General Constrained Dynamic (GCD) models with intertemporal utility functions

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Abstract

In economics balance identities as e.g. C+K'-Y(L,K)=0 must always apply. Therefore, they are called constraints. This means that variables C,K,L cannot change independently of each other. In general equilibrium theory (GE) the solution for the equilibrium is obtained as an optimisation under the above or similar constraints. The standard method for modelling dynamics in macroeconomics are Dynamic Stochastic General Equilibrium (DSGE) models. Dynamics in DSGE models result from the maximisation of an intertemporal utility function that results in the Euler-Lagrange equations. The Euler-Lagrange equations are differential equations that determine the dynamics of the system. In Glötzl, Glötzl, und Richters (2019) we have introduced an alternative method to model dynamics, which constitutes a natural extension of GE theory. This approach is based on the standard method for modelling dynamics under constraints in physics. We therefore call models of this type "General Constrained Dynamic (GCD)" models. In Glötzl (2022b) this modelling method is described for non-intertemporal utility functions in macroeconomics. Since intertemporal utility functions are, however, essential for many economic models, this paper sets out to extend the GCD modelling framework to intertemporal GCD models, referred to as IGCD models in the following. This paper sets out to define the principles of formulating IGCD models and show how IGCD can be understood as a generalisation and alternative to DSGE models.

Contents

1. Introduction

The standard method in macroeconomics for modelling dynamics are DSGE models (Dynamic Stochastic General Equilibrium). Dynamics in DSGE models originate from maximising an intertemporal utility function, which leads to the Euler-Lagrange equations. The Euler-Lagrange equations are differential equations which the dynamics must satisfy.

Recently there has been a renewed interest in alternative approaches in macroeconomics. In Zaman (2020) four different methodological principles are presented which lie outside the framework of the conventional approach. One of these concepts is called GCD (General Constrained Dynamics) and is based on the standard method of physics for modelling a dynamic under constraints. It can be seen as a natural extension of the GE theory for modelling dynamics in economics and can be thought of as an alternative to DSGE. The method was first introduced in Glötzl (2015) under the name Newtonian Constrained Dynamics, a name that was later changed to General Constrained Dynamics. The principles of GCD, an encompassing review of the literature and an application of GCD to the microeconomic Edgeworth box model are presented in Glötzl, Glötzl, und Richters (2019). In Richters und Glötzl (2020) it is shown that SFC models (stock flow consistent models (Godley und Lavoie 2012)) can be understood as special forms of GCD models. In Richters (2021) a more complex macroeconomic model is used to show that GCD models converge to the classical equilibrium solution under some assumptions. In Glötzl (2022c) we show how macroeconomic GCD models can be built in a systematic way and how they can be used for macroeconomic analysis. In this respect, we want to point out that all calculations for all GCD models with non-intertemporal utility functions can be performed easily and conveniently with the open source program GCDconfigurator, which is published in GitHub (Glötzl und Binter 2022) and can be downloaded under

https://github.com/lbinter/gcd

All Mathematica program codes used for calculations of the various GCD models can be downloaded under

https://www.dropbox.com/sh/npis47xjqkecggv/AAAMzCVhmhDYIIhoB5MfATFya?dl=

In contrast to DSGE models, all previously published GCD models were based on non-intertemporal utility functions. Since intertemporal utility functions are essential in many applications and DSGE models only use intertemporal utility functions, it is essential to extend the GCD approach to intertemporal utility functions. Such models are called intertemporal GCD models (IGCD models). This paper describes the principles of formulating IGCD models and shows that IGCD models can be seen as a generalisation and alternative to DSGE models.

Moreover, DSGE models are typically used to analyse economic shocks. Therefore, another paper (Glötzl 2022a) describes how any type of economic shock, e.g. demand, supply or price shocks, can be modelled with GCD.

A discussion of the introduction of expectations in SFC models can be found in Kappes und Milan (2020). In principle the arguments laid out there also hold for GCD and IGCD models.

Notably, non-intertemporal GCD models and intertemporal IGCD models can be seen as an essential contribution to solving problem 8 of the 18 major problems of dynamics listed by Steve Smale in 1991 (Smale 1991; 1997; 1998; Smale Institute 2003).

In **chapter 2** we give a brief introduction to GCD models with non-intertemporal utility functions.

In **chapter 3** we present the extension of GCD models to IGCD models, i.e. GCD models with intertemporal utility functions.

Since the Ramsey model is the simplest model with an intertemporal utility function, **chapter 4** first demonstrates basic principles of IGCD models using the intertemporal utility function of the Ramsey model.

Chapter 5 compares IGCD models with DSGE models.

Using model A1, it is shown how an IGCD model (with intertemporal utility functions) can be established as an extension of a GCD model (with non-intertemporal utility functions). A1 is a simple macroeconomic model with 1 household, 1 firm and 1 good, described in detail in Glötzl (2022b). We therefore first introduce this GCD model in **chapter 6**.

Finally, in **chapter 7** we describe how to set up the corresponding IGCD model $A1^{int}$.

In **chapter 8** we give a summary.

2. GCD models with non-intertemporal utility functions

In general, a dynamic economic model is described by agents and variables that describe any stock or flow of goods, resources, financial liabilities or other variables or parameters such as prices or interest rates. The behaviour of these variables is described by behavioural equations. The behaviour of these variables can be restricted by economic constraints, which are described by additional equations. In particular, all balance sheet identities are subject to such constraints. In general, the introduction of additional constraints to the behavioural equations can lead to the system of equations becoming overdetermined and thus unsolvable. The GCD method is a "closure" method to make a system of equations solvable by introducing additional Lagrange multipliers. It can also be understood as a method to transfer the concept of Lagrange multipliers from optimization problems under constraints to dynamic systems under constraints. This is done in analogy to what is done in classical mechanics.

The GCD method is described in detail in Glötzl, Glötzl, und Richters (2019). We will therefore limit ourselves to the explanation of a simple example with 2 agents (each with 1 non-intertemporal utility function) and 1 constraint.

We explain the principle for 2 agents A,B and 2 variables x_1,x_2.
The utility functions of A,B are $U^A(x_1,x_2), U^B(x_1,x_2)$. The interest of A is to change x_1,x_2 so that the increase of his utility function is maximal. This is given, if the change of x_1,x_2 is done in the direction of the gradient of $U^A(x_1,x_2)$, i.e.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \textit{ proportional to } \begin{pmatrix} \dfrac{\partial U^A}{\partial x_1} \\ \dfrac{\partial U^A}{\partial x_2} \end{pmatrix}$$

The interest of A in a change of the variables does not lead alone to an actual change, because the household must have also the power and/or possibility of actually implementing its change desire. For example, a household cannot or can only partially enforce its additional consumption desire, e.g., to go to the cinema or go on vacation, because it is possibly quarantined or the borders are closed. This limitation of the possibility to enforce his consumption change requests is described by a (possibly time-dependent and endogenously determined) "power factor" μ_C^H. In general, the change request for each of the variables is described by "power factors" $\mu_{x_1}^H, \mu_{x_2}^H$. Considering the power factors, the following applies to the change of x_1, x_2 (due to the interest of A and the power of A to enforce this interest)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \textit{ proportional to } \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix}$$

Just as A has an interest, to change x_1, x_2, also B has an interest to change these two variables. The actual change is therefore the result of the two individual efforts to change, weighted with the power factors. We therefore call this behaviour **"individual utility optimization"**.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} \qquad \text{<2.1>}$$

This equation of motion <2.1> describe the temporal development of (x_1, x_2) under the condition that there are no constraints that restrict the temporal development. It is therefore referred to as the **ex-ante equation of motion.**

If a constraint

$$Z(x_1, x_2) = 0$$

exists, there arises an additional constraint force f^Z to the ex-ante force which ensures that the constraint is fulfilled at all times. In physics, this constraint force is perpendicular to the constraint at all times due to the so-called d'Alembert principle, i.e.

$$f^Z(x_1, x_2) = \begin{pmatrix} f_1^Z(x_1, x_2) \\ f_1^Z(x_1, x_2) \end{pmatrix} = \lambda \begin{pmatrix} \dfrac{\partial Z(x_1, x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1, x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<2.2>}$$

The time-dependent factor $\lambda = \lambda(t)$ is called Lagrange multiplier, as in the case of optimisation under constraints.

In economic models, the constraint force does not necessarily have to be perpendicular to the constraint at any point in time due to a special economic principle as in physics, but in most cases it is plausible to model constraint forces in a similar way to physics, namely perpendicular to the constraint.

From <2.1> and <2.2> we get the equation of motion, which is called **ex post equation of motion**:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu^A_{x_1} \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu^A_{x_2} \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu^B_{x_1} \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu^B_{x_2} \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<2.3>}$$

For J agents with the designations $j = 1,2,...,J$
I Variables with the designations x_i $i = 1,2,...,I$ $x = (x_1, x_2,..., x_I)$
K Constraints with the designations Z_k $k = 1,2,...,K$
the general GCD model equations result analogously

$$x_i' = \sum_{j=1}^{J} \mu^j_{x_i} \frac{\partial U^j(x)}{\partial x_i} + \sum_{k=1}^{K} \lambda_k \frac{\partial Z^k(x)}{\partial x_i} \qquad i = 1,2,...,I \qquad \text{<2.4>}$$

Remark: If constraint conditions depend on time derivatives of variables
If a constraint depends not only on $x = (x_1, x_2,....x_I)$ but also on $x' = (x_1', x_2',....x_I')$ or higher derivatives $x'' = (x_1'', x_2'',....x_I'')$, ...,i.e.
$0 = Z(x, x', x'',....)$
the constraint forces are always to be derived from the highest time derivative of the variables (Flannery 2011), i.e.

$$\frac{\partial Z(x,x')}{\partial x_i'} \; instead \; of \; \frac{\partial Z(x,x')}{\partial x_i} \qquad resp. \quad \frac{\partial Z(x,x',x'')}{\partial x_i'} \; instead \; of \; \frac{\partial Z(x,x',x'')}{\partial x_i}$$

3. IGCD: Intertemporal General Constrained Dynamics

For sake of simplicity most is described for 2 agents A, B and 1 constraint Z .

