Capturing Income Distributions and Inequality Indices Using NETs (Negative Extremal Transfers)

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Capturing Income Distributions and Inequality Indices Using NETs (Negative Extremal Transfers)*

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Abstract

We introduce the concept of *negative extremal transfers (NETs)*, which are transfers from the poorest individuals to the richest individuals. This family of transfers alone is rich enough to describe the entire space of income distributions: our first result is that any income distribution can be obtained as an expansion from the uniform distribution by applying a sequence of NETs. In other words, NETs constitute a mathematical basis of the space of income distributions. Our second representation theorem establishes that one can describe any given inequality index based on the weight it attaches to all possible NETs.

These results allow one to observe how much importance a given inequality index attaches to poverty concerns in addition to inequality concerns. Anecdotally, we find that indices used in practice lie in a relatively small region of the index space: our NET representation theorem can serve as a guide to proposing new inequality indices. Practitioners will find this representation result useful to quantify the contribution of a given quantile or subgroup to the population's inequality level as well as to guide policy toward the most effective transfers to lower the inequality measure.

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1 Introduction

The issue of income inequality has been a major topic of economics for over a century, and has taken the political center stage again since the early 2010s.[1] This concern has given rise to a prolific literature on how to measure inequality in a population (e.g., Silber, 1999, Cowell, 2011, and references therein). Typically, measures of inequality take information about the income distribution in the population and return a number—an index—usually normalized between 0 (for full equality) and 1 (when a single individual hoards all of the available income). Naturally, there are many ways to translate an income distribution—an n-dimensional object, where n is the size of the population—into a one-dimensional index. Accordingly, many different indices have been proposed in the literature.

We propose a unifying representation of all inequality indices. The key concept behind our results is that of *negative extremal transfers (NETs)*, which are transfers from the poorest individuals to the richest individuals.[2] This concept alone is rich enough to describe the space of income distributions. Indeed, our first result (Theorem 1) is that any income distribution can be obtained as an expansion from the uniform distribution by applying a sequence of NETs.

For example, consider five individuals with the following income profile:

$$\mathbf{y} = (y_1, y_2, y_3, y_4, y_5) = (2, 3, 5, 5, 10)\,.$$

The average income is $\bar{y} = 5$. Consider the uniform distribution $\mathbf{y}_0 = (\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y}) = (5, 5, 5, 5, 5)$ (Figure 1a). We now construct the unique sequence of NETs from $\mathbf{y}_0$ to $\mathbf{y}$. Starting from $\mathbf{y}_0$, begin by transferring income from the individuals who will end up poorer than the mean income ($y_i < \bar{y}$) to those that will be richer than the mean income ($y_i > \bar{y}$). The "magnitude" of the transfer depends

[1] Income inequality is but one form of inequality in a society. On may instead be concerned with inequality in well-being (Adler, 2019), in essential ressources (Rawls, 1971), or in opportunities (Sen, 1992), among others; we refer the reader to Fleurbaey and Blanchet (2013) for an overview. Throughout this work, we shall talk mainly about income inequality, but our approach applies to other types of inequality, provided they track a transferable one-dimensional amount.

[2] The term is in relation to the foundational Transfer Principle (Dalton, 1920), which considers transfers from a rich*er* individual to a poor*er* individual. We use the word "extremal" to highlight the fact that we consider only transfers between the poor*est* and rich*est* individuals, and the word "negative" as a reminder that the transfer reduces equality (i.e., from poor to rich).

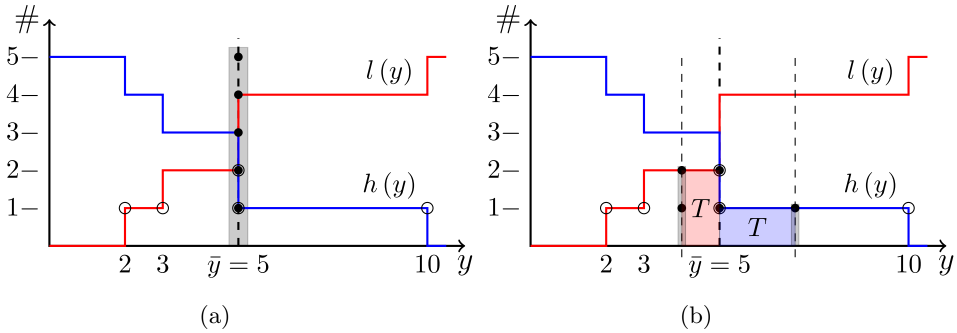


Figure 1: NET expansion sequence from the uniform distribution $\mathbf{y}_0 = (\bar{y}, \bar{y}, \bar{y}, \bar{y}, \bar{y}) = (5, 5, 5, 5, 5)$ to the final distribution $\mathbf{y} = (2, 3, 5, 5, 10)$. The black dots represent the provisional income profile and the white circles represent the final income profile.

on the number of "rich" and "poor" so as to keep total income constant: if there are l poor and h rich, a total transfer of T dollars means that the poor each give T/l to the rich whom, in turn, each receive T/h. In our example, $l = 2$ and $h = 1$ for small values of T, so that each poor person gives $T/2$ to the rich person, who receives T (Figure 1b). Keep increasing T until a first individual meets her final income y_i—so that $y_i = \bar{y} - T/l$ if the first income to be met turns out to be below the mean, or $y_i = \bar{y} + T/h$ if it is above the mean. In our example, $T = 4$ and the first person to reach her final income is Person 2, at which point the (provisional) income profile is $(3, 3, 5, 5, 9)$. This marks the end of the first sequence of NETs. The second sequence of NETs carries on the mean-preserving spread of income by transferring income from the poor who have not yet reached their final income—Person 1 in our example—to the rich who have not yet reached their final income—Person 5— (Figure 1c), and stops whenever the income level of another individual is reached. And so on, until all incomes have been reached, meaning that we reach the final income profile $\mathbf{y}$ (Figure 1d). The procedure reaches the incomes of the richest and poorest individuals simultaneously.

