank base rate r_{leit}
$\mu_{N^{ZB}}^{ZB}>0$	Influence on money creation N^{ZB}

A central bank can try to achieve the target of inflation in 2 different ways. Through interest rate policy (we characterise this by $\delta=1$) or through money creation policy (we characterise this by $\delta=0$). This behaviour of the central bank can be described by the following term in the utility function

$$\left(-\delta\, r_{leit} + (1-\delta)N^{ZB}\right)(\hat{p} - \frac{ps}{p})$$

with $0 \le \delta \le 1$

$\delta = 1$ (pure interest rate policy)

$\delta = 0$ (pure money creation policy)

r_{leit}	*central bank base rate*
N^{ZB}	*money creation* (*flow variable*!)
$\hat{p}$	*inflation target*
p	*price*
ps	*temporal price change (due to constraint 0 = ps - p')*
	because of chapter 3.9.2

It should be noted that the central bank has no direct influence on the price p, but can again only influence p and ps indirectly via the central bank base rate and money creation. This means

$\mu_{p}^{ZB} = 0$	Influence on the price p
$\mu_{ps}^{ZB} = 0$	Influence on the change of the price ps
$\mu_{r_{leit}}^{ZB} > 0$	Influence on the central bank base rate r_{leit}
$\mu_{N^{ZB}}^{ZB} > 0$	Influence on money creation N^{ZB}

The utility function

$$U^{ZB} = \left(-\delta\, r + (1-\delta)N^{ZB}\right)(\hat{p} - \frac{ps}{p}) - \left(\hat{L} - L\right)^2 \qquad \text{with constraint } 0 = ps - p'$$

because of chapter 3.9.2

leads (in addition to the other terms from the utility functions of other agents and the constraints) in the general GCD - model equations **Fehler! Verweisquelle konnte nicht gefunden werden.** to

$$r' = \mu_r^{ZB} \frac{\partial U^{ZB}}{\partial r} + ... \qquad = -\mu_r^{ZB}\,\delta\,(\hat{p} - \frac{ps}{p})$$

$$N^{ZB}{}' = \mu_{N^{ZB}}^{ZB} \frac{\partial U^{ZB}}{\partial N^{ZB}} + ... \qquad = +\mu_{N^{ZB}}^{ZB}\,(1-\delta)(\hat{p} - \frac{ps}{p})$$

$$p' = \mu_p^{ZB} \frac{\partial U^{ZB}}{\partial p} + ... \qquad = 0 + ... \qquad \textit{because of } \mu_p^{ZB} = 0$$

$$ps' = \mu_{ps}^{ZB} \frac{\partial U^{ZB}}{\partial ps} + ... \qquad = 0 + ... \qquad \textit{because of } \mu_{ps}^{ZB} = 0$$

$$L' = \mu_L^{ZB} \frac{\partial U^{ZB}}{\partial L} + ... \qquad = 0 + ... \qquad \textit{because of } \mu_L^{ZB} = 0$$

The term $-\mu_r^{ZB}\,\delta\,(\hat{p} - \frac{ps}{p})$ means: If the central bank pursues an interest rate policy ($\delta = 1$ *bzw.* $\delta > 0$), it exerts a force on the interest rate r such that r grows (i.e. $r' > 0$), if the actual inflation is greater than the targeted inflation $\frac{ps}{p}$. The same is true in reverse.

The term $+\mu_{N^{ZB}}^{ZB}\,(1-\delta)(\hat{p} - \frac{ps}{p})$ means: If the central bank pursues an interest rate policy ($\delta = 0$ *bzw.* $\delta < 1$), it exerts a force on the interest rate r such that r grows (i.e. $r' > 0$), if the actual inflation is smaller than the targeted inflation $\frac{ps}{p}$. The same is true in reverse.

4.4.2. Taylor rule

The **Taylor rule** is a monetary policy rule for setting the central bank base rate by a central bank. It reads:

$$\begin{aligned} \textit{base rate} \;=\; & \text{real equilibrium interest rate} + \text{inflation} + \\ & + \sigma_1 \,\text{inflation gap} + \sigma_2 \,\text{growth rate gap} \end{aligned} \tag{4.1}$$

Thereby, the weighting factors σ_1, σ_2 are derived from the actual behaviour of the central bank. If both gaps are equal to 0, the Taylor rule is equivalent to Fisher's rule *base rate* = real equilibrium interest rate + inflation <4.2>

We make the following simplifying assumptions:

Assumption 1: The economy is in equilibrium; therefore, it is reasonable to assume that the real equilibrium interest rate is equal to the real growth rate $\frac{Y'}{Y}$.

Assumption 2: Full employment of the economy prevails exactly when the actual labour L is equal to the targeted labour $\hat{L}$, i.e.

Production at full employment $\hat{Y} = \beta K^{(1-\alpha)}\hat{L}^{\alpha}$

growth rate at full employment $= \frac{\hat{Y}'}{Y}$

If $\hat{p}$ denotes the targeted inflation rate, this results in

$$r_{leit} = \frac{Y'}{Y} + \frac{p'}{p} + \sigma_1\left(\frac{p'}{p} - \hat{p}\right) + \sigma_2\left(\frac{Y'}{Y} - \frac{\hat{Y}'}{\hat{Y}}\right) \quad (4.3)$$

Interpretation: The interest rate is higher if the inflation rate $\frac{p'}{p}$ is higher than the target inflation rate $\hat{p}$ and/or the growth rate $\frac{Y'}{Y}$ is higher than the (target) growth rate at full employment.

If one inserts and simplifies one obtains

$$r_{leit} = \frac{p'}{p} + \sigma_1\left(\frac{p'}{p} - \hat{p}\right) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha\frac{L'}{L} \quad (4.4)$$

In terms of the GCD methodology, the Taylor rule sets the value of the policy rate as an algebraically defined variable. If the central bank acts only according to the Taylor rule, it does not act in the sense of optimising a utility function, but according to empirical values that have proven themselves in the past. In this case, one can therefore set the utility function of the central bank equal to 0.

4.4.3. Modified Taylor rule: Consideration of the interest rate premium on the key interest rate

The Fischer rule does not actually refer to the central bank's base interest rate, but to the lending rate. This consists of the base interest rate plus a premium. In economic equilibrium, this results in

$$\text{Loan interest rate} = \text{base rate} + \text{premium} = \text{growth rate} + \text{inflation} \qquad \langle 4.5\rangle$$

Under these assumptions, this results in the **modified Taylor rule**

$$\boxed{\begin{aligned}&\text{base rate} = \\ &= \text{growth rate} - \textit{premium} + \text{inflation} + \sigma_1\, \textit{inflation gap} + \sigma_2\, \textit{growth gap} \qquad \langle 4.6\rangle\end{aligned}}$$

4.5. Government

The government pursues the following targets, for example.

- **Government expenditure target**: Government expenditure serves to fulfil government tasks and is often also referred to as government consumption. For simplicity's sake, we assume that the government behaves like a household. Its desire to consume even more is smaller the more it consumes anyway.

- **Government debt target**: e.g., target government debt in the sense of the Maastricht criteria (60% of GDP).

- **Employment target**: The government has the target of full employment, as does the Fed in the USA.

- **Tax ratio target**: for the sake of simplicity, we will not discuss this further below.

- **Growth target**: for the sake of simplicity, we will not discuss this further below.

The stated targets of the government can be expressed, for example, in the following utility function.

$$U^G = (C^G)^{\gamma_G} - (\hat{D}^G Y - D^G)^2 - (\hat{L} - L)^2$$

where the "algebraicly" defined variable is used :

$$Y := \beta L^a K^{1-a}$$

insert and you get the dependence of the "differentialy" defined variables, i.e the variables defined by equation (3.7) :

$$U^G(C^G, L, K, D^G) = (C^G)^{\gamma_G} - (\hat{D}^G \beta L^a K^{1-a} - D^G)^2 - (\hat{L} - L)^2$$

with parameters	γ_G	*Cobb – Douglas parameter für governmental consumption*
	$\hat{D}^G = -0.6$	*Maastricht factor*
	$\hat{L}$	*targeted labour*

5. What insights can be gained from the modeling of GCD macro models

5.1. Practical insights: Causes and pattern of business cycles, analysis of measures to achieve economic policy targets

The simplest macroeconomic model imaginable consists of 2 agents: 1 company that produces 1 good and 1 household that works for the company and buys or consumes this good.

Even this simplest macroeconomic model shows that under certain assumptions about the power relations between household and firm and assumptions about the other parameters of the model, business cycles occur. This means that the individual variables show an approximately cyclical behaviour and the phase shifts between the individual variables remain approximately the same.

In chapter 7 we present and analyse this simple model and present some basic results.

As an example for measures to achieve economic policy targets in model B1, B2 and C1,C2 we analyse in a simple way the different effects for possible central bank policies: monetary supply policy, interest policy or behaviour in the sense of the Taylor rule.

The most **important tasks** that need to be done in the future to be able to use GCD models for practical problems in economics are:

a) Adjustment of parameters to describe real circumstances and comparison of model results with real business cycle trends.

b) Extend GCD models to multiple households, firms, and goods, and in particular to commodity and financial markets. For a first approach see Richters (2021)

c) In the long run, develop a more complex, real-world model to enable better economic forecasting and test measures to achieve economic policy targets.

d) Elaborate GCD models with economic shocks in detail.

e) Elaborate GCD models with intertemporal utility functions in detail.

5.2. Theoretical insight: Different macroeconomic theories differ in their assumptions of different power factors

A. Sen has shown in (Sen 1963) that

- the basic neoclassical model of macroeconomics
- the macroeconomic model of Kaldor
- the macroeconomic model of Johansen
- and the Keynesian model

differ only in their assumptions about which variables are exogenous and which variables are endogenous.

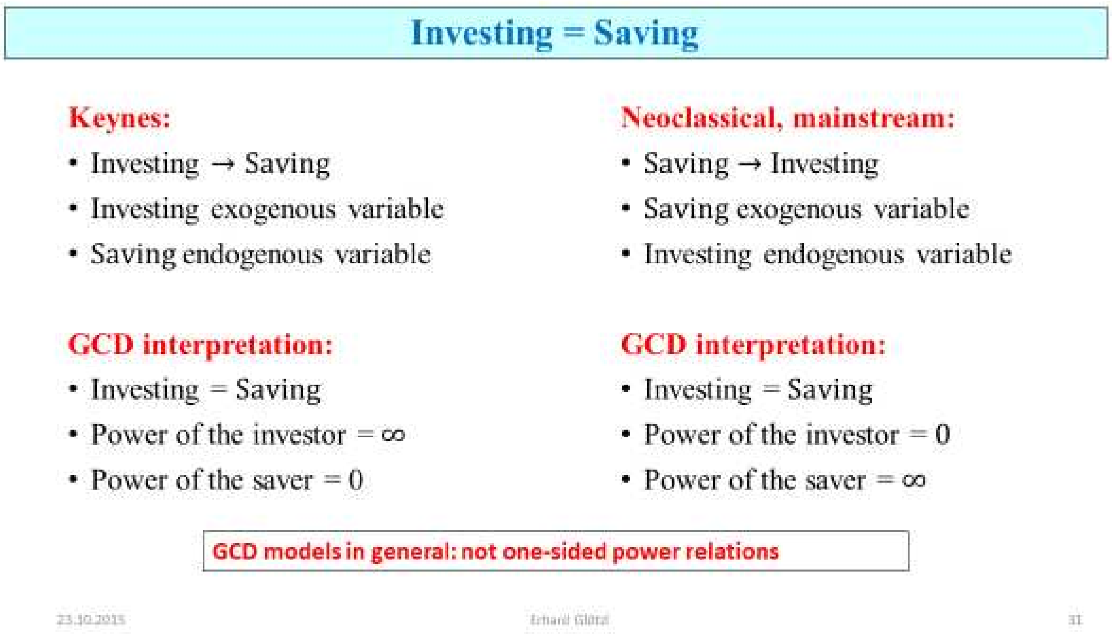
In the methodology of the GCD models it holds:

The variable x is exogenously determined $\Leftrightarrow$ There is an agent A with $\mu_x^A = \infty$

The variable x is endogenously determined $\Leftrightarrow$ For all agents $\mu_x^{A_i} = 0$

This means that the economic models described by Sen always assume one-sided power relations. Since in the GCD models the power factors can assume all values between 0 and ∞, i.e. that also not one-sided power relations are possible, all hybrid forms of economic theories can also be modeled within the framework of GCD models. This means that a continuous transition from one economic theory to another economic theory can be represented by the continuous transition of the various power factors from $0 \rightarrow \infty$ or. $\infty \rightarrow 0$. Since one-sided power relations hardly ever occur in reality, reality can therefore be better described with GCD models. In chapter 14 we describe in detail examples of corresponding theories and the corresponding models.

We show, for example, that even the theoretical assumptions about the causal relationship between "saving" and "investing", which differ from a neoclassical and a Keynesian perspective, can be understood as assumptions about one-sided power relations from the perspective of GCD models:



In Chapter 14.2 we describe the corresponding models and their interpretation as GCD models in detail.

6. The open source programme "GCDconfigurator”

In order to facilitate the concrete application to any complex GCD models (with non-intertemporal utility functions), we have written the open-source program "GCDconfigurator", with which any GCD model can be programmed very comfortably and solved numerically with the help of MATHEMATICA.

Essentially, it is sufficient to enter the following:

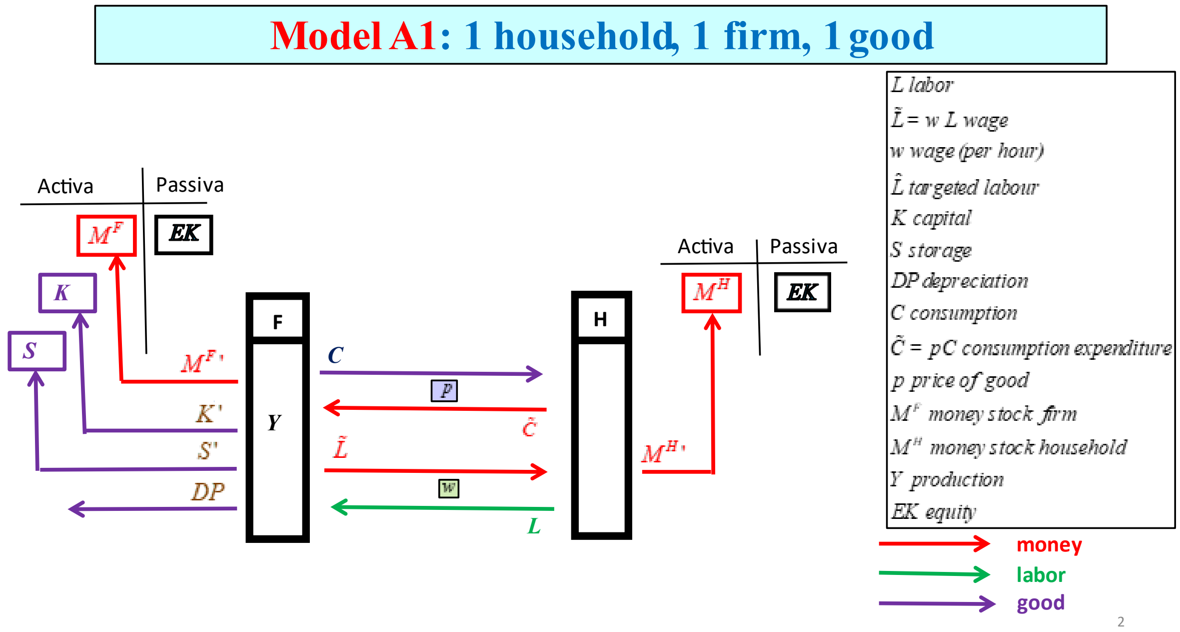
- The algebraically defined variables
- The utility functions for each agent
- The constraints

The output is the time evolution of all variables depending on the freely variable size of the power factors, the other parameters and the initial conditions.

The programme requires the installation of JAVA and MATHEMATICA. It can be downloaded from GitHub with the corresponding instructions (Glötzl und Binter 2022).

7. Model A1, (1 household, 1 firm, 1 good, without interest)

7.1. Overview of the setup



Model A1: basic equations

algebraically defined variables

$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$ "production function"

$DP(K) = \quad \widehat{dp}\, K$ "depreciation"

utility functions

$U^{H}(C,L,MH) = \quad C^{\gamma} - \left(\hat{L} - L\right)^{2} - (\hat{M}^{H} - M^{H})^{2}$ "utility function household"

$U^{F}(Y,L,S) = \quad pY - wL - (\hat{S} - S)^{2}$ "utility function firm"

constraints

$Z^{H} = 0 = wL - pC - M^{H\,\prime}$ for money of household H

$Z^{F} = 0 = pC - wL - M^{F\,\prime}$ for money of firm F

$Z_{1} = 0 = Y(L,K) - C - K' - S' - DP$ for good 1 of firm F

3

With the aid of the GCDconfigurator programme, the differential-algebraic equation system of the A1 model is calculated as follows:

Model A1 : diff.-alg. equation system

$$uF[t] == -(sdach - s[t])^2 - l[t] \times w[t] + p[t] \times y[t]$$
$$uH[t] == cH[t]^{\gamma} - (ldach - l[t])^2 - (mHdach - mH[t])^2$$

$$dp[t] == dpdach\, k[t]$$
$$inv[t] == k'[t]$$
$$y[t] == \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}$$

$$cH'[t] == \gamma\, \mu HcH\, cH[t]^{-1+\gamma} + p[t]\, \lambda_1[t] - p[t]\, \lambda_2[t] - \lambda_3[t]$$
$$k'[t] == (1-\alpha)\, \beta\, \mu Fk\, k[t]^{-\alpha}\, l[t]^{\alpha}\, p[t] - \lambda_3[t]$$
$$l'[t] == 2\, \mu Hl\, (ldach - l[t]) + \mu Fl\, \left(\alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, p[t] - w[t]\right) - w[t]\, \lambda_1[t] + w[t]\, \lambda_2[t] + \alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, \lambda_3[t]$$
$$mF'[t] == -\lambda_1[t]$$
$$mH'[t] == 2\, \mu HmH\, (mHdach - mH[t]) - \lambda_2[t]$$
$$p'[t] == \beta\, \mu Fp\, k[t]^{1-\alpha}\, l[t]^{\alpha} + cH[t]\, \lambda_1[t] - cH[t]\, \lambda_2[t]$$
$$s'[t] == 2\, \mu Fs\, (sdach - s[t]) - \lambda_3[t]$$
$$w'[t] == -\mu Fw\, l[t] - l[t]\, \lambda_1[t] + l[t]\, \lambda_2[t]$$

$$0 == cH[t] \times p[t] - l[t] \times w[t] - mF'[t]$$
$$0 == -cH[t] \times p[t] + l[t] \times w[t] - mH'[t]$$
$$0 == -cH[t] - dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - k'[t] - s'[t]$$

$$cH[0] == k0^{1-\alpha}\, l0^{\alpha}\, \beta$$
$$k[0] == k0$$
$$l[0] == l0$$
$$mF[0] == mF0$$
$$mH[0] == mH0$$
$$p[0] == p0$$
$$s[0] == s0$$
$$w[0] == w0$$

7.2. Description of the A1 model in detail

The one good serves as both a consumption good and an investment good. We assume that vertical constraint forces occur.

Since the target is first to show the principle, we choose the production function and the utility functions as simple as possible.

We choose a simple Cobb-Douglas production function as the production function, and the goods excreted per year (depreciation) are proportional to the capital stock. This results in the 2 necessary algebraically defined variables. They are necessary because they occur in the utility functions or constraints.

$$Y(L,K) = \beta L^{\alpha} K^{1-\alpha} \qquad \beta > 0,\ 0 < \alpha < 1$$
$$DP(K) = dp\, K \qquad 0 \le dp \le 1 \qquad \text{<7.1>}$$

In addition, one can be interested, for example, in net investment, for which one defines as a further algebraically defined variable

$$inv(K) = K' \qquad \text{<7.2>}$$

Households want to consume with decreasing marginal utility. Consumption of consumer goods C leads to a utility for households in the amount of C^{γ} with $0 < \gamma < 1$. They strive for a desired working time $\hat{L}$. Deviations from the desired working time $\hat{L}$ lead to a reduction of utility by $(L - \hat{L})^2$. In addition, households aim to keep cash in the amount of $\hat{M}^H$. Deviations from the desired cash position $\hat{M}^H$ lead to a reduction in utility by $(\hat{M}^H - M^H)^2$. This leads to the **utility function for the household**

$$U^H = C^{\gamma} - (\hat{L} - L)^2 - (\hat{M}^H - M^H)^2 \qquad 0 < \gamma < 1 \qquad \text{<7.3>}$$

For the company, in the simplest case, the utility initially consists of the goods produced, which are valued at the selling price, i.e. pY. The produced goods are used for:

C Sales = Consumption

S' change in inventory

K' changes in productive capital stock

In principle, it would be possible to weight the utility of these uses differently. For the sake of simplicity, we will refrain from doing so. Therefore, this utility is reduced by the cost of labor and the cost of storage, which we evaluate through the deviations from the planned inventory. For simplicity, we assume that holding money in cash has no influence on the utility. This leads to the **utility function for the firm**

$$U^F = pY(L,K) - wL - (\hat{S} - S)^2 = p\beta L^{\alpha} K^{1-\alpha} - wL - (\hat{S} - S)^2 \qquad \text{<7.4>}$$

From the model graph, it can be seen that the following **constraints** must be satisfied:

$$Z_1 = 0 = wL - pC - M^{H}{}' \qquad \textit{for money of household } H$$
$$Z_2 = 0 = pC - wL - M^{F}{}' \qquad \textit{for money of firm } F \qquad \text{<7.5>}$$
$$Z_3 = 0 = Y(L,K) - C - K' - S' \qquad \textit{for good } 1 \textit{ of firm } F$$

According to the methodology of GCD models, the interest or desire of households to change consumption is the greater the more the utility changes when consumption changes, i.e., the interest is proportional to $\frac{\partial U^H}{\partial C}$. However, the interest in changing consumption does not in itself lead to an actual change in consumption, because the household must also have the power or opportunity to actually implement its desire to change consumption. For example, a household cannot or can only partially enforce its additional consumption wish, e.g., to go to the cinema or on holiday, because it is in quarantine or the borders are closed. This restriction of the possibility to enforce his or her consumption change wishes is described by a (possibly time-dependent) "power factor" μ_C^H. Analogously, the firm could have an interest $\frac{\partial U^F}{\partial C}$ and power μ_C^F to influence consumption. In the specific case $\frac{\partial U^F}{\partial C} = 0$. This results in the following behavioural equation for the **ex-ante planned change in consumption**

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \mu_C^F \frac{\partial U^F}{\partial C} = \mu_C^H \gamma C^{\gamma-1} \qquad \text{<7.6>}$$

The same considerations apply to labour L as to consumption. Even the household's wish to increase or reduce working time does not in itself lead to an actual change in working time, because the household must also have the power or possibility to actually implement its wish to change. For example, a household might not be able to enforce its wish to increase working time, or only partially, because it is on short-time working or unemployed, or it might not be able to enforce its wish to reduce working time because it is contractually obliged to work overtime. This restriction of the possibility to enforce his wishes for a change in working time is also described by a (possibly time-dependent) power factor, which we denote with μ_L^H. The same applies to the firm's ability to influence working time.

Therefore, the behavioural equation for the **ex-ante planned change in working time** is as follows

$$L' = \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} = 2\mu_L^H (\hat{L} - L) + \mu_L^F (p\beta\alpha L^{\alpha-1} K^{1-\alpha} - w)$$

The ex-ante behavioural equations for the other variables result analogously.

However, the plans of the 2 agents household and firm to change consumption C, labour L and the other variables cannot be enforced independently of each other, because the constraints

$$
\begin{aligned}
&Z_1 = 0 = wL - pC - M^{H\,'} && \textit{für Geld von Haushalt } H \\
&Z_2 = 0 = pC - wL - M^{F\,'} && \textit{für Geld von Firma } F \qquad \text{<7.7>} \\
&Z_3 = 0 = Y(L,K) - C - K' - S' - DP && \textit{für Gut } 1 \textit{ von Firma } F
\end{aligned}
$$

lead to constraint forces, which we assume are vertical constraint forces. The constraint force for the change in consumption therefore results in

$$\lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = -\lambda_1 p + \lambda_2 p - \lambda_3$$

The behavioural equation for the actual **ex-post change in consumption** is therefore

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3 \qquad \text{<7.8>}$$

Analogously, the actual **ex-post change in labour** is as follows

$$
\begin{aligned}
L' &= \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} = \\
&= 2\mu_L^H (\hat{L} - L) + \mu_L^F (p\beta\alpha L^{\alpha-1} K^{1-\alpha} - w) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta L^{\alpha-1} K^{1-\alpha}
\end{aligned}
$$

This also applies analogously to the company's investments. In the case of the company, too, the actual implementation of ex-ante planned investment increases can be prevented by real restrictions, e.g. by interruptions in supply chains. In the same way, a desired reduction in investment may not be possible to the desired extent because the project is a large-scale project of many years' duration. These restrictions can in turn be described by a (possibly time-dependent) power factor μ_K^B. This results in the following behavioural equation for the actual ex-post change in capital

$$K' = \mu_K^F \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} = \mu_K^F p\beta(1-\alpha) L^{\alpha} K^{-\alpha} - \lambda_3 \qquad \text{<7.9>}$$

Note that we have to use $\frac{\partial Z_3}{\partial K'}$ instead of $\frac{\partial Z_3}{\partial K}$ because the constraint forces are always derived from the highest time derivative of the variables (see chapter 3.9.2 and (Flannery 2011)).

The equations of behaviour for M^H, M^F, S, p, w are derived analogously. In sum, this results in the model equations

differentiell behavioural equations

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \mu_C^F \frac{\partial U^F}{\partial C} + \lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} =$$
$$= \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3$$
$$L' = \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} =$$
$$= \mu_L^H (\hat{L} - L) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta L^{\alpha-1} K^{1-\alpha}$$
$$K' = \mu_K^H \frac{\partial U^H}{\partial K} + \mu_K^F \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} =$$
$$= \mu_K^F p\beta(1-\alpha) L^\alpha K^{-\alpha} - \lambda_3$$
$$M^{H\,\prime} = \mu_{M^H}^H \frac{\partial U^H}{\partial M^H} + \mu_{M^H}^F \frac{\partial U^F}{\partial M^H} + \lambda_1 \frac{\partial Z_1}{\partial M^{H\,\prime}} + \lambda_2 \frac{\partial Z_2}{\partial M^H} + \lambda_3 \frac{\partial Z_3}{\partial M^H} =$$
$$= 2\mu_{M^H}^H \left(\hat{M}^H - M^H\right) - \lambda^H$$
$$M^{F\,\prime} = \mu_{M^F}^H \frac{\partial U^H}{\partial M^F} + \mu_{M^F}^F \frac{\partial U^F}{\partial M^F} + \lambda^H \frac{\partial Z^H}{\partial M^{F\,\prime}} + \lambda^B \frac{\partial Z^B}{\partial M^F} + \lambda_1 \frac{\partial Z_1}{\partial M^F} =$$
$$= -\lambda_2$$
$$S' = \mu_S^H \frac{\partial U^H}{\partial S} + \mu_S^F \frac{\partial U^F}{\partial S} + \lambda_1 \frac{\partial Z_1}{\partial S} + \lambda_2 \frac{\partial Z_2}{\partial S} + \lambda_3 \frac{\partial Z_3}{\partial S'} =$$
$$= \mu_S^F \left(\hat{S} - S\right) - \lambda_3$$
$$p' = \mu_p^H \frac{\partial U^H}{\partial p} + \mu_p^F \frac{\partial U^F}{\partial p} + \lambda_1 \frac{\partial Z_1}{\partial p} + \lambda_2 \frac{\partial Z_2}{\partial p} + \lambda_3 \frac{\partial Z_3}{\partial p} =$$
$$= \mu_p^F \beta K^{1-\alpha} L^\alpha - \lambda_1 c + \lambda_2 c$$
$$w' = \mu_w^H \frac{\partial U^H}{\partial w} + \mu_w^F \frac{\partial U^F}{\partial w} + \lambda_1 \frac{\partial Z_1}{\partial w} + \lambda_2 \frac{\partial Z_2}{\partial w} + \lambda_3 \frac{\partial Z_3}{\partial w} =$$
$$= -\mu_w^F L + \lambda_1 L - \lambda_2 L$$

Or written in a clearer way

differentiell behavioural equations

$$C' = \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3$$

$$L' = 2\mu_L^H (\hat{L} - L) + \mu_L^F \left(\alpha\beta K^{1-\alpha} L^{-1+\alpha} p - w\right) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta K^{1-\alpha} L^{-1+\alpha}$$

$$K' = \mu_K^F \beta (1-\alpha) L^\alpha K^{-\alpha} p - \lambda_3$$

$$M^{H}{}' = 2\mu_{M^H}^H \left(\hat{M}^H - M^H\right) - \lambda_1$$

$$M^{F}{}' = -\lambda_2$$

$$S' = \mu_S^F \left(\hat{S} - S\right) - \lambda_3$$

$$p' = \mu_p^F \beta K^{1-\alpha} L^\alpha - \lambda_1 c + \lambda_2 c$$

$$w' = -\mu_w^F L + \lambda_1 L - \lambda_2 L$$

7.3. Calculation results of model A1

Depending on the choice of parameters, the system converges to a stationary state (see figure 1) or the system describes the occurrence of business cycles (see figure 2). A change in the parameters usually only changes the frequency and amplitude of the business cycle fluctuations. This means that the qualitative sequence of business cycles over a wide range of parameters is independent of the specific choice of parameters. For example, it can be seen that the minima or maxima of the variables typically occur in the following order (see figure 2):

	Minima	**Maxima**
1	Profit	Price
2	Price	Profit
3	Investment	Employment
4	Employment	Investment
5	BIP	BIP
6	Capital	Money stock of the company
7	Money stock of the company	Storage goods
8	Storage goods	Capital

9	Consumption	Consumption
10	Wages	Wages
11	Money stock of the household	Money stock of the household

Existing business cycle theories each assume certain cause-and-effect relationships between different variables. In contrast, in GCD models, business cycle fluctuations can only be explained by assumptions

- on the behaviour or utility functions of agents
- and about the balance of power between the agents.