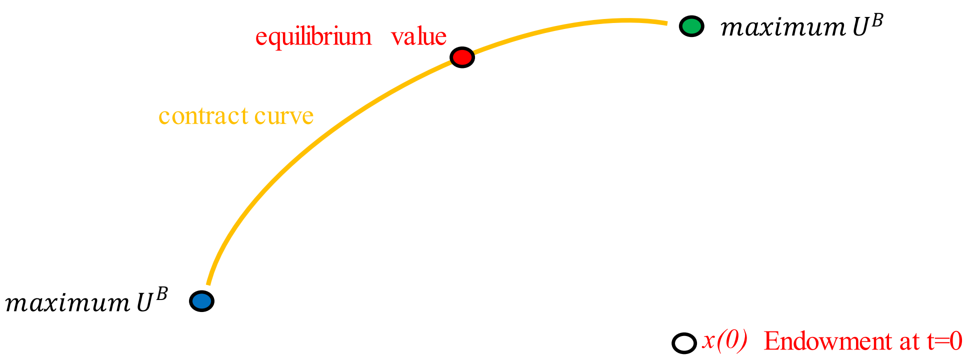
3.1. Comparison of the basic ideas

3.1.1. GE (for non-intertemporal utility functions)

The economic system jumps from endowment at t = 0 along an unspecified tatonnement curve to equilibrium value as symbolically is shown in the following graphic.

GE

The economic system jumps from endowment at t=0 along an unspecified tatonnement curve to equilibrium value



3.1.2. GCD (for non-intertemporal utility functions)

The basic idea of the **GCD** method for **non-intertemporal** utility functions is that each agent tries to change the variables in the direction in which the change in its individual utility function is maximum at any given time. In other words, every agent tries to change the variables in the direction of the gradient of its individual utility function:

$$\begin{pmatrix} \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \textit{resp.} \qquad \begin{pmatrix} \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

His desire for change is limited by his power to enforce his interest. This is expressed by the power factors $(\mu^A_{x_1}, \mu^B_{x_1}, \mu^A_{x_2}, \mu^B_{x_2})$. $\mu^A_{x_1}$ describes the power of the agent A to influence the variable x_1 and $\mu^A_{x_1}\dfrac{\partial U^A(x_1,x_2)}{\partial x_1}$ describes the effective force exerted by the agent on the change of the variable. This results in the effective forces

$$\begin{pmatrix} \mu^A_{x_1}\dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu^A_{x_2}\dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \textit{resp.} \qquad \begin{pmatrix} \mu^B_{x_1}\dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu^B_{x_2}\dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

Since normally the desires and the power of different agents are different, the system develops ex-ante according to the resultant of the two effective forces:

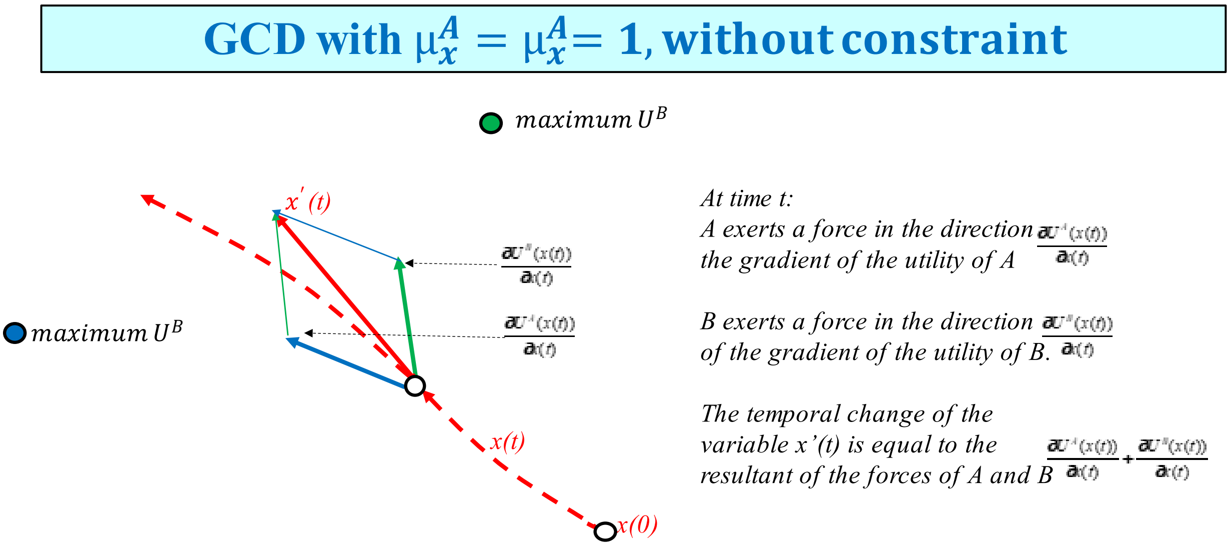
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu^A_{x_1}\dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu^A_{x_2}\dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu^B_{x_1}\dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu^B_{x_2}\dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

Considering the constraint Z , we obtain the GCD equation system for the ex-post dynamics:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu^A_{x_1}\dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu^A_{x_2}\dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu^B_{x_1}\dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu^B_{x_2}\dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

$$0 = Z(x_1, x_2)$$

The dynamics of a GCD model symbolically is shown in the following graphic.



3.1.3. GE for intertemporal utility functions

GE models are characterised by the fact that an objective function is maximised for a specific point in time. In GE models, in contrast to GCD models, it must therefore always be assumed that the individual utility functions can be aggregated to a master utility function MU , which then serves as an objective function, because maximisation is only ever possible for one objective function and not for several at the same time. In the case of intertemporal GE models, such as the Ramsey model or DSGE models, this objective function is the time integral over a master utility function MU discounted at a discount rate r, which is maximised. The model equations therefore result from the requirement

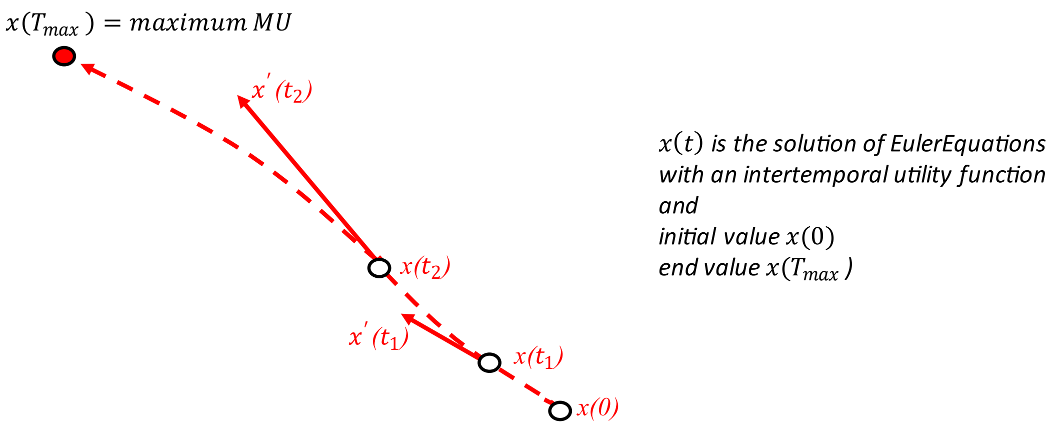
$$U^{int} = \int_0^\infty e^{-r\tau} MU(x_1(\tau), x_2(\tau))\, d\tau \to max$$

or, in the case of a constraint arising from the requirement

$$U^{int} = \int_0^\infty e^{-r\tau} \big(MU(x_1(\tau), x_2(\tau)) - \lambda(\tau) Z(x_1(\tau), x_2(\tau)) \big) d\tau \to max$$

These variation problems lead to the Euler-Lagrange equation system for ex-ante respectively ex-post dynamics. This is a differential equation system which the solutions for the intertemporal GE model must fulfil in any case. The Euler-Lagrange equation system thus describes the dynamics of an intertemporal GE model in the same way as the GCD equation system <2.3> does for a GCD model. The dynamics of the Ramsey model is shown in illustrative form in the following graphic.

Ramsey



3.1.4. IGCD: GCD with intertemporal utility functions

The basic idea of the **GCD** method for **intertemporal** utility functions is that each agent solves its own variational problem at any given time t. In other words, each agent looks for the solution that maximises his or her individual intertemporal utility function at the time t :

$$U^{A\,int\,t} = \int_0^\infty e^{-r(t+\tau)} U^A(x_1^{A\,int\,t}(t+\tau), x_2^{A\,int\,t}(t+\tau))\,d\tau \rightarrow max$$

respectively

$$U^{A\,int\,t} = \int_0^\infty e^{-r(t+\tau)} U^B(x_1^{B\,int\,t}(t+\tau), x_2^{B\,int\,t}(t+\tau))\,d\tau \rightarrow max$$

Thus

$$(x_1^{A\,int\,t}(t+\tau), x_2^{A\,int\,t}(t+\tau))$$
$$(x_1^{B\,int\,t}(t+\tau), x_2^{B\,int\,t}(t+\tau))$$

denotes the solutions of the independent variational problems for A and B at time t, which depends on the future time τ.

In non-intertemporal GCD models the agents try to change the variables in the direction of the gradient of his or her utility functions,

$$\begin{pmatrix} \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \textit{resp.} \qquad \begin{pmatrix} \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

which, taking their individual economic powers $\mu_{x_1}^A, \mu_{x_2}^A, \mu_{x_1}^B, \mu_{x_2}^B$ into account, leads to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

In intertemporal IGCD models the agents try to change the variables in the direction they assume to be optimal for their intertemporal utility, that is just the time derivative of their individual solutions of the variational problem

$$\begin{pmatrix} \left.\dfrac{d\, x_1^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \left.\dfrac{d\, x_2^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} \qquad \textit{resp.} \qquad \begin{pmatrix} \left.\dfrac{d\, x_1^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \left.\dfrac{d\, x_2^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix}$$

Assuming that their power to enforce their interests in such a way is proportional to their relative individual powers

$$\frac{\mu_1^A}{\mu_1^A+\mu_1^B}, \frac{\mu_2^A}{\mu_2^A+\mu_2^B}, \frac{\mu_1^B}{\mu_1^A+\mu_1^B}, \frac{\mu_2^B}{\mu_2^A+\mu_2^B}$$

leads to the **ex-ante IGCD equation** (for intertemporal utility functions)

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\mu_1^A}{\mu_1^A+\mu_1^B} \left.\dfrac{d\, x_1^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \dfrac{\mu_2^A}{\mu_2^A+\mu_2^B} \left.\dfrac{d\, x_2^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} + \begin{pmatrix} \dfrac{\mu_1^B}{\mu_1^A+\mu_1^B} \left.\dfrac{d\, x_1^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \dfrac{\mu_2^B}{\mu_2^A+\mu_2^B} \left.\dfrac{d\, x_2^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} \qquad \langle 3.1 \rangle$$

For sake of simplicity, in the following we also denote the solutions of the variational problems with constraint Z by

