Comment on "Labor- and Capital-augmenting technical change": Does the stability of balanced growth path depend on the elasticity of factor substitution?

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Comment on "Labor- and Capital-augmenting technical change": Does the stability of balanced growth path depend on the elasticity of factor substitution?

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Abstract: In a classic paper, Acemoglu (2003) developed a growth model where firms can undertake both labor- and capital-augmenting technological improvements. According to that paper the balanced growth path with purely labor-augmenting technical change is the unique asymptotic (noncycling) equilibrium, and is stable only when capital and labor are gross complements, i.e., only when the elasticity of substitution between these two factors is no greater than 1. Otherwise, the model not only has two other asymptotic steady-state paths, but also the balanced growth path is unstable. The current comment points out that Acemoglu's conclusion ignores the crowding effect in innovation sector that he has proposed due to the assumption of perfect mobility of scientists between sectors. By replacing the perfect mobility assumption with a smooth adjustment process, implicitly invoking the presence of some adjustment costs, this comment not only points out that the factors affecting the direction of technological progress include both the demand side of innovations (relative price and relative market size) and the supply side of innovations (relative marginal productivity of innovation), but also proves that regardless of whether the substitution elasticity is greater than 1 or less than 1, the balanced growth path is not only unique, but also at least locally saddle-path stable.

Key Words: elasticity of substitution, crowding effect of innovation, scientist migration function, balanced growth path, direction of technological progress

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Comment on "Labor- and Capital-augmenting technical change": Does the stability of balanced growth path depend on the elasticity of factor substitution?

In a classic paper, Acemoglu (2003) developed a growth model, in which firms can undertake both labor- and capital-augmenting technological improvements. In the long run, the economy resembles the standard neoclassical growth model with purely labor-augmenting technical change. Although the Acemoglu paper does not reveal the reason why only purely labor-augmenting technical change is consistent with steady-state equilibrium, it provides a micro foundation for the basic neoclassical growth model with labor-augmenting technical change. However, the paper holds that only when capital and labor are gross complements, i.e., only when the elasticity of substitution between these two factors is no greater than 1, the balanced growth path with purely labor-augmenting technical change is the unique asymptotic (noncycling) equilibrium, and is stable. Otherwise, the model not only has two other asymptotic steady-state paths, but also the balanced growth path is unstable, and the economy will converge to the non-balanced asymptotic path. Because empirical research (Karabarbounis and Neiman, 2014) shows that the real substitution elasticity may be greater than 1, this conclusion casts doubt on the balanced growth path.

However, the current comment argues that Acemoglu's conclusion ignores the crowding effect in innovation sector that he has proposed due to the assumption of perfect mobility of scientists between sectors. By replacing the perfect mobility assumption with a smooth adjustment process, implicitly invoking the presence of some adjustment costs, this comment proves that regardless of whether the substitution elasticity is greater than 1 or less than 1, the balanced growth path is not only unique, but also at least locally saddle-path stable.

Although this result is contrary to the original conclusion of Acemoglu (2003), its economic intuition is not complicated. If, like other economic activities, scientists' innovation activities are incentivized by market return that makes them always enter sectors with higher relative return, then the distribution of scientists in different innovation sectors will depend on the relative market value of different inventions which determined by the relative price of factors (proposed by Hicks 1932) and the relative market size (proposed by Acemoglu 2002), and relative marginal productivity of scientist which will decrease with the increase of the number of scientists in a sector if there is crowding effect of innovation. Obviously, the market return of scientists is determined by the two factors. Even if the market value of the invention is high, if the marginal output is too small, scientists may still be reluctant to enter this sector. Since the direction of technological progress depends on the distribution of scientists in different sectors, relative prices, relative market size and relative marginal productivity are the key factors affecting the direction of technological progress. Although the relative market size effect induces innovation to concentrate in the sector with larger market when the substitution elasticity is greater than 1 and consequently causes technological progress to deviate from the balanced growth path, the relative price effect and the crowding effect of innovation always lead innovation to converge to the balanced growth path. It is precisely the crowding effect which leads to the comprehensive consequence of the three effects is that, whether the substitution elasticity is greater than 1 or less than 1, the balanced growth path is unique and locally saddle-path stable. It is just because, Acemoglu (2003) ignored the crowding effect of innovation due to the assumption of perfect mobility of scientists between sectors that he was led to arrive at the conclusion that the stability of balanced growth path depends on the factor substitution elasticity.

The rest of the comment is organized as follows: In Section I, we model the migration of scientists between sectors in line with the original ideas of Acemoglu's paper but considering the crowding effect; Section II proves the existence, uniqueness of the balanced growth path of Acemoglu (2003)'s model; Section III shows that the balanced growth path of the model is at least locally saddle point stable; The fourth section concludes.