In this context, the following remark seems **important**: In economics, there is usually a very complex interplay of the various variables. This complex interaction can be modeled well by systems of differential equations. However, the complex behaviour of differential equation systems cannot usually be described by simple cause-effect relationships. **Simple cause-effect relationships are therefore generally not suitable for correctly reflecting economic interactions.**

Figure 1: model A1

https://www.dropbox.com/s/dc3kb2cb1d018uv/Modell%20A1%20Version%2011.ndsolve.nb?dl=0

Figure 2: model A1, business cycle analysis

https://www.dropbox.com/s/ng1y7g0u52egale/Modell%20A1%20Version%207%2C%20Konjunkturanalyse%20V5.ndsolve.nb?dl=0

Figure 1: model A1

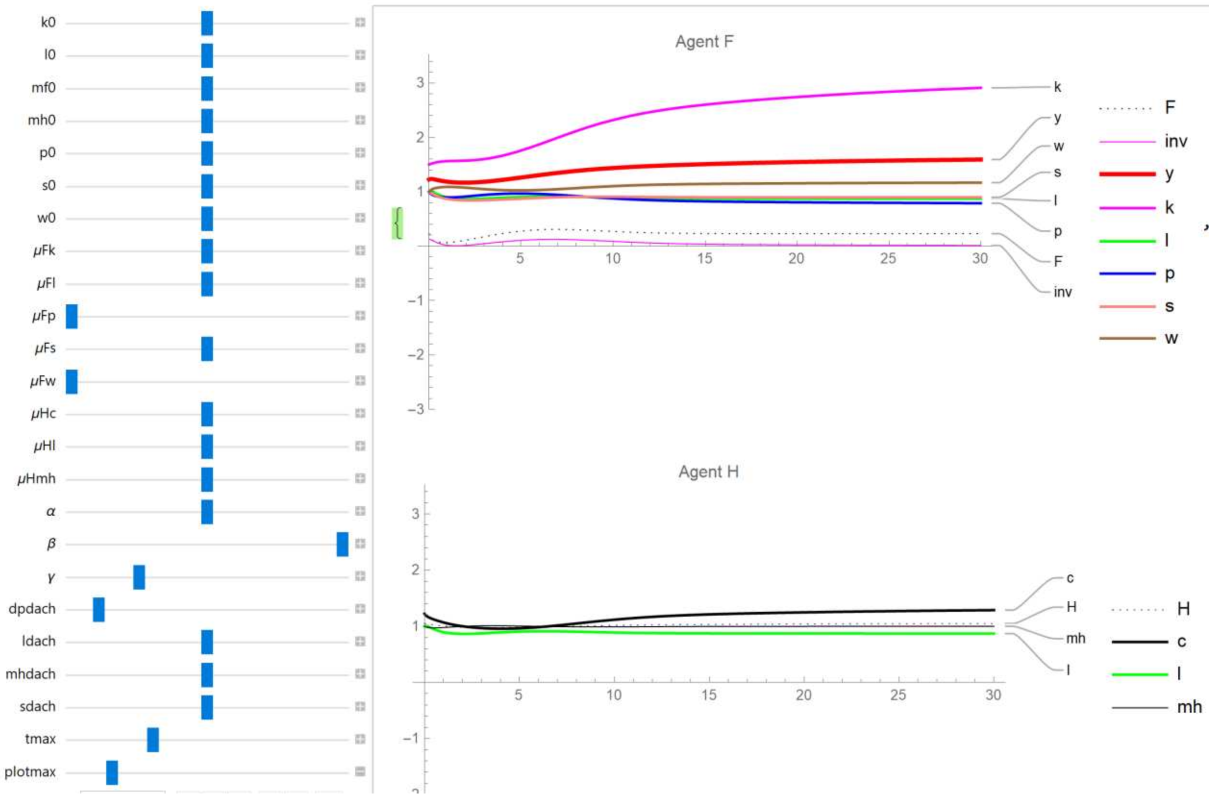
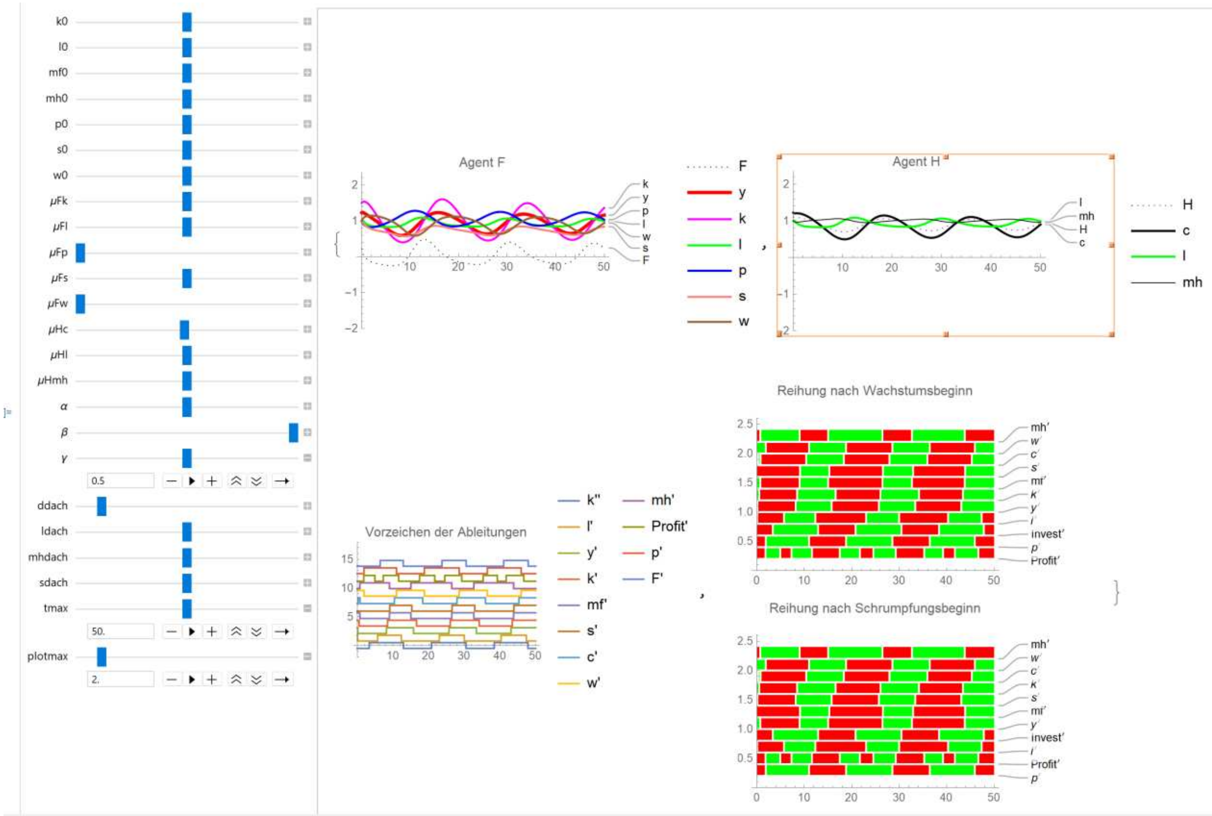
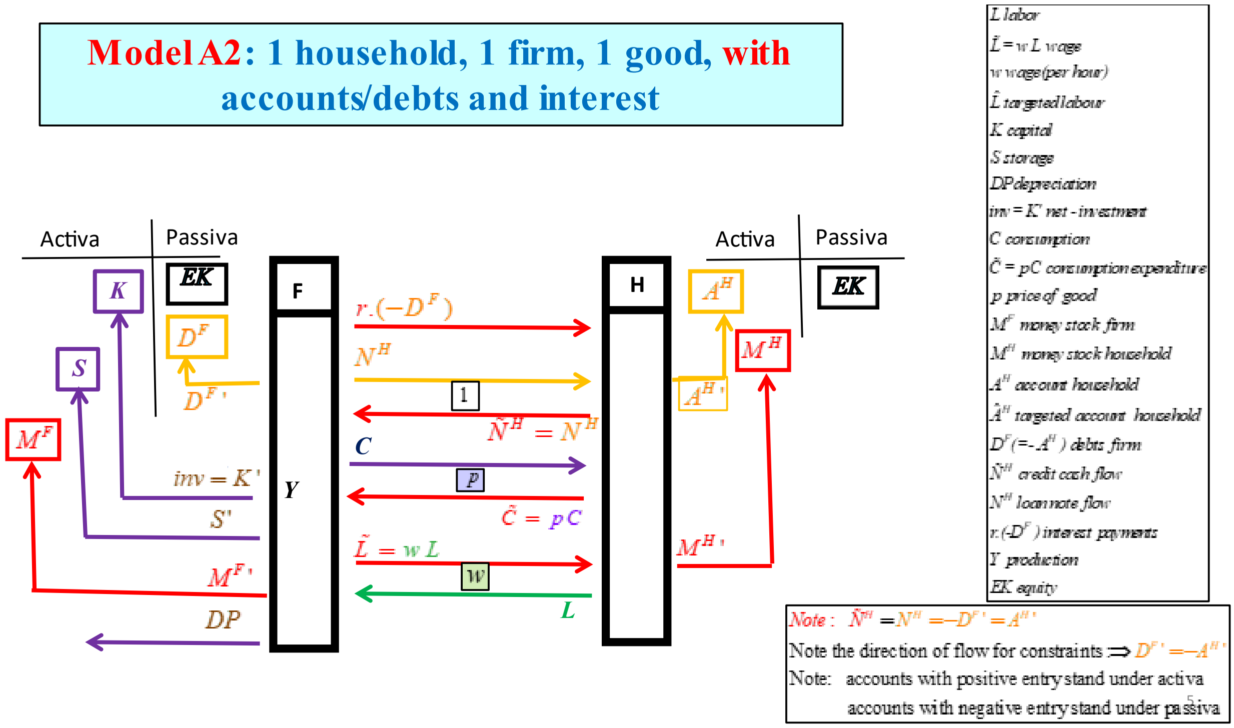


Figure 2: model A1, business cycle analysis



8. Model A2: Model A2: 1 household, 1 firm, 1 good, with accounts/debts and interest

8.1. Overview of the setup



Model A2: basic equations

algebraically defined variables

$$Y(L,K) = \beta L^{\alpha} K^{1-\alpha} \qquad \text{"production function"}$$

$$DP(K) = \widehat{dp}\, K \qquad \text{"depreciation"}$$

utility functions

$$U^H(C,L,M^H) = C^{\gamma} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 + rA^H \qquad \text{"utility function household"}$$

$$U^F(Y,L,S) = pY - wL - (\hat{S} - S)^2 - r(-D^F) \qquad \text{"utility function firm"}$$

constraints

$$Z_1 = 0 = wL - pC + rA^H - N^H - M^{H\,\prime} \qquad \text{for money flow of household } H$$

$$Z_2 = 0 = pC - wL - r(-D^F) + N^H - M^{F\,\prime} \qquad \text{for money flow of firm } F$$

$$Z_3 = 0 = Y(L,K) - C - S' - DP - K' \qquad \text{for good 1 flow of firm } F$$

$$Z^H = 0 = N^H - A^{H\,\prime} \qquad \text{for accounts / debts flow of } H$$

$$Z^F = 0 = -N^H - D^{F\,\prime} \qquad \text{for accounts / debts flow of } F$$

Assuming vertical constraints, the differential-algebraic equation system of model A2 is calculated from this with the help of the GCDconfigurator.

Model A2: diff.-alg. equation system for vertical constraints

$$
\begin{aligned}
&uF[t] == r\,dF[t] - (sdach - s[t])^2 - l[t] \times w[t] + p[t] \times y[t]\\
&uH[t] == r\,aH[t] + cH[t]^{\gamma} - (ldach - l[t])^2 - (mHdach - mH[t])^2
\end{aligned}
$$

$$
\begin{aligned}
&dp[t] == dpdach\,k[t]\\
&inv[t] == k'[t]\\
&y[t] == \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&aH'[t] == r\,\mu HaH + r\,\lambda_2[t] - \lambda_5[t]\\
&cH'[t] == \gamma\,\mu HcH\,cH[t]^{-1+\gamma} + p[t]\,\lambda_1[t] - p[t]\,\lambda_2[t] - \lambda_3[t]\\
&dF'[t] == r\,\mu FdF + r\,\lambda_1[t] - \lambda_4[t]\\
&k'[t] == (1-\alpha)\,\beta\,\mu Fk\,k[t]^{-\alpha}\,l[t]^{\alpha}\,p[t] - \lambda_3[t]\\
&l'[t] == 2\,\mu Hl\,(ldach - l[t]) + \mu Fl\left(\alpha\,\beta\,k[t]^{1-\alpha}\,l[t]^{-1+\alpha}\,p[t] - w[t]\right) - w[t]\,\lambda_1[t] +\\
&\quad w[t]\,\lambda_2[t] + \alpha\,\beta\,k[t]^{1-\alpha}\,l[t]^{-1+\alpha}\,\lambda_3[t]\\
&mF'[t] == -\lambda_1[t]\\
&mH'[t] == 2\,\mu HmH\,(mHdach - mH[t]) - \lambda_2[t]\\
&nH'[t] == \lambda_1[t] - \lambda_2[t] - \lambda_4[t] + \lambda_5[t]\\
&p'[t] == \beta\,\mu Fp\,k[t]^{1-\alpha}\,l[t]^{\alpha} + cH[t]\,\lambda_1[t] - cH[t]\,\lambda_2[t]\\
&s'[t] == 2\,\mu Fs\,(sdach - s[t]) - \lambda_3[t]\\
&w'[t] == -\mu Fw\,l[t] - l[t]\,\lambda_1[t] + l[t]\,\lambda_2[t]
\end{aligned}
$$

$$
\begin{aligned}
&0 == r\,dF[t] + nH[t] + cH[t] \times p[t] - l[t] \times w[t] - mF'[t]\\
&0 == r\,aH[t] - nH[t] - cH[t] \times p[t] + l[t] \times w[t] - mH'[t]\\
&0 == -cH[t] - dpdach\,k[t] + \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha} - k'[t] - s'[t]\\
&0 == -nH[t] - dF'[t]\\
&0 == nH[t] - aH'[t]
\end{aligned}
$$

$$
\begin{aligned}
&aH[0] == aH0\\
&cH[0] == k0^{1-\alpha}\,l0^{\alpha}\,\beta\\
&dF[0] == dF0\\
&k[0] == k0\\
&l[0] == l0\\
&mF[0] == mF0\\
&mH[0] == mH0\\
&nH[0] == nH0\\
&p[0] == p0\\
&s[0] == s0\\
&w[0] == w0
\end{aligned}
$$

8.2. Systematic derivation of constraints from the model graph

Using the A2 model, we show how to systematically derive the relevant constraints.

Arrows represent flows. In model A2 there are 3 different flows.

- The flow of the good (violet)
- The flw of money (red)
- The flow of debt notes when money is given as credit (light brown)
- The flow of labour (green)

Each flow leads to a decrease in the corresponding balance sheet item (stock) in the balance sheet of the agent from which the flow originates and to an increase in the corresponding balance sheet item (stock) in the balance sheet of the agent to which the flow goes.

In addition, there are source terms, such as production by the company, or sinks, such as actual consumption of consumer goods by the household. This sink for consumer goods at home is not shown in the graph for the sake of clarity and because it leads to a trivial constraint under the assumption that everything is consumed immediately.

Thus, for each agent and each flow there is a constraint in the form

$inflow - outflow - stock\ change = 0$	<8.1>

e.g., this results in a constraint Z_2 for the flow of money at the firm

$$Z_2 = 0 = pC - wL - rA^H + N - M^{F'}$$

When considering the direction of flow and the sign of variables on the liabilities side of the balance sheet (passive), one must **respect the convention** we use, namely that entries on the liabilities side of the balance sheet have a negative sign. This results, for example, in a constraint on the flow of debt notes in the company

$$Z^F = 0 = -N - D^{F'}$$

For interpretation: if the bank gives the company a loan of $\tilde{N} = 10$ €, this means that

- $\tilde{N} = +10$ money (red arrow) flows from the bank to the firm
- $N = \tilde{N} = +10$ debt notes flow from the firm to the bank (light brown arrow) if a debt note is issued for every euro
- that the debt increases and thus, due to the sign convention, the debt account on the liabilities side is reduced by 10, i.e. $D^{F'} = -10$

This results in

- debt note inflow to the firm = 0
- outflow of debt notes to the bank $N = 10$
- outflow of debt notes to the balance sheet $D^{F'} = -10$

$$Z^F = \textit{promissory note inflow} - \textit{outflow of promissory notes to the bank} -$$
$$- \textit{outflow of promissory notes to the balance sheet} =$$
$$= 0 - N - D^{F}\,' = 0 - 10 - (-10) = 0$$

If C denotes the inflow of consumption goods to the household and $\vec{C}$ denotes actual consumption and hence the destruction of consumption goods, then, assuming immediate consumption, the following applies $\vec{C} = C$.

Under the given assumption this is nothing else but the algebraically given behavioural equation for actual consumption $\vec{C}$. The constraint for the flow of consumption to the household $0 = C - \vec{C}$ is therefore equivalent to the algebraic definition equation of $\vec{C}$. Since $\vec{C}$ does not occur in the utility functions, this constraint is superfluous.

Analogously, the following constraints therefore arise:

$Z_1 = 0 = wL - pC + rA^H - N^H - M^{H}\,'$ *für Geld von Haushalt H*

$Z_2 = 0 = pC - wL - r(-D^F) + N^H - M^{F}\,'$ *für Geld von Firma F*

$Z_3 = 0 = Y(L,K) - C - S' - DP - K'$ *für Gut* 1 *von Firma F*

$Z^H = 0 = N^H - A^{H}\,'$ *für Forderungen / Verbindlichkeiten von H*

$Z^F = 0 = -N^H - D^{F}\,'$

8.3. Systematic derivation of constraints from the transaction matrices

The constraints can also be derived from the transaction matrices used to describe SFC models. It should be noted that this always results in linearly dependent constraints that can be omitted.

The relevant constraints are marked in red.

Transaction matrices of model A2

money	constraint→	Z_1	Z_2	$Z_{\text{money balance}}$
	agent→	H	F	
	stock→	M^H	M^F	
flow↓	wage	$+\tilde{L} = +w.L$	$-\tilde{L} = -w.L$	0
	consumption	$-\tilde{C} = -p.C$	$+\tilde{C} = +p.C$	0
	credit	$-\tilde{N} = -1.N$	$+\tilde{N} = +1.N$	0
	interest	$+\tilde{Z} = +r.A^H$	$-\tilde{Z} = -r.A^H$	0
	sum	$\sum = M^{H\prime}$	$\sum = M^{F\prime}$	$\sum\sum = M^{H\prime} + M^{F\prime}$

$$Z_1 = 0 = wL - pC - N + rA^H - M^{H\prime}$$

$$Z_2 = 0 = -wL + pC + N - rA^H - M^{F\prime}$$

$Z_{\text{money balance}} = 0 = M^{H\prime} + M^{F\prime}$ linearly dependent on Z_1 and Z_2

debt note	constraint→	Z_3	Z_4	$Z_{\text{debt note balance}}$
	agent→	H	F	
	stock→	A^H	D^F	
flow↓	credit	$+N$	$-N$	0
	sum	$\sum = A^{H\prime}$	$\sum = D^{F\prime}$	$\sum\sum = 0$

$Z_3 = 0 = N - A^{H\prime}$

$Z_4 = 0 = N - D^{F\prime}$

$Z_{\text{debt note balance}} = 0 = A^{H\prime} + D^{F\prime}$ linearly dependent on Z_3 and Z_4

In the case of the good, we consider the following stocks:

K "Capital”
S "Storage goods"
CS "Consumption stock" (all goods consumed by the household)

good	constraint→	Z_5	Z_6	Z_7	$Z_{\text{good balance}}$
	agent→	H	F	F	
	stock→	CS	K	S	
flow↓	production		$+Y$		$+Y$
	storage goods		$-S'$	$+S'$	0
	depreciation		$-DP$		$-DP$
	Consumption goods	$+C$	$-C$		0
	use of C	$-C$			$-C$
	sum	$\sum - CS' = 0$	$\sum - K' = 0$	$\sum - S' = 0$	$\sum\sum - CS' - K' - S' = 0$

$Z_5 = 0 = C - C - CS'$ trivial

$Z_6 = 0 = Y - S' - DP - C - K'$

$Z_7 = 0 = S' - S'$ trivial

$Z_{good\ balance} = 0 = -CS' - K' - S' + Y - DP - C)$ linearly dependent

No non-trivial constraint arises for the labour L. Therefore, only the constraints coloured red remain. These are the same as those that resulted from the model graph in chapter 8.2.

8.4. Calculation results of model A2

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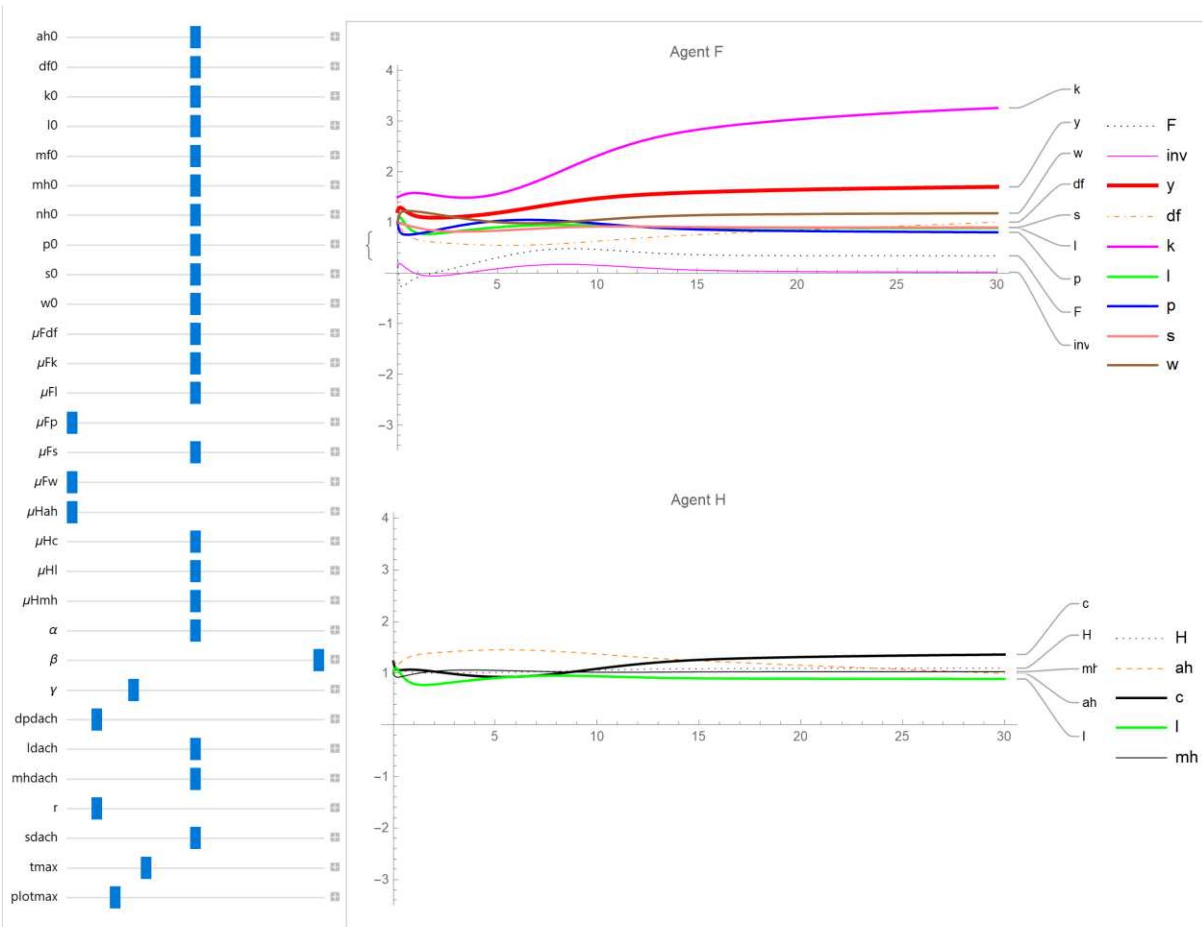


9. Model B1, (1 household, 1 firm, 1 good, 1 banking system), Interest rate policy versus monetary policy

9.1. Overview of the setup

The target of models B1 and B2 is to model the money creation process by the central bank in a simplified way.