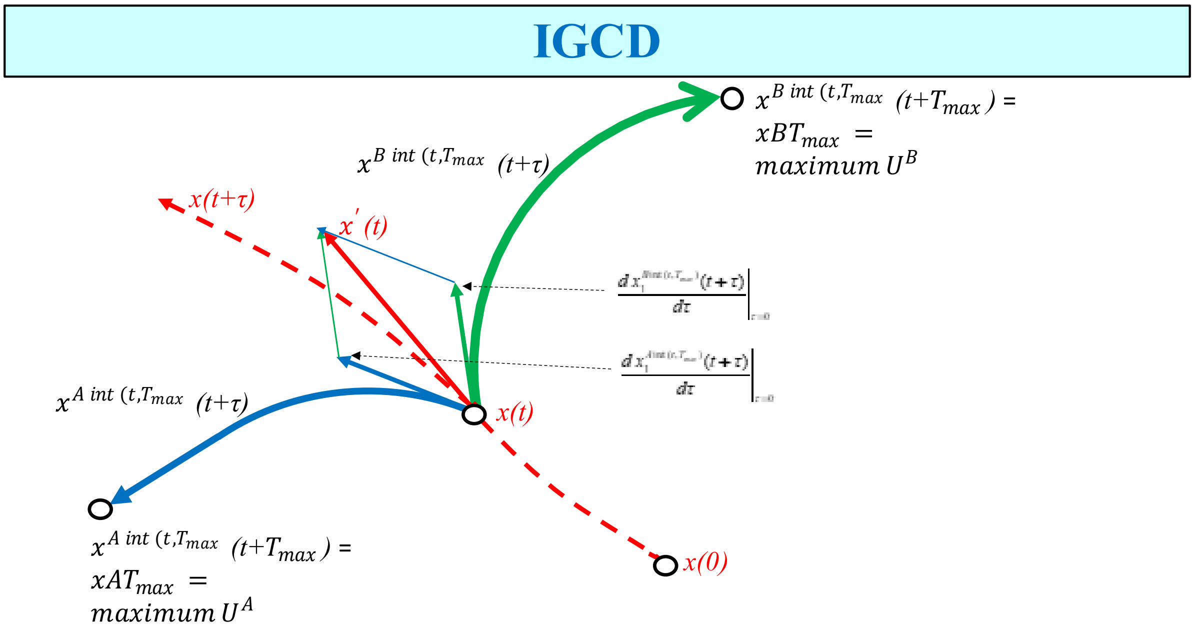
$(x_1^{A\,\mathrm{int}\,t,Z}(t+\tau), x_2^{A\,\mathrm{int}\,t,Z}(t+\tau))$

$(x_1^{B\,\mathrm{int}\,t,Z}(t+\tau), x_2^{B\,\mathrm{int}\,t,Z}(t+\tau))$

Then the **ex-post IGCD equation** (for intertemporal utility functions) reads formally the same as <3.1>

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \dfrac{\mu_1^A}{\mu_1^A + \mu_1^B} & \left.\dfrac{d\, x_1^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \dfrac{\mu_2^A}{\mu_2^A + \mu_2^B} & \left.\dfrac{d\, x_2^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} + \begin{pmatrix} \dfrac{\mu_1^B}{\mu_1^A + \mu_1^B} & \left.\dfrac{d\, x_1^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \dfrac{\mu_2^B}{\mu_2^A + \mu_2^B} & \left.\dfrac{d\, x_2^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} \qquad \text{<3.2>}$$

The dynamics of an IGCD model is shown illustratively in the following graphic.



3.2. Definition of IGCD in detail:

For the sake of clarity and simplicity, we rewrite the GCD-system of equations for two agents A,B with the **non-intertemporal** utility functions U^A, U^B, the 2 variables x_1, x_2 and the constraint Z.

$$x_1' = \mu_{x_1}^A \frac{\partial U^A}{\partial x_1} + \mu_{x_1}^B \frac{\partial U^B}{\partial x_1} + \lambda \frac{\partial Z}{\partial x_1}$$

$$x_2' = \mu_{x_{x_2}}^A \frac{\partial U^A}{\partial x_2} + \mu_{x_1}^B \frac{\partial U^B}{\partial x_2} + \lambda \frac{\partial Z}{\partial x_2}$$

$$0 = Z(x_1, x_1', x_2, x_2')$$

Designate T_{max} the end time and for each $t \in [0, T_{max}]$ designate $U^{A\,\mathrm{int}\,(t,T_{max})}, U^{B\,\mathrm{int}\,(t,T_{max})}$ the intertemporal utility functions of the two agents A,B for optimization in the period from t to T_{max} with discount rates r^A, r^B and describe $x_1(t+\tau),\ x_2(t+\tau)$ the time evolution of x_1, x_2 as a function of $\tau \in [t, T_{max}]$. The intertemporal utility functions are given by

$$U^{A\,int(t,T_{max})}(x_1,x_2) = \int_0^{T_{max}} e^{-r^A (t+\tau)} \Big(U^A(x_1(t+\tau),\, x_2(t+\tau)) - \lambda\, Z(x_1(t+\tau), x_2(t+\tau)) \Big) d\tau$$

$$U^{B\,int(t,T_{max})}(x_1,x_2) = \int_0^{T_{max}} e^{-r^B (t+\tau)} \Big(U^B(x_1(t+\tau),\, x_2(t+\tau)) - \lambda\, Z(x_1(t+\tau), x_2(t+\tau)) \Big) d\tau$$

At each point in time t, both agents **independently** try to maximise their intertemporal utilities under the constraint Z. The initial conditions must correspond to the values of the variables at the current time. The final condition is chosen by each agent individually according to his individual interest.

$$x_1^{A\,int(t,T_{max})}(t) = x_1(t) \qquad x_1^{A\,int(t,T_{max})}(T_{max}) = x1AT_{max}$$
$$x_2^{A\,int(t,T_{max})}(t) = x_2(t) \qquad x_2^{A\,int(t,T_{max})}(T_{max}) = x2AT_{max}$$

$$x_1^{B\,int(t,T_{max})}(t) = x_1(t) \qquad x_1^{B\,int(t,T_{max})}(T_{max}) = x1BT_{max}$$
$$x_2^{B\,int(t,T_{max})}(t) = x_2(t) \qquad x_2^{B\,int(t,T_{max})}(T_{max}) = x2BT_{max}$$

This gives for each fixed point in time t and for each agent for the period of time from t until T_{max} the intertemporal optimal solutions which are designated by $x_1^{A\,int(t,T_{max})}(t+\tau),\, x_2^{A\,int(t,T_{max})}(t+\tau)$ respectively $x_1^{B\,int(t,T_{max})}(t+\tau),\, x_2^{B\,int(t,T_{max})}(t+\tau)$.
The solutions $x_1^{A\,int(t,T_{max})}(t+\tau), x_2^{A\,int(t,T_{max})}(t+\tau)$ respectively $x_1^{B\,int(t,T_{max})}(t+\tau), x_2^{B\,int(t,T_{max})}(t+\tau)$ result from the Euler equations[1] of the two variation problems with constraints and with the corresponding initial and final conditions and thus for each fixed t and T_{max} are functions of $\tau \in [t, T_{max}]$:

$$\text{EulerEquations}\Big[e^{-r^A (t+\tau)} U^A (x_1^{A\,int(t,T_{max})}(t+\tau), x_2^{A\,int(t,T_{max})}(t+\tau)) + \lambda Z(x_1^{A\,int(t,T_{max})}(t+\tau), x_2^{A\,int(t,T_{max})}(t+\tau)), \Big\{ x_1^{A\,int(t,T_{max})}(t+\tau), x_2^{A\,int(t,T_{max})}(t+\tau) \Big\}, \tau \Big]$$

with intial and end values

$$x_1^{A\,int(t,T_{max})}(t) = x_1(t) \qquad x_1^{A\,int(t,T_{max})}(T_{max}) = x1AT_{max}$$
$$x_2^{A\,int(t,T_{max})}(t) = x_2(t) \qquad x_2^{A\,int(t,T_{max})}(T_{max}) = x2AT_{max}$$

$$\text{EulerEquations}\Big[e^{-r^B (t+\tau)} U^B (x_1^{B\,int(t,T_{max})}(t+\tau), x_2^{B\,int(t,T_{max})}(t+\tau)) + \lambda Z(x_1^{B\,int(t,T_{max})}(t+\tau), x_2^{B\,int(t,T_{max})}(t+\tau)), \Big\{ x_1^{B\,int(t,T_{max})}(t+\tau), x_2^{B\,int(t,T_{max})}(t+\tau) \Big\}, \tau \Big]$$

with intial and end values

$$x_1^{B\,int(t,T_{max})}(t) = x_1(t) \qquad x_1^{B\,int(t,T_{max})}(T_{max}) = x1BT_{max}$$
$$x_2^{B\,int(t,T_{max})}(t) = x_2(t) \qquad x_2^{B\,int(t,T_{max})}(T_{max}) = x2BT_{max}$$

Typically, the constraint does not depend on $x_2'(t)$, i.e.

[1] Be careful: to use "EulerEquations" in Mathematica correctly, one has to define $\vec{x}^t(\tau) := x(t+\tau)$ and use $\vec{x}^t(\tau)$ instead of $x(t+\tau)$

$0 = Z(x_1(t), x_1'(t), x_2(t))$

and the variable $x_2(t)$ can be expressed as a function of $x_1(t)$ and inserted into the utility function. This is what we will always assume in the following, because this simplifies the problem considerably. This is explained using the Ramsey model as an example (see chapters 4.1, 4.2). It leads to the fact that the Lagrange multiplier $\lambda(t)$ drops out and the variational problem with constraint is simplified to a variational problem without constraint and the utility function only depends on $x_1(t)$,. The variational problem to be solved is then

$$\text{EulerEquations}\left[e^{-r^A(t+\tau)} U^A(x_1^{A\,\text{int}(t,T_{max})}(t+\tau)), \{x_1^{A\,\text{int}(t,T_{max})}(t+\tau)\}, \tau\right]$$

with initial and end values

$$x_1^{A\,\text{int}(t,T_{max})}(t) = x_1(t) \qquad x_1^{A\,\text{int}(t,T_{max})}(T_{max}) = x1HT_{max}$$

$$\text{EulerEquations}\left[e^{-r^B(t+\tau)} U^B(x_1^{B\,\text{int}(t,T_{max})}(t+\tau)), \{x_1^{B\,\text{int}(t,T_{max})}(t+\tau)\}, \tau\right]$$

with initial and end values

$$x_1^{B\,\text{int}(t,T_{max})}(t) = x_1(t) \qquad x_1^{B\,\text{int}(t,T_{max})}(T_{max}) = x1BT_{max}$$

The end values can be selected freely.
Assuming the trajectory of x_1 until t is $x_1(s)$ with $s < t$.
In order to follow its optimal path for the future, agent A must try to set the temporal change at time t equal to the temporal change of its intertemporal maximised trajectory, i.e.

$$x_1'(t) = \left.\frac{d\, x_1^{A\,\text{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0}$$

But also, the agent B must try to set the temporal change $x_1(t)$ equal to the temporal change of his intertemporal maximized course, i.e.

$$x_1'(t) = \left.\frac{d\, x_1^{B\,\text{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0}$$

But these two wishes cannot both be fulfilled at the same time. The actual temporal change of $x_1(t)$ at the time t therefore results in retrospect on the one hand as a mixture of the wishes of A and B (weighted with their relative power relations) and on the other hand from the fact that the constraint at the time must also be fulfilled. This results in

$$x_1'(t) = \frac{\mu_1^A}{\mu_1^A + \mu_1^B} \left.\frac{d\, x_1^{A\,\text{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0} + \frac{\mu_1^B}{\mu_1^A + \mu_1^B} \left.\frac{d\, x_1^{B\,\text{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0} + \lambda(t)\frac{\partial Z(x_1(t), x_2(t))}{\partial x_1(t)}$$

Since we have assumed the simplifying case and expressed $x_2(t)$ through $x_1(t)$, the constraint is always fulfilled and the last term falls away. This results in

$$x_1'(t) = \frac{\mu_1^A}{\mu_1^A + \mu_1^B} \left. \frac{d\, x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} + \frac{\mu_1^B}{\mu_1^A + \mu_1^B} \left. \frac{d\, x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0}$$

This equation describes the temporal behaviour of x_1 as function of t. The initial condition $x_1(0)$ results from the model assumptions for the time $t=0$.