I. Migration of scientist between sectors

Acemoglu (2003) assumes that firms can undertake both labor- and capital-augmenting technological improvements, whereby invention is the result of scientists' efforts. The relative return of innovation allocates scientists into different innovative sectors, and thus affects the direction of technological progress. The return scientists obtain depends on their marginal productivity and on the market value of each invention. In equilibrium, scientists either receive equal returns at the two innovative sectors, or completely focus on either labor- or capital-augmenting technological improvement.

Acemoglu (2003) assumes the existence of two different sets of intermediate goods, n of which are produced solely with labor, and m are produced using capital only. An increase in n - an expansion in the set of labor-intensive intermediates - corresponds to labor-augmenting technical change, while an increase in m corresponds to capital-augmenting technical change. Changes in n and m are the result of scientific effort.

The economy has an exogenously given total number of scientists S. Let S_l and S_k denote, respectively, the number of scientists working to discover new labor-intensive and capital-intensive intermediates, with the market clearing condition $S_l + S_k = S$. Acemoglu assumes further that innovation has a crowding effect, that is, the more scientists are present in a sector, the lower the average productivity of these scientists. The innovation functions in Acemoglu (2003) are:

$$\frac{\dot{n}}{n} = b_l \phi(S_l) S_l - \delta \text{ and } \frac{\dot{m}}{m} = b_k \phi(S_k) S_k - \delta \tag{1}$$

where b_l, b_k, and δ are strictly positive constants, and $\phi(.)$is a continuously differentiable function reflecting the crowding effect of innovation, therefore, $\phi'(.) < 0$. Acemoglu further assumes that the crowding effect is not internalized by individual R&D firms, therefore, each R&D firm takes the productivity of allocating one more scientist to each of the two sectors, $b_l\phi(S_l)$ or $b_k\phi(S_k)$, as given when deciding which sector to enter.[4] While Acemoglu assumes $\phi(0) < \infty$, we assume $\phi(0) \to \infty$, which is an Inada-like condition to assure that in equilibrium, scientists are present in both sectors, thus excluding scientists from completely focusing on one sector and ensuring the uniqueness of the equilibrium. Equation (1) implies the marginal productivity of scientists in the two innovative sectors are: $\frac{\partial \dot{n}}{\partial S_l} = b_l \phi(S_l) n$ and $\frac{\partial \dot{m}}{\partial S_k} = b_k \phi(S_k) m$.

[4] If the crowding effect were to be internalized by individual R&D firms, the marginal output of scientists of equations (2) would be $b_i\phi'(S_i)S_i + b_i\phi(S_i)$ rather than $b_i\phi(S_i)$, $i = l \text{ or } k$.

Acemoglu assumes that the inventor has a permanent patent for each invention. The flow of profits from the sale of labor- and capital-intensive intermediate goods turn out to be $\pi_l = \frac{1-\beta}{\beta}\frac{wL}{n}$ and $\pi_k = \frac{1-\beta}{\beta}\frac{rK}{m}$, where $0 < \beta < 1$ determines the elasticity of substitution between intermediate goods in the production of final products, $\frac{1}{1-\beta}$.[5] L represents labor, w is the wage of labor, K represents capital and r is the interest rate. Therefore, the market value of a patent is the present value of the respective profit streams V_l and V_k , where $V_f(t) = \int_t^{\infty} exp\left[-\int_t^{s}(r(\omega) + \delta)d\omega\right]\pi_f(v)dv$, $f = l$ or k, $r(t)$ is the interest rate at date t, and δ is the depreciation (obsolescence) rate of existing intermediate inputs.

Suppose that the wage of scientists engaged in the labor-augmenting (capital-augmenting) technological innovation is ω_{sl} (ω_{sk}) and both are given by the market value of the marginal product of scientist, that is, $\omega_{sl} = b_l\phi(S_l)nV_l$ and $\omega_{sk} = b_k\phi(S_k)mV_k$. Acemoglu assumes that scientists are homogeneous, therefore in equilibrium the wage of scientists is the higher of the two, that is, $\omega_s = \max\{b_l\phi(S_l)nV_l, b_k\phi(S_k)mV_k\}$. This shows that there are three possible equilibria in the scientist market: First, scientists are active in both sectors and receive equal wage, that is, $0 < S_l < S, 0 < S_k < S, b_l\phi(S_l)nV_l = b_k\phi(S_k)mV_k$; Second, all scientists focus on the innovation of labor-intensive intermediates and their wage is determined by the value of innovations in the sector of labor-intensive intermediate inputs, that is, $S_l = S$, $S_k = 0$, $\omega_s = b_l\phi(S_l)nV_l > b_k\phi(S_k)mV_k$; Third, all scientists focus on the innovation of capital-intensive intermediate inputs and their wage is given by the value of innovations in that sector, that is, $S_l = 0$, $S_k = S$, $\omega_s = b_k\phi(S_k)mV_k > b_l\phi(S_l)nV_l$.