By construction, a sequence of NETs from $\mathbf{y}_0$ exists, and is unique to $\mathbf{y}$. Conversely, it is obvious that any given sequence of NETs from a uniform distribution leads to a unique income distribution. In other words, the set of NETs constitutes a mathematical basis of the space of income distributions.

Our next representation theorem (Theorem 2) builds upon the NET decomposition of income distributions to describe any given inequality index based on how it responds to all possible NETs. Put differently, it is enough to know how an index behaves on NETs to fully describe it. Specifically, any inequality index, ι, can be written in the following form:

$$\iota(\mathbf{y}) = \int_0^1 \varrho_L \alpha\left(z_L, z_H, \varrho_L, \varrho_H, \tilde{\mathbf{y}}\right) dz_L \tag{1}$$

where $\mathbf{y}$ is the income distribution of interest and $\varrho_L \equiv \varrho_L(\mathbf{y}, z_L)$ is the fraction of the population whose (relative) income is no greater than the fraction z_L of mean income $\bar{y}$. The function α is what actually defines the index by giving a weight to the NET where the expansion of $\mathbf{y}$ reaches income $z_L\bar{y}$. The weight attributed can depend on z_L, on $z_H \equiv z_H(\mathbf{y}, z_L)$—representing the (relative) income of the recipients of the NET in question—, and on the fractions of the population whose income is lower or higher than all involved

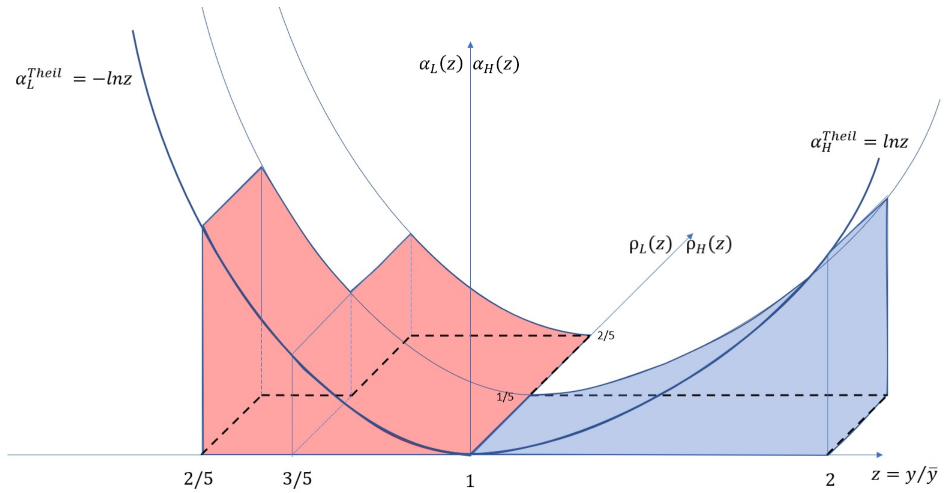


Figure 2: Volume representing inequality in the distribution $\mathbf{y} = (2, 3, 5, 5, 10)$ as measured by the Theil index, for which $\alpha_{Theil} = \ln(z_H/z_L)$. The L (resp. H) subscript relates to the range $z < 0$ (resp. $z > 1$). On those ranges, one can write $\alpha_L^{Theil} = -\ln(z_L)$ and $\alpha_H^{Theil} = \ln(z_H)$, respectively.

in the NET—$\varrho_L(\mathbf{y}, z_L)$ and $\varrho_H \equiv (\mathbf{y}, z_H(\mathbf{y}, z_L))$, respectively. The argument $\tilde{\mathbf{y}} \equiv \tilde{\mathbf{y}}(\mathbf{y}, z_L)$ describes the provisional income profile when the expansion of $\mathbf{y}$ reaches $z_L\bar{y}$.

Graphically, Expression (1) amounts to computing the total volume delimited by the shaded areas of Figure 1d and the height determined by the weight function α. Figure 2 illustrates the Theil index on the profile in example, $\mathbf{y} = (2, 3, 5, 5, 10)$.

Expression (1) allows one to easily compute the weighting functions α of many well-known inequality measures, which take on simple expressions. For example, the variance index uses $\alpha_{var} \equiv 2\bar{y}^2(z_H - z_L)$ and the well-known Gini coefficient uses $\alpha_{Gini} \equiv 2 - (\varrho_H + \varrho_L)$, which is equal to one plus the fraction of people strictly between incomes $z_L\bar{y}$ and $z_H\bar{y}$. We can therefore see clearly that the variance attaches a weight to a NET that is proportional to its spread, $z_H - z_L$, whereas α_{Gini} only indirectly depends on the values of z_L and z_H through the number of agents within the bracket $[z_L\bar{y}, z_H\bar{y}]$. Hence, this representation highlights the fact that the Gini coefficient is insensitive to actual poverty levels. By contrast, the Pietra index, which is the amount of income to be distributed from the rich to the poor to achieve equality, is associated with

$\alpha_{Pietra} \equiv 1$, thus attaching a unit weight to all NETs.