Thus, taking into account the final values $x1AT_{max}, x1BT_{max}$ assumed by the agents for their variational problem and the initial value $x_1(0) = x10$, the following **IGCD** (intertemporal GCD) **equation system** results:

behavioural equation for $x_1(t)$

$$x_1'(t) = \frac{\mu_1^A}{\mu_1^A + \mu_1^B} \left. \frac{d\, x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} + \frac{\mu_1^B}{\mu_1^A + \mu_1^B} \left. \frac{d\, x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0}$$

initial value for $x_1(t)$

$$x_1(0) = x10$$

Euler equations for $x_1^{A\,\mathrm{int}(t,T_{max})}(\tau)$ for A with initial and final values

$$\mathrm{EulerEquations}\left[e^{-r^A(t+\tau)} U^A(x_1^{H\,\mathrm{int}(t,T_{max})}(t+\tau)), \left\{ x_1^{H\,\mathrm{int}(t,T_{max})}(t+\tau) \right\}, \tau \right] \qquad \text{<3.3>}$$

$$x_1^{A\,\mathrm{int}(t,T_{max})}(t) = x_1(t) \qquad x_1^{A\,\mathrm{int}(t,T_{max})}(T_{max}) = x1AT_{max}$$

Euler equations for $x_1^{B\,\mathrm{int}(t,T_{max})}(\tau)$ for B with initial and final values

$$\mathrm{EulerEquations}\left[e^{-r^B(t+\tau)} U^B(x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)), \left\{ x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau) \right\}, \tau \right]$$

$$x_1^{B\,\mathrm{int}(t,T_{max})}(t) = x_1(t) \qquad x_1^{B\,\mathrm{int}(t,T_{max})}(T_{max}) = x1BT_{max}$$

With n variables and m constraints, the number of variables is reduced to $k = n - m$ variables $x_1, x_2, ..., x_k$ respectively $x_i, i = 1,2,...,k$. This results in the **IGCD** (intertemporal GCD) **equation system for 2 agents and k variables**: <3.4>

$for\ all\ i = 1,2,...,k$

behavioural equations for $x_1(t)$

$$x_i'(t) = \frac{\mu_i^A}{\mu_i^A + \mu_i^B} \left. \frac{d\, x_i^{A\,\text{int}\,(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} + \frac{\mu_i^B}{\mu_i^A + \mu_i^B} \left. \frac{d\, x_i^{B\,\text{int}\,(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0}$$

initial values for $x_i(t)$

$$x_i(0) = xi0$$

Euler equations for $x_i^{A\,\text{int}\,(t,T_{max})}(\tau)$ *for A with initial and final values*

$$\text{EulerEquations}\left[e^{-r^A (t+\tau)} U^A (x_1^{A\,\text{int}\,(t,T_{max})}(t+\tau),...,x_k^{A\,\text{int}\,T_{max}}(t+\tau)), \left\{ (x_1^{A\,\text{int}\,T_{max}}(t+\tau),...,x_k^{A\,\text{int}\,T_{max}}(t+\tau)) \right\}, \tau \right]$$

$$x_i^{A\,\text{int}\,(t,T_{max})}(t) = x_i(t) \qquad x_i^{A\,\text{int}\,(t,T_{max})}(T_{max}) = xiAT_{max}$$

Euler equations for $x_i^{B\,\text{int}\,(t,T_{max})}(\tau)$ *for B with initial and final values*

$$\text{EulerEquations}\left[e^{-r^B (t+\tau)} U^B (x_1^{B\,\text{int}\,(t,T_{max})}(\tau),...,x_k^{B\,\text{int}\,T_{max}}(\tau)), \left\{ (x_1^{B\,\text{int}\,T_{max}}(\tau),...,x_k^{B\,\text{int}\,T_{max}}(\tau)) \right\}, \tau \right]$$

$$x_i^{B\,\text{int}\,(t,T_{max})}(t) = x_i(t) \qquad x_i^{B\,\text{int}\,(t,T_{max})}(T_{max}) = xiBT_{max}$$

Note

(a) Up to now we have set fixed end values for the end time T_{max} for intertemporal optimisation. For other end conditions (e.g. "free" or "greater than") these conditions can be replaced by the corresponding so-called transversality conditions.

(b) The intertemporal optimisation for infinite time intervals can be approximated by large T_{max} .

3.3. Numerical solution

This differential equation system <3.4> cannot be solved directly with NDSolve from Mathematica. For the numerical solution the interval $[0, T_{max}]$ must be divided into N intervals with the points in time $t_0 = 0, t_1, t_2, ..., t_N = T_{max}$. Proceed step by step as follows:

(1) solve the Euler equations for the interval $[0, T_{max}]$ with initial and final values

$$x_i^{A\,\text{int}\,(t_0,T_{max})}(t_0) = xi0 \qquad x_i^{A\,\text{int}\,(t_0,T_{max})}(T_{max}) = xiAT_{max}$$
$$x_i^{B\,\text{int}\,(t_0,T_{max})}(t_0) = xi0 \qquad x_i^{B\,\text{int}\,(t_0,T_{max})}(T_{max}) = xiBT_{max}$$

(2) Calculate $x_i'(t_0)$

$$x_i'(t_0) = \frac{\mu_i^A}{\mu_i^A + \mu_i^B} \frac{d\, x_i^{A\,\mathrm{int}\,(t_0,T_{max})}(t_0+\tau)}{d\tau}\Bigg|_{\tau=0} + \frac{\mu_i^B}{\mu_i^A + \mu_i^B} \frac{d\, x_i^{B\,\mathrm{int}\,(t_0,T_{max})}(t_0+\tau)}{d\tau}\Bigg|_{\tau=0}$$

(3) Calculate $x_i(t_1)$

either as a linear approximation: $x_i(t_1) = x_i(t_0) + x_i'(t_0)(t_1 - t_0)$

or as an exponential approximation: $x_i(t_1) = x_i(t_0)\, e^{x_i'(t_0)\,(t_1-t_0)}$

(4) Solve the Euler equations for the interval $[t_1, T_{max}]$ with initial or final values

$x_i^{A\,\mathrm{int}\,(t_1,T_{max})}(t_1) = x_i(t_1)$ $\quad x_i^{A\,\mathrm{int}\,(t_1,T_{max})}(T_{max}) = xiAT_{max}$

$x_i^{B\,\mathrm{int}\,(t_1,T_{max})}(t_1) = x_i(t_1)$ $\quad x_i^{B\,\mathrm{int}\,(t_1,T_{max})}(T_{max}) = xiBT_{max}$

(5) Calculate $x_i'(t_1)$

$$x_i'(t_1) = \frac{\mu_i^A}{\mu_i^A + \mu_i^B} \frac{d\, x_i^{A\,\mathrm{int}\,(t_1,T_{max})}(t_1+\tau)}{d\tau}\Bigg|_{\tau=0} + \frac{\mu_i^B}{\mu_i^A + \mu_i^B} \frac{d\, x_i^{B\,\mathrm{int}\,(t_1,T_{max})}(t_1+\tau)}{d\tau}\Bigg|_{\tau=0}$$

(7) Calculate $x_i(t_2)$

either as a linear approximation: $x_i(t_2) = x_i(t_1) + x_i'(t_1)(t_2 - t_1)$

or as an exponential approximation: $x_i(t_2) = x_i(t_1)\, e^{x_i'(t_1)\,(t_2-t_1)}$

(8) Solve the Euler equations for the interval $[t_2, T_{max}]$ with initial or final values

$x_i^{A\,\mathrm{int}\,(t_2,T_{max})}(t_2) = x_i(t_2)$ $\quad x_i^{A\,\mathrm{int}\,(t_2,T_{max})}(T_{max}) = xiAT_{max}$

$x_i^{B\,\mathrm{int}\,(t_2,T_{max})}(t_2) = x_i(t_2)$ $\quad x_i^{B\,\mathrm{int}\,(t_2,T_{max})}(T_{max}) = xiBT_{max}$

(9) etc.

3.4. The relationship between the dynamics of GCD models (with non-intertemporal utility functions) and the dynamics of GE models with intertemporal utility functions

3.4.1. Basic principles

In simplified terms, non-intertemporal GCD models behave at $t=0$ the same as intertemporal GE models, in which the future is increasingly devalued by shortening the optimisation period. It should be noted that intertemporal GE models require that the utility functions can be aggregated. Therefore, the relationship between these two models can only be established for utility functions that can be aggregated. For simplicity, we describe everything for 2 agents A, B, 2 goods (x_1, x_2) and 1 constraint $Z(x_1, x_2) = 0$

2 utility functions U^A, U^B are called aggregable if there is a utility function MU so that

$$\mu_{x_1}^H \frac{\partial U^H}{\partial x_1} + \mu_{x_1}^B \frac{\partial U^B}{\partial x_1} = \frac{\partial MU}{\partial x_1}$$
$$\mu_{x_2}^H \frac{\partial U^H}{\partial x_2} + \mu_{x_2}^B \frac{\partial U^B}{\partial x_2} = \frac{\partial MU}{\partial x_2}$$

The non-intertemporal GCD model is described ex-ante (i.e. without considering the constraints) by

$$x_1' = \frac{\partial MU}{\partial x_1}$$
$$x_2' = \frac{\partial MU}{\partial x_2}$$

and ex-post (i.e. taking into account the constraints) described by

$$x_1' = \frac{\partial MU}{\partial x_1} + \lambda \frac{\partial Z}{\partial x_1}$$
$$x_2' = \frac{\partial MU}{\partial x_2} + \lambda \frac{\partial Z}{\partial x_2}$$
$$0 = Z(x_1, x_2)$$

The GE non-intertemporal model is described ex-ante (i.e. without considering the constraints) by:

$$\int_0^{T_{max}} MU(x_1(t), x_2(t))\,dt \to max$$

and ex-post (i.e. with consideration of the constraint) described by

$$\int_0^{T_{max}} \left(MU(x_1(t), x_2(t)) + \lambda(t) Z(x_1(t), x_2(t))\right) dt \to max$$

A necessary condition that must be fulfilled by x_1, x_2 such that the integrals become maximum are the Euler-Lagrange equations.