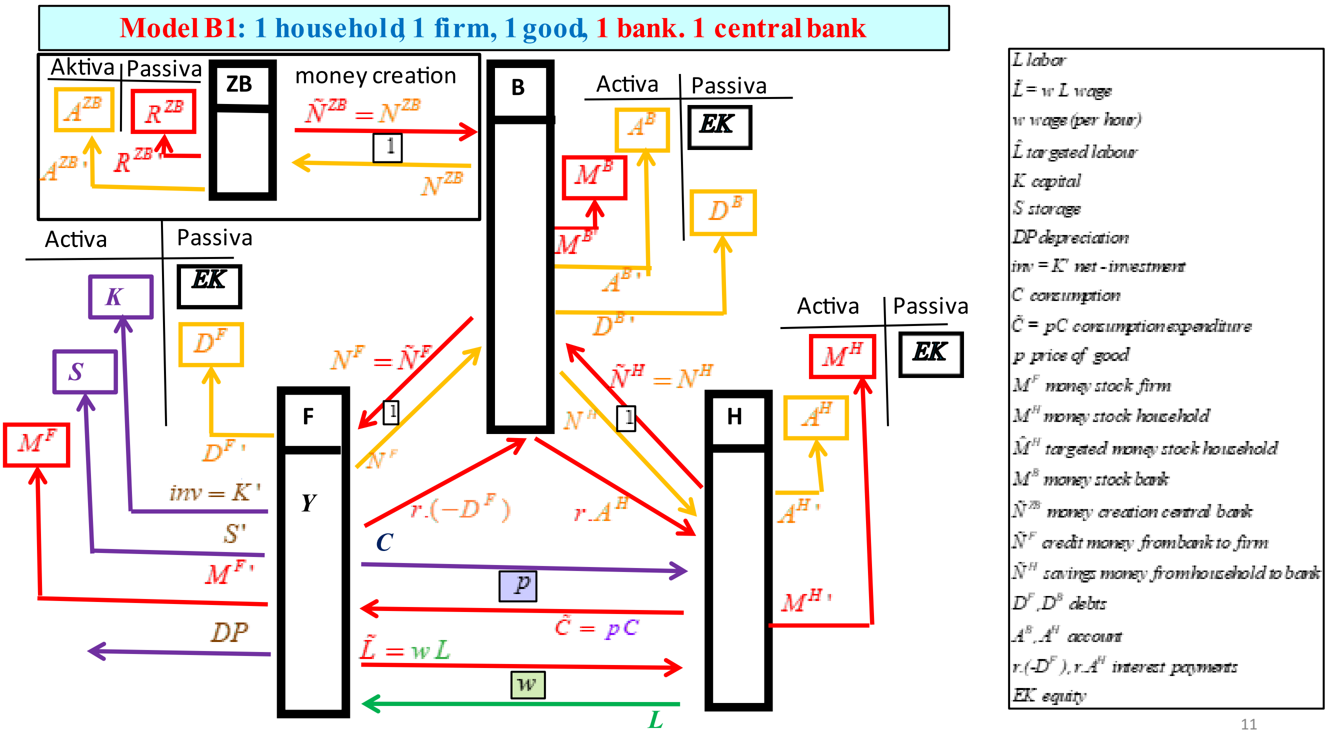
In model B1, the central bank is seen as an endogenous money creator and the bank is seen as an endogenous credit creator. The central bank's target is to keep inflation $\frac{p'}{p}$ at the target inflation $\hat{p} = 0.02$ i.e. 2% by means of interest rate policy ($\delta = 1$) and monetary-supply policy($\delta = 0$).

In this model B1, the central bank's interest rate policy is still modeled in a very simplified way. We assume that the policy rate is constant 0 (banks do not pay interest to the central bank) and that the central bank can, however, influence the interest rate directly. That the policy rate is constant 0 is possible and does not cause the bank to borrow arbitrarily from the central bank, since the bank is assumed to have a constant 0 utility function. This means that the bank has no particular interest in lending to firms or receiving savings deposits from households. Thus, the bank lends endogenously and accepts savings deposits endogenously.

In model B2, we will model the behaviour of the central bank according to the Taylor rule.

All these simplifying restrictions regarding money creation, we will still keep in models C1, C2. This is because in models C1, C2, we are concerned with modeling the government.

It is only in the much more comprehensive model D2 that we will largely abandon the restrictions on the modeling of money creation and the modeling of the government.



Pay attention when establishing the constraints:

(1) Claims A have a positive sign, liabilities D have a negative sign

(2) Banks' equity capital is 0. They do not make profits.

Model B1 : basic equations

algebraically defined variables

$Y(L,K) =$	$\beta L^{\alpha} K^{1-\alpha}$	*"production function"*
$DP(K) =$	$\widehat{dp}\, K$	*"depreciation"*

utility functions

$U^{H}(C,L,M^{H}) =$	$C^{\gamma} - \left(\hat{L} - L\right)^{2} - (\hat{M}^{H} - M^{H})^{2} + r\, A^{H}$	*"utility function household"*
$U^{F}(Y,L,S) =$	$pY - w\,L - (\hat{S} - S)^{2} - r(-D^{F})$	*"utility function firm"*
$U^{B} = 0$		*"utility functionbank"*
$U^{ZB}(r,p,N^{B}) =$	$\left(-\delta r + (1-\delta) N^{ZB}\right)(\hat{p} - \frac{p'}{p})$	*"utility function central bank"*

constraints

$Z_1 = 0 = w\,L - pC + r\,A^{H} - N^{H} - M^{H\,\prime}$	*for money flow of household H*
$Z_2 = 0 = -w\,L + pC - r(-D^{F}) + N^{F} - M^{F\,\prime}$	*for money flow of firm F*
$Z_3 = 0 = N^{ZB} - N^{F} + r(-D^{F}) - r\,A^{H} + N^{H} - M^{B\,\prime}$	*for money flow of bank B*
$Z_4 = 0 = -N^{ZB} - R^{ZB\,\prime}$	*for money flow of central bank ZB*
$Z_5 = 0 = Y(L,K) - C - S' - DP - K'$	*for flow of good 1 of firm F*
$Z_6 = 0 = N^{H} - A^{H\,\prime}$	*for accounts / debts flow of household H*
$Z_7 = 0 = -N^{F} - D^{F\,\prime}$	*for accounts / debts flow of firm F*
$Z_8 = 0 = -N^{ZB} + N^{F} - N^{H} - D^{B\,\prime} - A^{B\,\prime}$	*for accounts / debts flow of of bank B*
$Z_9 = 0 = N^{ZB} - A^{ZB\,\prime}$	*for accounts / debts flow of central bank ZB*

Model B1 : diff.-alg. equation system

```
uB[t] == 0
uF[t] == dF[t] × r[t] - (sdach - s[t])^2 - l[t] × w[t] + p[t] × y[t]
uH[t] == cH[t]^γ - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] × r[t]
uZB[t] == (pdach - ps[t]/p[t]) ((1 - δ) nZB[t] - δ r[t])

dp[t] == dpdach k[t]
inflation[t] == ps[t]/p[t]
inv[t] == k'[t]
y[t] == β k[t]^(1-α) l[t]^α

aB'[t] == -λ3[t]
aH'[t] == μHaH r[t] - λ4[t] - r[t] λ6[t] + r[t] λ8[t]
aZB'[t] == -λ5[t]
cH'[t] == γ μHcH cH[t]^(-1+γ) + p[t] λ7[t] - p[t] λ8[t] - λ9[t]
dB'[t] == -λ2[t]
dF'[t] == μFdF r[t] - λ1[t] - r[t] λ6[t] + r[t] λ7[t]
k'[t] == (1 - α) β μFk k[t]^(-α) l[t]^α p[t] - λ9[t]
l'[t] == 2 μHl (ldach - l[t]) + μFl (α β k[t]^(1-α) l[t]^(-1+α) p[t] - w[t]) - w[t] λ7[t] +
   w[t] λ8[t] + α β k[t]^(1-α) l[t]^(-1+α) λ9[t]
mB'[t] == -λ6[t]
mF'[t] == -λ7[t]
mH'[t] == 2 μHmH (mHdach - mH[t]) - λ8[t]
nF'[t] == -λ1[t] + λ2[t] - λ6[t] + λ7[t]
nH'[t] == -λ2[t] + λ4[t] + λ6[t] - λ8[t]
nZB'[t] == (1 - δ) μZBnZB (pdach - ps[t]/p[t]) - λ2[t] - λ3[t] + λ5[t] + λ6[t]
p'[t] == β μFp k[t]^(1-α) l[t]^α + μZBp ps[t] ((1-δ) nZB[t] - δ r[t]) / p[t]^2 + cH[t] λ7[t] - cH[t] λ8[t] - λ10[t]
ps'[t] == -μZBps ((1-δ) nZB[t] - δ r[t]) / p[t] + λ10[t]
r'[t] == μHr aH[t] + μFr dF[t] - δ μZBr (pdach - ps[t]/p[t]) + (-aH[t] - dF[t]) λ6[t] +
   dF[t] λ7[t] + aH[t] λ8[t]
rZB'[t] == -λ3[t]
s'[t] == 2 μFs (sdach - s[t]) - λ9[t]
w'[t] == -μFw l[t] - l[t] λ7[t] + l[t] λ8[t]

0 == -nF[t] - dF'[t]
0 == nF[t] - nH[t] - nZB[t] - aB'[t] - dB'[t]
0 == -nZB[t] - rZB'[t]
0 == nH[t] - aH'[t]
0 == nZB[t] - aZB'[t]
0 == -nF[t] + nH[t] + nZB[t] - aH[t] × r[t] - dF[t] × r[t] - mB'[t]
0 == nF[t] + cH[t] × p[t] + dF[t] × r[t] - l[t] × w[t] - mF'[t]
0 == -nH[t] - cH[t] × p[t] + aH[t] × r[t] + l[t] × w[t] - mH'[t]
0 == -cH[t] - dpdach k[t] + β k[t]^(1-α) l[t]^α - k'[t] - s'[t]
0 == ps[t] - p'[t]

aB[0] == aB0
aH[0] == aH0
aZB[0] == aZB0
cH[0] == 1/2 k0^(1-α) l0^α β
dB[0] == dB0
dF[0] == dF0
k[0] == k0
l[0] == l0
mB[0] == mB0
mF[0] == mF0
mH[0] == mH0
nF[0] == nF0
nH[0] == nH0
nZB[0] == nZB0
p[0] == 2 k0^(-1+α) l0^(-α) (-nH0 + aH0 r0 + l0 w0) / β
ps[0] == ps0
r[0] == r0
rZB[0] == rZB0
s[0] == s0
w[0] == w0
```

9.2. Calculation results of model B1

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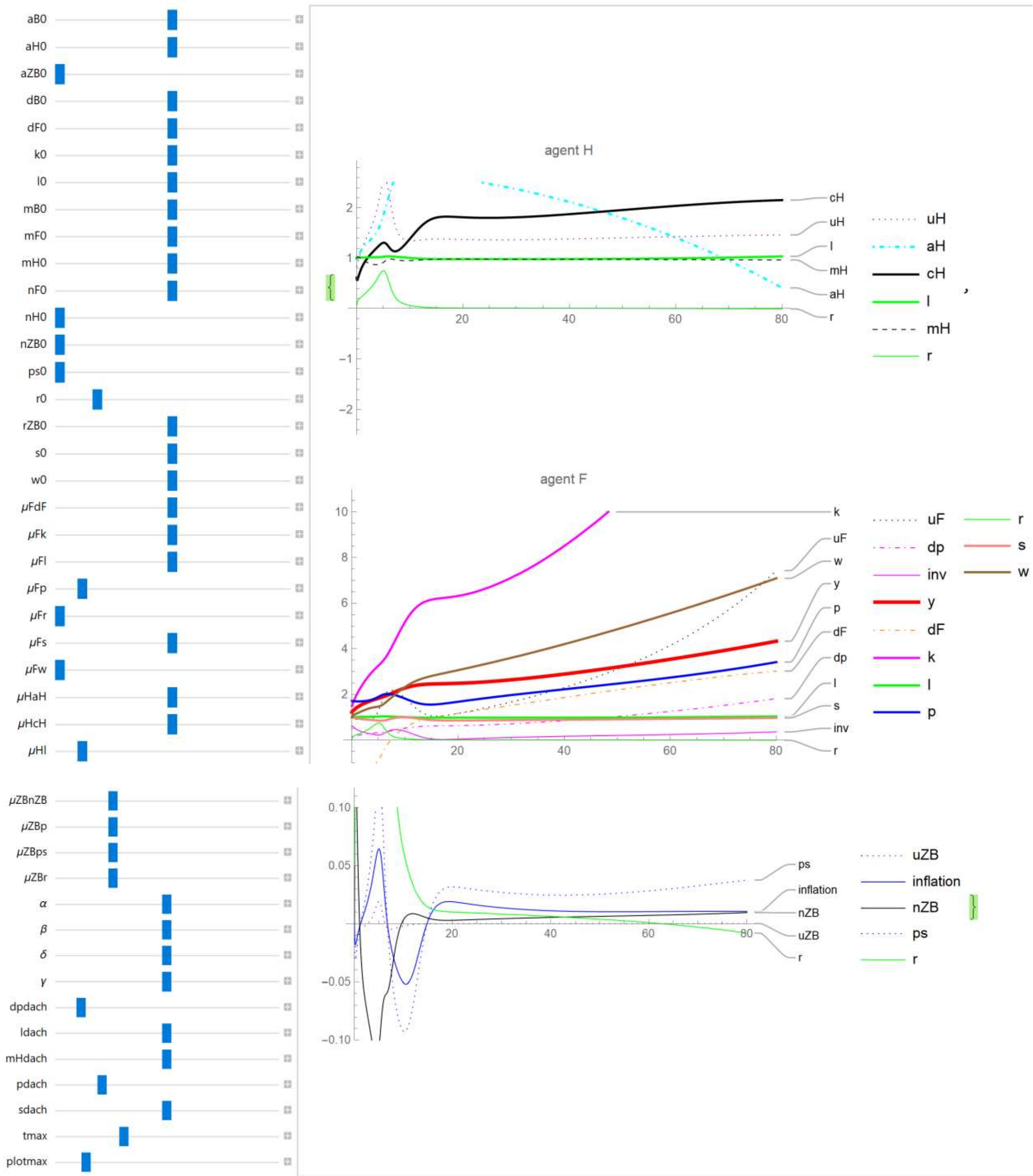
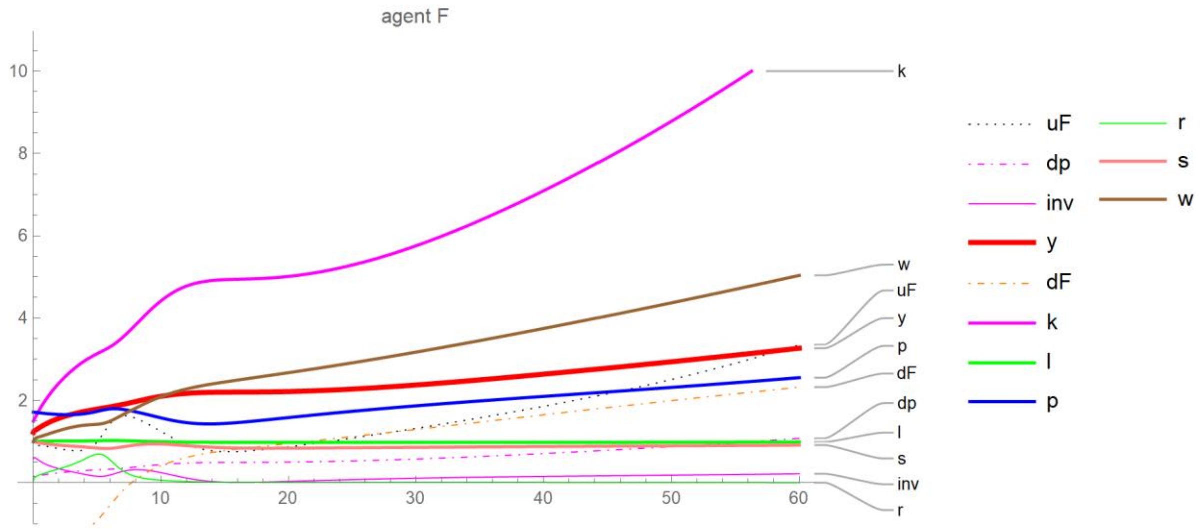
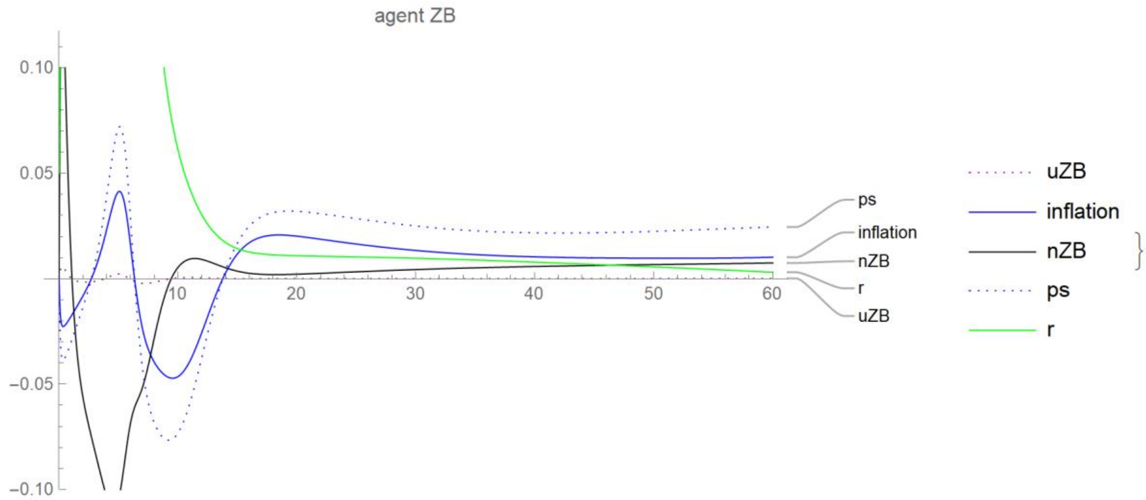


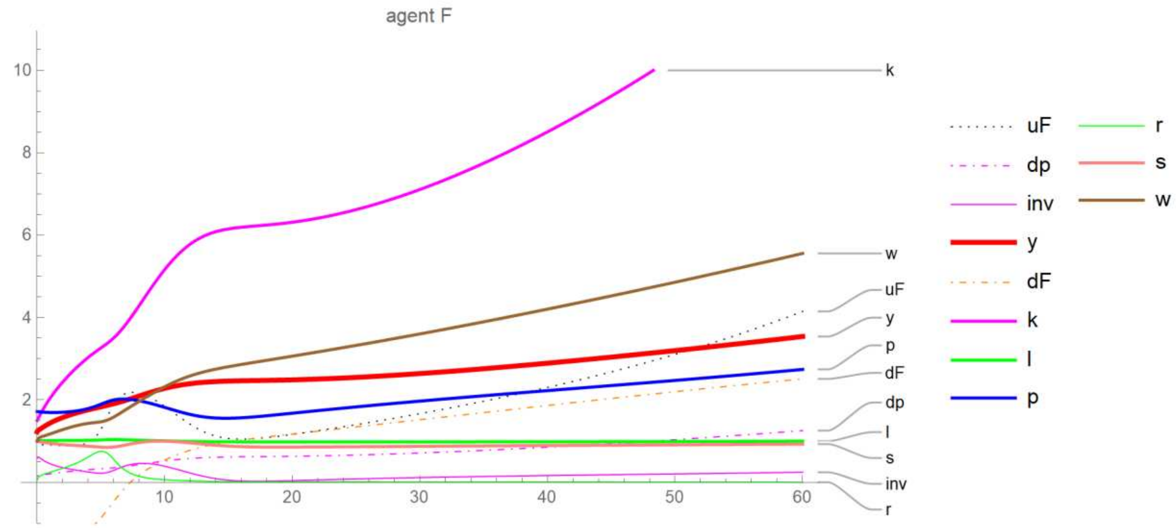
<u>Comparison of</u>

pure money supply policy $\delta = 0$





mixed money supply-interest rate policy $\delta = 0.5$



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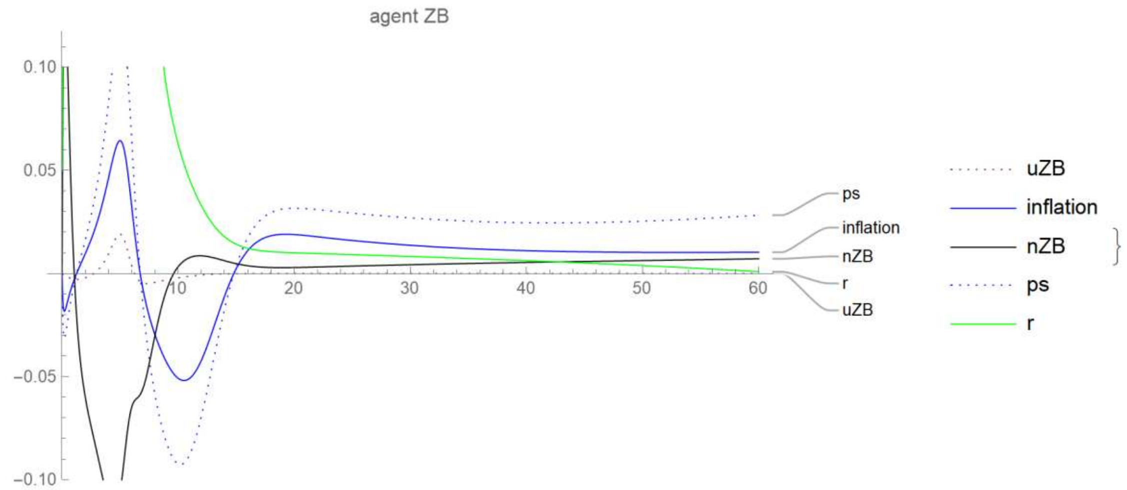
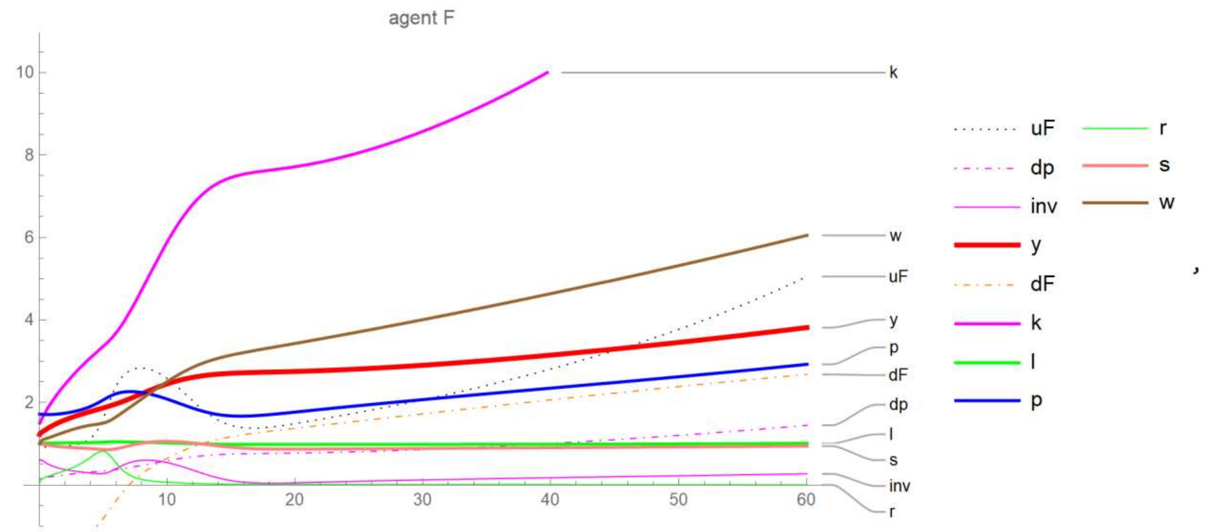


pure interest rate policy $\delta = 1$



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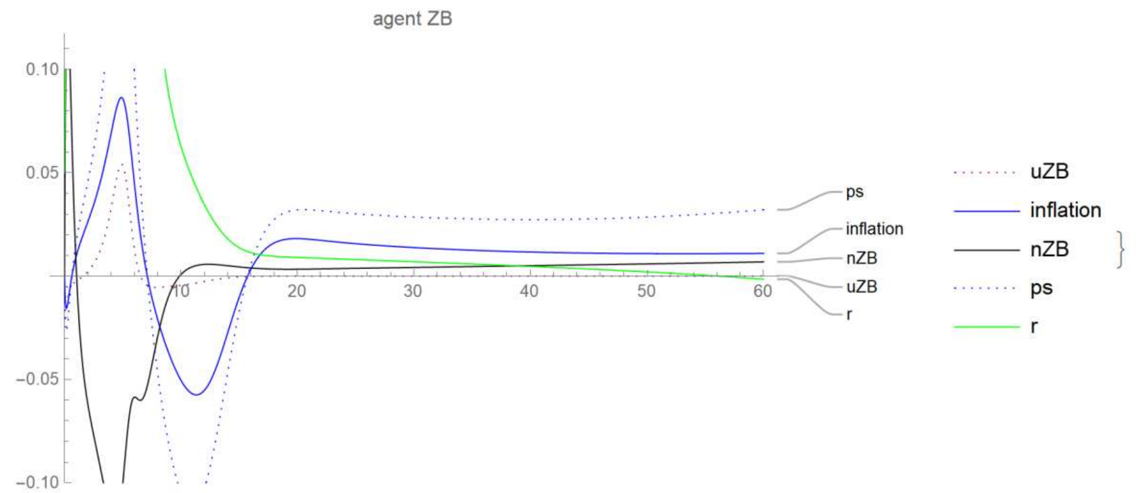

10. Model B2, (1 household, 1 firm, 1 good, 1 bank, 1 central bank) Taylor rule

10.1. Set up

Model B2 differs from model B1 only in the assumption that the central bank acts according to the Taylor rule.

In terms of the GCD methodology, the Taylor rule sets the value of the policy rate as an algebraically defined variable (see chapter 3.11).

If $\hat{p}$ denotes the target inflation rate, this results in

$$r = \frac{Y'}{Y} + \frac{p'}{p} + \sigma_1\left(\frac{p'}{p} - \hat{p}\right) + \sigma_2\left(\frac{Y'}{Y} - \frac{\hat{Y}'}{\hat{Y}}\right)$$

(For simplicity we write r instead of r_{leit}).

If you insert and simplify you get

$$r = \frac{p'}{p} + \sigma_1\left(\frac{p'}{p} - \hat{p}\right) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha\frac{L'}{L} \qquad \text{<10.1>}$$

If the central bank acts only according to the Taylor rule, it does not act in the sense of optimizing a utility function, but according to empirical values that have proven their worth in the past. In this case, therefore, the utility function of the central bank can be set equal to 0.