The representation theorems—Theorem 1 and Theorem 2—suggest useful graphical representations to compare inequality indices (Section 3). For a given income distribution, one can visualize precisely where each index allocates more weight—i.e., where α is the largest.

Moreover, additive separability of the α function—which is often the case for well-known indices—allows for finer decomposability, depending on the extent to which α is additively separable. For example, if α is additively separable in z_L and ϱ_L on one side and z_H and ϱ_H on the other—a property we coin "inner-NET additivity"—it becomes a simple matter to decompose the value of the inequality index ι into an inequality at the bottom, ι_L, and an inequality at the top, ι_H so that $\iota = \iota_L + \iota_H$ (Proposition 1). In addition to this rich-poor decomposition, one can use our representation results to compute the contribution to inequality of a given household (Proposition 2). It then becomes straightforward to compute the contribution of any collection of individuals. Of particular interest to practitioners might be the ability to measure the contribution to inequality of any given income quantile (Proposition 4).

Finally, if α is independent of the position in the distribution (i.e., independent of ϱ_L and ϱ_H), the contribution to the inequality measure of a single household is actually independent of the distribution itself (other than through its relative income), Proposition 3. This is not the case for the Gini coefficient, but is the case for many other well-known indices, like the variance, the Pietra index, the Theil index, the mean-log deviation index and, more generally, the entire family of generalized entropy indices.

Section 4 discusses several implications of our representation results for the design of inequality indices and redistributive policies. We introduce the *NET Principle* as a minimal property for an index to be considered a reasonable inequality measure: an index should worsen as a result of a transfer from poorest to richest. Though logically weaker than Dalton's Transfer Principle, which requires the inequality measure to worsen as a result of any transfer from poor to rich, the NET Principle is also more morally intuitive, making it less likely to be rejected empirically. We show that the NET Principle is equivalent to α being strictly positive (Proposition 5) and is thus very simple to check.

Some inequality measures are not solely concerned with inequality per se—in the sense of income spreads—but are also sensitive to whether the inequality occurs at the top or at the bottom of the income distribution. Other measures, like the Gini coefficient, are *symmetric*: they assign the same value to a distribution and to its symmetric with respect to the mean income (when it exists). Proposition 6 establishes that the symmetry of the index is related to a symmetry property of the underlying α function.

Lastly, our NET representation theorem gives clear guidance for what are the most effective (budget-neutral) transfers to lower the inequality index. If the index is such that α increases as NETs involve incomes that are farther and farther away from the mean income, then the most effective income transfer at reducing the index is to transfer income from the richest to the poorest (Proposition 8). Hence, efficiently reducing inequality is very different from the usual progressivity of the income tax for redistribution purposes. Instead, it requires taking money from the richest so as to reduce their income to a common value (effectively an income cap) and transfer the funds to the poorest (effectively creating an income floor). In particular, it should be noted that mid-income households are not involved in this efficient redistribution scheme.

The literature on inequality measures is large and contains many strands, ranging from the axiomatic—where the goal is to identify inequality measures that satisfy desirable (moral) properties—to the statisitical, where computability given limited datasets is paramount. Many works have focused on the decomposition of inequality indices (Shorrocks, 1980, 1982, 1984, 2013; Foster et al. 1984; Chantreuil and Trannoy, 1999), whether it is the contribution of subgroups contributing to inequality or different factor components like gender and ethnicity. While our analysis also focuses on decomposition, it does so in a very different way. We do not decompose indices into the contributions of different factors to inequality.[3] Rather, we specify how inequality indices attach weights to transfers between various points of the distribution. In other words, we decompose the space of distributions into a mathematical basis of infinetisimal transfers (NETs) and represent inequality indices by their description using this new mathematical basis.

The remainder of the paper is organized as follows. Section 2 introduces the decomposition result of any income distribution into a sequence of NETs (Theorem 1) and the representation theorem for inequality indices (Theorem

[3]Although, in principle, our approach could be adapted to handle such decompositions. As we have seen, when the index is inner-NET additive, we can quantify the contribution to inequality of any individual—and, therefore, of any set of individuals grouped according to some characteristic.

2). Section 3 defines several degrees of additive separability of inequality indices (Section 3.1) and details the resulting decomposability of the observed inequality (Section 3.2). Section 4 introduces the NET principle (Section 4.1), a condition on α to determine whether the index is concerned with poverty in addition to only inequality (Section 4.2) and a policy rule to efficiently reduce the inequality index (Section 4.3). Section 5 concludes.

2 A NET-space for representations

2.1 Negative Extremal Transfers (NETs)

Let n be the number of individuals in the population, and let $\mathbf{y} \in \mathbb{R}_+^n$ be the *income profile* or *income distribution* of the population. Denote by $Y = \sum_i y_i$ the total income and by $\bar{y} = Y/n$ the mean income of the population.