Acemoglu (2003) does not clearly point out how scientists move between the two innovation sectors when the scientist market is out of equilibrium, a process that would affect the dynamic adjustment of the direction of technological progress. The wage rates of the two sectors are clearly equal along a balanced growth path but Acemoglu assumes scientists instantaneously move between sectors to keep the wage rates of the two sectors always equal. Accordingly, he takes the time derivative on both sides of the equation $b_l\phi(S - S_k)nV_l = b_k\phi(S_k)mV_k$ to obtain the equation of $\frac{\dot{S_k}}{S_k}$ in transitional dynamics. We replace the perfect mobility assumption with a smooth adjustment process, implicitly invoking the presence of some adjustment costs. Moreover, we show that it is the setting of scientists' migration function in the dynamic process that leads to the instability of the balanced growth path of the Acemoglu model.

Specifically, we assume that scientists are incentivized by wage difference to move from one

[5] See equation (21) in Acemoglu (2003).

sector to the other. This becomes the main micro mechanism which changes the direction of technological progress in the transitional dynamics. Therefore, we assume the following scientist's migration function:

$$\frac{\dot{S_k}}{S_k} = G\left[\frac{\omega_{sk}}{\omega_{sl}}\right], \tag{2}$$

There are three possible equilibria which imply $\frac{\dot{S_k}}{S_k} = 0$: (1) when $\frac{\omega_{sk}}{\omega_{sl}} = 1$; (2) $\frac{\omega_{sk}}{\omega_{sl}} < 1$ and $S_k = 0$, $S_l = S$; (3) when $\frac{\omega_{sk}}{\omega_{sl}} > 1$ and $S_k = S, S_l = 0$. We further assume that $G'(.) > 0$, that is, the greater the wage difference, the faster the migration of scientists.

Substituting $\omega_{sl} = b_l \phi(S_l) n V_l$ and $\omega_{sk} = b_k \phi(S_k) m V_k$ into equation (2) we obtain:

$$\frac{\dot{S_k}}{S_k} = G\left[\frac{b_k \phi(S_k) m V_k}{b_l \phi(S - S_k) n V_l}\right], \tag{3}$$

and further substituting for π_l and π_k into equation (3) we yield:

$$\frac{\dot{S_k}}{S_k} = G\left[\frac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \cdot \frac{r}{w} \cdot \frac{K}{L}\right] \tag{4}$$

When $\frac{\dot{S_k}}{S_k} > 0$, more scientists are entering the capital-augmenting sector, $\frac{\dot{m}}{m}$ rises relative to $\frac{\dot{n}}{n}$, and technological progress will be more capital-augmenting. If the opposite holds, it will be more labor- augmenting. Therefore, equation (4) shows that there are three factors affecting the direction of technological progress: *the relative factor price* $\frac{r}{w}$ and *the relative market size* $\frac{K}{L}$, both representing the demand side factors affecting the direction of technological progress as emphasized by Acemoglu (2002, 2003); The third factor is *the relative marginal productivity of scientists* $\frac{b_k \phi(S_k)}{b_l \phi(S - S_k)}$, which is a supply side factor affecting the direction of technological progress. It is the migration equation of scientists proposed in this comment that can clearly reveal the impact of the relative marginal productivity of innovation on the direction of technological progress. Although Acemoglu (2002, 2003) too has pointed this factor out, given the assumption that the wage rates of scientists in different sectors are always equal, its impact is ignored. In addition, as we show below, it is also the key factor causing the economy to converge to a unique balanced growth path.

Next, let $N \equiv n^{\frac{1-\beta}{\beta}}$ and $M \equiv m^{\frac{1-\beta}{\beta}}$, $k \equiv \frac{MK}{NL}$, so that k represents effective units of capital. In addition, specify a CES production function $Y = \left[\gamma (NL)^{(\varepsilon-1)/\varepsilon} + (1-\gamma)(MK)^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}$, $f(k) \equiv \frac{Y}{NL} = \left[\gamma + (1-\gamma)(k)^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}$. Provided r and w are the marginal product of K and L, then $\frac{rK}{wL} = \frac{(1-\gamma)}{\gamma} k^{(\varepsilon-1)/\varepsilon}$. Substituting this into equation (4) we obtain:

$$\frac{\dot{S_k}}{S_k} = G\left[\frac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \frac{(1-\gamma)}{\gamma} k^{(\varepsilon-1)/\varepsilon}\right] \tag{5}$$

Altogether, unlike Acemoglu (2003), this comment uses equation (5) as the migration equation of scientists between sectors to discuss on the steady state and stability of steady state in the model.