3.4.2. A non-intertemporal GCD model behaves at time t = 0 in the same way as an intertemporal GE model with a very short optimisation interval

Looking at the ex-ante behaviour of a GE model with a non-intertemporal utility function, it follows

$$\Rightarrow \int_0^{T_{max}} e^{-rt} MU(x_1(t), x_2(t))\,dt \to max$$

Assume that $T_{\max}$ and r are very small. If one carries out a series expansion of $e^{-r\tau}$ and MU with respect to t at point $t = 0$ one obtains the following

$$\int_0^{T_{max}} e^{-rt} MU(x_1(t),x_2(t))\,dt =$$

$$= \int_0^{T_{max}} \left(1-rt+....\right)\left(MU(x_1(0),x_2(0)) + \left.\frac{d\,MU}{dt}\right|_{t=0} .t + \frac{1}{2}\left.\frac{d^2MU}{dt^2}\right|_{t=0} .t^2 + ... \right)dt \approx$$

$$\approx \int_0^{T_{max}} \left(MU(x_1(0),x_2(0)) + \left.\frac{d\,MU}{dt}\right|_{t=0} .t + ... \right)dt$$

because of the assumption r is small and for small T_{max} t is small

$$= T_{max}\, MU(x_1(0),x_2(0)) + \int_0^{T_{max}} \left(\left.\frac{\partial MU(x_1(t),x_2(t))}{\partial x_1(t)}\right|_{x_1(t)=x_1(0)} \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} + \left.\frac{\partial MU(x_1(t),x_2(t))}{\partial x_2(t)}\right|_{x_2(t)=x_2(0)} \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} \right).t\,dt + ...$$

$$\approx T_{max}\, MU(x_1(0),x_2(0)) + \left(\frac{\partial MU}{\partial x_1} \left. x_1' \right|_{t=0} + \frac{\partial MU}{\partial x_2} \left. x_2' \right|_{t=0} \right) \frac{{T_{max}}^2}{2} \; for \;\; small \;\; T_{max}$$

The first term is constant, the second term becomes maximal exactly when the vector $(\frac{\partial MU}{\partial x_1},\frac{\partial MU}{\partial x_2})$ and the vector (x_1', x_2') at the time $t=0$ point in the same direction, i.e. there is a $\mu \in \mathbb{R}$ such that

$$\left.\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\right|_{t=0} = \mu \left.\begin{pmatrix} \frac{\partial MU}{\partial x_1} \\ \frac{\partial MU}{\partial x_2} \end{pmatrix}\right|_{t=0} = \left.\begin{pmatrix} \frac{\partial \mu MU}{\partial x_1} \\ \frac{\partial \mu MU}{\partial x_2} \end{pmatrix}\right|_{t=0}$$

This means that it applies to small r and small T_{max}:

A GE model with an intertemporal utility function

$$U^{\mathrm{int}(0,T_{\max})} = \int_0^{T_{max}} e^{-rt} MU(x_1(t),x_2(t))\,dt$$

behaves at the time $t=0$ ex-ante (i.e. without considering the constraint) similar to a non-intertemporal GCD model with the utility function μMU.

4. The principles of IGCD are first presented using the Ramsey model as an example

4.1. The Ramsey model

The standard Ramsey model consists of 1 agent (household) that attempts to maximize the intertemporal utility of consumption C over the period from $t_0 = 0 \; to \; T_{\max}$. The utility function U^A is

$$U^A(C(t)) = C(t)^\gamma \qquad 0 \le \gamma \le 1$$

The constraint Z is given by

$$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$

The intertemporal utility function of the household $U^{A\text{int}}$ is given by

$$U^{A\text{int}(0,T_{\max})}(C) = \int_0^{T_{\max}} e^{-r\,\tau} U^A(C(\tau))d\tau = \int_0^{T_{\max}} e^{-r\,\tau} C(\tau)^\gamma d\tau$$

Calculate $C(t)$ from the constraint and insert in U^A . This results in the variation problem

$$U^{A\text{int}(0,T_{\max})}(K) = \int_0^{T_{\max}} e^{-r\,\tau} (K(\tau))^{(1-\alpha)} - K'(\tau))^\gamma d\tau \to \max$$

$$with \; K(0) = k0 \qquad K(T_{\max}) = kT_{\max}$$

The solution is obtained by solving the Euler equation with initial and final values:

$$\text{EulerEquations}\left[e^{-r\tau}(K(\tau)^{(1-\alpha)} - K'(\tau))^\gamma, \{K(\tau)\}, \tau\right]$$

$$with \; K(0) = k0 \qquad K(T_{\max}) = kT_{\max}$$

which result in the differential equation system to be solved

$$\begin{aligned} 0 &= (-1+\alpha)K(\tau) + r\,K'(\tau)^{(1+\alpha)} + (-1+\alpha)(-2+\gamma)K(\tau)^\alpha K'(\tau) + \\ &\quad + K(\tau)^{2\alpha}(-r\,K'(\tau) + (-1+\gamma)K''(\tau)) \\ K(0) &= k0 \\ K(T_{\max}) &= kT_{\max} \end{aligned} \qquad \text{<4.1>}$$

4.2. The Ramsey model (modeled with Lagrange function with constraint)

To model the standard Ramsey model, you can also use the Lagrange function with constraint and proceed as follows:

As before the standard Ramsey model consists of 1 agent (household) that attempts to maximize the intertemporal utility of consumption C over the period from $t_0 = 0$ to $T_{\max}$. The utility function U^A is

$U^A(C(t)) = C(t)^\gamma \qquad 0 \le \gamma \le 1$

The constraint Z is given by

$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$

The intertemporal utility function of the household $U^{A\,\mathrm{int}}$ is given by

$$U^{A\,\mathrm{int}(0,T_{\max})}(C) = \int_0^{T_{\max}} e^{-r\,\tau}\, U^A(C(\tau))d\tau = \int_0^{T_{\max}} e^{-r\,\tau}\, C(\tau)^\gamma d\tau$$

Instead of using the constraint, we use the Lagrange function with constraint. This results in the variation problem

$$\int_0^{T_{\max}} \left(e^{-r\tau}C(\tau)^\gamma + \lambda(\tau)(K(\tau)^{(1-\alpha)} - K'(\tau) - C(\tau)) \right) d\tau \to \max$$

$with\ K(0) = k0 \qquad K(T_{\max}) = kT_{\max}$

This results in the Euler equation with initial and final values:

$$\text{EulerEquations}\left[\left(e^{-r\tau}C(\tau)^\gamma + \lambda(\tau)(K(\tau)^{(1-\alpha)} - K'(\tau) - C(\tau)) \right), \{C(\tau), K(\tau), \lambda(\tau)\}, \tau \right]$$

$K(0) = k0$

$K(T_{\max}) = kT_{\max}$

which result in the following differential equation system

$e^{-r\,\tau}\gamma C(\tau)^{(-1+\gamma)} - \lambda(\tau) = 0$

$-(-1+\alpha)K(\tau)^{-\alpha}\lambda(\tau) + \lambda'(\tau) = 0$

$-C(\tau) + K(\tau)^{(1-\alpha)} - K'(\tau) = 0$

$K(0) = K0$

$K(T_{\max}) = KT_{\max}$

The differentiation of the first and third equations and the addition of these equations to the equation system results in the differential equation system

(1) $\quad e^{-r\tau}\gamma C(\tau)^{(-1+\gamma)} - \lambda(\tau) = 0$

(2) $\quad -e^{-r\tau}rC(\tau)^{(-1+\gamma)} + e^{-r\tau}(-1+\gamma)\gamma C(\tau)^{(-2+\gamma)}C'(\tau) - \lambda'(\tau) = 0$

(3) $\quad -(-1+\alpha)K(\tau)^{-\alpha}\lambda(\tau) + \lambda'(\tau) = 0$

(4) $\quad -C(\tau) + K(\tau)^{(1-\alpha)} - K'(\tau) = 0 \qquad$ <4.2>

(5) $\quad -C'(\tau) + (1-\alpha)K(\tau)^{-\alpha}K'(\tau) - K''(\tau) = 0$

(6) $\quad K(0) = K0$

(7) $\quad K(T_{\max}) = KT_{\max}$

The solution of this (complicated) differential-algebraic equation is much more complicated than the solution of the equation system <4.1> in chapter 4.1.

To show that both systems of equations are equivalent, the following steps are taken:

Calculate $\lambda'(\tau)$ from (3) and leave out (3)

Insert $\lambda'(\tau)$ into (2)
Calculate $\lambda(\tau)$ from (1) and leave out (1)
Insert $\lambda(\tau)$ into (2)
Calculate $C(\tau)$ from (4) and leave out (4)
Insert $C(\tau)$ into (2)
Calculate $C'(\tau)$ from (5) and leave out (5)
Insert $C'(\tau)$ into (2)
Simplify under the condition $r > 0 \; and \; K(\tau) - K(\tau)^{\alpha} K'(\tau) \neq 0$

This again results in <4.1>

$$0 = (-1+\alpha)K(\tau) + r\,K'(\tau)^{(1+\alpha)} + (-1+\alpha)(-2+\gamma)K(\tau)^{\alpha} K'(\tau) +$$
$$+K(\tau)^{2\alpha}(-r\,K'(\tau) + (-1+\gamma)K''(\tau))$$
$$K(0) = k0$$
$$K(T_{\max}) = kT_{\max}$$

The conclusion from all this: It is much more convenient to use the constraint to calculate $C(\tau)$ and to eliminate $\lambda(\tau)$ and to use the Lagrange function without constraint than the Lagrange function with constraint.