For any profile $\mathbf{y} \in \mathbb{R}_+^n$ and any income level $y \in \mathbb{R}_+$, define $L(\mathbf{y}, y) = \{i | y_i \leq y\}$ and $H(\mathbf{y}, y) = \{j | y_j \geq y\}$ the sets of agents who earn no more and no less than y, respectively. Define also the cardinality of these sets: $l(\mathbf{y}, y) = \#L(\mathbf{y}, y)$ and $h(\mathbf{y}, y) = \#H(\mathbf{y}, y)$. It follows that $L(\mathbf{y}, \min_i y_i)$ and $H(\mathbf{y}, \max_i y_i)$ are the set of the poorest and the wealthiest individuals in the population, respectively.

Let $\mathcal{I}(Y, n)$ be the set of distributions of total income Y between n individuals:

$$\mathcal{I}(Y, n) = \left\{ \mathbf{y} \in \mathbb{R}_+^n | \sum_{i=1}^{n} y_i = Y \right\}.$$

To measure income inequality, our main object of study will be inequality indices, broadly defined[4]:

Definition 1. A*n inequality index* (or *index*, for short) is a continuously differentiable function $\iota : \{\mathcal{I}(Y, n) | Y \in \mathbb{R}_+, n \in \mathbb{N}\} \to \mathbb{R}_+$ such that :

- ι is invariant to permutations of the incomes of individuals (anonymity),
- $\iota(\mathbf{y}_0) = 0$ where $\mathbf{y}_0$ is the uniform distribution, $\mathbf{y}_0 = (\bar{y}, ..., \bar{y}) \in \mathcal{I}(Y, n)$.

An important concept throughout will be that of a *negative extremal transfer (NET)*, which is a transfer from some (or all) of the poorest individuals to some (or all) of the richest individuals.

[4]Our definition of what constitutes an inequality index is very broad. It is merely a non-negative symmetric function that takes on its lowest value when all of its coordinates are equal.

Definition 2. ***(Negative Extremal Transfer, NET)*** Let $\mathbf{y}, \mathbf{y}' \in \mathcal{I}(Y, n)$. Distribution $\mathbf{y}'$ is obtained from distribution $\mathbf{y}$ by a *negative extremal transfer (NET)* of size $T > 0$ if, for some of the poorest individuals in $\mathbf{y}$, $i \in L(\mathbf{y}, \min_i y_i) \cap L(\mathbf{y}', \min_i y_i)$,

$$y_i' = y_i - \frac{T}{l(\mathbf{y}, \min_i y_i)},$$

and if

$$y_j' = y_j + \frac{T}{h(\mathbf{y}, \max_i y_i)}$$

for some of the richest individuals in $\mathbf{y}$, $j \in H(\mathbf{y}, \max_i y_i) \cap H(\mathbf{y}', \max_i y_i)$, while all non-extremal agents see their incomes unchanged: $y_k' = y_k$ for all $k \notin L(\mathbf{y}', \min_i y_i) \cup H(\mathbf{y}', \max_i y_i)$.

Given Y and n, any NET is fully described by its starting income relative to the mean income, $z_L \equiv y_L/\bar{y} \leq 1$, its ending relative income, $z_H \equiv y_H/\bar{y} \geq 1$, the fraction of the population who gives up income ($\varrho_L \equiv l/n$), the fraction of the population who receive income from the poor ($\varrho_H \equiv h/n$) and the size of the transfer, T. When considering transfers of infinitesimal size, which we will do throughout the remainder of the paper, a NET is fully described by z_L, z_H, ϱ_L, and ϱ_H. In fact we shall denote by $\mathcal{NET}(Y, n)$ the set of all possible 4-tuples $(z_L, z_H, \varrho_L, \varrho_H)$ that define a NET in an economy where a population of n individuals owns a total income of Y.

2.2 NET-representation of income distributions

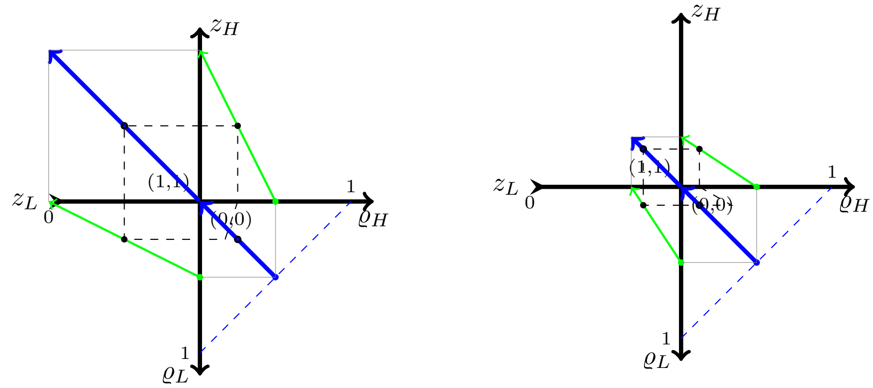
The following theorem establishes that every income distribution, $\mathbf{y}$, is the result of a unique sequence of NETs from the egalitarian distribution.