II. The existence and uniqueness of the balanced growth path

In order to discuss the existence and stability of the steady-state equilibrium of the model, the dynamic equations of the model must be provided. To complete the setup, Acemoglu (2003) assume that the economy is populated by a representative consumer with the usual constant relative risk aversion (CRRA) preference as $\int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt$, where θ is the elasticity of marginal utility, and ρ is the time discount rate. From these preferences, the familiar consumption Euler equation, $\theta\frac{\dot{C}}{C} = r - \rho$, can be obtained. Using the Euler equation, the innovation equations (1), the migration equation (5), the definitions of N, M and k，and further defining $c \equiv C/K$, the dynamics of the amended Acemoglu model are summarized by:

$$\begin{cases} \dfrac{\dot{c}}{c} = M\left(\dfrac{\beta}{\theta}f'(k) - \dfrac{f(k)}{k}\right) + c - \dfrac{\rho}{\theta} \\ \dfrac{\dot{k}}{k} = \left(M\dfrac{f(k)}{k} - c\right) + \dfrac{1-\beta}{\beta}[b_k\phi(S_k)S_k - b_l\phi(S-S_k)(S-S_k)] \\ \dfrac{\dot{M}}{M} = \dfrac{1-\beta}{\beta}[b_k\phi(S_k)S_k - \delta] \\ \dfrac{\dot{S_k}}{S_k} = G\left[\dfrac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\dfrac{(1-\gamma)}{\gamma}k^{(\varepsilon-1)/\varepsilon}\right] \end{cases} \tag{6}$$

From equations (6) we can then obtain the following Proposition:

Proposition 1: The system described by equations (6) possesses a unique steady-state equilibrium, described by equations (7) as follow:

$$\begin{cases} {S_k}^* = \dfrac{\delta}{b_k\phi({S_k}^*)} \\ {S_l}^* = S - {S_k}^* \\ k^* = \left[\dfrac{b_k\phi({S_k}^*)}{b_l\phi(S-{S_k}^*)}\dfrac{(1-\gamma)}{\gamma}\right]^{\frac{-\varepsilon}{\varepsilon-1}} \\ M^* = \dfrac{\beta\rho - \theta(1-\beta)}{\beta\beta f'(k^*)}[b_k\phi({S_k}^*){S_k}^* - b_l\phi(S-{S_k}^*)(S-{S_k}^*)] \\ c^* = \left[\dfrac{\beta\rho-\theta(1-\beta)}{\beta\beta}\dfrac{f(k^*)}{k^*f'(k^*)} + \dfrac{1-\beta}{\beta}\right][b_k\phi({S_k}^*){S_k}^* - b_l\phi(S-{S_k}^*)(S-{S_k}^*)] \\ \dfrac{\dot{N}}{N} = \dfrac{\dot{K}}{K} = \dfrac{\dot{C}}{C} = \dfrac{\dot{I}}{I} = \dfrac{\dot{Y}}{Y} = \dfrac{1-\beta}{\beta}[b_l\phi(S-{S_k}^*)(S-{S_k}^*) - \delta] \\ \dfrac{\dot{M}}{M} = 0 \end{cases} \tag{7}$$

Proof:

First, we prove existence of a steady-state equilibrium.

Set $\frac{\dot{M}}{M} = \frac{\dot{S_k}}{S_k} = \frac{\dot{c}}{c} = \frac{\dot{k}}{k} = 0$. From equations (6) and using $S = S_l + S_k$, $Y = NLf(k)$, we can obtain equations (7), which shows that there is a steady-state equilibrium of the model. Technological progress in equilibrium is purely labor-augmenting, $\frac{\dot{M}}{M} = 0$. This is the core conclusion of Acemoglu (2003).

Second, we prove that the steady-state equilibrium is unique, regardless of whether the substitution elasticity is greater or less than 1. It has been proved that $S_k{}^* = \frac{\delta}{b_k \phi(S_k{}^*)}$ is a steady-state equilibrium of the model implying $\frac{\dot{M}}{M} = 0$. What we need to prove is that there is no other steady-state equilibrium. This requires to prove, contrary to the argument in Acemoglu (2003), that $S_k = S$ and $S_k = 0$ cannot be equilibria. Further, it needs to be shown that there is no other equilibrium $0 < S_k{}' \neq \frac{\delta}{b_k \phi(S_k{}^*)} < S$. We prove this conclusion by contradiction.