Model B2 : basic equations for standard Taylor rule

algebraically defined variables

$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$ "*production function*"

$DP(K) = \quad \widehat{dp}\, K$ "*depreciation*"

$r = \frac{p'}{p} + \sigma_1 (\frac{p'}{p} - \hat{p}) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha \frac{L'}{L}$ "*standard Taylor rule*"

utility functions

$U^H(C,L,M^H) = \quad C^{\gamma} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 + r A^H$ "*utility function household*"

$U^F(Y,L,S) = \quad pY - wL - (\hat{S} - S)^2 - r(-D^F)$ "*utility function firm*"

$U^B = 0$ "*utility function bank*"

$U^{ZB} = \ 0$ "*utility function central bank*"

constraints

$Z_1 = 0 = wL - pC + rA^H - N^H - M^{H\,\prime}$ *for money flow of household H*

$Z_2 = 0 = -wL + pC - r(-D^F) + N^F - M^{F\,\prime}$ *for money flow of firm F*

$Z_3 = 0 = N^{ZB} - N^F + r(-D^F) - rA^H + N^H - M^{B\,\prime}$ *for money flow of bank B*

$Z_4 = 0 = -N^{ZB} - R^{ZB\,\prime}$ *for money flow of central bank ZB*

$Z_5 = 0 = Y(L,K) - C - S' - DP - K'$ *for flow of good 1 of firm F*

$Z_6 = 0 = N^H - A^{H\,\prime}$ *for accounts / debts flow of household H*

$Z_7 = 0 = -N^F - D^{F\,\prime}$ *for accounts / debts flow of firm F*

$Z_8 = 0 = -N^{ZB} + N^F - N^H - D^{B\,\prime} - A^{B\,\prime}$ *for accounts / debts flow of of bank B*

$Z_9 = 0 = N^{ZB} - A^{ZB\,\prime}$ *for accounts / debts flow of central bank ZB*

Model B2 : diff.-alg. equation system standard Taylor rule

```
uB[t] == 0
uF[t] == dF[t] × r[t] - (sdach - s[t])^2 - l[t] × w[t] + p[t] × y[t]
uH[t] == cH[t]^γ - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] × r[t]
uZB[t] == 0

dp[t] == dpdach k[t]
inflation[t] == ps[t] / p[t]
inv[t] == k'[t]
r[t] == -pdach σ1 + σ1 ps[t] / p[t] - (-1+α) k'[t] / k[t] + α (1+σ2) l'[t] / l[t]
y[t] == β k[t]^(1-α) l[t]^α
```

$$aB'[t] == -\lambda_2[t]$$

$$aH'[t] == -\lambda_4[t] + \lambda_6[t]\left(pdach\ \sigma 1 - \frac{\sigma 1\ ps[t]}{p[t]} + \frac{(-1+\alpha)\ k'[t]}{k[t]} - \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) + \mu HaH\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) + \lambda_8[t]\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right)$$

$$aZB'[t] == -\lambda_5[t]$$

$$cH'[t] == \gamma\ \mu HcH\ cH[t]^{-1+\gamma} + p[t]\ \lambda_7[t] - p[t]\ \lambda_8[t] - \lambda_9[t]$$

$$dB'[t] == -\lambda_2[t]$$

$$dF'[t] == -\lambda_1[t] + \lambda_6[t]\left(pdach\ \sigma 1 - \frac{\sigma 1\ ps[t]}{p[t]} + \frac{(-1+\alpha)\ k'[t]}{k[t]} - \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) + \mu FdF\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) + \lambda_7[t]\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right)$$

$$k'[t] = \mathrm{If}\left[\frac{(-1+\alpha)\ aH[t]}{k[t]} + \frac{(-1+\alpha)\ dF[t]}{k[t]} \neq 0,\ \frac{(-1+\alpha)\ aH[t]}{k[t]} + \frac{(-1+\alpha)\ dF[t]}{k[t]},\ -\frac{(-1+\alpha)\ aH[t]\ k'[t]}{k[t]^2} - \frac{(-1+\alpha)\ dF[t]\ k'[t]}{k[t]^2}\right]\lambda_6[t] + \mathrm{If}\left[-\frac{(-1+\alpha)\ dF[t]}{k[t]} \neq 0,\ -\frac{(-1+\alpha)\ dF[t]}{k[t]},\ \frac{(-1+\alpha)\ dF[t]\ k'[t]}{k[t]^2}\right]\lambda_7[t] + \mathrm{If}\left[-\frac{(-1+\alpha)\ aH[t]}{k[t]} \neq 0,\ -\frac{(-1+\alpha)\ aH[t]}{k[t]},\ \frac{(-1+\alpha)\ aH[t]\ k'[t]}{k[t]^2}\right]\lambda_8[t] - \lambda_9[t] + \frac{(-1+\alpha)\ \mu Hk\ aH[t]\ k'[t]}{k[t]^2} + \mu Fk\left((1-\alpha)\ \beta\ k[t]^{-\alpha}\ l[t]^{\alpha}\ p[t] + \frac{(-1+\alpha)\ dF[t]\ k'[t]}{k[t]^2}\right)$$

$$l'[t] == \mathrm{If}\left[-\frac{\alpha\ (1+\sigma 2)\ aH[t]}{l[t]} - \frac{\alpha\ (1+\sigma 2)\ dF[t]}{l[t]} \neq 0,\ -\frac{\alpha\ (1+\sigma 2)\ aH[t]}{l[t]} - \frac{\alpha\ (1+\sigma 2)\ dF[t]}{l[t]},\ \frac{\alpha\ (1+\sigma 2)\ aH[t]\ l'[t]}{l[t]^2} + \frac{\alpha\ (1+\sigma 2)\ dF[t]\ l'[t]}{l[t]^2}\right]\lambda_6[t] + \mathrm{If}\left[\frac{\alpha\ (1+\sigma 2)\ dF[t]}{l[t]} \neq 0,\ \frac{\alpha\ (1+\sigma 2)\ dF[t]}{l[t]},\ -w[t] - \frac{\alpha\ (1+\sigma 2)\ dF[t]\ l'[t]}{l[t]^2}\right]\lambda_7[t] + \mathrm{If}\left[\frac{\alpha\ (1+\sigma 2)\ aH[t]}{l[t]} \neq 0,\ \frac{\alpha\ (1+\sigma 2)\ aH[t]}{l[t]},\ w[t] - \frac{\alpha\ (1+\sigma 2)\ aH[t]\ l'[t]}{l[t]^2}\right]\lambda_8[t] + \alpha\ \beta\ k[t]^{1-\alpha}\ l[t]^{-1+\alpha}\ \lambda_9[t] + \mu Hl\left(2\ (ldach - l[t]) - \frac{\alpha\ (1+\sigma 2)\ aH[t]\ l'[t]}{l[t]^2}\right) + \mu Fl\left(\alpha\ \beta\ k[t]^{1-\alpha}\ l[t]^{-1+\alpha}\ p[t] - w[t] - \frac{\alpha\ (1+\sigma 2)\ dF[t]\ l'[t]}{l[t]^2}\right)$$

$$mB'[t] == -\lambda_6[t]$$

$$mF'[t] == -\lambda_7[t]$$

$$mH'[t] == 2\ \mu HmH\ (mHdach - mH[t]) - \lambda_8[t]$$

$$nF'[t] == -\lambda_1[t] + \lambda_2[t] - \lambda_6[t] + \lambda_7[t]$$

$$nH'[t] == -\lambda_2[t] + \lambda_4[t] + \lambda_6[t] - \lambda_8[t]$$

$$nZB'[t] == -\lambda_2[t] - \lambda_3[t] + \lambda_5[t] + \lambda_6[t]$$

$$p'[t] == -\frac{\mu Hp\ \sigma 1\ aH[t] \times ps[t]}{p[t]^2} + \mu Fp\left(\beta\ k[t]^{1-\alpha}\ l[t]^{\alpha} - \frac{\sigma 1\ dF[t] \times ps[t]}{p[t]^2}\right) + \left(\frac{\sigma 1\ aH[t] \times ps[t]}{p[t]^2} + \frac{\sigma 1\ dF[t] \times ps[t]}{p[t]^2}\right)\lambda_6[t] + \left(cH[t] - \frac{\sigma 1\ dF[t] \times ps[t]}{p[t]^2}\right)\lambda_7[t] + \left(-cH[t] - \frac{\sigma 1\ aH[t] \times ps[t]}{p[t]^2}\right)\lambda_8[t] - \lambda_{10}[t]$$

$$ps'[t] == \frac{\mu Hps\ \sigma 1\ aH[t]}{p[t]} + \frac{\mu Fps\ \sigma 1\ dF[t]}{p[t]} + \left(-\frac{\sigma 1\ aH[t]}{p[t]} - \frac{\sigma 1\ dF[t]}{p[t]}\right)\lambda_6[t] + + \frac{\sigma 1\ dF[t]\ \lambda_7[t]}{p[t]} + \frac{\sigma 1\ aH[t]\ \lambda_8[t]}{p[t]} + \lambda_{10}[t]$$

$$rZB'[t] == -\lambda_3[t]$$

$$s'[t] == 2\ \mu Fs\ (sdach - s[t]) - \lambda_9[t]$$

$$w'[t] == -\mu Fw\ l[t] - l[t]\ \lambda_7[t] + l[t]\ \lambda_8[t]$$

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$$0 == -nF[t] - dF'[t]$$

$$0 == nF[t] - nH[t] - nZB[t] - aB'[t] - dB'[t]$$

$$0 == -nZB[t] - rZB'[t]$$

$$0 == nH[t] - aH'[t]$$

$$0 == nZB[t] - aZB'[t]$$

$$0 == -nF[t] + nH[t] + nZB[t] - aH[t]\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) - dF[t]\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) - mB'[t]$$

$$0 == nF[t] + cH[t] \times p[t] - l[t] \times w[t] + dF[t]\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) - mF'[t]$$

$$0 == -nH[t] - cH[t] \times p[t] + l[t] \times w[t] + aH[t]\left(-pdach\ \sigma 1 + \frac{\sigma 1\ ps[t]}{p[t]} - \frac{(-1+\alpha)\ k'[t]}{k[t]} + \frac{\alpha\ (1+\sigma 2)\ l'[t]}{l[t]}\right) - mH'[t]$$

$$0 == -cH[t] - dpdach\ k[t] + \beta\ k[t]^{1-\alpha}\ l[t]^{\alpha} - k'[t] - s'[t]$$

$$0 == ps[t] - p'[t]$$

$$aB[0] == aB0$$
$$aH[0] == aH0$$
$$aZB[0] == aZB0$$
$$cH[0] == \frac{1}{2}\ k0^{1-\alpha}\ l0^{\alpha}\ \beta$$
$$dB[0] == dB0$$
$$dF[0] == dF0$$
$$k[0] == k0$$
$$l[0] == l0$$
$$mB[0] == mB0$$
$$mF[0] == mF0$$
$$mH[0] == mH0$$
$$nF[0] == nF0$$
$$nH[0] == nH0$$
$$nZB[0] == nZB0$$
$$p[0] == p0$$
$$ps[0] == ps0$$
$$rZB[0] == rZB0$$
$$s[0] == s0$$
$$w[0] == w0$$

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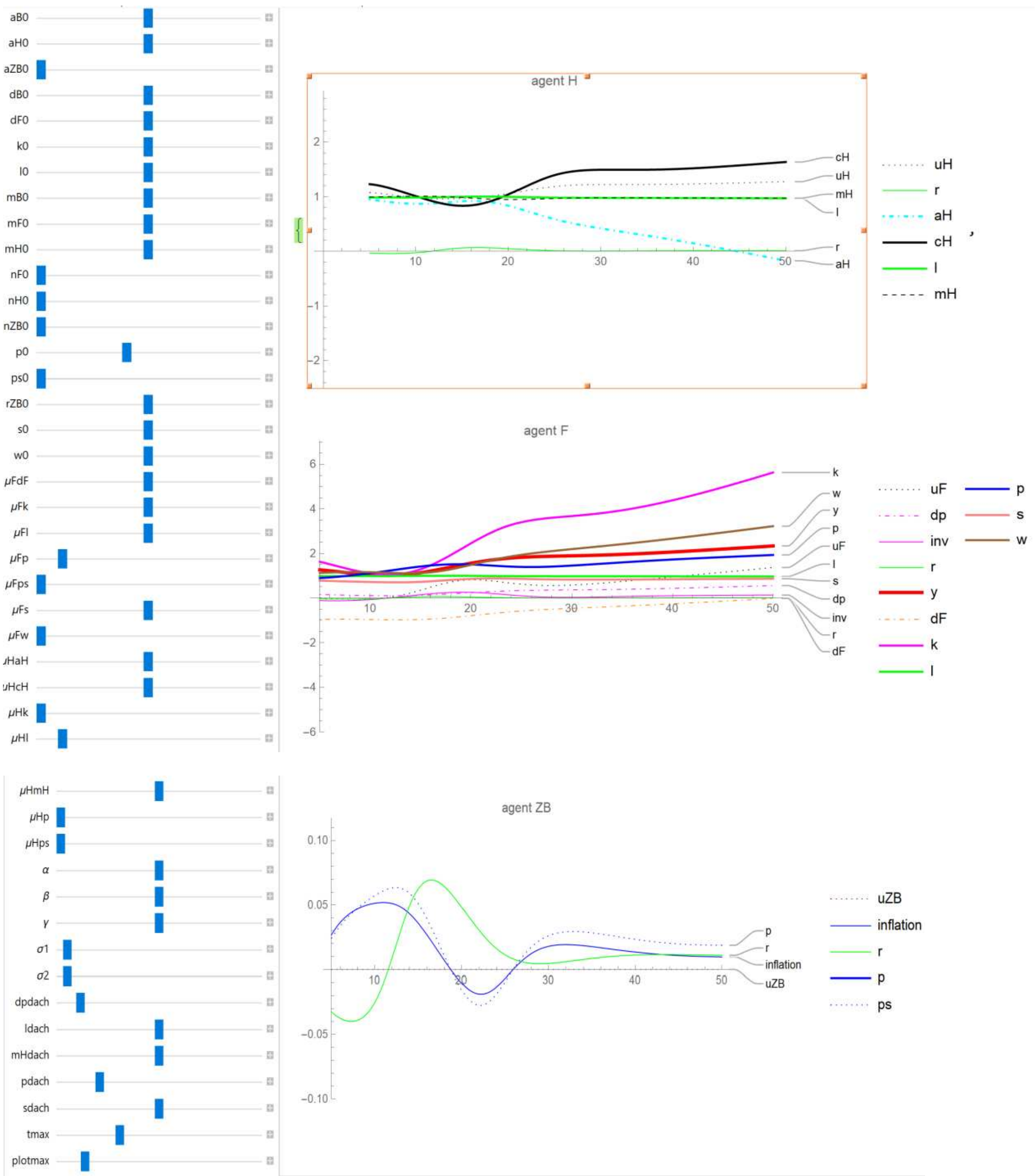
10.2. Calculation results of model B2

https://www.dropbox.com/s/wtlgtbxqlf38cnw/Modell%20B2%20Version%203.ndsolve.nb?dl=0

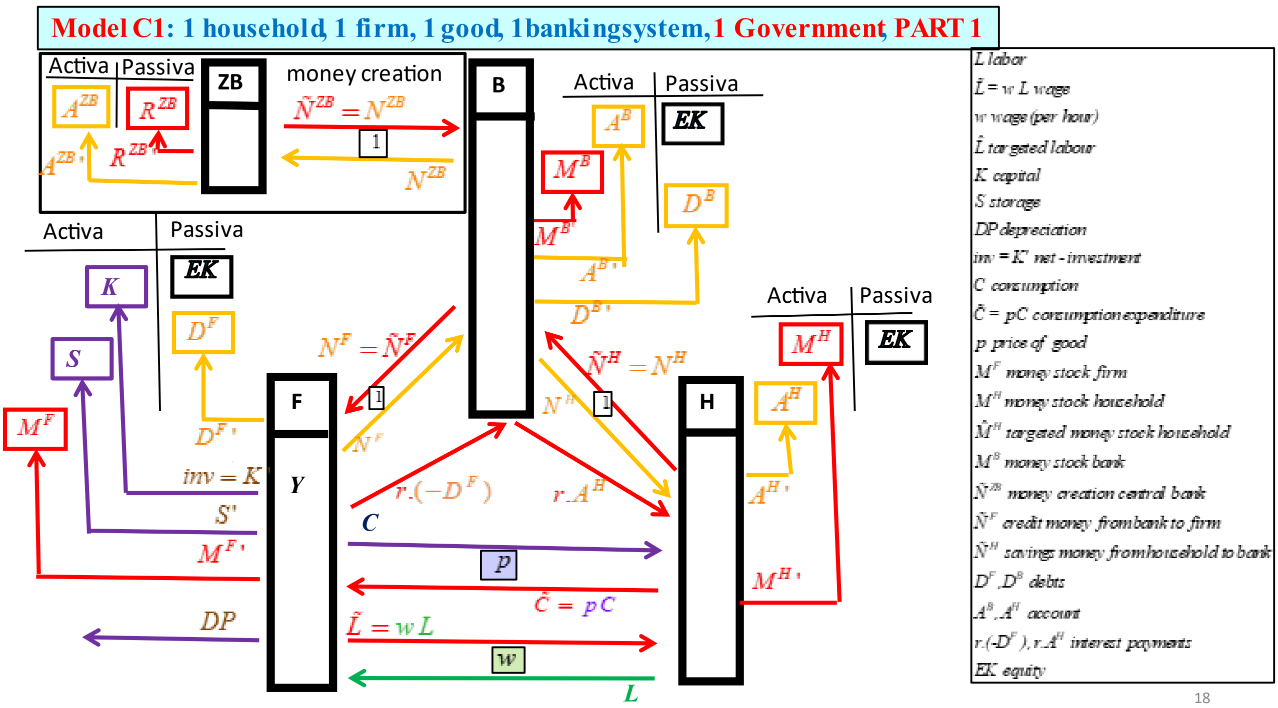


11. Model C1, (1 household, 1 firm, 1 good, 1 banking system, 1 government) interest rate policy versus money supply policy

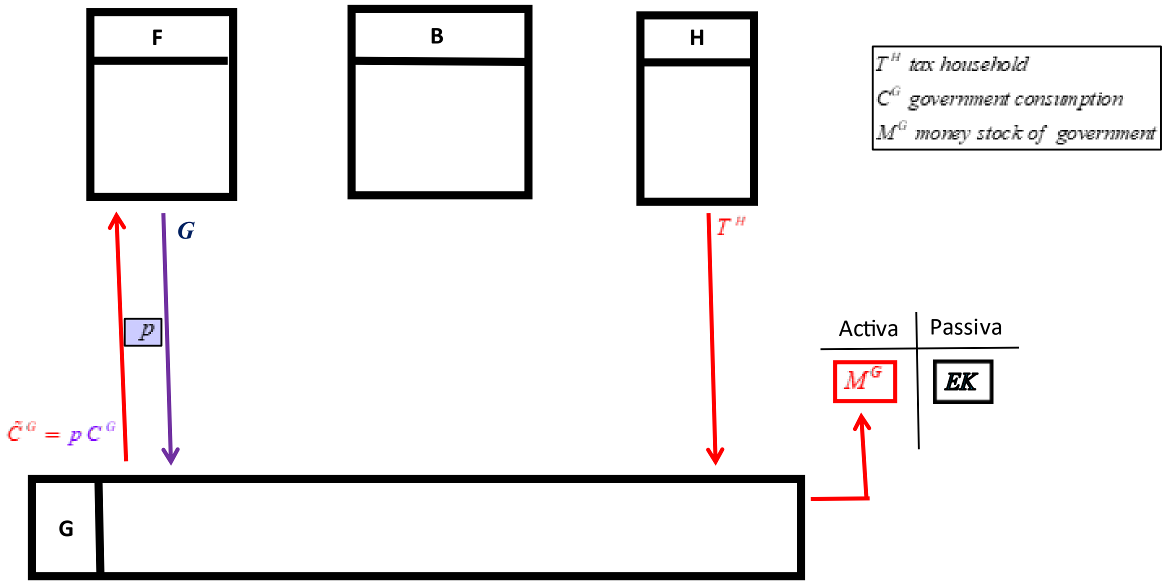
11.1. Set up

The target of model C1 is to extend model B1 by the government G as an agent in a simple form.

The government has a utility function analogous to that of the household. It collects an income tax from the household, which either flows to its money stock M^G or is used for government consumption C^G.



Model C1: 1 household, 1 firm, 1 good, 1 bankingsystem, 1 Government, PART 2



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Model C1 : basic equations

algebraically defined variables	
$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$	"production function"
$DP(K) = \quad \overline{dp} K$	"depreciation"
$T^H(w,L) = \quad \tau^H w L$	"income tax household"
utility functions	
$U^H = C^{\gamma_H} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 + r A^H$	"utility function household"
$U^F = pY - wL - (\hat{S} - S)^2 - r(-D^F)$	"utility function firm"
$U^B = 0$	"utility function bank"
$U^{ZB} = \left(-\delta r + (1-\delta) N^{ZB}\right)(\hat{p} - \frac{ps}{p})$	"utility function central bank"
$U^G = G^{\gamma_G}$	"utility function government"
constraints	
$Z_1 = 0 = wL - pC + rA^H - N^H - T^H - M^{H\prime}$	*for money flow of household H*
$Z_2 = 0 = -wL + pC - r(-D^F) + N^F + pG - M^{F\prime}$	*for money flow of firm F*
$Z_3 = 0 = N^{ZB} - N^F + r(-D^F) - rA^H + N^H - M^{B\prime}$	*for money flow of bank B*
$Z_4 = 0 = -N^{ZB} - R^{ZB\prime}$	*for money flow of central bank ZB*
$Z_5 = 0 = Y(L,K) - C - G - S' - DP - K'$	*for flow of good 1 of firm F*
$Z_6 = 0 = N^H - A^{H\prime}$	*for accounts / debts flow of household H*
$Z_7 = 0 = -N^F - D^{F\prime}$	*for accounts / debts flow of firm F*
$Z_8 = 0 = -N^{ZB} + N^F - N^H - D^{B\prime} - A^{B\prime}$	*for accounts / debts flow of of bank B*
$Z_9 = 0 = N^{ZB} - A^{ZB\prime}$	*for accounts / debts flow of central bank ZB*
$Z_{10} = 0 = -pG + T^H - M^{G\prime}$	*for money flow of government*
$Z_{11} = 0 = ps - p'$	*because "no derivation in utility function of ZB"*

ModelC1: diff.-alg. equation system

$uB[t] = 0$
$uF[t] = dF[t] \times r[t] - (sdach - s[t])^2 - l[t] \times w[t] + p[t] \times y[t]$
$uG[t] = cG[t]^{\gamma G}$
$uH[t] = cH[t]^{\gamma H} - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] \times r[t]$
$uZB[t] = \left(pdach - \frac{ps[t]}{p[t]}\right) ((1-\delta)\, nZB[t] - \delta\, r[t])$

$cGschlange[t] = cG[t] \times p[t]$
$dp[t] = dpdach\, k[t]$
$inflation[t] = \frac{ps[t]}{p[t]}$
$inv[t] = k'[t]$
$taxH[t] = \tau H\, l[t] \times w[t]$
$y[t] = \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}$

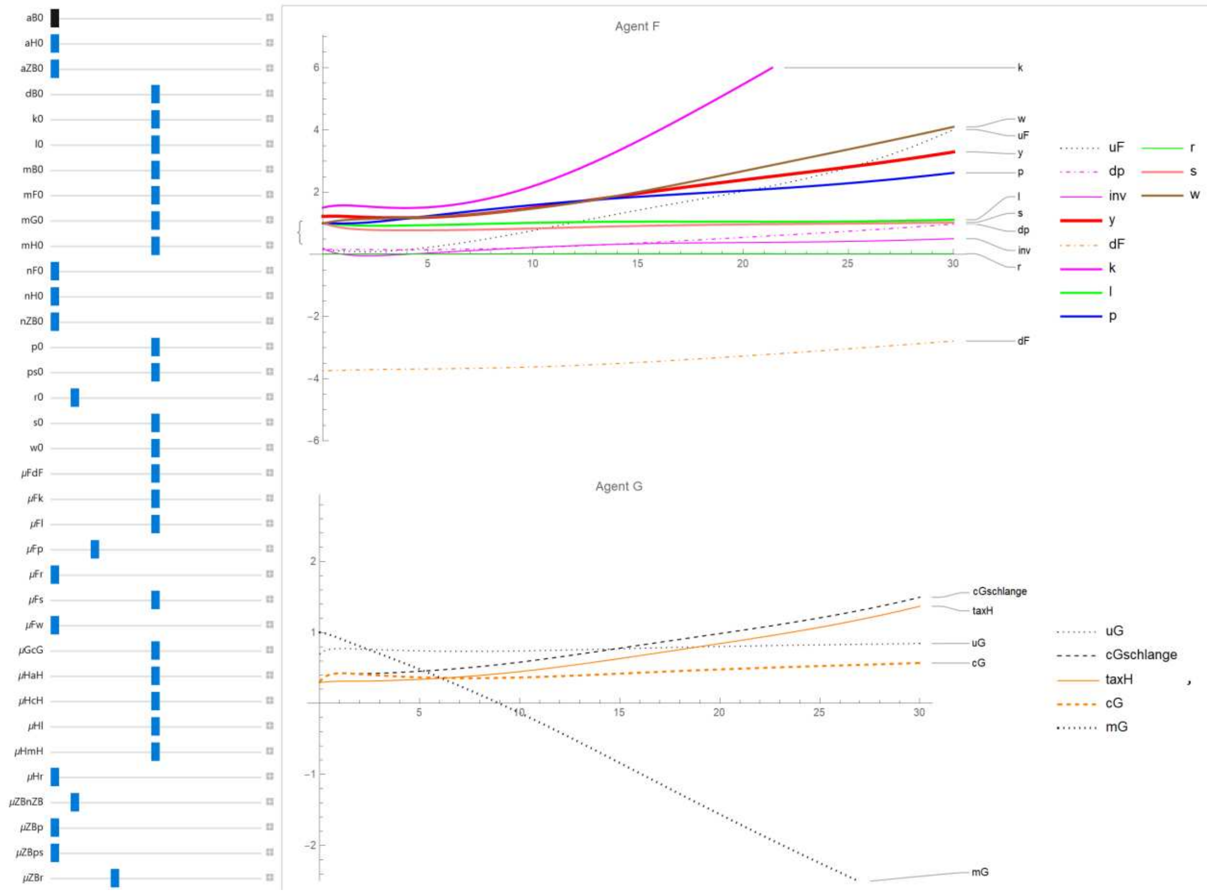
$aB'[t] = -\lambda_8[t]$
$aH'[t] = \mu HaH\, r[t] + r[t]\, \lambda_1[t] - r[t]\, \lambda_3[t] - \lambda_6[t]$
$aZB'[t] = -\lambda_9[t]$
$cG'[t] = \gamma G\, \mu GcG\, cG[t]^{-1+\gamma G} + p[t]\, \lambda_2[t] - \lambda_6[t] - p[t]\, \lambda_{10}[t]$
$cH'[t] = \gamma H\, \mu HcH\, cH[t]^{-1+\gamma H} - p[t]\, \lambda_1[t] + p[t]\, \lambda_2[t] - \lambda_5[t]$
$dB'[t] = -\lambda_8[t]$
$dF'[t] = \mu FdF\, r[t] + r[t]\, \lambda_2[t] - r[t]\, \lambda_3[t] - \lambda_7[t]$
$k'[t] = (1-\alpha)\, \beta\, \mu Fk\, k[t]^{-\alpha}\, l[t]^{\alpha}\, p[t] - \lambda_5[t]$
$l'[t] = 2\, \mu Hl\, (ldach - l[t]) + \mu Fl \left(\alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, p[t] - w[t]\right) + (w[t] - \tau H\, w[t])\, \lambda_1[t] -$
$\quad w[t]\, \lambda_2[t] + \alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, \lambda_5[t] + \tau H\, w[t]\, \lambda_{10}[t]$
$mB'[t] = -\lambda_3[t]$
$mF'[t] = -\lambda_2[t]$
$mG'[t] = -\lambda_{10}[t]$
$mH'[t] = 2\, \mu HmH\, (mHdach - mH[t]) - \lambda_1[t]$
$nF'[t] = \lambda_2[t] - \lambda_3[t] - \lambda_7[t] + \lambda_8[t]$
$nH'[t] = -\lambda_1[t] + \lambda_3[t] + \lambda_6[t] - \lambda_8[t]$
$nZB'[t] = (1-\delta)\, \mu ZBnZB \left(pdach - \frac{ps[t]}{p[t]}\right) + \lambda_3[t] - \lambda_4[t] - \lambda_8[t] + \lambda_9[t]$
$p'[t] = \beta\, \mu Fp\, k[t]^{1-\alpha}\, l[t]^{\alpha} + \frac{\mu ZBp\, ps[t]\, ((1-\delta)\, nZB[t] - \delta\, r[t])}{p[t]^2} - cH[t]\, \lambda_1[t] + (cG[t] + cH[t])\, \lambda_2[t] -$
$\quad cG[t]\, \lambda_{10}[t] - \lambda_{11}[t]$
$ps'[t] = -\frac{\mu ZBps\, ((1-\delta)\, nZB[t] - \delta\, r[t])}{p[t]} + \lambda_{11}[t]$
$r'[t] = \mu Hr\, aH[t] + \mu Fr\, dF[t] - \delta\, \mu ZBr \left(pdach - \frac{ps[t]}{p[t]}\right) + aH[t]\, \lambda_1[t] + dF[t]\, \lambda_2[t] +$
$\quad (-aH[t] - dF[t])\, \lambda_3[t]$
$rZB'[t] = -\lambda_4[t]$
$s'[t] = 2\, \mu Fs\, (sdach - s[t]) - \lambda_5[t]$
$w'[t] = -\mu Fw\, l[t] + (l[t] - \tau H\, l[t])\, \lambda_1[t] - l[t]\, \lambda_2[t] + \tau H\, l[t]\, \lambda_{10}[t]$