Theorem 1. *Any $\mathbf{y} \in \mathcal{I}(Y, n)$ can be obtained from $\mathbf{y}_0 = (\bar{y}, ..., \bar{y}) \in \mathcal{I}(Y, n)$ by a sequence of NETs. Conversely, any sequence of NETs from $\mathbf{y}_0 = (\bar{y}, ..., \bar{y}) \in \mathcal{I}(Y, n)$ leads to a unique distribution $\mathbf{y} \in \mathcal{I}(Y, n)$.*

Proof. In Appendix A.1. □

By establishing that any income distribution can be represented by a unique sequence of NETs, Theorem 1 invites a graphical representation of income distributions in the 4-dimensional space of NETs. For instance, the projection of the path onto the (z_L, z_H)-quadrant is a downward-sloping curve ending at $(1, 1)$, the slope of which is equal to $-\varrho_L/\varrho_H$.[5]

For example, uniform distributions yield straight lines with slope equal to -1 in both the (z_L, z_H)-quadrant and the (ϱ_L, ϱ_H)-quadrants, but of different lengths, depending on the spread of the distribution (Figure 3). The Pareto I distribution is a convex curve (Fig 4).



(a) Uniform distribution over $[0, 2\bar{y}]$

(b) Uniform distribution over $[(2/3)\bar{y}, (4/3)\bar{y}]$

Figure 3: NET paths associated with two uniform distributions

2.3 NET-representation of inequality indices

An immediate consequence of Theorem 1 is that the set of potential NETs is a mathematical basis of $\mathcal{I}(Y, n)$. It follows that an inequality index is completely represented by how it behaves on the set of NETs. This is precisely the essence of the next representation theorem.

To state Theorem 2, it will be useful to introduce some notation. Our first main result, Theorem 1, established that any income profile, $\mathbf{y}$, could be obtained from a unique sequence of mean-preserving spreads, coined NETs, from the egalitarian distribution. This sequence of NETs forms a path which, given $\mathbf{y}$, can be parametrized by any one of the four components defining a NET: z_L,

[5]The expression for the slope at any point comes from the money conservation equation in a NET: $-l dy_L = h dy_H$. Dividing troughout by $n\bar{y}$ and rearranging leads to $dz_H/dz_L = -\varrho_L/\varrho_H$.

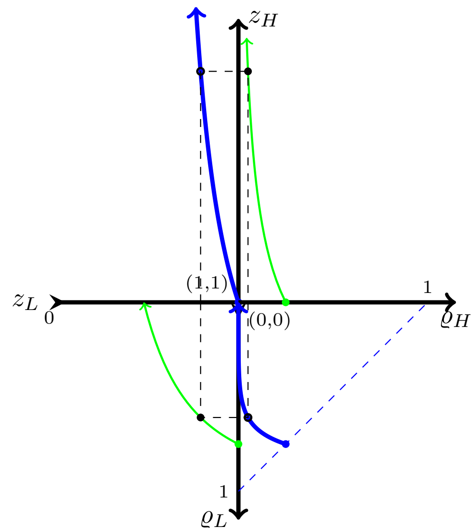


Figure 4: NET path associated with the Pareto I distribution

z_H, ϱ_L, or ϱ_H, the other three being uniquely defined. We shall choose z_L as the parametrizing variable so that, given income profile $\mathbf{y}$ and z_L, we can define the corresponding $z_H(\mathbf{y}, z_L)$, $\varrho_L(\mathbf{y}, z_L)$, and $\varrho_H(\mathbf{y}, z_L)$.[6]

Finally, we denote by $\tilde{\mathbf{y}}(\mathbf{y}, z_L)$ the profile when the path of NETs reaches z_L :

$$\tilde{\mathbf{y}}(\mathbf{y}, z_L) = \begin{cases} \tilde{\mathbf{y}}_i(\mathbf{y}, z_L) = y_L & \text{for all } i \text{ such that } y_i < \bar{y} z_L, \\ \tilde{\mathbf{y}}_k(\mathbf{y}, z_L) = y_k & \text{for all } k \text{ such that } \bar{y} z_L \leq y_k \leq \bar{y} z_H(\mathbf{y}, z_L), \\ \tilde{\mathbf{y}}_j(\mathbf{y}, z_L) = y_H & \text{for all } j \text{ such that } y_j > \bar{y} z_H(\mathbf{y}, z_L). \end{cases}$$

In words, $\tilde{\mathbf{y}}(\mathbf{y}, z_L)$ is a snapshot of the construction of $\mathbf{y}$ along the mean-preserving expansion at the stage where the lowest income is $\bar{y} z_L$ (recall Figure 1).

Theorem 2. *Any inequality index, ι, can be written as follows:*

[6]Formally, $\varrho_L(\mathbf{y}, z_L) = l(\mathbf{y}, \bar{y} z_L)/n$ and $\varrho_H(\mathbf{y}, z_L) = h(\mathbf{y}, \bar{y} z_H(\mathbf{y}, z_L))/n$, where $z_H(\mathbf{y}, z_L)$ is the value $z \geq 1$ such that $\sum_i (y_i - \min\{y_i, \bar{y} z\}) = \sum_i (\max\{y_i, \bar{y} z_L\} - y_i)$, meaning that the income left to distribute at the top equals the income left to withdraw from the bottom. See Appendix A.1 for more detail.