On the one hand, we rule out steady-state equilibria in which scientists focus on one of the sectors, that is, $S_k = S$ or $S_k = 0$ cannot be steady-state equilibria.

Suppose to the contrary that scientists are completely concentrated in one sector, that is, $S_k = S$ or $S_k = 0$, is also a steady-state equilibrium. Then, when $S_k = S$, $\phi(S - S_k) \to \infty$ and $\frac{b_k \phi(S)}{b_l \phi(S - S_k)} \frac{(1-\gamma)}{\gamma} k^{(\varepsilon-1)/\varepsilon} < 1$ and hence $\frac{\dot{S_k}}{S_k} < 0$. Therefore, $S_k = S$ cannot be a steady-state equilibrium. Analogously, when $S_k = 0$, $\phi(0) \to \infty$ and $\frac{b_k \phi(0)}{b_l \phi(S - S_k)} \frac{(1-\gamma)}{\gamma} k^{(\varepsilon-1)/\varepsilon} > 1$ and accordingly $\frac{\dot{S_k}}{S_k} > 0$, therefore $S_k = 0$ cannot be a steady-state equilibrium either.[6]

On the other hand, we now show that there is no another steady-state equilibrium in which $0 < S_k{}' \neq \frac{\delta}{b_k \phi(S_k{}^*)} < S$.

Suppose to the contrary there is another such steady-state equilibrium in which $0 < S_k{}' \neq \frac{\delta}{b_k \phi(S_k{}^*)} < S$. By construction, $\frac{\dot{M}}{M} \neq 0$ and $0 < S_l{}' = S - S_k{}' < S$. Since the number of scientists in both sectors is greater than zero, the wage rates of scientists in the two sectors must be equal in steady state, that is $\frac{b_k \phi(S_k{}')}{b_l \phi(S - S_k{}')} \frac{(1-\gamma)}{\gamma} k'^{\frac{\varepsilon-1}{\varepsilon}} = 1$.Therefore, from the wage equality k' is a finite constant which satisfies $k' = \left[\frac{b_k \phi(S_k{}')}{b_l \phi(S - S_k{}')} \frac{(1-\gamma)}{\gamma}\right]^{\frac{-\varepsilon}{\varepsilon-1}}$. Accordingly, given the definition of k and $Y = NLf(k')$ we can obtain

[6] As mentioned above, Acemoglu (2003) assumes $\phi(0) < \infty$. However, as long as $\phi(0)$ is not "too small", $\frac{b_k \phi(S)}{b_l \phi(0)} \frac{(1-\gamma)}{\gamma} k^{(\varepsilon-1)/\varepsilon} < 1$ and $\frac{b_k \phi(0)}{b_l \phi(S)} \frac{(1-\gamma)}{\gamma} k^{(\varepsilon-1)/\varepsilon} > 1$ will still hold, and the model will still have a unique steady-state equilibrium.

$$\frac{\dot{M}}{M}+\frac{\dot{K}}{K}=\frac{\dot{N}}{N}+\frac{\dot{L}}{L}=\frac{\dot{Y}}{Y} \tag{8}$$

The capital accumulation equation $\dot{K}=I=Y-C$ implies that in steady state

$$\frac{\dot{Y}}{Y}=\frac{\dot{C}}{C}=\frac{\dot{I}}{I}=\frac{\dot{K}}{K} \tag{9}$$

Substituting equation (9) into equation (8), we then obtain $\frac{\dot{M}}{M}=0$, which is consistent only with ${S_k}'={S_k}^*=\frac{\delta}{b_k\phi({S_k}^*)}$.

QED.

This proves that the model has a unique steady-state equilibrium given by equations (7), the capital-augmenting technological progress rate is equal to zero ($\frac{\dot{M}}{M}=0$), and technological progress is purely labor augmenting. The uniqueness of the steady-state equilibrium does not depend on whether $\varepsilon>1$ or $\varepsilon<1$. Therefore, **Proposition 2** of Acemoglu (2003), which argues that with $\varepsilon>1$, there are three asymptotic paths (Aps), does not hold unless $\phi'(.)=0$, that is, the marginal productivity of scientists remains unchanged, which contradicts the crowding effect Acemoglu assumed in his paper. Accordingly, it is the properly specified crowding effect (and the Inada-like condition that we added) that guarantee the equilibrium's uniqueness.