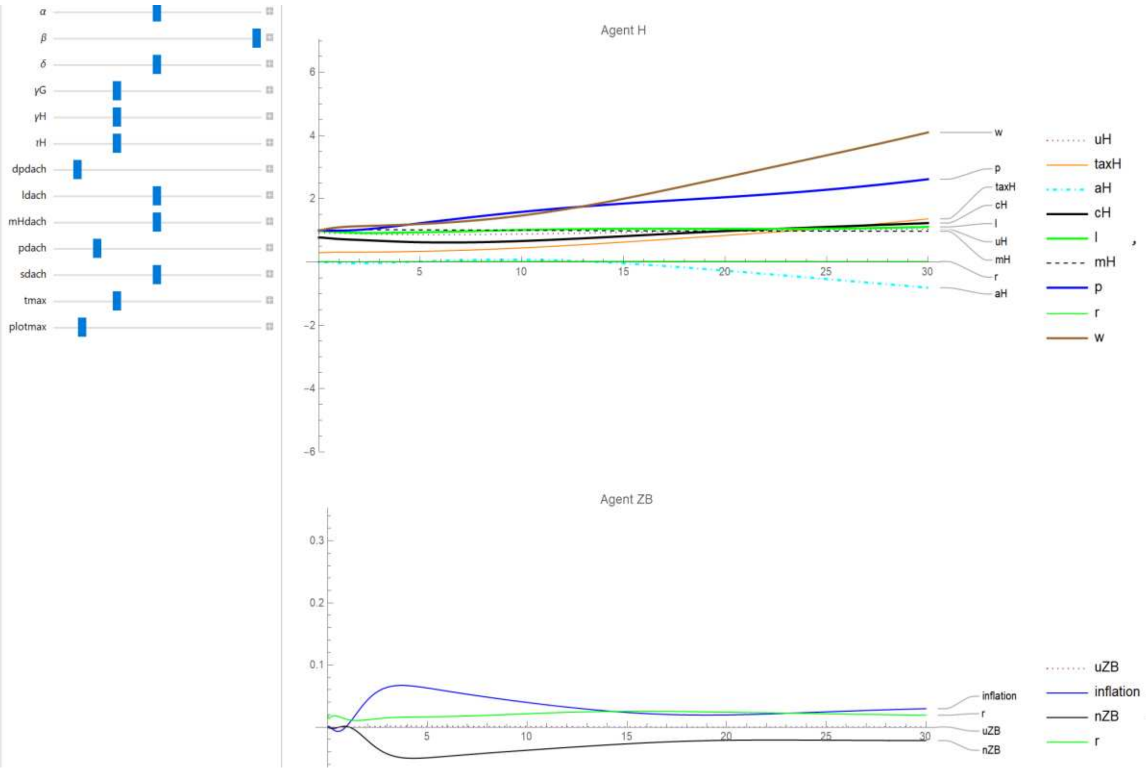
$0 = -nH[t] - cH[t] \times p[t] + aH[t] \times r[t] + l[t] \times w[t] - \tau H\, l[t] \times w[t] - mH'[t]$
$0 = nF[t] + cG[t] \times p[t] + cH[t] \times p[t] + dF[t] \times r[t] - l[t] \times w[t] - mF'[t]$
$0 = -nF[t] + nH[t] + nZB[t] - aH[t] \times r[t] - dF[t] \times r[t] - mB'[t]$
$0 = -nZB[t] - rZB'[t]$
$0 = -cG[t] - cH[t] - dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - k'[t] - s'[t]$
$0 = nH[t] - aH'[t]$
$0 = -nF[t] - dF'[t]$
$0 = nF[t] - nH[t] - nZB[t] - aB'[t] - dB'[t]$
$0 = nZB[t] - aZB'[t]$
$0 = -cG[t] \times p[t] + \tau H\, l[t] \times w[t] - mG'[t]$
$0 = ps[t] - p'[t]$

$aB[0] = aB0$
$aH[0] = aH0$
$aZB[0] = aZB0$
$cG[0] = \frac{l0\, w0\, \tau H}{p0}$
$cH[0] = \frac{-dpdach\, k0\, p0 + k0^{1-\alpha}\, l0^{\alpha}\, p0\, \beta - l0\, w0\, \tau H}{p0}$
$dB[0] = dB0$
$dF[0] = -\frac{-dpdach\, k0\, p0 - l0\, w0 + k0^{1-\alpha}\, l0^{\alpha}\, p0\, \beta}{r0}$
$k[0] = k0$
$l[0] = l0$
$mB[0] = mB0$
$mF[0] = mF0$
$mG[0] = mG0$
$mH[0] = mH0$
$nF[0] = nF0$
$nH[0] = nH0$
$nZB[0] = nZB0$
$p[0] = p0$
$ps[0] = ps0$
$r[0] = r0$
$rZB[0] = -mB0 - mF0 - mG0 - mH0$
$s[0] = s0$
$w[0] = w0$

11.2. Calculation results of model C1

https://www.dropbox.com/s/vh378ffs1t9oik4/Modell%20C1%20Version%2019.ndsolve.nb?dl=0

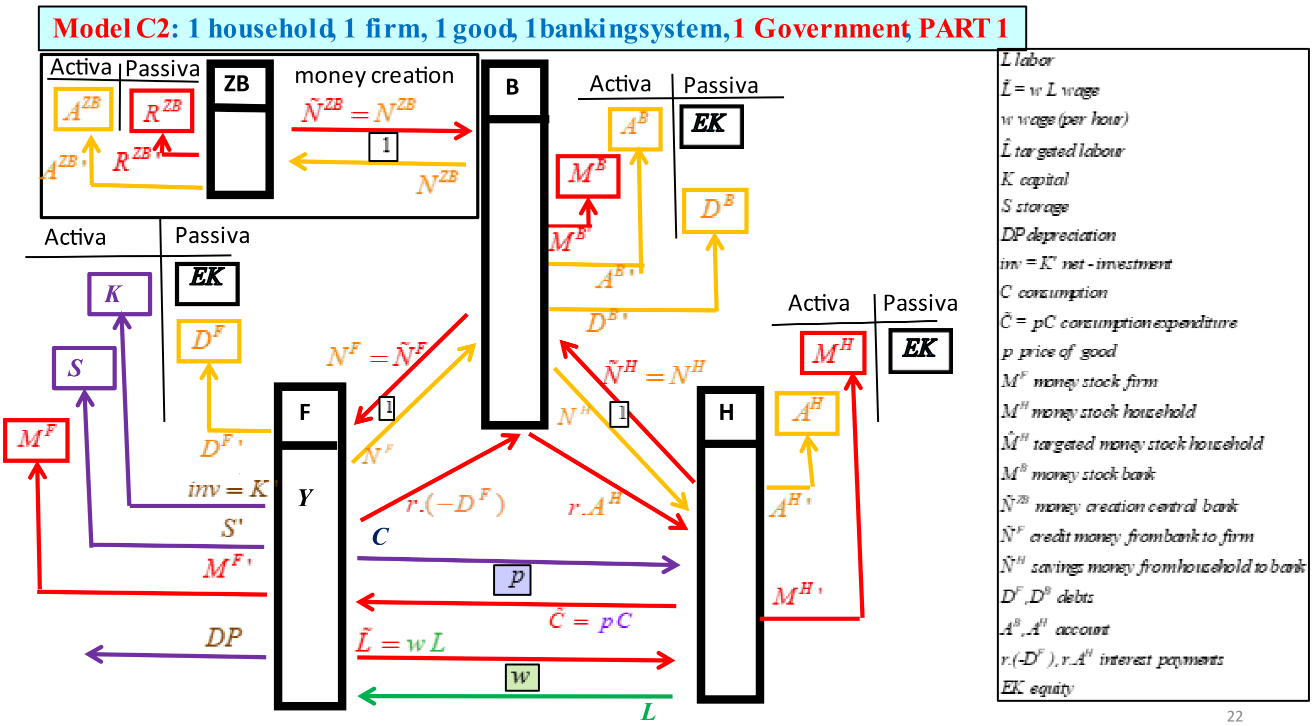



α
β
δ
γG
γH
τH
dpdach
ldach
mHdach
pdach
sdach
tmax
plotmax
Agent H
w
p
taxH
cH
l
uH
mH
r
aH
uH
taxH
aH
cH
l
mH
p
r
w
Agent ZB
inflation
r
uZB
nZB
uZB
inflation
nZB
r

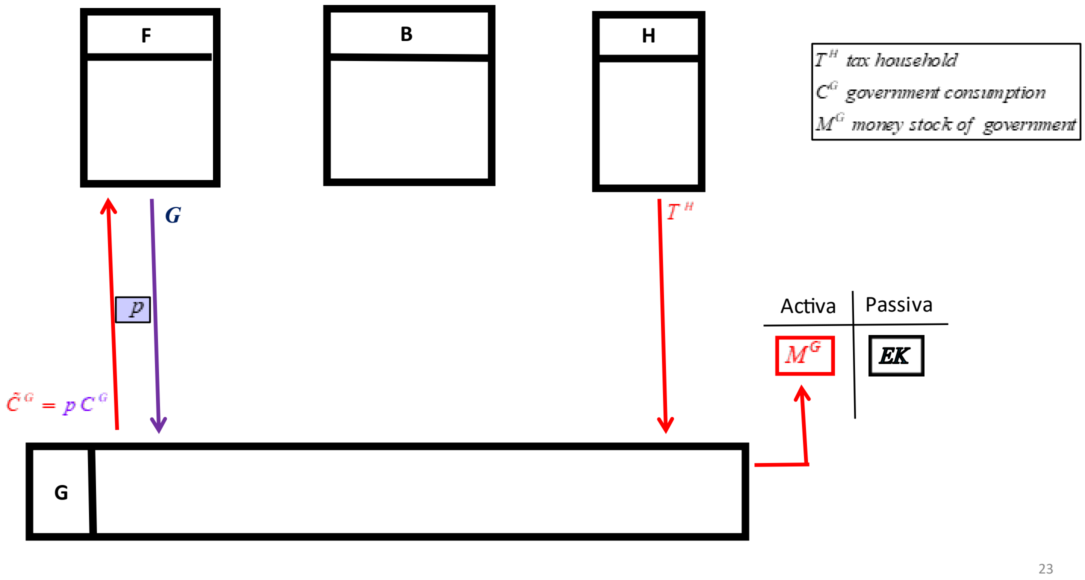
12. Model C2, (1 household, 1 firm, 1 good, 1 banking system, 1 government)), standard Taylor rule

12.1. Set up

Model C2 corresponds to the extension of model B2 by the agent government in the sense of model C1 and corresponds to model C1 with the change that the central bank acts in the sense of the standard Taylor rule.



Model C2: 1 household, 1 firm, 1 good, 1 bankingsystem, 1 Government, PART 2



Model C2 : basic equations

algebraically defined variables

$Y(L,K) =$	$\beta L^{\alpha} K^{1-\alpha}$	"production function"
$DP(K) =$	dpK	"depreciation"
$T^H(w,L) =$	$\tau^H w L$	"income tax household"

$$r = \frac{p'}{p} + \sigma_1\left(\frac{p'}{p} - \hat{p}\right) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha\frac{L'}{L}$$ "standard Taylor rule"

utility functions

$U^H = C^{\gamma_H} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 + rA^H$	"utility function household"
$U^F = pY - wL - (\hat{S} - S)^2 - r(-D^F)$	"utility function firm"
$U^B = 0$	"utility function bank"
$U^{ZB} = 0$	"utility function central bank"
$U^G = G^{\gamma_G}$	"utility function government"

constraints

$Z_1 = 0 = wL - pC + rA^H - N^H - T^H - M^{H\,\prime}$	for money flow of household H
$Z_2 = 0 = -wL + pC - r(-D^F) + N^F + pG - M^{F\,\prime}$	for money flow of firm F
$Z_3 = 0 = N^{ZB} - N^F + r(-D^F) - rA^H + N^H - M^{B\,\prime}$	for money flow of bank B
$Z_4 = 0 = -N^{ZB} - R^{ZB\,\prime}$	for money flow of central bank ZB
$Z_5 = 0 = Y(L,K) - C - G - S' - DP - K'$	for flow of good 1 of firm F
$Z_6 = 0 = N^H - A^{H\,\prime}$	for accounts / debts flow of household H
$Z_7 = 0 = -N^F - D^{F\,\prime}$	for accounts / debts flow of firm F
$Z_8 = 0 = -N^{ZB} + N^F - N^H - D^{B\,\prime} - A^{B\,\prime}$	for accounts / debts flow of of bank B
$Z_9 = 0 = N^{ZB} - A^{ZB\,\prime}$	for accounts / debts flow of central bank ZB
$Z_{10} = 0 = -pG + T^H - M^{G\,\prime}$	for money flow of government
$Z_{11} = 0 = ps - p'$	because "no derivation in utility function of ZB"

Model C2: diff.-alg. equation system

$$uB[t] == 0$$
$$uF[t] == dF[t] \times r[t] - (sdach - s[t])^2 - l[t] \times w[t] + p[t] \times y[t]$$
$$uG[t] == cG[t]^{\gamma G}$$
$$uH[t] == cH[t]^{\gamma H} - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] \times r[t]$$
$$uZB[t] == 0$$

$$cGschlange[t] == cG[t] \times p[t]$$
$$dp[t] == dpdach\, k[t]$$
$$inflation[t] == \frac{ps[t]}{p[t]}$$
$$inv[t] == k'[t]$$
$$r[t] == -pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}$$
$$taxH[t] == \tau H\, l[t] \times w[t]$$
$$y[t] == \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}$$

$$aB'[t] == -\lambda_8[t]$$
$$aH'[t] == -\lambda_6[t] + \lambda_5[t] \left(pdach\, \sigma 1 - \frac{\sigma 1\, ps[t]}{p[t]} + \frac{(-1+\alpha)\, k'[t]}{k[t]} - \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \mu HaH \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \lambda_1[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)$$
$$aZB'[t] == -\lambda_9[t]$$
$$cG'[t] == \gamma G\, \mu GcG\, cG[t]^{-1+\gamma G} + p[t]\, \lambda_2[t] - \lambda_5[t] - p[t]\, \lambda_{10}[t]$$
$$cH'[t] == \gamma H\, \mu HcH\, cH[t]^{-1+\gamma H} - p[t]\, \lambda_1[t] + p[t]\, \lambda_2[t] - \lambda_5[t]$$
$$dB'[t] == -\lambda_8[t]$$
$$dF'[t] == -\lambda_7[t] + \lambda_5[t] \left(pdach\, \sigma 1 - \frac{\sigma 1\, ps[t]}{p[t]} + \frac{(-1+\alpha)\, k'[t]}{k[t]} - \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \mu FdF \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \lambda_2[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)$$
$$k'[t] == If\left[-\frac{(-1+\alpha)\, aH[t]}{k[t]} \neq 0,\ -\frac{(-1+\alpha)\, aH[t]}{k[t]},\ \frac{(-1+\alpha)\, aH[t]\, k'[t]}{k[t]^2}\right] \lambda_1[t] + If\left[-\frac{(-1+\alpha)\, dF[t]}{k[t]} \neq 0,\ -\frac{(-1+\alpha)\, dF[t]}{k[t]},\ \frac{(-1+\alpha)\, dF[t]\, k'[t]}{k[t]^2}\right] \lambda_2[t] + If\left[\frac{(-1+\alpha)\, aH[t]}{k[t]} + \frac{(-1+\alpha)\, dF[t]}{k[t]} \neq 0,\ \frac{(-1+\alpha)\, aH[t]}{k[t]} + \frac{(-1+\alpha)\, dF[t]}{k[t]},\ -\frac{(-1+\alpha)\, aH[t]\, k'[t]}{k[t]^2} - \frac{(-1+\alpha)\, dF[t]\, k'[t]}{k[t]^2}\right] \lambda_3[t] - \lambda_5[t] + \frac{(-1+\alpha)\, \mu Hk\, aH[t]\, k'[t]}{k[t]^2} + \mu Fk \left((1-\alpha)\, \beta\, k[t]^{-\alpha}\, l[t]^{\alpha}\, p[t] + \frac{(-1+\alpha)\, dF[t]\, k'[t]}{k[t]^2}\right)$$
$$l'[t] == If\left[\frac{\alpha\, (1+\sigma 2)\, aH[t]}{l[t]} \neq 0,\ \frac{\alpha\, (1+\sigma 2)\, aH[t]}{l[t]},\ w[t] - \tau H\, w[t] - \frac{\alpha\, (1+\sigma 2)\, aH[t]\, l'[t]}{l[t]^2}\right] \lambda_1[t] + If\left[\frac{\alpha\, (1+\sigma 2)\, dF[t]}{l[t]} \neq 0,\ \frac{\alpha\, (1+\sigma 2)\, dF[t]}{l[t]},\ -w[t] - \frac{\alpha\, (1+\sigma 2)\, dF[t]\, l'[t]}{l[t]^2}\right] \lambda_2[t] + If\left[-\frac{\alpha\, (1+\sigma 2)\, aH[t]}{l[t]} - \frac{\alpha\, (1+\sigma 2)\, dF[t]}{l[t]} \neq 0,\ -\frac{\alpha\, (1+\sigma 2)\, aH[t]}{l[t]} - \frac{\alpha\, (1+\sigma 2)\, dF[t]}{l[t]},\ \frac{\alpha\, (1+\sigma 2)\, aH[t]\, l'[t]}{l[t]^2} + \frac{\alpha\, (1+\sigma 2)\, dF[t]\, l'[t]}{l[t]^2}\right] \lambda_3[t] + \alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, \lambda_5[t] + \tau H\, w[t]\, \lambda_{10}[t] + \mu Hl \left(2\, (ldach - l[t]) - \frac{\alpha\, (1+\sigma 2)\, aH[t]\, l'[t]}{l[t]^2}\right) + \mu Fl \left(\alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, p[t] - w[t] - \frac{\alpha\, (1+\sigma 2)\, dF[t]\, l'[t]}{l[t]^2}\right)$$

$$mB'[t] == -\lambda_3[t]$$
$$mF'[t] == -\lambda_2[t]$$
$$mG'[t] == -\lambda_{10}[t]$$
$$mH'[t] == 2\, \mu HmH\, (mHdach - mH[t]) - \lambda_1[t]$$
$$nF'[t] == \lambda_2[t] - \lambda_3[t] - \lambda_7[t] + \lambda_8[t]$$
$$nH'[t] == -\lambda_1[t] + \lambda_3[t] + \lambda_6[t] - \lambda_8[t]$$
$$nZB'[t] == \lambda_3[t] - \lambda_4[t] - \lambda_8[t] + \lambda_9[t]$$
$$p'[t] == -\frac{\mu Hp\, \sigma 1\, aH[t] \times ps[t]}{p[t]^2} + \mu Fp \left(\beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - \frac{\sigma 1\, dF[t] \times ps[t]}{p[t]^2}\right) + \left(-cH[t] - \frac{\sigma 1\, aH[t] \times ps[t]}{p[t]^2}\right) \lambda_1[t] + \left(cG[t] + cH[t] - \frac{\sigma 1\, dF[t] \times ps[t]}{p[t]^2}\right) \lambda_2[t] + \left(\frac{\sigma 1\, aH[t] \times ps[t]}{p[t]^2} + \frac{\sigma 1\, dF[t] \times ps[t]}{p[t]^2}\right) \lambda_3[t] - cG[t]\, \lambda_{10}[t] - \lambda_{11}[t]$$
$$ps'[t] == \frac{\mu Hps\, \sigma 1\, aH[t]}{p[t]} + \frac{\mu Fps\, \sigma 1\, dF[t]}{p[t]} + \frac{\sigma 1\, aH[t]\, \lambda_1[t]}{p[t]} + \frac{\sigma 1\, dF[t]\, \lambda_2[t]}{p[t]} + \left(-\frac{\sigma 1\, aH[t]}{p[t]} - \frac{\sigma 1\, dF[t]}{p[t]}\right) \lambda_3[t] + \lambda_{11}[t]$$
$$rZB'[t] == -\lambda_4[t]$$
$$s'[t] == 2\, \mu Fs\, (sdach - s[t]) - \lambda_5[t]$$
$$w'[t] == -\mu Fw\, l[t] + (l[t] - \tau H\, l[t])\, \lambda_1[t] - l[t]\, \lambda_2[t] + \tau H\, l[t]\, \lambda_{10}[t]$$

$$0 = -\mathrm{nH}[t] - \mathrm{cH}[t] \times p[t] + l[t] \times w[t] - \tau H\, l[t] \times w[t] + \mathrm{aH}[t] \left(-\mathrm{pdach}\, \sigma 1 + \frac{\sigma 1\, \mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \mathrm{mH}'[t]$$

$$0 = \mathrm{nF}[t] + \mathrm{cG}[t] \times p[t] + \mathrm{cH}[t] \times p[t] - l[t] \times w[t] + \mathrm{dF}[t] \left(-\mathrm{pdach}\, \sigma 1 + \frac{\sigma 1\, \mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \mathrm{mF}'[t]$$

$$0 = -\mathrm{nF}[t] + \mathrm{nH}[t] + \mathrm{nZB}[t] - \mathrm{aH}[t] \left(-\mathrm{pdach}\, \sigma 1 + \frac{\sigma 1\, \mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \mathrm{dF}[t] \left(-\mathrm{pdach}\, \sigma 1 + \frac{\sigma 1\, \mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \mathrm{mB}'[t]$$

$$0 = -\mathrm{nZB}[t] - \mathrm{rZB}'[t]$$

$$0 = -\mathrm{cG}[t] - \mathrm{cH}[t] - \mathrm{dpdach}\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - k'[t] - s'[t]$$

$$0 = \mathrm{nH}[t] - \mathrm{aH}'[t]$$

$$0 = -\mathrm{nF}[t] - \mathrm{dF}'[t]$$

$$0 = \mathrm{nF}[t] - \mathrm{nH}[t] - \mathrm{nZB}[t] - \mathrm{aB}'[t] - \mathrm{dB}'[t]$$

$$0 = \mathrm{nZB}[t] - \mathrm{aZB}'[t]$$

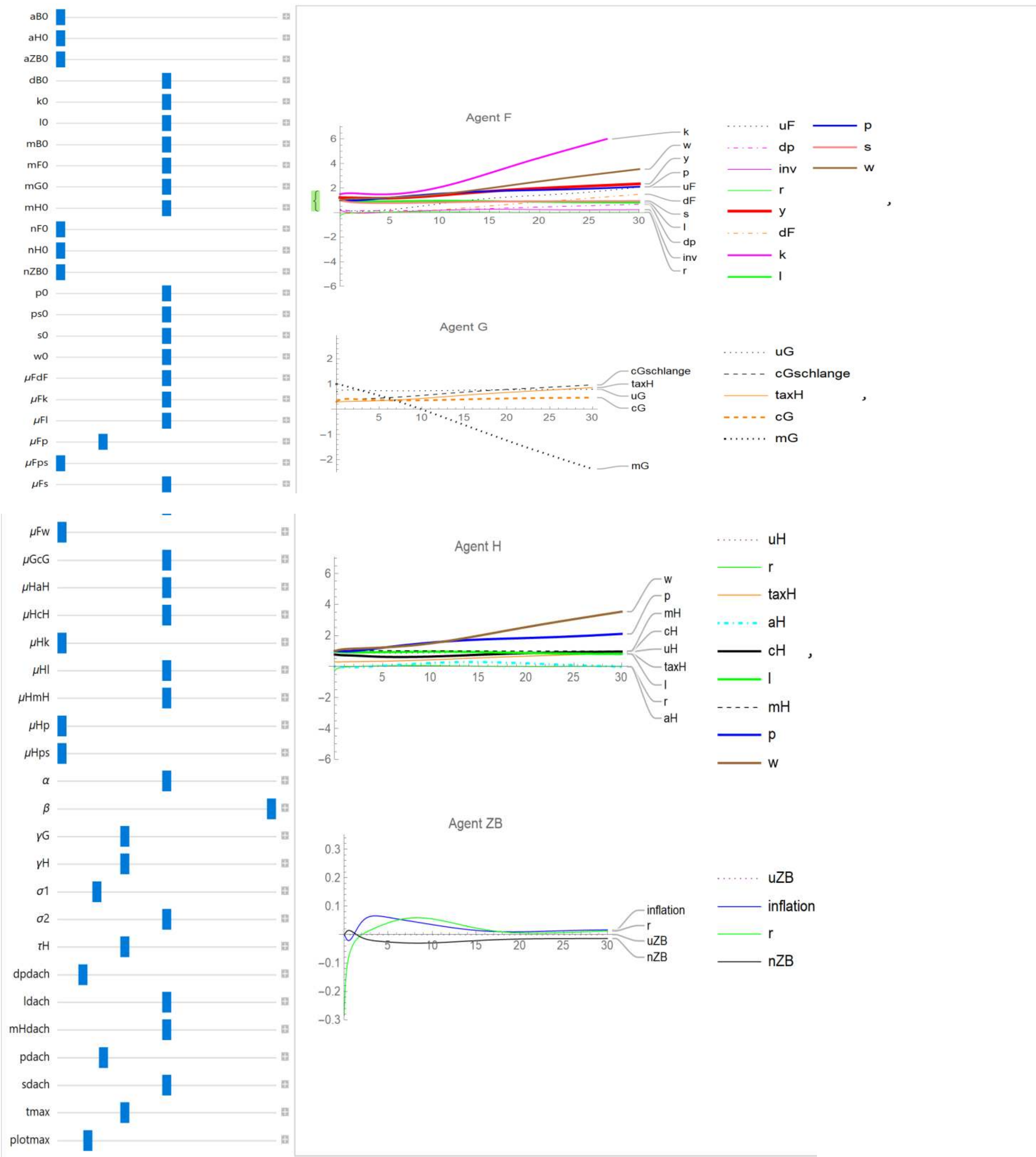
$$0 = -\mathrm{cG}[t] \times p[t] + \tau H\, l[t] \times w[t] - \mathrm{mG}'[t]$$

$$0 = \mathrm{ps}[t] - p'[t]$$

$$\mathrm{aB}[0] = \mathrm{aB0}$$
$$\mathrm{aH}[0] = \mathrm{aH0}$$
$$\mathrm{aZB}[0] = \mathrm{aZB0}$$
$$\mathrm{cG}[0] = \frac{l0\, w0\, \tau H}{p0}$$
$$\mathrm{cH}[0] = \frac{-\mathrm{dpdach}\, k0\, p0 + k0^{1-\alpha}\, l0^{\alpha}\, p0\, \beta - l0\, w0\, \tau H}{p0}$$
$$\mathrm{dB}[0] = \mathrm{dB0}$$
$$\mathrm{dF}[0] = \mathrm{dpdach}\, k0\, p0 + l0\, w0 - k0^{1-\alpha}\, l0^{\alpha}\, p0\, \beta$$
$$k[0] = k0$$
$$l[0] = l0$$
$$\mathrm{mB}[0] = \mathrm{mB0}$$
$$\mathrm{mF}[0] = \mathrm{mF0}$$
$$\mathrm{mG}[0] = \mathrm{mG0}$$
$$\mathrm{mH}[0] = \mathrm{mH0}$$
$$\mathrm{nF}[0] = \mathrm{nF0}$$
$$\mathrm{nH}[0] = \mathrm{nH0}$$
$$\mathrm{nZB}[0] = \mathrm{nZB0}$$
$$p[0] = p0$$
$$\mathrm{ps}[0] = \mathrm{ps0}$$
$$\mathrm{rZB}[0] = -\mathrm{mB0} - \mathrm{mF0} - \mathrm{mG0} - \mathrm{mH0}$$
$$s[0] = s0$$
$$w[0] = w0$$

12.2. Calculation results of model C2

https://www.dropbox.com/s/e630zq6ta6jmva6/Modell%20C2%20Version%201%20%28Taylor-Regel%29.ndsolve.nb?dl=0



13. Model D2, comprehensive model

13.1. Setup

We extend the C2 model in the following aspects:

- The interest rate is formulated in more detail: in central bank policy rate, premium for lending rates, premium for savings rates r_{leit}, r_D, r_A

- The central bank distributes the profit to the government, the bank distributes the profit to the household, the firm distributes a part of the profit to the household

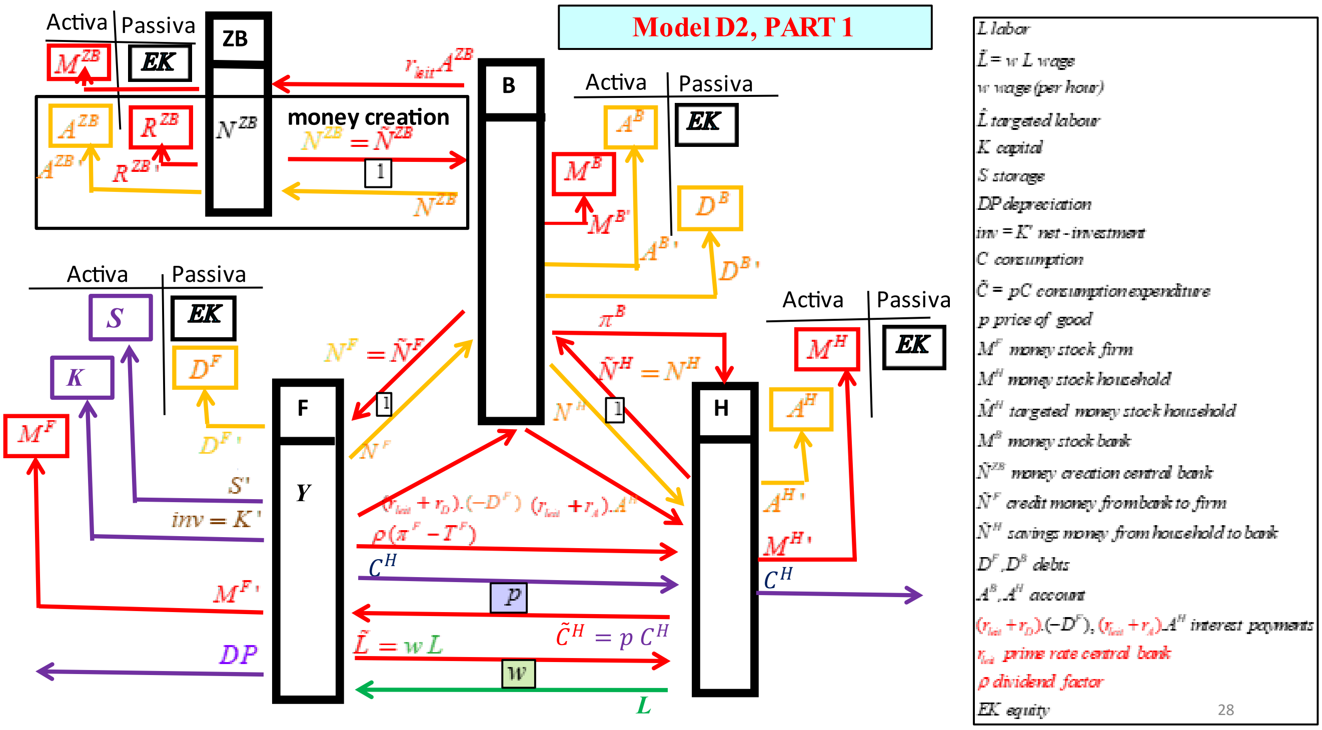
- Taxes are composed of income and property taxes for household and firm.