4.3. The GCD Ramsey model (with non-intertemporal utility function)

The utility function U^A can be used not only to construct the (intertemporal) standard Ramsey model (see Chapter 4.1), but also to construct a (non-intertemporal) standard GCD model, which we call the GCD Ramsey model. This makes it possible to show the different dynamic behaviour of these two models (see chapter 4.5)
The utility function U^A and the constraints are the same as in the standard Ramsey model

$$U^A(C(t)) = C(t)^{\gamma} \qquad\qquad 0 \leq \gamma \leq 1$$
$$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$

So we have 2 variables $C(t), K(t)$ and 1 constraint. The corresponding differential-algebraic GCD equation system consists of 2 differential equations (the behavioural equations for the 2 variables) and 1 algebraic equation (the constraint). For the sake of simplicity, we set all power factors to 1. The ex-ante behavioural equations describe that the household tries to change the consumption $C(t)$ along the partial derivation of $U^A(C(t).K(t))$ with respect to $C(t)$ and tries to change the capital $K(t)$ along the partial derivation of $U^A(C(t).K(t))$ with respect to $K(t)$. The ex-post equation for the behavioural equation for $C(t)$ is obtained by adding the constraint forces given by $\lambda(t)$ multiplied by the partial derivative from Z with respect to $C(t)$. In the same way, the ex-post behavioural equation is obtained for $K(t)$ by adding the constraint force given by $\lambda(t)$ multiplied by the partial derivative from Z with respect to $K(t)$. Together with the constraint, the differential algebraic GCD equation system to be solved is obtained:

$$C'(t) = \frac{\partial U^A(C(t),K(t))}{\partial C(t)} + \lambda(t)\frac{\partial Z(C(t),K(t),K'(t))}{\partial C(t)} = \gamma C(t)^{(\gamma-1)} - \lambda(t)$$

$$K'(t) = \frac{\partial U^A(C(t),K(t))}{\partial K(t)} + \lambda(t)\frac{\partial Z(C(t),K(t),K'(t))}{\partial K(t)} = 0 - \lambda(t)$$

$$0 = Z(C(t),K(t),K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t)$$

This results in

$$C'(t) = \gamma C(t)^{(\gamma-1)} - \lambda(t)$$

$$K'(t) = -\lambda(t)$$

$$0 = K(t)^{(1-\alpha)} - K'(t) - C(t)$$

4.4. The IGCD Ramsey model (with intertemporal utility function)

The utility function U^A, the intertemporal utility function $U^{A\,\mathrm{int}}$ and the constraints are the same as in the standard Ramsey model

$$U^A(C(t)) = C(t)^\gamma \qquad 0 \le \gamma \le 1$$

$$Z(C(t),K(t),K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$

$$U^{A\,\mathrm{int}(0,T_{\max})}(C) = \int_0^{T_{\max}} e^{-r\,\tau}\, U^A(C(t))d\tau = \int_0^{T_{\max}} e^{-r\,\tau}\, C(t)^\gamma\, d\tau$$

We calculate $C(\tau)$ from the constraint and insert it into U^A. This results in the intertemporal utility function $U^{A\,\mathrm{int}(0,T_{\max})}$

$$U^{A\,\mathrm{int}(0,T_{\max})}(K) = \int_0^{T_{\max}} e^{-r\,\tau}(K(t))^{(1-\alpha)} - K'(t))^\gamma\, d\tau \rightarrow \max$$

$$\textit{with}\ K(0) = k0 \qquad K(T_{\max}) = kT_{\max}$$

Since we only have 1 agent A and 1 variable $x_1 = K$, <3.3> reduces to

(1) $K'(t) = \left. \dfrac{d\,K^{A\,\mathrm{int}\,(t,T_{max})}(\tau)}{d\tau} \right|_{\tau=t}$

(2) $k(0) = k0$

(3) $\mathrm{EulerEquations}[e^{-r\tau}\left(K^{A\,\mathrm{int}\,(t,T_{max1})}(t+\tau)^{(1-\alpha)} - K^{A\,\mathrm{int}\,(t,T_{max})}\,'(t+\tau)\right)^{\gamma}, \left\{K^{A\,\mathrm{int}\,(t,T_{max1})}(\tau)\right\}, \tau]$

with initial and final value

$K^{A\,\mathrm{int}\,(t,T_{max})}(t) = K(t)$

$K^{A\,\mathrm{int}\,(t,T_{max})}(T_{max}) = KT_{max}$

From the uniqueness theorem for differential equations, it follows that the **standard Ramsey model and the IGCD-Ramsey model have the same solutions** if they have the same initial values:

- Designate $K_R(t) = K_{R,(0,T_{\max})}(t)$ the solution of the classic Ramsey model and $K_G(t)$ the solution of the IGCD-Ramsey model with the initial condition $K_R(0) = K_G(0) = K0$ and the final condition $K_R(T_{\max}) = K_G(T_{\max}) = KT_{\max}$
- Let $t_a \in [0, T_{\max}]$ and designate $K_{R,(t_a,T_{\max})}(t)$ the solution of the classical Ramsey model with the initial condition $K_{R,(t_a,T_{\max})}(t_a) = K_R(t_a)$ and the final condition $K_{R,(t_a,T_{\max})}(T_{\max}) = KT_{\max}$
- then the following applies:

a) $K_R(t_a+\tau) = K_{R,(t_a,T_{\max})}(t_a+\tau)$ for all $\tau \in [0, T_{\max} - t_a]$

Because a variational problem for a part of the whole interval gives the same solution as the variational problem for the whole interval, if the initial and final values correspond to the solution values of the variational problem for the whole interval.

b) $K_R{}'(t_a) = K_{R,(t_a,T_{\max})}{}'(t_a) =$ *because of a*)

$= K_{G,(t_a,T_{\max})}{}'(t_a) =$ *because* $K_{R,(t_a,T_{\max})} = K_{G,(t_a,T_{\max})}$ *because of* (3)

$= K_G{}'(t_a)$ *because of* (1)

c) $K_R{}'(t) = K_G{}'(t)$ *because of* (*b*) *and because* t_a *can be chosen arbitrary*

d) $K_R(0) = K_G(0)$ *because of preconditions*

e) $K_R = K_G$ *because of c*), *d*) *and the uniqueness theorem of diiferential equations*

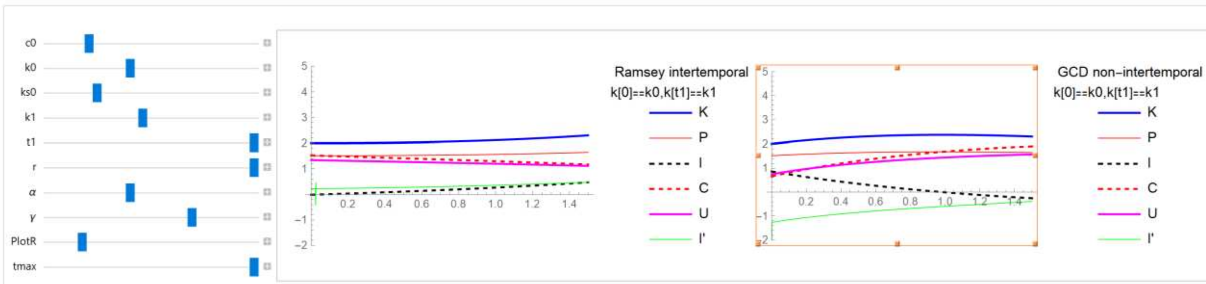
Of course, this only applies to this special case, where there is only 1 agent. Of course, this does not apply if there are several agents.

4.5. Numerical calculations and comparison of Ramsey model and GCD Ramsey model

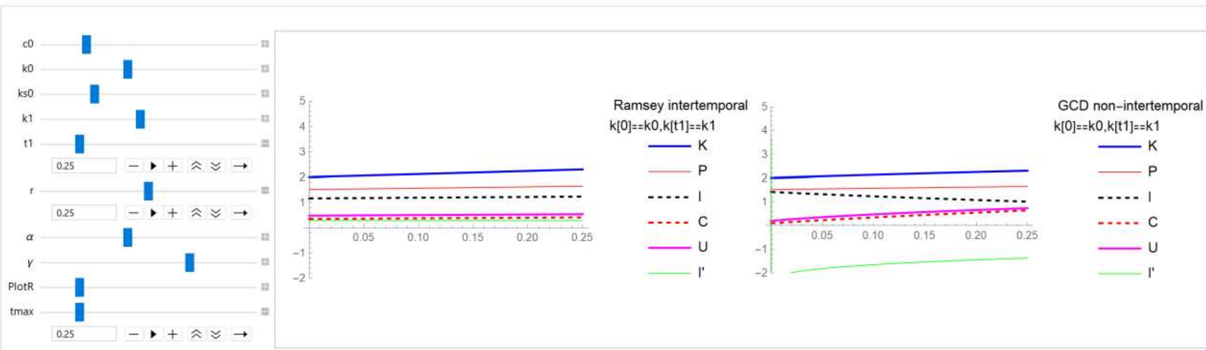
The Ramsey model is equivalent to a IGCD Ramsey model, because there is only 1 agent involved (see chapter 4.4). We therefore only compare the Ramsey model with the GCD Ramsey model.

https://www.dropbox.com/s/075cucs48ginqit/Vergleich%20Ramsey%20klassisch%2C%20GCD%20klassisch%2C%20GCD%20intertemp%20Version%2010.nb?dl=0

Results of the calculations for "large" discount rate $r = 0.5$ and $T_{max} = t_1 = 1.5$ you can find in the next graphic. It shows the difference in the dynamics of the standard (intertemporal) Ramsey model and the non-intertemporal GCD-Ramsey model.



A calculation for the "small" discount rate $r = 0.25$ and small $T_{max} = t_1 = 0.25$ gives the expected result shown in 3.4.2.: If discount rate and optimisation interval are small, the two models (standard (intertemporal) Ramsey and non-intertemporal GCD-Ramsey) are similar.



5. Comparison of IGCD models with DSGE models

DSGE models require utility functions that can be aggregated to a master utility function. In the case of GCD and IGCD models, the utility functions do not need to be able to be aggregated.

Due to the use of a master utility function, DSGE models consist of only 1 variational problem and the economic system is in principle controlled by only 1 agent. The simplest ancestor to modern DSGE models is the Ramsey model. In chapter 3.4 we showed that the classic Ramsey model is equivalent to the corresponding IGCD Ramsey model. As a main result, we therefore propose (that it should be possible to show) that (non-stochastic, expectation-free) DSGE models are in principle equivalent to (non-stochastic, expectation-free) IGCD models with only 1 agent.

But although this is not currently done, GCD and IGCD models can in principle be extended by stochasticity and expectations in the same way as DSGE models. Thus, DSGE models should in principle be equivalent to IGCD models with only 1 agent.