III. Steady-state equilibrium is at least locally saddle-path stable

Acemoglu (2003) argues that when $\varepsilon>1$ there are multiple steady-state equilibria. Moreover, he argues that in this case, the balanced growth path is unstable, and the economy will inevitably converge to the asymptotic equilibrium with ${S_k}^*=S$ or ${S_k}^*=0$. As a result, the balanced growth path of purely labor-augmenting technological progress may be of no practical significance. However, as pointed out in section I, Acemoglu's conclusion is due to the assumption of perfect mobility of scientists between sectors. We prove next that for the model described by equations (6), the balanced growth paths are at least locally saddle-path stable regardless of the value of the elasticity of substitution. Like the original paper of Acemoglu (2003), we also examine the case which $\theta=0$ (risk neutral preferences) first, and then consider a more general situation.

1. Let $\theta=0$. When $\varepsilon<1$, the steady state is locally stable, and when $\varepsilon>1$, it is locally saddle point stable.

When $\theta=0$, the consumption Euler equation degenerates to

$$M\beta f'(k)=\rho \tag{10}$$

From equation (10), we obtain $\mathrm{k}=\mathrm{k(M)}$ with $\frac{\mathrm{dk}}{\mathrm{dM}}>0$, that is, k is an increasing function of M. Using this condition, equations (6) simplify into the following pair of dynamic equations:

$$\begin{cases} \dfrac{\dot{S_k}}{S_k} = G\left[\dfrac{b_k \phi(S_k)}{b_l \phi(S - S_k)} \dfrac{(1-\gamma)}{\gamma} k(M)^{(\varepsilon - 1)/\varepsilon}\right] \\ \dfrac{\dot{M}}{M} = \dfrac{1-\beta}{\beta} b_k \phi(S_k) S_k - \delta \end{cases} \tag{11}$$

Linearizing equations (11) near the equilibrium point yields:

$$\begin{cases} \dfrac{\dot{S_k}}{S_k} = a_{ss}(S_k - {S_k}^*) + a_{sm}(M - M^*) \\ \dfrac{\dot{M}}{M} = a_{ms}(S_k - {S_k}^*) \end{cases} \tag{12}$$

where $a_{ss} \equiv \frac{\partial \frac{\dot{S_k}}{S_k}}{\partial S_k} = G'.\frac{b_k}{b_l}\frac{(1-\gamma)}{\gamma} k(M)^{\frac{\varepsilon-1}{\varepsilon}} \frac{\phi'(S_k)\phi(S-S_k)+\phi(S_k)\phi'(S-S_k)}{[\phi(S-S_k)]^2} < 0$, $a_{sm} \equiv \frac{\partial \frac{\dot{S_k}}{S_k}}{\partial M} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma} k(M)^{\frac{-1}{\varepsilon}} \frac{dk}{dM}$ the sign of which depends on the value of ε, and $a_{ms} \equiv \frac{\partial \frac{\dot{M}}{M}}{\partial S_k} = \frac{1-\beta}{\beta} b_k \phi(S_k) > 0$.[7]

The characteristic equation of the model is:

$$det\begin{vmatrix} a_{ss} - \lambda & a_{sm} \\ a_{ms} & -\lambda \end{vmatrix} = 0 \ , \tag{13}$$

leading to:

$$\lambda^2 - \lambda a_{ss} - a_{sm}a_{ms} = 0 \, , \tag{14}$$

By using the Vieta theorem we obtain:

$$\begin{cases} \lambda_1\lambda_2 = -a_{sm}a_{ms} \\ \lambda_1 + \lambda_2 = a_{ss} < 0 \end{cases} \tag{15}$$

When $\varepsilon < 1$， $a_{sm} < 0$ and $a_{ms} > 0$, so that $-a_{sm}a_{ms} > 0$. Equation (15) shows that equation (14) must have two negative roots, $\lambda_1 < 0$ and $\lambda_2 < 0$. In this case, the steady-state equilibrium of the model is locally stable.

When $\varepsilon > 1$, then $a_{sm} > 0$ and $a_{ms} > 0$ Since $\lambda_1\lambda_2 = -a_{sm}a_{ms} < 0$, there must be a positive root and a negative root, so the steady-state equilibrium of the model is locally saddle point stable.[8] This stands in contrast to **Proposition 6** in Acemoglu (2003), arguing that the steady-state equilibrium of the model is unstable when $\varepsilon > 1$.