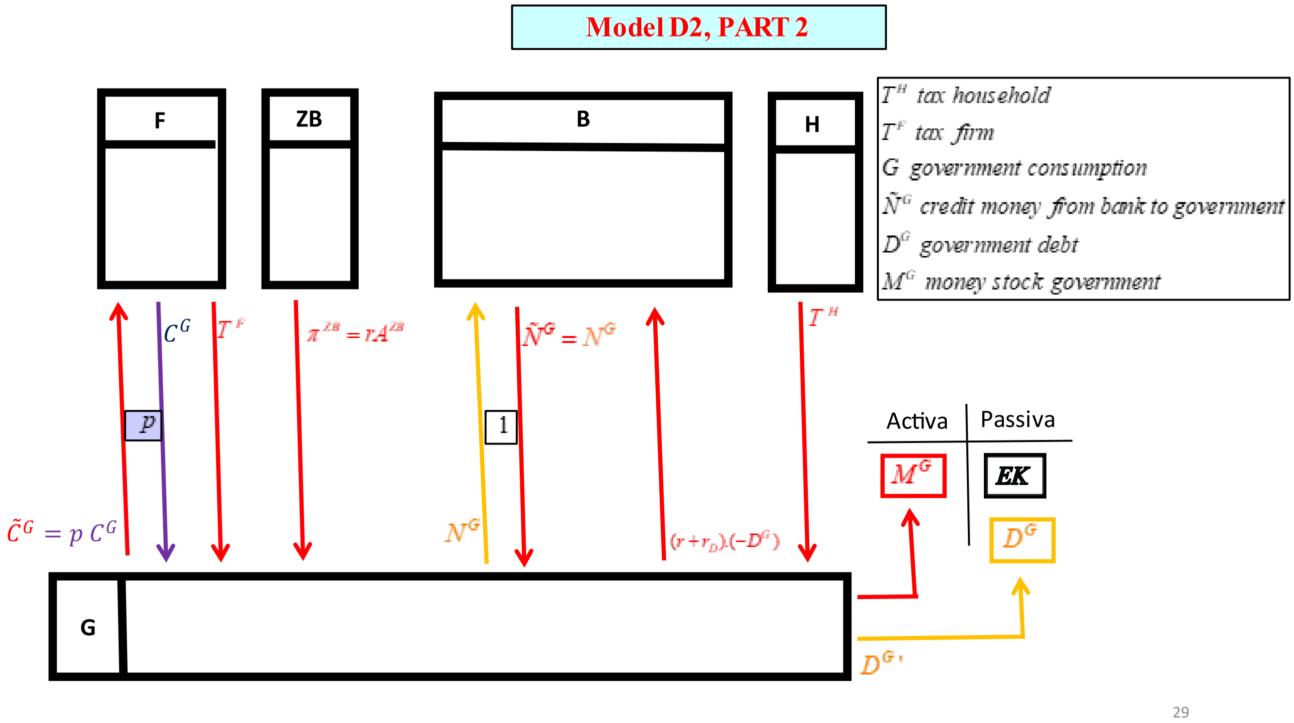
- The government aims to achieve a debt level of 60% of GDP in line with the Maastricht criterion.

- The central bank acts according to the modified Taylor rule

- The target level of the firm's investment and the target level of the household's money stock are interest rate dependent.

- The targeted level of the government's money stock is not interest rate dependent


Model D2, PART 1
Activa
Passiva
ZB
M^{ZB}
EK
A^{ZB}
R^{ZB}
N^{ZB}
money creation
$N^{ZB} = \tilde{N}^{ZB}$
$r_{leit} A^{ZB}$
B
A^{B}
M^{B}
D^{B}
π^{B}
F
Y
S
K
M^{F}
D^{F}
S'
$inv = K'$
DP
$N^{F} = \tilde{N}^{F}$
H
M^{H}
A^{H}
$\tilde{N}^{H} = N^{H}$
C^{H}
p
$\tilde{L} = w\,L$
$\tilde{C}^{H} = p\,C^{H}$
w
L
$\rho(\pi^{F} - T^{F})$
L labor
$\tilde{L} = w\,L$ wage
w wage (per hour)
$\hat{L}$ targeted labour
K capital
S storage
DP depreciation
$inv = K'$ net-investment
C consumption
$\tilde{C} = pC$ consumption expenditure
p price of good
M^{F} money stock firm
M^{H} money stock household
$\hat{M}^{H}$ targeted money stock household
M^{B} money stock bank
$\tilde{N}^{ZB}$ money creation central bank
$\tilde{N}^{F}$ credit money from bank to firm
$\tilde{N}^{H}$ savings money from household to bank
D^{F}, D^{B} debts
A^{B}, A^{H} account
$(r_{leit}+r_{D}).(-D^{F}),\ (r_{leit}+r_{A}).A^{H}$ interest payments
r_{leit} prime rate central bank
ρ dividend factor
EK equity
28


Model D2, PART 2
F
ZB
B
H
G
T^{H} tax household
T^{F} tax firm
G government consumption
$\tilde{N}^{G}$ credit money from bank to government
D^{G} government debt
M^{G} money stock government
C^{G}
T^{F}
$\pi^{ZB} = rA^{ZB}$
$\tilde{N}^{G} = N^{G}$
T^{H}
p
1
$\tilde{C}^{G} = p\,C^{G}$
N^{G}
$(r+r_{D}).(-D^{G})$
Activa
Passiva
M^{G}
EK
D^{G}
$D^{G}{}'$
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Model D2 : basic equations

algebraically defined variables

$DP = \overline{dp}\, K$ "depreciation"

$inflation = \frac{p'}{p}$ "inflation"

$inv = K'$ "net investment"

$invmax = 0.1\, K$ "maximal investment (inv if $r_{last} + r_D = 0$)"

$r = -r_D + \frac{p'}{p} + \sigma_1 (\frac{p'}{p} - \hat{p}) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha \frac{L'}{L}$ "modified Taylor rule"

$T^H = \tau^H (wL + (r + r_A) A^H) + \upsilon^H (A^H + M^H)$ "income tax + asset tax H"

$T^F = \tau^F \left(p\beta L^\alpha K^{1-\alpha} - wL - (r + r_D)(-D^F) - DP \right) +$

$\upsilon^F \left(M^F + S + K + D^F \right)$ "income tax + asset tax F"

$\pi^B = profit^B = +(r + r_D).(-D^F) + (r + r_D).(-D^G) - rA^{ZB} - (r + r_A).A^H$ "profit before taxes B"

$\pi^F = profit^F = p\beta L^\alpha K^{1-\alpha} - wL - (r + r_D)(-D^F) - DP$ "profit before taxes F"

$\pi^{ZB} = profit^{ZB} = r\, A^{ZB}$ "profit ZB"

$Y = \beta L^\alpha K^{1-\alpha}$ "production function"

Nutzenfunktionen

$U^B = profit^B$ "utility function bank B"

$U^F = profit^F - (\hat{S} - S)^2 - \left(invmax(1 - \eta(r + r_D)) - inv \right)^2$ "utility function firm F"

$U^G = (C^G)^{\gamma_G} - (\hat{D}^G Y - D^G)^2$ "utility function government G"

$U^H = (C^H)^{\gamma_H} - \left(\hat{L} - L \right)^2 - (mhmax(1 - \theta(r + r_A)) - M^H)^2 + (r + r_A)A^H$ "utility function H"

$U^{ZB} = 0$ "utility function central bank ZB"

constraints

$Z_1 = 0 = -r\, A^{ZB} + N^{ZB} - N^F + (r + r_D)(-D^F) + (r + r_D).(-D^G)$

$\quad -(r + r_A)A^H + N^H - \pi^B - N^G - M^{B\,\prime}$ money flow B

$Z_2 = 0 = -wL + pC^H - \rho(\pi^F - T^F) - (r + r_D)(-D^F)$

$\quad +N^F + pC^G - T^F - M^{F\,\prime}$ money flow F

$Z_3 = 0 = -pC^G + T^F + \pi^{ZB} + N^G - (r + r_D).(-D^G) + T^H - M^{G\,\prime}$ money flow G

$Z_4 = 0 = wL - pC^H + \rho(\pi^F - T^F) + (r + r_A)A^H - N^H + \pi^B - T^H - M^{H\,\prime}$ money flow H

$Z_5 = 0 = r\, A^{ZB} - \pi^{ZB} - M^{ZB\,\prime}$ money flow ZB

$Z_6 = 0 = -N^{ZB} - R^{ZB\,\prime}$ reserve flow ZB

$Z_7 = 0 = Y - DP - C^H - C^G - K' - S'$ flow of good F

$Z_8 = 0 = -N^{ZB} + N^F - N^H + N^G - A^{B\,\prime} - D^{B\,\prime}$ accounts / debts flow B

$Z_9 = 0 = -N^F - D^{F\,\prime}$ accounts / debts flow F

$Z_{10} = 0 = -N^G - D^{G\,\prime}$ accounts / debts flow G

$Z_{11} = 0 = N^H - A^{H\,\prime}$ accounts / debts flow H

$Z_{12} = 0 = N^{ZB} - A^{ZB\,\prime}$ accounts / debts flow ZB

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$$\mathrm{uB}[t] == \mathrm{profitB}[t]$$
$$\mathrm{uF}[t] == \mathrm{profitF}[t] - (-\mathrm{inv}[t] + \mathrm{invmax}[t]\,(1 - \eta\,(\mathrm{rd}[t] + \mathrm{rleit}[t])))^2 - (\mathrm{sdach} - \mathrm{s}[t])^2$$
$$\mathrm{uG}[t] == \mathrm{cG}[t]^{\gamma G} - (\mathrm{mGdach} - \mathrm{mG}[t])^2 - (-\mathrm{dG}[t] + \mathrm{dGdach}\,\mathrm{y}[t])^2$$
$$\mathrm{uH}[t] == \mathrm{cH}[t]^{\gamma H} - (\mathrm{ldach} - \mathrm{l}[t])^2 + \mathrm{aH}[t]\,(\mathrm{ra}[t] + \mathrm{rleit}[t]) - (-\mathrm{mH}[t] + \mathrm{mHmax}\,(1 - \theta\,(\mathrm{ra}[t] + \mathrm{rleit}[t])))^2$$
$$\mathrm{uZB}[t] == 0$$

$$\mathrm{dp}[t] == \mathrm{dpdach}\,\mathrm{k}[t]$$
$$\mathrm{inflation}[t] == \frac{\mathrm{p}'[t]}{\mathrm{p}[t]}$$
$$\mathrm{inv}[t] == \mathrm{k}'[t]$$
$$\mathrm{invmax}[t] == 0.1\text{`}\ \mathrm{k}[t]$$
$$\mathrm{profitB}[t] == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitF}[t] == -\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitZB}[t] == \mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{rleit}[t] == -\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}$$
$$\mathrm{tF}[t] == \nu F\,(\mathrm{dF}[t] + \mathrm{k}[t] + \mathrm{mF}[t] + \mathrm{s}[t]) + \tau F\left(-\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{tH}[t] == \nu H\,(\mathrm{aH}[t] + \mathrm{mH}[t]) + \tau H\left(\mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{y}[t] == \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}$$

$$\mathrm{uB}[t] == \mathrm{profitB}[t]$$
$$\mathrm{uF}[t] == \mathrm{profitF}[t] - (-\mathrm{inv}[t] + \mathrm{invmax}[t]\,(1 - \eta\,(\mathrm{rd}[t] + \mathrm{rleit}[t])))^2 - (\mathrm{sdach} - \mathrm{s}[t])^2$$
$$\mathrm{uG}[t] == \mathrm{cG}[t]^{\gamma G} - (\mathrm{mGdach} - \mathrm{mG}[t])^2 - (-\mathrm{dG}[t] + \mathrm{dGdach}\,\mathrm{y}[t])^2$$
$$\mathrm{uH}[t] == \mathrm{cH}[t]^{\gamma H} - (\mathrm{ldach} - \mathrm{l}[t])^2 + \mathrm{aH}[t]\,(\mathrm{ra}[t] + \mathrm{rleit}[t]) - (-\mathrm{mH}[t] + \mathrm{mHmax}\,(1 - \theta\,(\mathrm{ra}[t] + \mathrm{rleit}[t])))^2$$
$$\mathrm{uZB}[t] == 0$$

$$\mathrm{dp}[t] == \mathrm{dpdach}\,\mathrm{k}[t]$$
$$\mathrm{inflation}[t] == \frac{\mathrm{p}'[t]}{\mathrm{p}[t]}$$
$$\mathrm{inv}[t] == \mathrm{k}'[t]$$
$$\mathrm{invmax}[t] == 0.1\text{`}\ \mathrm{k}[t]$$
$$\mathrm{profitB}[t] == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitF}[t] == -\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitZB}[t] == \mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{rleit}[t] == -\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}$$
$$\mathrm{tF}[t] == \nu F\,(\mathrm{dF}[t] + \mathrm{k}[t] + \mathrm{mF}[t] + \mathrm{s}[t]) + \tau F\left(-\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{tH}[t] == \nu H\,(\mathrm{aH}[t] + \mathrm{mH}[t]) + \tau H\left(\mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{y}[t] == \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}$$

$$\mathrm{uB}[t] == \mathrm{profitB}[t]$$
$$\mathrm{uF}[t] == \mathrm{profitF}[t] - (-\mathrm{inv}[t] + \mathrm{invmax}[t]\,(1 - \eta\,(\mathrm{rd}[t] + \mathrm{rleit}[t])))^2 - (\mathrm{sdach} - \mathrm{s}[t])^2$$
$$\mathrm{uG}[t] == \mathrm{cG}[t]^{\gamma G} - (\mathrm{mGdach} - \mathrm{mG}[t])^2 - (-\mathrm{dG}[t] + \mathrm{dGdach}\,\mathrm{y}[t])^2$$
$$\mathrm{uH}[t] == \mathrm{cH}[t]^{\gamma H} - (\mathrm{ldach} - \mathrm{l}[t])^2 + \mathrm{aH}[t]\,(\mathrm{ra}[t] + \mathrm{rleit}[t]) - (-\mathrm{mH}[t] + \mathrm{mHmax}\,(1 - \theta\,(\mathrm{ra}[t] + \mathrm{rleit}[t])))^2$$
$$\mathrm{uZB}[t] == 0$$

$$\mathrm{dp}[t] == \mathrm{dpdach}\,\mathrm{k}[t]$$
$$\mathrm{inflation}[t] == \frac{\mathrm{p}'[t]}{\mathrm{p}[t]}$$
$$\mathrm{inv}[t] == \mathrm{k}'[t]$$
$$\mathrm{invmax}[t] == 0.1\grave{}\ \mathrm{k}[t]$$
$$\mathrm{profitB}[t] == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitF}[t] == -\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitZB}[t] == \mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{rleit}[t] == -\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}$$
$$\mathrm{tF}[t] == \nu\mathrm{F}\,(\mathrm{dF}[t] + \mathrm{k}[t] + \mathrm{mF}[t] + \mathrm{s}[t]) + \tau\mathrm{F}\left(-\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{tH}[t] == \nu\mathrm{H}\,(\mathrm{aH}[t] + \mathrm{mH}[t]) + \tau\mathrm{H}\left(\mathrm{l}[t]\times\mathrm{w}[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{y}[t] == \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}$$

$$
\begin{aligned}
ps'[t] =\,& \mu Bps \left(-\frac{\sigma 1\, aH[t]}{p[t]} - \frac{\sigma 1\, aZB[t]}{p[t]} - \frac{\sigma 1\, dF[t]}{p[t]} - \frac{\sigma 1\, dG[t]}{p[t]}\right) + \\
& \left(\rho \left(\frac{\sigma 1\, dF[t]}{p[t]} - \frac{\sigma 1\, \tau F\, dF[t]}{p[t]}\right) - \frac{\sigma 1\, \tau H\, aH[t]}{p[t]} - \frac{\sigma 1\, aZB[t]}{p[t]} - \frac{\sigma 1\, dF[t]}{p[t]} - \frac{\sigma 1\, dG[t]}{p[t]}\right) \lambda_2[t] + \\
& \left(\frac{\sigma 1\, \tau H\, aH[t]}{p[t]} + \frac{\sigma 1\, aZB[t]}{p[t]} + \frac{\sigma 1\, \tau F\, dF[t]}{p[t]} + \frac{\sigma 1\, dG[t]}{p[t]}\right) \lambda_7[t] + \\
& \left(-\rho \left(\frac{\sigma 1\, dF[t]}{p[t]} - \frac{\sigma 1\, \tau F\, dF[t]}{p[t]}\right) + \frac{\sigma 1\, dF[t]}{p[t]} - \frac{\sigma 1\, \tau F\, dF[t]}{p[t]}\right) \lambda_9[t] + \lambda_{13}[t] + \\
& \mu Fps \left(\frac{\sigma 1\, dF[t]}{p[t]} + \frac{0.2`\, \eta\, \sigma 1\, k[t] \left(-k'[t] + 0.1`\, k[t] \left(1 - \eta \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\right)\right)}{p[t]}\right) + \\
& \mu Hps \left(\frac{\sigma 1\, aH[t]}{p[t]} + \frac{2\, mHmax\, \theta\, \sigma 1 \left(-mH[t] + mHmax \left(1 - \theta \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\right)\right)}{p[t]}\right)
\end{aligned}
$$

$$rZB'[t] = -\lambda_6[t]$$

$$
\begin{aligned}
ra'[t] =\,& -\mu Bra\, aH[t] - \tau H\, aH[t]\, \lambda_2[t] + \tau H\, aH[t]\, \lambda_7[t] + \\
& \mu Hra \\
& \quad \Bigg(aH[t] + \\
& \qquad 2\, mHmax\, \theta \\
& \qquad \left(-mH[t] + mHmax \left(1 - \theta \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\right)\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
rd'[t] =\,& \mu Frd\, dF[t] + \mu Brd\, (-dF[t] - dG[t]) - aZB[t]\, \lambda_1[t] + \\
& (-aH[t] - dF[t] + \rho\, (dF[t] - \tau F\, dF[t]) - dG[t])\, \lambda_2[t] + \tau F\, dF[t]\, \lambda_7[t] + \\
& (aH[t] + aZB[t] + dF[t] + dG[t])\, \lambda_8[t] + (-\tau F\, dF[t] - \rho\, (dF[t] - \tau F\, dF[t]))\, \lambda_9[t] + \\
& \mu Hrd \\
& \quad \Bigg(-aH[t] - \\
& \qquad 2\, mHmax\, \theta \\
& \qquad \left(-mH[t] + mHmax \left(1 - \theta \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\right)\right)\Bigg)
\end{aligned}
$$

$$s'[t] = 2\, \mu Fs\, (sdach - s[t]) - vF\, \rho\, \lambda_2[t] + vF\, \lambda_7[t] + (-vF + vF\, \rho)\, \lambda_9[t] - \lambda_{12}[t]$$

$$
\begin{aligned}
w'[t] =\,& -\mu Fw\, l[t] + (l[t] - \tau H\, l[t] + \rho\, (-l[t] + \tau F\, l[t]))\, \lambda_2[t] + (-\tau F\, l[t] + \tau H\, l[t])\, \lambda_7[t] + \\
& (-l[t] + \tau F\, l[t] - \rho\, (-l[t] + \tau F\, l[t]))\, \lambda_9[t]
\end{aligned}
$$

$$
\begin{aligned}
0 =\,& -aZB[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& aZB[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - mZB'[t]
\end{aligned}
$$

$$
\begin{aligned}
0 =\,& -vH\, (aH[t] + mH[t]) - nH[t] - cH[t] \times p[t] + l[t] \times w[t] - \\
& aZB[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& aH[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& aH[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& dG[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& \tau H \left(l[t] \times w[t] + aH[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\right) + \\
& \rho \Bigg(-dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}\, p[t] - vF\, (dF[t] + k[t] + mF[t] + s[t]) - l[t] \times w[t] + \\
& \quad dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& \quad \tau F \Bigg(-dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}\, p[t] - l[t] \times w[t] + \\
& \qquad dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\Bigg)\Bigg) - mH'[t]
\end{aligned}
$$

$$0 = -nF[t] - dF'[t]$$

$$0 = -nG[t] - dG'[t]$$

$$0 = nF[t] + nG[t] - nH[t] - nZB[t] - aB'[t] - dB'[t]$$

$$0 = -nZB[t] - rZB'[t]$$

$$
\begin{aligned}
0 =\,& vH\, (aH[t] + mH[t]) + nG[t] - cG[t] \times p[t] + vF\, (dF[t] + k[t] + mF[t] + s[t]) + \\
& aZB[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& dG[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& \tau H \left(l[t] \times w[t] + aH[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\right) + \\
& \tau F \Bigg(-dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}\, p[t] - l[t] \times w[t] + \\
& \quad dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\Bigg) - mG'[t]
\end{aligned}
$$

$$
\begin{aligned}
0 =\,& -nF[t] - nG[t] + nH[t] + nZB[t] + aZB[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& dG[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& aH[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& aZB[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& aH[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + ra[t] - rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) + \\
& dG[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - mB'[t]
\end{aligned}
$$

$$
\begin{aligned}
0 =\,& nF[t] + cG[t] \times p[t] + cH[t] \times p[t] - vF\, (dF[t] + k[t] + mF[t] + s[t]) - l[t] \times w[t] + \\
& dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& \tau F \Bigg(-dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}\, p[t] - l[t] \times w[t] + \\
& \quad dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\Bigg) - \\
& \rho \Bigg(-dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}\, p[t] - vF\, (dF[t] + k[t] + mF[t] + s[t]) - l[t] \times w[t] + \\
& \quad dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right) - \\
& \quad \tau F \Bigg(-dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}\, p[t] - l[t] \times w[t] + \\
& \qquad dF[t] \left(-pdach\, \sigma 1 + \frac{\sigma 1\, ps[t]}{p[t]} + rd[t] - \frac{(-1+\alpha)\, k'[t]}{k[t]} + \frac{\alpha\, (1+\sigma 2)\, l'[t]}{l[t]}\right)\Bigg)\Bigg) - mF'[t]
\end{aligned}
$$

$$0 = nH[t] - aH'[t]$$

$$0 = nZB[t] - aZB'[t]$$

$$0 = -cG[t] - cH[t] - dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - k'[t] - s'[t]$$

$$0 = ps[t] - p'[t]$$

```
aB[0] == aB0
aH[0] == aH0
aZB[0] == aZB0
cG[0] == cG0
cH[0] == cH0
dB[0] == dB0
dF[0] == dF0
dG[0] == dG0
k[0] == k0
l[0] == l0
mB[0] == mB0
mF[0] == mF0
mG[0] == mG0
```

```
mG[0] == mG0
mH[0] == mH0
mZB[0] == mZB0
nF[0] == nF0
nG[0] == nG0
nH[0] == nH0
nZB[0] == nZB0
p[0] == p0
ps[0] == ps0
rZB[0] == -mB0 - mF0 - mG0 - mH0
ra[0] == ra0
rd[0] == rd0
s[0] == s0
w[0] == w0
```

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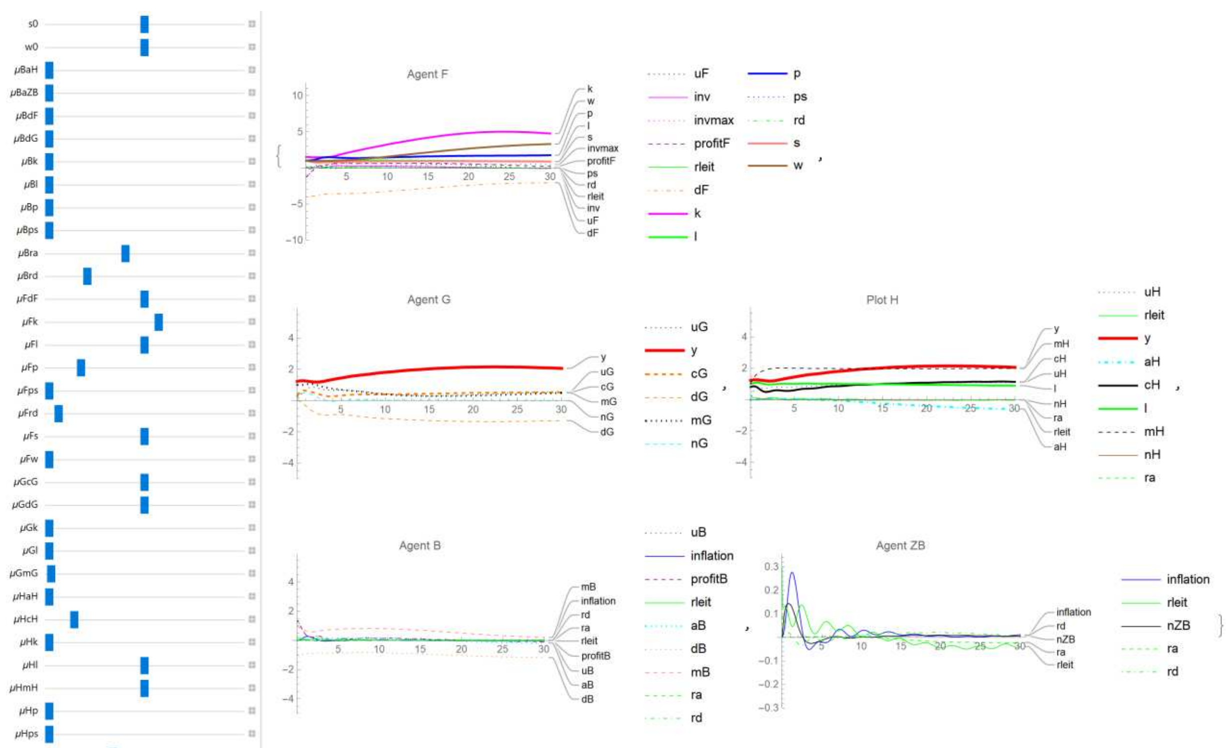
13.2. Calculation results of model D2

https://www.dropbox.com/s/za8t1h14bgae8po/Modell%20D2%20Version%2010.ndsolve.nb?dl=0



14. Different economic theories differ only by different assumptions about the power of agents

14.1. Basic idea

The basic idea of GCD models can also be formulated in the following way: With GCD models, the supposedly insurmountable opposition of different economic models can be eliminated in the sense that they can be understood as versions of a single model that differ from each other only by different one-sided power relations or adjustment speeds. On the other hand, GCD models offer the possibility of better representing reality, because mixed power relations usually correspond better to reality than one-sided power relations.