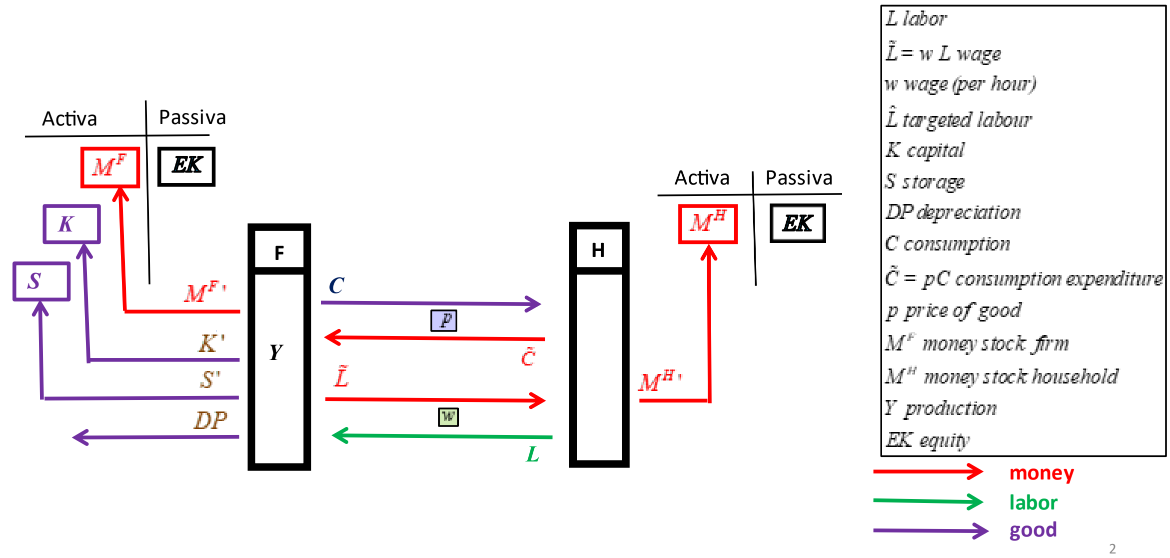
DSGE models are essentially equilibrium models. The dynamics in DSGE models arise from the maximisation of an intertemporal master utility function leading to the Euler-Lagrange equations. The dynamics after a shock is caused by the swing back to the equilibrium state. However, non-intertemporal GCD and (intertemporal) IGCD models are "true" dynamic models that can be formulated independently of any equilibrium states. Both can also be used to model economic shocks (Glötzl 2022a).

In summarising, IGCD models can therefore be seen as a generalisation or alternative to DSGE models.

6. Model *A1* , (1 household, 1 firm, 1 good, without interest)

6.1. Overview of the setup

Model A1: 1 household, 1 firm, 1 good



Model A1: basic equations

algebraically defined variables

$$Y(L,K) = \beta L^{\alpha} K^{1-\alpha} \qquad \text{"production function"}$$

$$DP(K) = \widehat{dp}\, K \qquad \text{"depreciation"}$$

utility functions

$$U^H(C,L,MH) = C^{\gamma} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 \qquad \text{"utility function household"}$$

$$U^F(Y,L,S) = pY - wL - (\hat{S} - S)^2 \qquad \text{"utility function firm"}$$

constraints

$$Z^H = 0 = wL - pC - M^{H}{}' \qquad \text{for money of household } H$$

$$Z^F = 0 = pC - wL - M^{F}{}' \qquad \text{for money of firm } F$$

$$Z_1 = 0 = Y(L,K) - C - K' - S' - DP \qquad \text{for good 1 of firm } F$$

3

With the aid of the GCDconfigurator programme, the differential-algebraic equation system of the A1 model is calculated as follows:

Model A1 : diff.-alg. equation system

$$uF[t] == -(\text{sdach} - s[t])^2 - l[t] \times w[t] + p[t] \times y[t]$$
$$uH[t] == cH[t]^{\gamma} - (\text{ldach} - l[t])^2 - (\text{mHdach} - mH[t])^2$$

$$dp[t] == \text{dpdach}\, k[t]$$
$$inv[t] == k'[t]$$
$$y[t] == \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}$$

$$cH'[t] == \gamma\, \mu HcH\, cH[t]^{-1+\gamma} + p[t]\, \lambda_1[t] - p[t]\, \lambda_2[t] - \lambda_3[t]$$
$$k'[t] == (1-\alpha)\, \beta\, \mu Fk\, k[t]^{-\alpha}\, l[t]^{\alpha}\, p[t] - \lambda_3[t]$$
$$l'[t] == 2\, \mu Hl\, (\text{ldach} - l[t]) + \mu Fl\, \left(\alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, p[t] - w[t]\right) - w[t]\, \lambda_1[t] + w[t]\, \lambda_2[t] + \alpha\, \beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, \lambda_3[t]$$
$$mF'[t] == -\lambda_1[t]$$
$$mH'[t] == 2\, \mu HmH\, (\text{mHdach} - mH[t]) - \lambda_2[t]$$
$$p'[t] == \beta\, \mu Fp\, k[t]^{1-\alpha}\, l[t]^{\alpha} + cH[t]\, \lambda_1[t] - cH[t]\, \lambda_2[t]$$
$$s'[t] == 2\, \mu Fs\, (\text{sdach} - s[t]) - \lambda_3[t]$$
$$w'[t] == -\mu Fw\, l[t] - l[t]\, \lambda_1[t] + l[t]\, \lambda_2[t]$$

$$0 == cH[t] \times p[t] - l[t] \times w[t] - mF'[t]$$
$$0 == -cH[t] \times p[t] + l[t] \times w[t] - mH'[t]$$
$$0 == -cH[t] - \text{dpdach}\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - k'[t] - s'[t]$$

$$cH[0] == k0^{1-\alpha}\, l0^{\alpha}\, \beta$$
$$k[0] == k0$$
$$l[0] == l0$$
$$mF[0] == mF0$$
$$mH[0] == mH0$$
$$p[0] == p0$$
$$s[0] == s0$$
$$w[0] == w0$$

6.2. Description of the A1 model in detail

The one good serves as both a consumption good and an investment good. We assume that vertical constraint forces occur.

Since the target is first to show the principle, we choose the production function and the utility functions as simple as possible.

We choose a simple Cobb-Douglas production function as the production function, and the goods excreted per year (depreciation) are proportional to the capital stock. This results in the 2 necessary algebraically defined variables. They are necessary because they occur in the utility functions or constraints.

$$Y(L,K) = \beta L^{\alpha} K^{1-\alpha} \qquad \beta > 0,\ 0 < \alpha < 1$$
$$DP(K) = dp\,K \qquad 0 \le dp \le 1 \qquad \text{<7.1>}$$

In addition, one can be interested, for example, in net investment, for which one defines as a further algebraically defined variable

$$inv(K) = K' \qquad \text{<7.2>}$$

Households want to consume with decreasing marginal utility. Consumption of consumer goods C leads to a utility for households in the amount of C^{γ} with $0 < \gamma < 1$. They strive for a desired working time $\hat{L}$. Deviations from the desired working time $\hat{L}$ lead to a reduction of utility by $(L - \hat{L})^2$. In addition, households aim to keep cash in the amount of $\hat{M}^H$. Deviations from the desired cash position $\hat{M}^H$ lead to a reduction in utility by $(\hat{M}^H - M^H)^2$. This leads to the **utility function for the household**

$$U^H = C^{\gamma} - (\hat{L} - L)^2 - (\hat{M}^H - M^H)^2 \qquad 0 < \gamma < 1 \qquad \text{<7.3>}$$

For the company, in the simplest case, the utility initially consists of the goods produced, which are valued at the selling price, i.e. pY. The produced goods are used for:

C Sales = Consumption
S' change in inventory
K' changes in productive capital stock

In principle, it would be possible to weight the utility of these uses differently. For the sake of simplicity, we will refrain from doing so. Therefore, this utility is reduced by the cost of labor and the cost of storage, which we evaluate through the deviations from the planned inventory. For simplicity, we assume that holding money in cash has no influence on the utility. This leads to the **utility function for the firm**

$$U^F = pY(L,K) - wL - (\hat{S} - S)^2 = p\beta L^{\alpha} K^{1-\alpha} - wL - (\hat{S} - S)^2 \qquad \text{<7.4>}$$

From the model graph, it can be seen that the following **constraints** must be satisfied:

$$Z_1 = 0 = wL - pC - M^{H\,\prime} \qquad \textit{for money of household } H$$
$$Z_2 = 0 = pC - wL - M^{F\,\prime} \qquad \textit{for money of firm } F \qquad \text{<7.5>}$$
$$Z_3 = 0 = Y(L,K) - C - K' - S' \qquad \textit{for good } 1 \textit{ of firm } F$$

According to the methodology of GCD models, the interest or desire of households to change consumption is the greater the more the utility changes when consumption changes, i.e., the interest is proportional to $\frac{\partial U^H}{\partial C}$. However, the interest in changing consumption does not in itself lead to an actual change in consumption, because the household must also have the power or opportunity to actually implement its desire to change consumption. For example, a household cannot or can only partially enforce its additional consumption wish, e.g., to go to the cinema or on holiday, because it is in quarantine or the borders are closed. This restriction of the possibility to enforce his or her consumption change wishes is described by a (possibly time-dependent) "power factor" μ_C^H. Analogously, the firm could have an interest $\frac{\partial U^F}{\partial C}$ and power μ_C^F to influence consumption. In the specific case $\frac{\partial U^F}{\partial C} = 0$. This results in the following behavioural equation for the **ex-ante planned change in consumption**

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \mu_C^F \frac{\partial U^F}{\partial C} = \mu_C^H \gamma C^{\gamma-1}$$

<7.6>

The same considerations apply to labour L as to consumption. Even the household's wish to increase or reduce working time does not in itself lead to an actual change in working time, because the household must also have the power or possibility to actually implement its wish to change. For example, a household might not be able to enforce its wish to increase working time, or only partially, because it is on short-time working or unemployed, or it might not be able to enforce its wish to reduce working time because it is contractually obliged to work overtime. This restriction of the possibility to enforce his wishes for a change in working time is also described by a (possibly time-dependent) power factor, which we denote with μ_L^H. The same applies to the firm's ability to influence working time.

Therefore, the behavioural equation for the **ex-ante planned change in working time** is as follows

$$L' = \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} = 2\mu_L^H (\hat{L} - L) + \mu_L^F (p\beta\alpha L^{\alpha-1} K^{1-\alpha} - w)$$

The ex-ante behavioural equations for the other variables result analogously.