2. The stability of steady-state equilibrium in the general case

The dynamic equation (6) of the model is linearly approximated to obtain:

[7] The sign of these coefficients holds in all circumstances, not just in the steady state.
[8] Because the Euler equation disappears, there is nothing to guarantee that the economy is on the saddle path initially. However, this problem disappears when one considers $\theta > 0$ and imposes the transversality conditions.

$$
\begin{cases}
\frac{\dot{c}}{c} = a_{cc}(c - c^*) + a_{ck}(k - k^*) + a_{cm}(M - M^*) \\
\frac{\dot{k}}{k} = a_{kc}(c - c^*) + a_{kk}(k - k^*) + a_{km}(M - M^*) + a_{ks}(S_k - S_k{}^*) \\
\frac{\dot{M}}{M} = a_{ms}(S_k - S_k{}^*) \\
\frac{\dot{S_k}}{S_k} = a_{sk}(k - k^*) + a_{ss}(S_k - S_k{}^*)
\end{cases} \tag{16}
$$

Where $a_{ss} = G'.\frac{b_k}{b_l}\frac{(1-\gamma)}{\gamma}k^{\frac{\varepsilon-1}{\varepsilon}}\frac{\phi'(S_k)\phi(S-S_k)+\phi(S_k)\phi'(S-S_k)}{[\phi(S-S_k)]^2} < 0$, $a_{sm} = a_{sc} = 0$, $a_{sk} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}}$ the sign of which depends on the value of substitution elasticity ε; $a_{ms} = \frac{1-\beta}{\beta}b_k\phi(S_k) > 0$, $a_{mm} = a_{mc} = a_{mk} = 0$; $a_{cs} = 0$, $a_{cm} = \left(\frac{\beta}{\theta}\frac{kf'(k)}{k} - \frac{f(k)}{k}\right)$ the sign of which is unknown, $a_{cc} = 1$, $a_{ck} = M\left(\frac{\beta}{\theta}f''(k) - \frac{kf'(k)-f(k)}{k*k}\right)$ the sign of which is also is unknown; $a_{ks} = \frac{1-\beta}{\beta}\left(b_k\phi(S_k) + b_l\phi(S - S_k)\right) > 0$, $a_{km} = \frac{f(k)}{k} > 0$, $a_{kc} = -1 < 0$, $a_{kk} = M\frac{kf'(k)-f(k)}{k*k} < 0$.

The characteristic equation is as follows:

$$
det\begin{vmatrix}
a_{cc} - \lambda & a_{ck} & a_{cm} & 0 \\
a_{kc} & a_{kk} - \lambda & a_{km} & a_{ks} \\
0 & 0 & -\lambda & a_{ms} \\
0 & a_{sk} & 0 & a_{ss} - \lambda
\end{vmatrix} = 0 \tag{17}
$$

Expanding of the characteristic equation yields:

$$
\begin{aligned}
&\lambda^4 - \lambda^3(a_{ss} + 1 + a_{kk}) + \lambda^2(a_{ss} + a_{ss}a_{kk} + a_{kk} + a_{ck} - a_{sk}a_{ks}) \\
&+ \lambda(-a_{ss}a_{ck} - a_{ss}a_{kk} + a_{sk}a_{ks} - a_{sk}a_{ms}a_{km}) + a_{sk}a_{ms}(a_{km} + a_{cm}) \\
&= 0
\end{aligned} \tag{18}
$$

From the Vieta theorem:

$$
\lambda_1\lambda_2\lambda_3\lambda_4 = a_{sk}a_{ms}(a_{km} + a_{cm}) \tag{19}
$$

When $\varepsilon > 1$, Proposition 5 of Acemoglu (2003) says that the equilibrium growth path is unstable, and the economy will converge to one of two unbalanced asymptotic equilibria. However, here we prove that the steady-state equilibrium is a locally saddle-path stable equilibrium, regardless of the elasticity of substitution's value relative to 1.

When $\varepsilon > 1$, $a_{sk} = \frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}} > 0$, $a_{ms} > 0$, $(a_{km} + a_{cm}) = \frac{\beta}{\theta}\frac{kf'(k)}{k} > 0$, therefore $\lambda_1\lambda_2\lambda_3\lambda_4 > 0$. Equation (19) shows that the characteristic equation must have 4 positive roots, or 4 negative roots, or two positive and two negative roots. If there are 4 positive roots, the steady state is unstable. If there are 4 negative roots, then the steady state is locally stable, if there are two positive and two negative roots, then the steady state is locally saddle-stable. Therefore, as long as we can rule out the case of four positive roots, the equilibrium growth path is at least saddle-path stable.

We prove by contradiction that not all four roots can be positive. That is, provided $\varepsilon > 1$, claiming that equation (18) has four positive roots results in a contradiction.