For all $\mathbf{y} \in \{\mathcal{I}(Y,n) \,|Y \in \mathbb{R}_+, n \in \mathbb{N}\}$,

$$\iota(\mathbf{y}) = \int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha\left(z_L, z_H(\mathbf{y}, z_L), \varrho_L(\mathbf{y}, z_L), \varrho_H(\mathbf{y}, z_H(\mathbf{y}, z_L)), Y, n, \tilde{\mathbf{y}}(\mathbf{y}, z_L)\right) dz_L, \tag{2}$$

for some function $\alpha : [0,1] \times [1, +\infty) \times [0,1] \times [0,1] \times \mathbb{R}_+ \times \mathbb{N} \times \mathbb{R}_+^n \to \mathbb{R}$.

Proof. In Appendix A.2. □

Theorem 2 establishes that an inequality index can be uniquely represented by a function α on the NET space. One can interpret $\alpha(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}})$ as the cost that the evaluator attaches to the NET $(z_L, z_H, \varrho_L, \varrho_H)$ on the path to the real distribution. The infinitesimal flow of income transferred by this NET is given by $\varrho_L(\mathbf{y}, z_L)\, dz_L$.

Remark 1. Because all the arguments of α in Expression (2) depend only on z_L and on the profile $\mathbf{y}$, a more compact expression would be to simply write

$$\iota(\mathbf{y}) = \int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha(\mathbf{y}, z_L)\, dz_L.$$

Instead, we find Expression (2) more useful because it narrows down the specific dimensions in which α depends on z_L and on $\mathbf{y}$; i.e., the specific information required to compute it.

Example 1 provides the expressions of the weight functions α for several well-known inequality indices. A simple method for computing α for any inequality index ι can be found in Appendix B.

Example 1. Expressions of α for well-known indices:

- The variance index,

$$\iota_{var}(\mathbf{y}) = \frac{1}{n} \sum_{i=1,\ldots,n} (y_i - \bar{y})^2,$$

corresponds to $\alpha_{var}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv 2\bar{y}^2 (z_H - z_L)$.

- The Gini coefficient,

$$\iota_{Gini}(\mathbf{y}) = \frac{2\sum_{i=1}^n iy_i}{nY} - \frac{n+1}{n},$$

corresponds to $\alpha_{Gini}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv 2 - (\varrho_H + \varrho_L)$, which is 1 plus the fraction of the population whose incomes belong to the open interval $(z_L\bar{y}, z_H\bar{y})$.

- The Pietra index is the share of income to be distributed from rich to poor to achieve equality:

$$\iota_{Pietra}(\mathbf{y}) = \frac{\sum_{\{i|y_i \geq \bar{y}\}} (y_i - \bar{y})}{Y},$$

corresponds to a constant α: $\alpha_{Pietra}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv 1$.

- The Theil index,

$$\iota_{Theil}(\mathbf{y}) = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{\bar{y}} \ln\left(\frac{y_i}{\bar{y}}\right),$$

corresponds to $\alpha_{Theil}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv \ln(z_H/z_L)$.

- The log deviation index,

$$\iota_{log}(\mathbf{y}) = \frac{1}{n} \sum_{i=1}^{n} \ln\left(\frac{y_i}{\bar{y}}\right),$$

corresponds to $\alpha_{log}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv 1/z_L - 1/z_H$.

- Generalized entropy indices are parametrized by a coefficent $\varepsilon \in \mathbb{R}_+ \setminus \{0, 1\}$:

$$\iota_{GE\varepsilon}(\mathbf{y}) = \frac{1}{\varepsilon(\varepsilon - 1)} \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{y_i}{\bar{y}}\right)^{\varepsilon} - 1\right],$$

and correspond to $\alpha_{GE\varepsilon}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv \left[(z_H)^{\varepsilon-1} - (z_L)^{\varepsilon-1}\right] / (\varepsilon - 1)$.

- The Atkinson index associated with utility function $u(y) = \left(y^{1-\varepsilon} - 1\right) / (1 - \varepsilon)$ for $\varepsilon \in \mathbb{R}_+ \setminus \{1\}$ (Atkinson, 1970), when given an empirical distribution, writes:

$$\iota_{Atk\varepsilon}(\mathbf{y}) = 1 - \frac{1}{\bar{y}} \left(\frac{1}{n} \sum_{i=1}^{n} y_i^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}},$$

which corresponds to

$$\alpha_{Atk\varepsilon}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}) \equiv \left[z_L^{-\varepsilon} - z_H^{-\varepsilon}\right] (1 - \iota_{Atk\varepsilon}(\tilde{\mathbf{y}}))^{\varepsilon}.$$

	z_L	z_H	ϱ_L	ϱ_H	Y	n	$\tilde{\mathbf{y}}$
α_{var}	✓	✓	-	-	✓	✓	-
α_{Gini}	-	-	✓	✓	-	-	-
α_{Pietra}	-	-	-	-	-	-	-
α_{Theil}	✓	✓	-	-	-	-	-
α_{log}	✓	✓	-	-	-	-	-
$\alpha_{GE\varepsilon}$	✓	✓	-	-	-	-	-
$\alpha_{Atk\varepsilon}$	✓	✓	-	-	-	-	✓

Table 1: Display of the relevant variables for the weight functions α of Example 1

3 Index additivity and inequality decomposition

This sections discusses some implications raised by the NET representation of inequality indices. It highlights the relative simplicity of the weighting function α of most inequality measures. Furthermore, we show that when the inequality meausre is additive, the value of the index is easily decomposed.

3.1 NET additivity and Inner-NET additivity

Example 1 displayed seven weight functions, α, corresponding to well-known inequality indices. Table 1 summarizes, for each one, which variables are actually relevant.