This is illustrated by the following 2 examples.

14.2. Savings → Investment (Neoclassics) or Investment → Savings (Keynes)

14.2.1. Problem description

The two economic schools of neoclassical economics and Keynesianism differ diametrically in their assumptions about the variables "saving" and "investing".

In the **Keynesian sense**, investing is an exogenous variable, saving is an endogenous variable and the cause-effect relationship applies

$Investing \Rightarrow Saving$

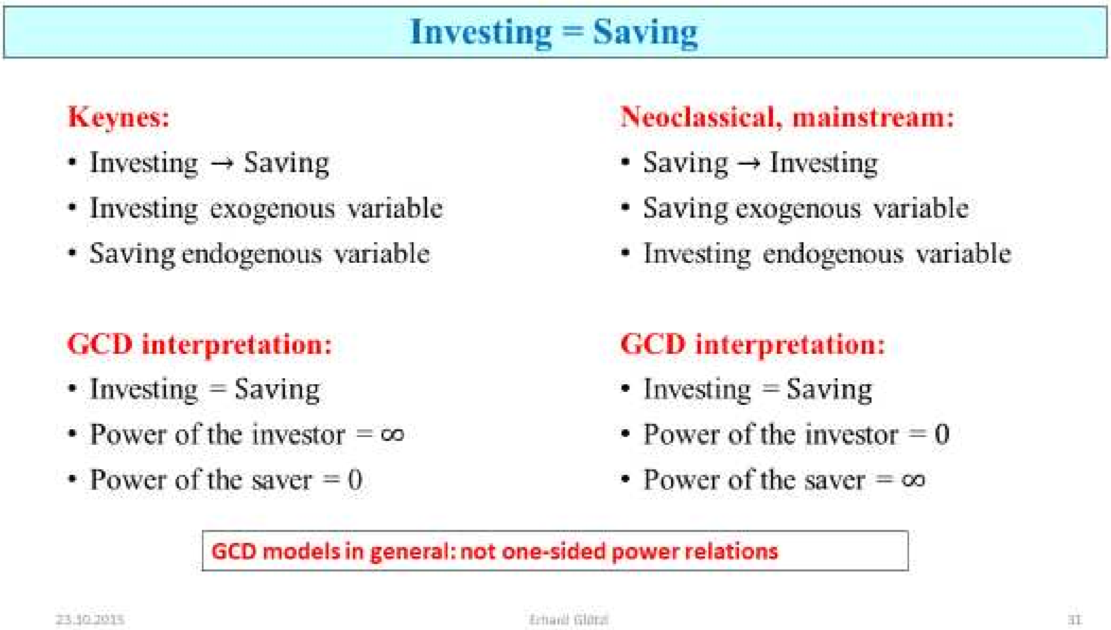
In the **neoclassical sense**, the opposite is true: investing is an endogenous variable, saving is an exogenous variable and the cause-effect relationship applies.

$Saving \Rightarrow Investing$

From the perspective of the GCD models, these seemingly insurmountable opposites can be overcome and resolved in the following sense. The statement that saving and investing must always be the same corresponds to an accounting identity that results from the definition of saving and investing. The two economic schools differ only in the different assumptions about the power of savers and investors.

The Keynesian cause-effect relationship results from the assumption that the power of investors is ∞ and the power of savers is 0. The neoclassical cause-effect relationship results from the opposite assumption that the power of investors is 0 and the power of savers is ∞.

In reality, however, these one-sided power relations do not usually occur, but rather mixed power relations. Therefore, reality can be better described with GCD models than with Keynesian or neoclassical models.



The model equations for the **Keynesian model** are

$$I = \hat{I}$$
$$S = I$$

The model equations for the **neoclassical model** are

$$S = \hat{S}$$
$$I = S$$

Furthermore, with the assumed master utility function

$$MU = \frac{1}{2}(\hat{I} - I)^2 + \frac{1}{2}(\hat{S} - S)^2$$

the **general equilibrium model** can be formulated as maximising MU under the constraint $Z(I,S) = I - S = 0$ in the following way:

$$0 = \frac{\partial MU}{\partial I} + \lambda \frac{\partial Z}{\partial I} = (\hat{I} - I) + \lambda$$
$$0 = \frac{\partial MU}{\partial S} + \lambda \frac{\partial Z}{\partial S} = (\hat{S} - S) - \lambda$$
$$0 = I - S$$

All these 3 models can be understood as special cases of the following GCD model:

utility functions

$$U^F = \frac{1}{2}(\hat{I} - I)^2 \qquad F \ firm, I \ investment, \ \hat{I} \ targeted \ investment$$

$$U^H = \frac{1}{2}(\hat{S} - S)^2 \qquad H \ household, S \ savings, \ \hat{S} \ targeted \ savings$$

constraints

$$0 = I - S$$

basic GCD - equations

$$(a) \qquad I' = \mu_I^F (\hat{I} - I) + \lambda$$
$$(b) \qquad S' = \mu_S^H (\hat{S} - S) - \lambda$$
$$(c) \qquad 0 = I - S$$

From this GCD model we get the Keynesian model with the assumptions

$$\mu_I^F = \infty$$
$$\mu_S^H = 0 \qquad (oder \ 0 \le \mu_S^H < \infty)$$

because it follows from equation (a)

$$I' = \mu_I^F (\hat{I} - I) + \lambda \Rightarrow \frac{I'}{\mu_I^F} = (\hat{I} - I) + \frac{\lambda}{\mu_I^F} \Rightarrow wegen \, \mu_I^F = \infty \quad 0 = (\hat{I} - I) + 0 \Rightarrow I = \hat{I}$$

and from equation (c)

$$S = I$$

Equation (b) is not needed. It would also be possible $0 \le \mu_S^H < \infty$.

Similarly, the neoclassical model results with the assumptions

$$\mu_I^F = 0 \qquad (oder\ 0 \le \mu_I^F < \infty)$$
$$\mu_S^H = \infty$$

and the general equilibrium model with the assumptions

$$I' = 0 \quad Annahme\ des\ stationären\ Gleichgewichts$$
$$S' = 0 \quad Annahme\ des\ stationären\ Gleichgewichts$$
$$\mu_I^F = 1$$
$$\mu_S^H = 1$$

GCD ~ mixed power **Keynesian,Neoclassic ~ one sided power**
Constraint GE ~ stationary

$GCD - Model$ "$Lagrange - Closure$"	$Keynes$	$Neoclassic$	$Constraint\ GE$
$I' = \mu_I^B(IF - I) + \lambda$	$I = IF$	$\times\times\times\times$	$0 = \frac{\partial MU}{\partial I} + \lambda = (IF - I) + \lambda$
$S' = \mu_S^H(SF - S) - \lambda$	$\times\times\times\times$	$S = SF$	$0 = \frac{\partial MU}{\partial S} - \lambda = (SF - S) - \lambda$
$I - S = 0$	$S = I$	$I = S$	$I - S = 0$
μ_I^B $Power\ of\ Business$	$\mu_I^B = \infty$	$\mu_I^B = 0$	$\mu_I^B = 1$
μ_S^H $Power\ of\ Households$	$\mu_S^H = 0$	$\mu_S^H = \infty$	$\mu_S^H = 1$
			$MU = \frac{1}{2}((IF - I)^2 + (SF - S)^2)$

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14.2.2. Formally analogous problems

Completely analogous to the accounting identity $I = S$, in a closed economy the accounting identity applies that the sum of the accounts A (receivables) is always equal to the sum of the debts D (liabilities), i.e. $A = D$ or with the convention used in this paper for the negative sign of liabilities $A = -D$. The development of these quantities over time depends on the one hand on the interests of the sum of creditors and the sum of debtors, and on the other hand on their power to enforce these interests[1] (Glötzl 1999; 2015).

The two models (investing/saving and liabilities/receivables) are not only formally mathematically completely equivalent to each other, but they are also formally completely equivalent to the movement on an inclined straight line inclined at 45 degrees, which is described by the constraint $x_1 = x_2$ (see Glötzl (2015)).

Similar Models

Model	variables		constraint condition
Inclined plane	x1	x2	x1=x2
Investment versus Saving	I	S	I=S
Creditors versus Debitors	R	D	R=D

14.2.3. Calculation results

The GCD equation system is given by:

---

[1] (Glötzl 1999; 2009) describes the "fundamental paradox of the monetary economy". It states that in an economy where credit is measured in monetary units, the power of the sum of creditors to increase their acounts is always greater than the power of the sum of debtors to reduce their debts. In other words, it describes the "powerlessness" of debtors relative to the "power of creditors””. These power relations are ultimately the cause of debt traps and the constant growth of accounts and debts.

```
uF[t] == -1/2 (idach - inv[t])^2
uH[t] == -1/2 (sdach - spar[t])^2

inv'[t] == μFinv (idach - inv[t]) + λ1[t]
spar'[t] == μHspar (sdach - spar[t]) - λ1[t]

0 == inv[t] - spar[t]

inv[0] == inv0
spar[0] == inv0
```

We assume that investors want to invest 4 units and savers want to save 2 units, i.e.

$$idach = 4$$
$$sdach = 2$$

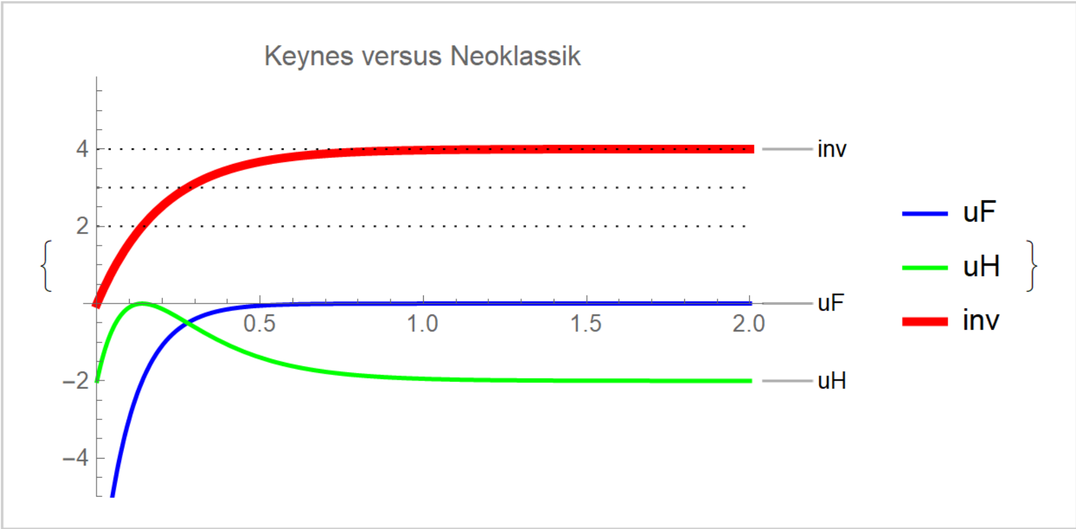
At the time $t = 0$ neither investing nor saving occurs, i.e.

$$inv[0] = spar[0] = 0$$

The following numerical calculations show the different behaviour for the different assumptions about the power factors.

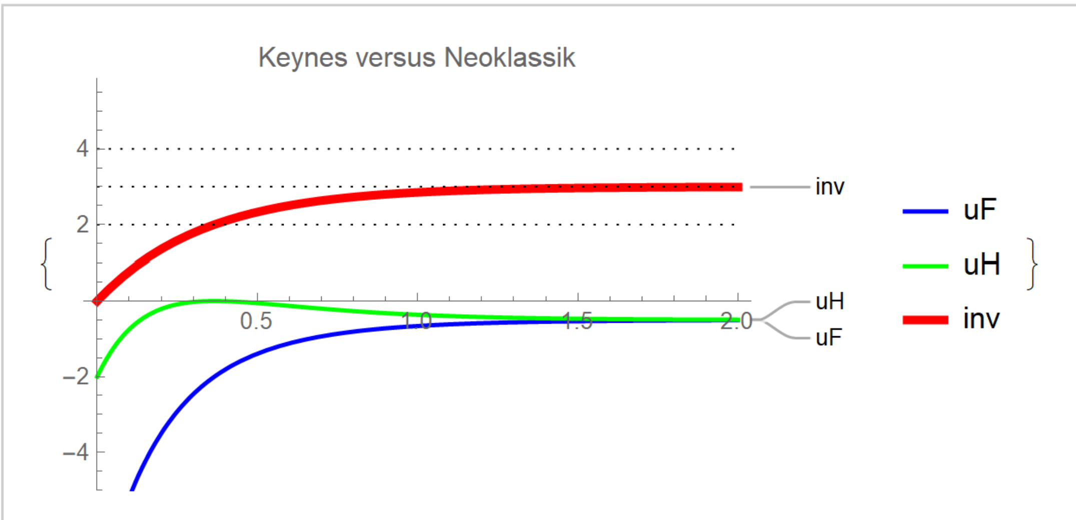
https://www.dropbox.com/s/1wo3f68pcfqzmcf/Keynes%20Version%205.ndsolve.nb?dl=0

Keynes $\mu Finv = \infty$ (*approximated by* $\mu Finv = 6$) $\mu Hspar = 0$



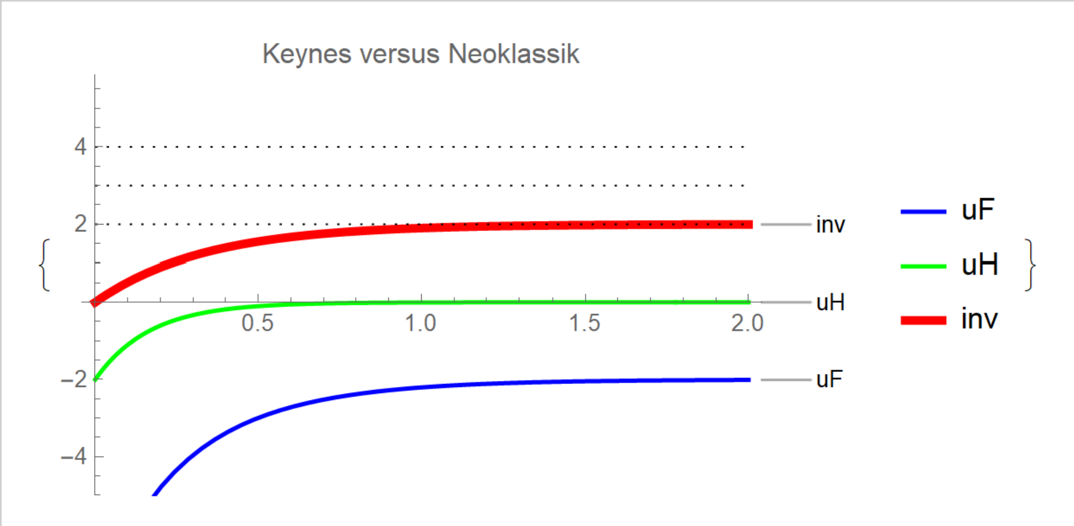
Investing (= saving) converge against the firm's targeted investments.

equal power $\mu Finv = 3$ $\mu Hspar = 3$



Investing(=saving) converges to a mixture of the investment targeted by the firm and the saving targeted by the household. The speed of convergence, depends on the level of the power factors, because the power factors can always be interpreted as speed-adjustment factors (see also chapter 3.7).

Neoclassics $\mu Finv = 0$ $\mu Hspar = \infty$ (*approximated by* $\mu Hspar = 6$)



Investing (=saving) converge against the saving targeted by the household.

14.2.4. On the relationship of "drop closure", "Lagrange closure", GCD and general equilibrium GE

We explain the relationship first with the simple example above and then in the following chapter 14.3 with the models of A. Sen (Sen 1963). More detailed information can be found in Glötzl (2015).

Based on the utility functions for F and H

$$U^F = -\frac{1}{2}(\hat{I} - I)^2 \qquad F\ Firma,\ I\ Investieren,\ \hat{I}\ angestrebtes\ Investieren$$

$$U^H = -\frac{1}{2}(\hat{S} - S)^2 \qquad H\ Haushalt,\ S\ Sparen,\ \hat{S}\ angestrebtes\ Sparen$$

the ex-ante behavioural equations (i.e., the behavioural equations without considering the constraint $0 = I - S$) are as follows

$$\begin{aligned} &(a) \qquad I' = \mu_I^F (\hat{I} - I) \wedge 2 \\ &(b) \qquad S' = \mu_S^H (\hat{S} - S) \wedge 2 \end{aligned} \qquad \text{<14.1>}$$

This system of equations has 2 variables (S, I) and 2 equations. It is therefore solvable with appropriate initial conditions.

However, these ex-ante solutions do not describe the reality, because they usually do not fulfill the constraint $0 = I - S$ which has to be fulfilled.

If the constraint is added to the ex-ante system of equations, the following is obtained

$$\begin{array}{ll} (a) & I' = \mu_I^F \frac{1}{2}(\hat{I} - I) \\ (b) & S' = \mu_S^H \frac{1}{2}(\hat{S} - S) \\ (c) & I = S \end{array}$$ <14.2>

This system of equations consists of 3 equations for 2 variables and is therefore usually not solvable. A method with which this system of equations is changed in such a way that it becomes solvable is called a closure method.

14.2.4.1. Drop closure

In the simplest case, one omits so many equations until the system of equations becomes solvable. This basic procedure is used by A. Sen in (Sen 1963)).

If in the case of equation system <14.2> equation (a) is omitted, the result is

$$\begin{array}{ll} (b) & S' = \mu_S^H \frac{1}{2}(\hat{S} - S) \\ (c) & I = S \end{array}$$ <14.3>

which corresponds exactly to the neoclassical approach and, in equilibrium ($S' = 0$)

$$\begin{array}{ll} (b) & S = \hat{S} \\ (c) & I = S \end{array}$$

results.

If we omit (b), we get

$$\begin{array}{ll} (a) & I' = \mu_I^F \frac{1}{2}(\hat{I} - I) \\ (c) & I = S \end{array}$$ <14.4>

which is exactly in line with the Keynesian approach and in equilibrium ($I' = 0$)

$$\begin{array}{ll} (a) & I = \hat{I} \\ (c) & I = S \end{array}$$

results.

14.2.4.2. Lagrange Closure, GCD, general equilibrium

In the case of Lagrange Closure, the opposite approach is taken: equations are not omitted, but new additional variables are introduced until the system of equations becomes solvable. In the concrete case, one adds the Lagrange multiplier λ as a new additional variable to the variables and the constraint forces to the behaviour equations in the sense of the GCD methodology. This results in the GCD equation system, which is usually solvable.

$$\begin{aligned}
&(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda \frac{\partial Z}{\partial I} = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda \\
&(b) \qquad S' = \mu_S^H \frac{1}{2}(\hat{S} - S) + \lambda \frac{\partial Z}{\partial S} = \mu_S^H \frac{1}{2}(\hat{S} - S) - \lambda \qquad\qquad <14.5> \\
&(c) \qquad Z = 0 = I - S
\end{aligned}$$

We show that in the Keynesian case, because of $\mu_S^H = 0$ this system of equations <14.5> transforms to the system of equations

$$\begin{aligned}
&(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda \qquad\qquad <14.6> \\
&(c) \qquad Z = 0 = I - S
\end{aligned}$$

This means that $\mu_S^H = 0$ leads to (b) becoming linearly dependent on (a) and (c) and can therefore be omitted in the sense of drop closure. This is discussed in more detail in Glötzl (2015).

In the case of the general equilibrium $I' = 0$, because of $\mu_I^F = \infty$ it follows that

$$\begin{aligned}
&(a) \qquad I = \hat{I} \\
&(c) \qquad I = S
\end{aligned}$$

Proof:

Because of $\mu_S^H = 0$ and because of (c), it follows from <14.5>

$$\begin{aligned}
&(a) \qquad I' == \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda \\
&(b1) \qquad S' == -\lambda \\
&(c) \qquad Z = 0 = I - S \\
&(d) \qquad Z' = 0 = I' - S'
\end{aligned}$$

If we apply (d) in (b1), we get

$$
\begin{array}{ll}
(a) & I' = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda \\
(b2) & I' = -\lambda \\
(c) & Z = 0 = I - S \\
(d) & Z' = 0 = I' - S'
\end{array}
$$

From (a) and (b2) we get

$$\lambda = -\frac{1}{2}\mu_I^F \frac{1}{2}(\hat{I} - I)$$

Inserting into (a) and (b1) results in

$$
\begin{array}{ll}
(a) & I' = \frac{1}{2}\mu_I^F \frac{1}{2}(\hat{I} - I) \\
(b) & S' = \frac{1}{2}\mu_I^F \frac{1}{2}(\hat{I} - I) \\
(c) & Z = 0 = I - S \\
(d) & Z' = 0 = I' - S'
\end{array}
$$

Thus, equation (b) is linearly dependent on (a) and (d) and can therefore be omitted.

In the case of the general equilibrium ($I' = 0$), this results in the following equations because of $\mu_I^F = \infty$

$$
\begin{array}{ll}
(a) & I = \hat{I} \\
(c) & I = S
\end{array}
$$

by bringing μ_I^F to the left side at first.

Summary: The Keynesian model results from the GCD model both by drop-closure, by omitting equation (b), and by setting the power factor $\mu_S^H = 0$. The power factor μ_I^F need only be $\mu_I^F > 0$, it can also be $\mu_I^F = \infty$. The magnitude of μ_I^F only determines the speed of convergence. For the neoclassical model, everything applies correspondingly.

14.3. A. Sen: different economic theories differ in their assumptions about the endogeneity or exogeneity of different variables.

14.3.1. Problem description

In 1963, Amartya Sen showed that neoclassical and Keynesian models can often be derived from the same system of equations and essentially differ only in which behavioural equations are dropped (Sen 1963). This also corresponds to a decision on the direction of causality within the model.

Similarly to the previous chapter, all models examined by Sen can be understood as special cases of a single GCD model and dropping certain equations is equivalent to assuming different one-sided power relations. Again, it is true that in reality, these one-sided power relations do not usually occur, but rather mixed power relations. Therefore, reality can be better described with GCD models than with the models cited by Sen.

The original system of equations of Sen is

(1)	$Y(L,K) = \beta L^a K^{1-a}$	*wir nehmen die Cobb – Douglas Produktionsfunktion an*	
(2)	$w = \dfrac{\partial Y}{\partial L}$	*w Lohn, L Arbeit*	
(3)	$Y = P + wL$	*P Profit*	<14.7>
(4)	$I = S_L\, L + S_P\, P$	*S_L Sparanteil vom Arbeitseinkommen*	
		S_P Sparanteil vom Profit	
(5)	$I = i_1 + i_2 Y$	*wir nehmen diese Standard - Investitionsfunktion an*	

For clarity, we also introduce the variable S for saving and the constraints $0 = I - S$ and $I = K'$. Implicitly, Sen assumes that L is exogenously given by $L = \hat{L}$. This yields the system of equations (14.8) which is equivalent to (14.7). We write it in our methodology as follows *algebraically defined variables*

(1) $Y(L,K) = \beta L^{a} K^{1-a}$

behavioural equations

(2) $w = \dfrac{\partial Y}{\partial L}$

(4) $S = S_L\, L + S_P\, P$

(5) $I = i_1 + i_2 Y$ <14.8>

(6) $L = \hat{L}$

constraints

(3) $0 = Y - P - wL$

(7) $0 = I - S$

(8) $0 = I - K'$

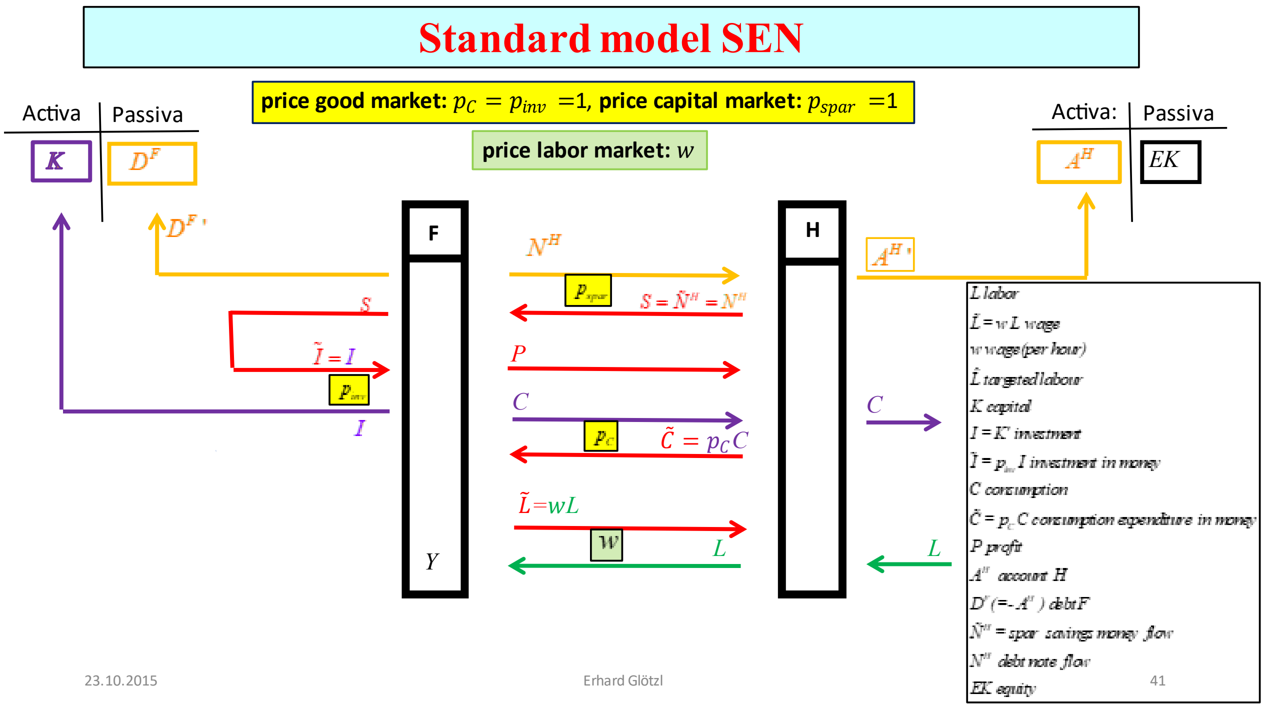
This system of equations consists of 8 equations for the 7 variables

$Y,\ L,\ K,\ w,\ S, P, I$

and is therefore generally not solvable. Sen shows that by dropping different equations (drop closure) different solvable economic models result:

omitting (5) results in the *neoclassical model*
omitting (2) results in the *Kaldor model* (*Neo – Keynesianisches Modell*) <14.9>
omitting (4) results in the *Johansen model*
omitting (6) results in the *Keynesian model of the General Theory*

We show below that the system of equations <14.8> and the various models <14.9> arise from a single GCD model through a specific choice of power factors in each case.



The variables N^H, D^F, A^H are only listed for the sake of completeness. They are omitted in the following (as by Sen), because due to the assumption $p_{spar} = 1$ immediately $spar = p_{spar} N^H = N^H$ is valid.