Use the Vieta theorem to obtain：

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{ss} + 1 + a_{kk} \tag{20}$$

If equation (18) has 4 positive roots, then from equation (20) we can obtain $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = (a_{ss} + 1 + a_{kk}) > 0$, implying that $1 + a_{kk} > -a_{ss} > 0$. From the Vieta theorem also the following equation holds:

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = a_{ss}(1 + a_{kk}) + (a_{kk} + a_{ck}) - a_{sk}a_{ks} \tag{21}$$

Owing to $a_{ss} < 0$，if $1 + a_{kk} > 0$, then $a_{ss}(1 + a_{kk}) < 0$; and $a_{kk} + a_{ck} = M\frac{\beta}{\theta}f''(k) < 0$; $a_{ks} = \frac{1-\beta}{\beta}\left(b_k\phi(S_k) + b_l\phi(S - S_k)\right) > 0$, but when $\varepsilon > 1$, $a_{sk} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}} > 0$. then $-a_{sk}a_{ks} < 0$. Therefore, the RHS of equation (21) is less than 0. However, if all four roots are positive, then the RHS of equation (21) should be greater than zero. Therefore, equation (18) cannot possess four positive roots, and can either have two positive roots and two negative roots, or four negative roots. If there are two positive roots and two negative roots, the steady-state equilibrium is locally saddle-path stable, if there are four negative roots, the steady-state equilibrium is locally stable. In a word, the steady-state equilibrium is at least saddle-path stable.

When $\varepsilon < 1$, $a_{sk} = G'.\frac{\varepsilon-1}{\varepsilon}\frac{b_k\phi(S_k)}{b_l\phi(S-S_k)}\frac{(1-\gamma)}{\gamma}k^{\frac{-1}{\varepsilon}} < 0$, $a_{ms} > 0$ and $(a_{km} + a_{cm}) = \frac{\beta}{\theta}\frac{kf'(k)}{k} > 0$ then $\lambda_1\lambda_2\lambda_3\lambda_4 = a_{sk}a_{ms}(a_{km} + a_{cm}) < 0$, so the equation must have negative roots and the steady-state equilibrium is also saddle point stable. Whether there is just one negative root or there are three negative roots cannot be established.

This proves the most important conclusion of this comment: even if the elasticity of substitution is greater than 1, i.e., $\varepsilon > 1$, the steady-state equilibrium of Acemoglu (2003) is at least saddle-path stable. Therefore, under our specification, **Proposition 5** of Acemoglu (2003) does not hold either.

IV. Conclusion

Acemoglu's classic paper (2003) developed a growth model with an endogenous direction of technological progress, in which firms can undertake both labor- and capital-augmenting technological improvements. In the long run, the economy resembles the standard growth model with purely labor-augmenting technical change. It provides a micro foundation for the purely labor-augmenting technological progress in the neoclassical growth model. However, Acemoglu (2003) argues that the balanced growth path is unstable when the substitution elasticity is greater than 1. In that case the economy will converge to the unbalanced growth path. This makes the notion of a balanced growth path questionable, given the empirical work that shows that the real substitution elasticity may be greater than 1 (e.g., Karabarbounis and Neiman, 2014). However, this comment proves that as long as scientists cannot move instantaneously between sectors with adjustment cost, and there is a crowding effect of scientist in innovation and the marginal productivity of scientist decreases, whether the substitution elasticity is greater than or less than 1, the balanced growth path is not only the unique steady-state equilibrium of the model, but also at least locally saddle-path stable.

This comment also points out that the factors affecting the direction of technological progress are not only coming from the demand side of innovations (relative price and relative market size), but also from the supply side (relative marginal productivity of innovation). It is the latter that makes the balanced growth path of Acemoglu (2003) model not only the unique steady-state equilibrium, but also at least locally saddle-path stable. It is Acemoglu's (2003) assumption that scientists' wages are always equal even in the transitional dynamics that led him to ignore the impact of relative marginal productivity of innovation on the direction of technological progress. As a consequence, Acemoglu came to the conclusion that the model has multiple asymptotic steady states and the balanced growth path is unstable when the factor substitution elasticity is greater than 1.

The relative marginal productivity of innovation not only affects the steady-state equilibrium of Acemoglu's (2003) model, but is also of great significance in understanding the direction of technological progress in reality. It shows that that direction depends not only on the relative market value of inventions, but also on the relative difficulty of inventions (describled by the relative marginal productivity of innovation). In particular, it may be the case that some kind innovations have greater market demand than others, but are relatively more difficult to obtain. As a result, enterprises may not necessarily be willing to invest in such research. This result has important policy implications, indicating that it may not be enough to regulate the direction of technological progress purely by changing relative market incentives. Effort should also be made to change the relative marginal productivity of innovations, e.g. through improved research and development conditions, innovative talent training, forming an environment conducive to innovation, and the like.

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