Table 1 allows us to make two observations regarding the space of inequality indices.

NET additivity

Looking at Table 1, a first observation is that the α of all indices shown depends only on a few of the potential seven variables. Only one of them—the Atkinson index—makes use of the $\tilde{\mathbf{y}}$ variable, which is the only variable to account for the path of NETs taken up to now in the expansion. In other words, all other indices shown exhibit a path-independent NET representation.

Definition 3. An inequality index, ι, is *NET additive* if its corresponding α does not depend on $\tilde{\mathbf{y}}$:

$$\alpha\left(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}\right) \equiv \alpha\left(z_L, z_H, \varrho_L, \varrho_H, Y, n\right). \tag{3}$$

It is interesting to note that, for NET-additive indices, the function α acts as a field on the space of NETs, on which any continuous path represents an income distribution. Figures 5 and 6 display such fields for the Gini coefficient, the Theil index and the log deviation index, respectively.

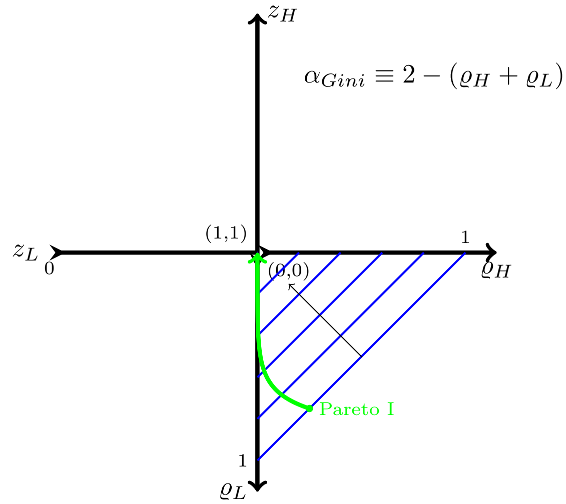


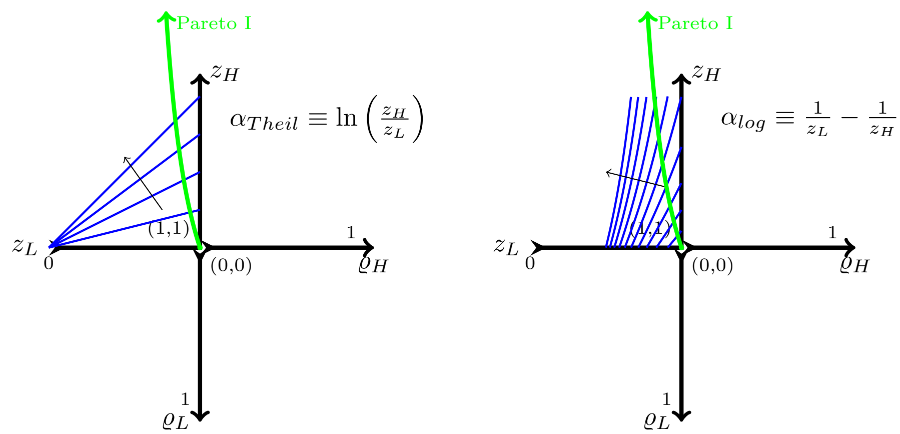
Figure 5: Mapping of the Gini coefficient in the NET-space. A Pareto I distribution is superimposed to illustrate the interaction between the distribution and the level curves of the α function in the four-dimensional NET space. For the Gini coefficient, only the ϱ_L and ϱ_H dimensions are relevant.

Inner-NET additivity

Following the previous observation regarding NET-additive indices, one can identify a further type of separability that will prove useful in various applications of inequality measures. Namely, when the lower and upper components of a NET can be disentangled, one can measure the contribution of each individual in the distribution to the overall inequality.

Definition 4. A NET-additive inequality index, ι, is *inner-NET additive* if its corresponding α is additively additive in the NET's lower and upper components:

$$\alpha\left(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}\right) \equiv \alpha_L\left(z_L, \varrho_L, Y, n\right) + \alpha_H\left(z_H, \varrho_H, Y, n\right) \tag{4}$$



(a) Theil Index (b) Log Deviation Index

Figure 6: Mappings of Inequality Indices in the NET-space. A Pareto I distribution is superimposed to illustrate the interaction between the distribution and the level curves of the α function in the four-dimensional NET space. For these two indices, only the z_L and z_H dimensions are relevant.

for some functions α_L and α_H.

As it turns out, other than the family of Atkinson indices, which are not NET additive, all indices mentioned thus far are not only NET additive, but also inner-NET additive. Even among those, which already lie in a restricted subspace of inequality indices, the Gini coefficient's weight function depends exclusively on ϱ_L and ϱ_H : $\alpha_{Gini,L} = 1 - \varrho_L$ and $\alpha_{Gini,H} = 1 - \varrho_H$. This highlights the already well-known fact that the Gini coefficient considers only the individuals' positions in the distribution but not the (relative) income inequality per se, as α_{Gini} does not depend on z_L nor z_H. By contrast, all remaining examples have their corresponding α depend only on z_L and z_H (and, in the case of the variance index, on Y/n).