Standard model SEN, neoclassic, Kaldor, Johansen, Keynes, GCD

SEN		neoclassic	Kaldor	Johansen	Keynes	GCD - Modell
overdetermined		drop (5)	drop (2)	drop (4)	drop (6)	"Lagrange – Closure"
7 variables: Y, L, K, w, S, P, I						
8 equations						
algebraically defined variables						
(1)	$Y(L,K) = \beta L^{a} K^{1-a}$					
behavioural equations						
(2)	$w = \frac{\partial Y}{\partial L}$	$\mu_w^F = \infty$	$\mu_w^F = 0$	$\mu_w^F = \infty$	$\mu_w^F = \infty$	$w' = \mu_w^F (\frac{\partial Y}{\partial L} - w) - \lambda_1 L$
(4)	$S = \hat{S}_L\, L + \hat{S}_P\, P$	$\mu_S^H = \infty$	$\mu_S^H = \infty$	$\mu_S^H = 0$	$\mu_S^H = \infty$	$S' = \mu_S^H (s_L wL + s_P P - S) - \lambda_2$
(5)	$I = i_1 + i_2 Y$	$\mu_I^F = 0$	$\mu_I^F = \infty$	$\mu_I^F = \infty$	$\mu_I^F = \infty$	$I' = \mu_I^F (i_1 + i_2 Y - I) + \lambda_2 + \lambda_3$
(6)	$L = \hat{L}$	$\mu_L^H = \infty$	$\mu_L^H = \infty$	$\mu_L^H = \infty$	$\mu_L^H = 0,$	$L' = \mu_L^H (\hat{L} - L) - .... + \lambda_1 (\frac{\partial Y}{\partial L} - w)$
constraints						
(3)	$0 = Y - P - wL$					
(7)	$0 = I - S$					
(8)	$0 = I - K'$					

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The GCD model SEN results from the following basic equations:

GCD-Modell SEN : Grundgleichungen

algebraically defined variables

$Y(L,K) = \beta L^{\alpha} K^{1-\alpha}$ "*production function*"

utility functions

$U^{H} = -\frac{1}{2}\left(\hat{L} - L\right)^{2} - \frac{1}{2}(\hat{S}_{L} L + \hat{S}_{P} P - S)^{2}$ "*utility function household*"

$U^{F} = -\frac{1}{2}(i_{1} + i_{2} Y - I)^{2} - \frac{1}{2}(\frac{\partial Y}{\partial L} - w)^{2}$ "*utility function firm*"

constraints

(3) $0 = Y - P - wL$

(7) $0 = I - S$

(8) $0 = I - K'$

The GCD equations are then

$$\text{uF}[t] == -\frac{1}{2}\left(\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}-\text{w}[t]\right)^2-\frac{1}{2}\left(\text{i1}-\text{inv}[t]+\text{i2}\,\text{y}[t]\right)^2$$
$$\text{uH}[t] == -\frac{1}{2}\left(\text{ldach}-\text{l}[t]\right)^2-\frac{1}{2}\left(\text{spardachl}\,\text{l}[t]+\text{spardachprofit}\,\text{profit}[t]-\text{spar}[t]\right)^2$$

$$\text{c}[t] == -\text{spar}[t]+\text{l}[t]\times\text{w}[t]$$
$$\text{y}[t] == \beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha}$$

$$\text{inv}'[t] == \mu\text{Finv}\left(\text{i1}-\text{inv}[t]+\text{i2}\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha}\right)+\lambda_2[t]+\lambda_3[t]$$
$$\begin{aligned}\text{k}'[t] ==\ & \mu\text{Fk}\Big(-\text{i2}\,(1-\alpha)\,\beta\,\text{k}[t]^{-\alpha}\,\text{l}[t]^{\alpha}\left(\text{i1}-\text{inv}[t]+\text{i2}\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha}\right)-\\ & (1-\alpha)\,\alpha\,\beta\,\text{k}[t]^{-\alpha}\,\text{l}[t]^{-1+\alpha}\left(\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}-\text{w}[t]\right)\Big)+(1-\alpha)\,\beta\,\text{k}[t]^{-\alpha}\,\text{l}[t]^{\alpha}\,\lambda_1[t]-\lambda_3[t]\end{aligned}$$
$$\begin{aligned}\text{l}'[t] ==\ & \mu\text{Hl}\left(\text{ldach}-\text{l}[t]-\text{spardachl}\left(\text{spardachl}\,\text{l}[t]+\text{spardachprofit}\,\text{profit}[t]-\text{spar}[t]\right)\right)+\\ & \mu\text{Fl}\Big(-\text{i2}\,\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}\left(\text{i1}-\text{inv}[t]+\text{i2}\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha}\right)-\\ & (-1+\alpha)\,\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-2+\alpha}\left(\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}-\text{w}[t]\right)\Big)+\left(\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}-\text{w}[t]\right)\lambda_1[t]\end{aligned}$$
$$\text{profit}'[t] == -\text{spardachprofit}\,\mu\text{Hprofit}\left(\text{spardachl}\,\text{l}[t]+\text{spardachprofit}\,\text{profit}[t]-\text{spar}[t]\right)-\lambda_1[t]$$
$$\text{spar}'[t] == \mu\text{Hspar}\left(\text{spardachl}\,\text{l}[t]+\text{spardachprofit}\,\text{profit}[t]-\text{spar}[t]\right)-\lambda_2[t]$$
$$\text{w}'[t] == \mu\text{Fw}\left(\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}-\text{w}[t]\right)-\text{l}[t]\,\lambda_1[t]$$

$$0 == \beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha}-\text{profit}[t]-\text{l}[t]\times\text{w}[t]$$
$$0 == \text{inv}[t]-\text{spar}[t]$$
$$0 == \text{inv}[t]-\text{k}'[t]$$

$$\text{inv}[0] == \text{inv0}$$
$$\text{k}[0] == \text{k0}$$
$$\text{l}[0] == \text{l0}$$
$$\text{profit}[0] == -\text{l0}\,\text{w0}+\text{k0}^{1-\alpha}\,\text{l0}^{\alpha}\,\beta$$
$$\text{spar}[0] == \text{inv0}$$
$$\text{w}[0] == \text{w0}$$

Dividing the differential equations for w,S,I,L by $\mu_w^F,\mu_S^H,\mu_I^F,\mu_L^H$ in each case and setting

$$\mu_w^F = \infty$$
$$\mu_S^H = \infty$$
$$\mu_I^F = \infty$$
$$\mu_L^H = \infty$$

we get the 4 behavioural equations of the standard model SEN

(2) $\quad w = \dfrac{\partial Y}{\partial L}$

(4) $\quad S = S_L\, L + S_P\, P$

(5) $\quad I = i_1 + i_2\, Y$

(6) $\quad L = \hat{L}$

In addition, one has the differential behavioural equations for K and P.

If one sets individual power factors equal to 0, this leads in an analogous way, as it was shown in chapter 14.2.4.1, to the fact that the corresponding differential equation becomes linearly dependent on the others and can therefore be omitted. More details can also be found in Glötzl (2015).

14.3.2. Calculation results

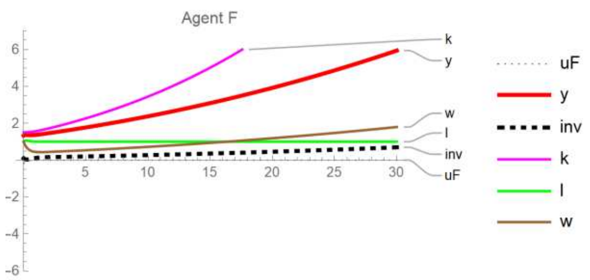
https://www.dropbox.com/s/nro0gczdek1ramn/Modell%20SEN%20Version%2010.ndsolve.nb?dl=0

In order to solve the differential-algebraic GCD equation system with NDSolve one has to use the method

Method→{IndexReduction→Automatic}

$\mu=\infty$ is always approximated by $\mu=6$.

Neoclassical model $\mu_w^F=\infty,\ \mu_S^H=\infty,\ \mu_I^F=0,\ \mu_L^H=\infty$



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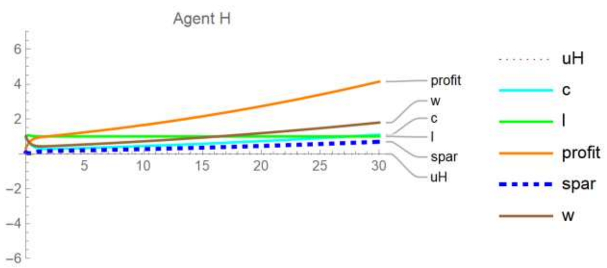
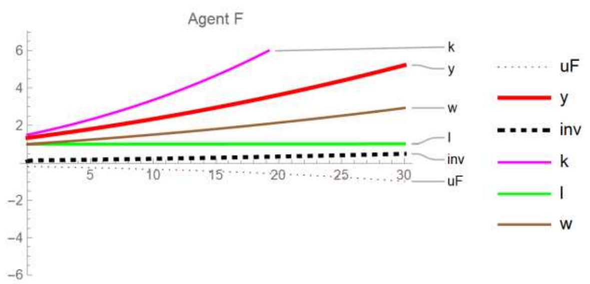


Kaldor model $\mu_w^F=0,\ \mu_S^H=\infty,\ \mu_I^F=\infty,\ \mu_L^H=\infty$



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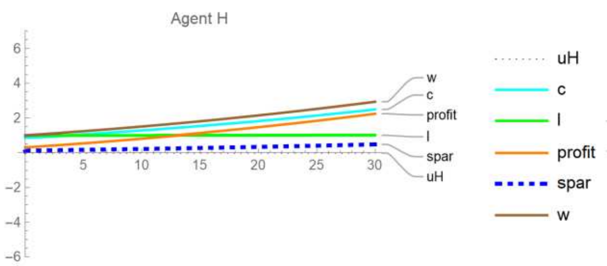
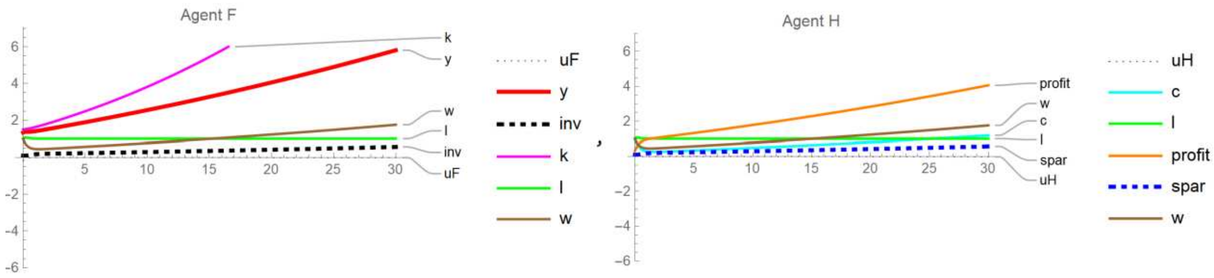
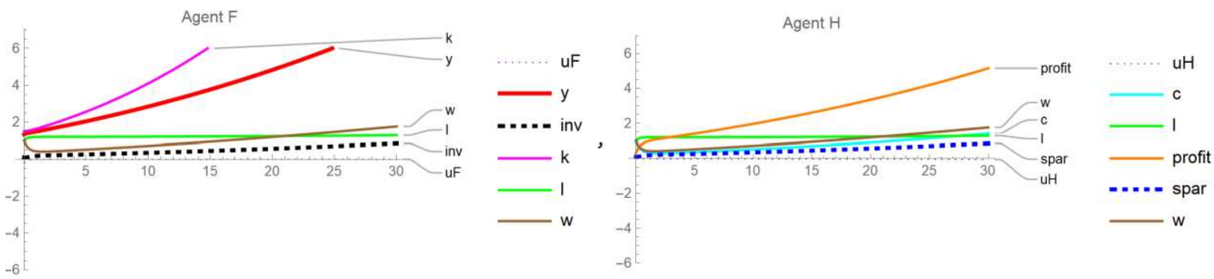


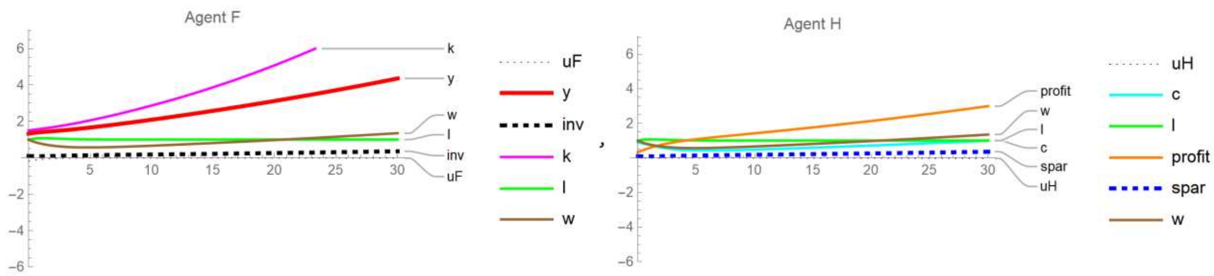
Johansen model $\mu_w^F = \infty,\ \mu_S^H = 0,\ \mu_I^F = \infty,\ \mu_L^H = \infty$



Keynes model $\mu_w^F = \infty,\ \mu_S^H = \infty,\ \mu_I^F = \infty,\ \mu_L^H = 0$



GCD-Modell with mixed parameters $\mu_w^F = 1,\ \mu_S^H = 1,\ \mu_I^F = 1,\ \mu_L^H = 1$



14.3.3. On the relationship between GCD models, General Constrained Equilibrium models (GCE model) and DSGE models.

A general equilibrium model can only start from 1 master utility function to be maximized (note: multiple utility functions cannot be maximized at the same time, they have to be combined to 1 master utility function, e.g. by weighting). A possible master utility function for a general constrained equilibrium (GCE) model would be:

$$\hat{U} = U^H + U^F = -\frac{1}{2}\left(\hat{L} - L\right)^2 - \frac{1}{2}(S_L L + S_P P - S)^2 - \frac{1}{2}(i_1 + i_2 Y - I)^2 - \frac{1}{2}(\frac{\partial Y}{\partial L} - w)^2$$

With the algebraically defined variable

$$Y(L,K) = \beta L^\alpha K^{1-\alpha}$$

this results in

$$\hat{U}(L,P,S,K,I,w) = -\frac{1}{2}\left(\hat{L} - L\right)^2 - \frac{1}{2}(S_L L + S_P P - S)^2 - \frac{1}{2}(i_1 + i_2 \beta L^\alpha K^{1-\alpha} - I)^2 - \frac{1}{2}(\beta \alpha L^{\alpha-1} K^{1-\alpha} - w)^2$$

The constraints remain the same:

$$\begin{aligned} Z_1 &= 0 = Y - P - wL \\ Z_2 &= 0 = I - S \\ Z_3 &= 0 = I - K' \end{aligned}$$

A maximum under constraints can only exist if the "first order" conditions are fulfilled, i.e.

$$\begin{aligned}
0 &= \frac{\partial \hat{U}}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} = \frac{\partial U^H}{\partial L} + \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} \\
0 &= \frac{\partial \hat{U}}{\partial P} + \lambda_1 \frac{\partial Z_1}{\partial P} + \lambda_2 \frac{\partial Z_2}{\partial P} + \lambda_3 \frac{\partial Z_3}{\partial P} = \frac{\partial U^H}{\partial P} + \frac{\partial U^F}{\partial P} + \lambda_1 \frac{\partial Z_1}{\partial P} + \lambda_2 \frac{\partial Z_2}{\partial P} + \lambda_3 \frac{\partial Z_3}{\partial P} \\
0 &= \frac{\partial \hat{U}}{\partial S} + \lambda_1 \frac{\partial Z_1}{\partial S} + \lambda_2 \frac{\partial Z_2}{\partial S} + \lambda_3 \frac{\partial Z_3}{\partial S} = \frac{\partial U^H}{\partial S} + \frac{\partial U^F}{\partial S} + \lambda_1 \frac{\partial Z_1}{\partial S} + \lambda_2 \frac{\partial Z_2}{\partial S} + \lambda_3 \frac{\partial Z_3}{\partial S} \\
0 &= \frac{\partial \hat{U}}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} = \frac{\partial U^H}{\partial K} + \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} \\
0 &= \frac{\partial \hat{U}}{\partial I} + \lambda_1 \frac{\partial Z_1}{\partial I} + \lambda_2 \frac{\partial Z_2}{\partial I} + \lambda_3 \frac{\partial Z_3}{\partial I} = \frac{\partial U^H}{\partial I} + \frac{\partial U^F}{\partial I} + \lambda_1 \frac{\partial Z_1}{\partial I} + \lambda_2 \frac{\partial Z_2}{\partial I} + \lambda_3 \frac{\partial Z_3}{\partial I} \\
0 &= \frac{\partial \hat{U}}{\partial w} + \lambda_1 \frac{\partial Z_1}{\partial w} + \lambda_2 \frac{\partial Z_2}{\partial w} + \lambda_3 \frac{\partial Z_3}{\partial w} = \frac{\partial U^H}{\partial w} + \frac{\partial U^F}{\partial w} + \lambda_1 \frac{\partial Z_1}{\partial w} + \lambda_2 \frac{\partial Z_2}{\partial w} + \lambda_3 \frac{\partial Z_3}{\partial w} \\
Z_1 &= 0 = Y - P - wL \\
Z_2 &= 0 = I - S \\
Z_3 &= 0 = I - K'
\end{aligned} \tag{14.10}$$

This system of equations <14.10> is obviously identical to the GCD system of equations in steady state, i.e. for

$$L' = P' = S' = K' = I' = w' = 0$$

In contrast to GCE models, in DSGE models in particular (apart from the stochastic terms) not a master utility function is maximized under constraints, but rather the master utility function discounted by the discount rate β is maximized

$$\hat{U}_\beta(t) = \int_0^t e^{-\beta t} \hat{U}(t) dt \to \max \quad \text{under constraints}$$

For holonomic constraints this problem can be solved by the variational problem with the Lagrange function

$$\hat{U}_\beta^Z(t) = \int_0^t e^{-\beta t} \left(\hat{U}(t) + \sum \lambda_j \frac{\partial Z_j}{\partial x_i} \right) dt \to \max$$

This leads to the corresponding Euler equations that describe the dynamics of the DSGE model.

Note: Without going into more detail here, we would like to point out the following: If the constraints are neither holonomic nor integrable nor linear, the two problems

(1) $$\hat{U}_\beta(t) = \int_0^t e^{-\beta t} \hat{U}(t) dt \to \max \quad \text{under constraints}$$

(2) $$\hat{U}_\beta^Z(t) = \int_0^t e^{-\beta t} \left(\hat{U}(t) + \sum \lambda_j \frac{\partial Z_j}{\partial x_i} \right) dt \to \max$$

are different and lead to different Euler equations and thus different dynamics. The dynamics to (1) is called "vakonomic mechanics". For more details see Glötzl (2018).

15. Obesity or consumption/environment model

In section 3.9.2 we referred to the special case where a utility function depends on variables $x=\{x_1,x_2, ..., x_I\}$ as well as on their antiderivatives $X=(X_1,X_2,...,X_I)$ and/or the derivatives $x'=(x'_1,x'_2,...,x'_I)$ of these variables. In these cases, both the antiderivatives $X=(X_1,X_2,...,X_I)$ and the derivatives $x'=(x'_1,x'_2,...,x'_I)$ are to be regarded as additional variables of their own and appropriate constraints are to be added describing the relations between antiderivatives, functions and derivatives of the function.

We will describe this situation using a simple example with only one variable x, where the utility function depends on a flow variable x as well as on some stock variable X.

A good illustrative example is that we all like to eat but do not want to be fat. Here, x describes the flow variable eat and X the stock variable, which describes the body weight.

utility function

$$U(x,X)=-(\hat{x}-x)^2-(\hat{X}-X)^2 \qquad oder \quad U=x^\gamma-(\hat{X}-X)^2$$

constraint

$$0=X'-x+\sigma X$$

The utility function describes the decreasing marginal benefit of eating and the increasing marginal cost of body weight. The constraint describes that eating increases weight and decreases it at the rate σ due to natural weight loss. If the parameter $\sigma=0$, then the constraint just describes the direct stock-flow relationship $X'=x$ between X and x.

This results in the following GCD equation system

$$uA[t] == x[t]^{\gamma 1} - xx[t]^{\gamma 2}$$

$$x'[t] == \gamma 1\, \mu Ax\, x[t]^{-1+\gamma 1} - \lambda_1[t]$$
$$xx'[t] == -\gamma 2\, \mu Axx\, xx[t]^{-1+\gamma 2} + \lambda_1[t]$$

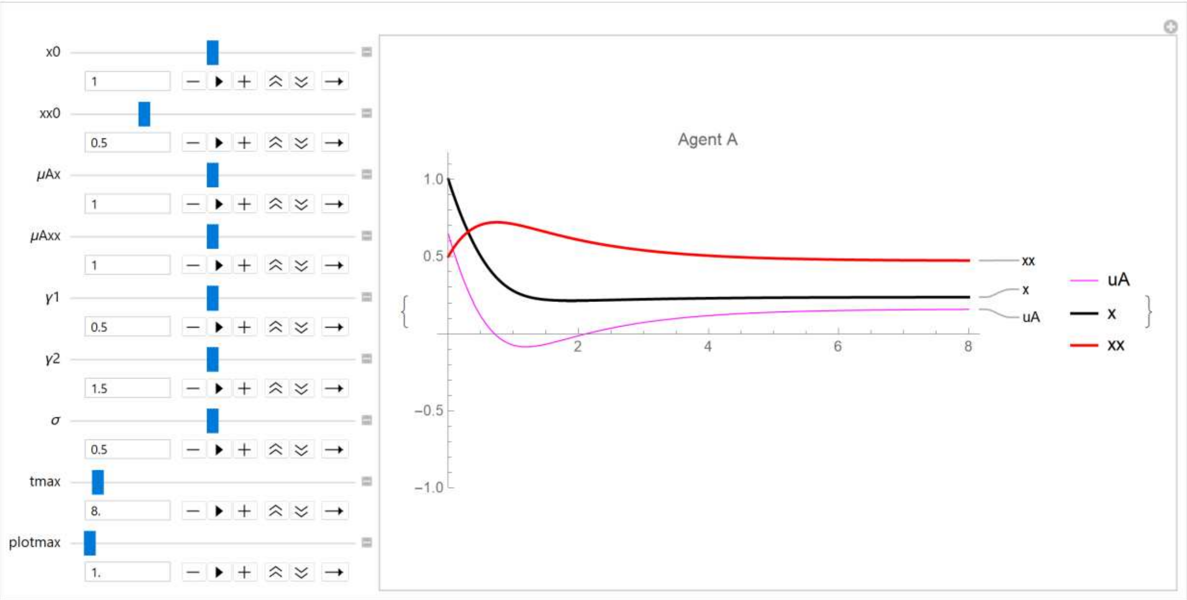
$$0 == -x[t] + \sigma\, xx[t] + xx'[t]$$

$$x[0] == x0$$
$$xx[0] == xx0$$

https://www.dropbox.com/s/7oetfsp9shlmzca/Modell%20Fresssack%20Version%202.ndsolve.nb?dl=0

The result of the calculation is, for example:

(In the plot stands xx for the stock variable X)



We give this simple example mainly because this contradictory behaviour of flow variable and stock variable is also relevant in many environmental problems. For example, the following other interpretations are also possible:

	Flow variable x	Stock variable X
Land consumption	Building	Built-up area
Waste	Production	Total waste
Plastic packaging	Consumption	Plastic waste in the sea
Carbon dioxide	Fossil fuel combustion	Carbon dioxide concentration in the air

Furthermore, this simple model serves as an example for a model in which the stock and flow variables occur simultaneously in the utility function. As already explained in chapter 3.9, in this case a separate variable must be introduced for the stock variable and the flow variable. The relationship between the two is described by a constraint $X' = x - \sigma X$. If the parameter $\sigma = 0$, then the constraint just describes the direct stock-flow relationship $X' = x$ between X and x.

16. Summary and conclusions

16.1. Principle of GCD

By using differential-algebraic equations in continuous time, the GCD approach extends existing analogies between classical mechanics and economics from constrained optimization to constrained dynamics.

16.2. Problem 8 by Stephen Smale

Problem 8 of the 18 problems published by Smale in 1998 (Smale 1991; 1997; 1998; Smale Institute 2003) is: introducing dynamics (adjustment of prices) in economic equilibrium theory (Arrow-Debreu equilibrium model). The problem arose from Smale's own involvement with mathematical economics.

GCD models describe the out-off-equilibrium dynamics of economic systems. They converge to the solutions of general equilibrium theory under certain conditions. They describe not only the dynamic adjustment of prices, but also of all other economic variables and thus may represent a solution to S. Smale's problem 8.

The method is based on the standard method for modeling dynamics under constraints in physics.

16.3. GCD is a fundamentally new methodology for modeling economic systems and, in a certain sense, can be seen as a metatheory of economic modeling

Simplified, so far there are 4 najor groups of methods for modeling economic systems:

16.3.1. Neoclassical general equilibrium theory (GE, DSGE)

This is essentially based on the maximization of an (overall) utility function under constraints (overall utility maximization). The existence of one overall utility function presupposes the aggregability of individual utility functions.

.

16.3.2. Post-Keynesian Models

These reject the use of individual utility functions and describe the aggregate variables via differential equations.

A special case of these are the Stock-Flow-Consistent (SFC) models.

16.3.3. Agent-Based Models (ABM)

These describe the behaviour of mostly many agents based on individual interactions among them.

16.3.4. The relation of the basic principles of GCD models to these types of economic models

- The dynamic evolution of the variables is determined in GCD models by the fact that each of the agents applies an individual force to these variables and the actual dynamics is determined by the resultant of these forces. These individual forces can be described (in most practical cases) as gradients of individual utility functions. The resulting dynamics can be called individual utility optimization as opposed to neoclassical overall utility maximization. A detailed discussion of the relationship between individual utility optimization and overall utility maximization can be found in Glötzl (2022b).

- Note on post-Keynesian models: Agents' forces do not necessarily arise as gradients of individual utility functions. Therefore, GCD models can also describe post-Keynesian models that cannot be described by utility functions. In principle, the forces (on the right-hand side of the differential equations of post-Keynesian models) can always be decomposed into a gradient component (resulting from a utility function) and a rotation component. This is called a Helmholtz decomposition, which is not only possible in 3 dimensions, as it usually occurs in physics, but is possible in any dimension (Glötzl und Richters 2021b; 2021a)

- GCD models are always stock-flow consistent (SFC). But not only (economic) accounting identities, but also any other relations or conservation laws like the 1st law of chemistry (conservation of mass) or the 1st law of thermodynamics (conservation of energy) can be used as constraints.

- GCD models are always agent-based and thus microfounded.

16.4. GCD models can be the bases for a new economic thinking in terms of: economic power, economic force, economic constraint force

Especially the concept of economic power is of fundamental importance for understanding economics (Rothschild 2002). With GCD models, this concept can also be formally incorporated into economic models. In comparison with classical mechanics in physics, power factors correspond to the reciprocal value of mass (Glötzl 2015). Conventional economic models usually describe one-sided power relations, which, however, rarely occur in reality. GCD models can also be used to better describe mixed power relations and thus reality.

GCD models can be the basis for a new theoretical understanding of e.g.:

- Economic growth
- Business cycles and economic crises
- Analogies between physics and economics

16.5. With the help of the GCD methodology, a formally clean definition of the terms ex-ante and ex-post is possible

16.6. Non-equilibrium dynamics

GCD models can be used to describe true disequilibrium dynamics. In particular, it is also possible to describe situations in which no equilibrium exists or situations in which the utility function is not concave.

16.7. Genuine competitive models

Apart from game-theoretic models, the other types of economic models mentioned cannot be used to describe genuine competition models, i.e. models in which the individual optimization strategy does not lead to an overall optimum. In reality, however, such situations, which are similar to the prisoner's dilemma, are very common. With GCD models, genuine competition models can be described very well.

16.8. Applications

GCD models and IGCD models can be used for many practical tasks such as economic forecasting, modeling the impacts of fiscal or monetary policy, modeling business cycle fluctuations and economic shocks.

16.9. GCD models are a generalisation and alternative to DSGE models

GCD models in principle can also be formulated with intertemporal utility functions called IGCD models (Glötzl 2022c). IGCD models can be seen as a generalisation or alternative to DSGE models.

16.10. What remains to be done in the future

a) Adjustment of parameters to describe real circumstances and comparison of model results with real business cycle trends.

b) Extend GCD models to multiple households, firms, and goods, and in particular to commodity and financial markets.

c) In the long run, develop a more complex, real-world model to enable better economic forecasting and test measures to achieve economic policy targets.

d) Elaborate GCD models with economic shocks in detail.

e) Elaborate IGCD models (with intertemporal utility functions) in detail.

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