However, the plans of the 2 agents household and firm to change consumption C, labour L and the other variables cannot be enforced independently of each other, because the constraints

$$Z_1 = 0 = wL - pC - M^{H}{}' \qquad \textit{für Geld von Haushalt H}$$

$$Z_2 = 0 = pC - wL - M^{F}{}' \qquad \textit{für Geld von Firma F} \qquad \text{<7.7>}$$

$$Z_3 = 0 = Y(L,K) - C - K' - S' - DP \qquad \textit{für Gut 1 von Firma F}$$

lead to constraint forces, which we assume are vertical constraint forces. The constraint force for the change in consumption therefore results in

$$\lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = -\lambda_1 p + \lambda_2 p - \lambda_3$$

The behavioural equation for the actual **ex-post change in consumption** is therefore

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3 \qquad \text{<7.8>}$$

Analogously, the actual **ex-post change in labour** is as follows

$$L' = \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} =$$

$$= 2\mu_L^H (\hat{L} - L) + \mu_L^F (p\beta\alpha L^{\alpha-1} K^{1-\alpha} - w) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta L^{\alpha-1} K^{1-\alpha}$$

This also applies analogously to the company's investments. In the case of the company, too, the actual implementation of ex-ante planned investment increases can be prevented by real restrictions, e.g. by interruptions in supply chains. In the same way, a desired reduction in investment may not be possible to the desired extent because the project is a large-scale project of many years' duration. These restrictions can in turn be described by a (possibly time-dependent) power factor μ_K^B. This results in the following behavioural equation for the actual ex-post change in capital

$$K' = \mu_K^F \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} = \mu_K^F p\beta(1-\alpha) L^{\alpha} K^{-\alpha} - \lambda_3 \qquad \text{<7.9>}$$

Note that we have to use $\frac{\partial Z_3}{\partial K'}$ instead of $\frac{\partial Z_3}{\partial K}$ because the constraint forces are always derived from the highest time derivative of the variables (see Remark in chapter 2 and Flannery (2011).

The equations of behaviour for M^H, M^F, S, p, w are derived analogously. In sum, this results in the model equations *differentiell behavioural equations*

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \mu_C^F \frac{\partial U^F}{\partial C} + \lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} =$$
$$= \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3$$

$$L' = \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} =$$
$$= \mu_L^H (\hat{L} - L) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta L^{\alpha-1} K^{1-\alpha}$$

$$K' = \mu_K^H \frac{\partial U^H}{\partial K} + \mu_K^F \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} =$$
$$= \mu_K^F p\beta(1-\alpha) L^\alpha K^{-\alpha} - \lambda_3$$

$$M^{H\,\prime} = \mu_{M^H}^H \frac{\partial U^H}{\partial M^H} + \mu_{M^H}^F \frac{\partial U^F}{\partial M^H} + \lambda_1 \frac{\partial Z_1}{\partial M^{H\,\prime}} + \lambda_2 \frac{\partial Z_2}{\partial M^H} + \lambda_3 \frac{\partial Z_3}{\partial M^H} =$$
$$= 2\mu_{M^H}^H \left(\hat{M}^H - M^H\right) - \lambda^H$$

$$M^{F\,\prime} = \mu_{M^F}^H \frac{\partial U^H}{\partial M^F} + \mu_{M^F}^F \frac{\partial U^F}{\partial M^F} + \lambda^H \frac{\partial Z^H}{\partial M^{F\,\prime}} + \lambda^B \frac{\partial Z^B}{\partial M^F} + \lambda_1 \frac{\partial Z_1}{\partial M^F} =$$
$$= -\lambda_2$$

$$S' = \mu_S^H \frac{\partial U^H}{\partial S} + \mu_S^F \frac{\partial U^F}{\partial S} + \lambda_1 \frac{\partial Z_1}{\partial S} + \lambda_2 \frac{\partial Z_2}{\partial S} + \lambda_3 \frac{\partial Z_3}{\partial S'} =$$
$$= \mu_S^F \left(\hat{S} - S\right) - \lambda_3$$

$$p' = \mu_p^H \frac{\partial U^H}{\partial p} + \mu_p^F \frac{\partial U^F}{\partial p} + \lambda_1 \frac{\partial Z_1}{\partial p} + \lambda_2 \frac{\partial Z_2}{\partial p} + \lambda_3 \frac{\partial Z_3}{\partial p} =$$
$$= \mu_p^F \beta K^{1-\alpha} L^\alpha - \lambda_1 c + \lambda_2 c$$

$$w' = \mu_w^H \frac{\partial U^H}{\partial w} + \mu_w^F \frac{\partial U^F}{\partial w} + \lambda_1 \frac{\partial Z_1}{\partial w} + \lambda_2 \frac{\partial Z_2}{\partial w} + \lambda_3 \frac{\partial Z_3}{\partial w} =$$
$$= -\mu_w^F L + \lambda_1 L - \lambda_2 L$$

Or written in a clearer way

differentiell behavioural equations

$$C' = \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3$$
$$L' = 2\mu_L^H (\hat{L} - L) + \mu_L^F \left(\alpha\beta K^{1-\alpha} L^{-1+\alpha} p - w\right) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta K^{1-\alpha} L^{-1+\alpha}$$
$$K' = \mu_K^F \beta\left(1-\alpha\right) L^\alpha K^{-\alpha} p - \lambda_3$$
$$M^{H\,\prime} = 2\mu_{M^H}^H \left(\hat{M}^H - M^H\right) - \lambda_1$$
$$M^{F\,\prime} = -\lambda_2$$
$$S' = \mu_S^F \left(\hat{S} - S\right) - \lambda_3$$
$$p' = \mu_p^F \beta K^{1-\alpha} L^\alpha - \lambda_1 c + \lambda_2 c$$
$$w' = -\mu_w^F L + \lambda_1 L - \lambda_2 L$$

7. Model $A1^{int}$: IGCD model corresponding to model A1

We develop the IGCD model from the non-intertemporal GCD model A1. For this purpose, we develop the intertemporal utility functions from the utility functions for the A1 model and specify the system of differential equations for the corresponding IGCD model in accordance with the definition in Chapter 3.2.

7.1. Intertemporal utility functions

The algebraically defined equations, utility functions and the constraints of model A1 are unchanged:

algebraically defined variables

$$Y(L,K) = \beta L^{\alpha} K^{1-\alpha} \qquad \beta > 0,\ 0 < \alpha < 1$$

utility function

$$U^H = C^{\gamma} - (L - \hat{L})^2 - (M^H - \hat{M}^H)^2 \qquad 0 < \gamma < 1$$

$$U^B = pY(L,K) - wL - (S - \hat{S})^2 = p\beta L^{\alpha} K^{1-\alpha} - wL - (S - \hat{S})^2$$

constraints

$$Z^H = 0 = wL - pC - M^{H}{}'$$

$$Z^B = 0 = pC - wL - M^{B}{}'$$

$$Z_1 = 0 = Y(L,K) - C - K' - S' = \beta L^{\alpha} K^{1-\alpha} - C - K' - S'$$

We simplify in the following steps

- Calculate $wL = pC + M^{H}{}'$ from the constraint Z^H and put in U^B and in Z^B
- Calculate $C = \beta L^{\alpha} K^{1-\alpha} - K' - S'$ from the constraint Z_1 and put in U^H and U^B and simplify.

This results in

$$U^H = (\beta L^{\alpha} K^{1-\alpha} - K' - S')^{\gamma} - (L - \hat{L})^2 - (M^H - \hat{M}^H)^2$$

$$U^B = K' + S' + M^{H}{}' - (S - \hat{S})^2$$

$$Z^B = 0 = M^{H}{}' + M^{B}{}'$$

The utility functions depend on the 4 variables

L, K, S, M^H

or their derivatives. The system is completely determined by these variables, because the variable M^B is completely determined by the variable M^H by $M^B{}' = -M^H{}'$ due to the constraint Z^B.
This gives the intertemporal utility functions

$$U^{H\,\mathrm{int}((t,T_{max})} = \int_t^{T_{max}} e^{-r^H(t+\tau)}\left((\beta L^\alpha(t+\tau)K^{1-\alpha}(t+\tau) - K'(t+\tau) - S'(t+\tau))^\gamma - (L(t+\tau)-\hat{L})^2 - (M^H(t+\tau)-\hat{M}^H)^2\right)d\tau$$

$$U^{B\,\mathrm{int}(t,T_{max})} = \int_t^{T_{max}} e^{-r^B(t+\tau)}\left(K'(t+\tau) + S'(t+\tau) + M^H{}'(t+\tau) - (S(t+\tau)-\hat{S})^2\right)d\tau$$

7.2. Intertemporal GCD-equations

The (intertemporal) IGCD equations are obtained according to <3.4>:

for all $i = 1,2,3,4$

denote $x_1 = L$

$x_2 = K$

$x_3 = S$

$x_4 = M^H$

behavioural equations for $x_i(t)$

$$x_i'(t) = \frac{\mu_i^A}{\mu_i^A+\mu_i^B}\left.\frac{d\,x_i^{A\,\mathrm{int}(t,T_{max1})}(t+\tau)}{d\tau}\right|_{\tau=0} + \frac{\mu_i^B}{\mu_i^A+\mu_i^B}\left.\frac{d\,x_i^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0}$$

initial values for $x_i(t)$

$x_i(0) = xi0$

Euler equations for A with intial and end values for $x_i^{A\,\mathrm{int}(t,T_{max})}(\tau)$

$$\mathrm{EulerEquations}[\int_t^{T_{max}} e^{-r^A(t+\tau)}U^H(x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau),\dots,x_4^{A\,\mathrm{int}(t,T_{max})}(t+\tau)),\left\{x_1^{A\,\mathrm{int}(t,T_{max})}(\tau),\dots,x_4^{A\,\mathrm{int}(t,T_{max1})}(\tau)\right\},\tau]$$

$x_i^{A\,\mathrm{int}(t,T_{max1})}(t) = x_i(t) \qquad x_i^{A\,\mathrm{int}(t,T_{max})}(t_1) = xiAT_{max}$

Euler equations for B with intial and end values for $x_i^{B\,\mathrm{int}(t,T_{max})}(\tau)$

$$\mathrm{EulerEquations}[\int_t^{T_{max}} e^{-r^B}(t+\tau)U^B(x_1^{B\,\mathrm{int}(t,T_{max1})}(t+\tau),\dots,x_4^{B\,\mathrm{int}(t,T_{max1})}(t+\tau)),\left\{(x_1^{B\,\mathrm{int}(t,T_{max1})}(\tau),\dots,x_4^{B\,\mathrm{int}(t,T_{max1})}(\tau))\right\},\tau]$$

$x_i^{B\,\mathrm{int}(t,T_{max1})}(t) = x_i(t) \qquad x_i^{B\,\mathrm{int}(t,T_{max1})}(T_{max}) = xiBT_{max}$

7.3. Numerical calculations

The numerical calculations can be performed as shown in chapter 3.3

8. Summary

This paper set out to show how the GCD modelling framework can be extended to also incorporate intertemporal utility functions, since intertemporal utility functions are essential in many types of economic models and at the core of the standard DSGE framework used in macroeconomic modelling.

The paper illustrates that GCD models are a natural extension of GE theory and that, similarly, IGCD models can be seen as a generalisation to the DSGE framework. IGCD models are based on the standard method for modelling dynamics under constraints in physics and constitute a valuable alternative to DSGE models.

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