While the list of indices mentioned as examples is far from exhaustive, one can make the following two observations. First, many indices used in practice are NET-additive. Second, many of those NET-additive index, if not all, are actually inner-NET additive. Moreover, it seems the richness of even the space of inner-NET additive inequality indices is strikingly underutilized, as one could have expected more indices to depend on both relative income (reflected in z_L and z_H) as well as ranking in the distribution (reflected in ϱ_L and ϱ_H).

One could very well imagine an (inner-NET additive) inequality index that would make explicit use of all four dimensions. In fact, the NET representation we develop in this article could provide a way to propose new inequality indices, simply by proposing new α functions, potentially using all variables.[7]

3.2 Inner-NET additivity and inequality decomposition

Inner-NET additivity is convenient because it allows for clean decompositions of the inequality index. We shall make explicit three such decompositions: rich-poor decomposition, an individual's contribution to the inequality measure, and the contribution of a given quantile to the inequality measure.

Rich-poor decomposition

The first decomposition of inequality is between rich and poor, which highlights whether the inequality is mainly due to the bottom or to the top of the distribution. This can be useful because indices typically do not explicitly convey the weight they attribute to low-income spreads relative to high-income spreads, or are even invariant to symmetry around the mean (like the Pietra index and the Gini coefficient, for example, see Section 4.2).

Proposition 1. *When the index ι is inner-NET additive, the value of the index can be decomposed into two parts:*

$$\iota(\mathbf{y}) = \iota_L(\mathbf{y}) + \iota_H(\mathbf{y})$$

with

$$\iota_L(\mathbf{y}) = \int_0^1 \varrho_L(\mathbf{y}, z)\, \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz \qquad \textit{and}$$
$$\iota_H(\mathbf{y}) = \int_1^{+\infty} \varrho_H(\mathbf{y}, z)\, \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz,$$

where $\iota_L(\mathbf{y})$ (resp. $\iota_H(\mathbf{y})$) can be interpreted as the contribution to the inequality of the low-income (resp. high-income) population.

Proof. Consider an inner-NET additive index, ι. From Theorem 2 and Expres-

[7] Although, regarding variables Y and n, it may be preferable that they only enter as the ratio Y/n (as in α_{var}) so as to ensure that the index is population invariant or, better, that these variables do not enter at all, so that the index is scale invariant (unlike α_{var}).

sion (4), it can be rewritten as

$$\begin{aligned}\iota(\mathbf{y}) = &\int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha_L(z_L, \varrho_L(\mathbf{y}, z_L), Y, n)\, dz_L \\ &+ \int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha_H(z_H(\mathbf{y}, z_L), \varrho_H(\mathbf{y}, z_H(\mathbf{y}, z_L)), Y, n)\, dz_L.\end{aligned}$$

By definition of a NET, we know that

$$\frac{dz_H}{dz_L} = \frac{dy_H}{dy_L} = -\frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)} = -\frac{\varrho_L(\mathbf{y}, z_L)}{\varrho_H(\mathbf{y}, z_H)}.$$

This says that $dz_L = -\left[\varrho_H(\mathbf{y}, z_H) / \varrho_L(\mathbf{y}, z_L)\right] dz_H$, so that a simple change of variable (from z_L to z_H) yields:

$$\begin{aligned}&\int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha_H(z_H(\mathbf{y}, z_L), \varrho_H(\mathbf{y}, z_H(\mathbf{y}, z_L)), Y, n)\, dz_L \\ &\qquad\qquad = -\int_{+\infty}^1 \varrho_H(\mathbf{y}, z_H)\, \alpha_H(z_H, \varrho_H(\mathbf{y}, z_H), Y, n)\, dz_H.\end{aligned}$$

Therefore,

$$\begin{aligned}\iota(\mathbf{y}) = &\int_0^1 \varrho_L(\mathbf{y}, z)\, \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz \\ &+ \int_1^{+\infty} \varrho_H(\mathbf{y}, z)\, \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz.\end{aligned} \tag{5}$$

Denoting

$$\iota_L(\mathbf{y}) = \int_0^1 \varrho_L(\mathbf{y}, z)\, \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz, \tag{6}$$

$$\iota_H(\mathbf{y}) = \int_1^{+\infty} \varrho_H(\mathbf{y}, z)\, \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz, \tag{7}$$

we obtain

$$\iota(\mathbf{y}) = \iota_L(\mathbf{y}) + \iota_H(\mathbf{y}).$$

□

Individual contribution to the inequality measure

The second decomposition we consider allows one to assess the contribution of a single individual household to the inequality index.

Proposition 2. *When the index ι is inner-NET additive, the contribution to the inequality measure of a single individual with income $y = z\bar{y}$ is equal to:*

$$\eta(\mathbf{y}, z_L) = \begin{cases} \frac{1}{n}\int_z^1 \alpha_L(u, \varrho_L(\mathbf{y}, u), Y, n)\,du & \text{if } z \in [0,1] \\ \frac{1}{n}\int_1^z \alpha_H(u, \varrho_H(\mathbf{y}, u), Y, n)\,du. & \text{if } z \geq 1 \end{cases}$$

Proof. Consider an inner-NET additive index, ι, and define the contribution to $\iota(\mathbf{y})$ of all households with (relative) income between $z_L \leq 1$ and 1, abusing notations slightly:

$$\iota_L(z_L) = \int_{z_L}^1 \varrho_L(\mathbf{y}, z)\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz, \tag{8}$$

(notice that $\iota_L(0) = \iota_L(\mathbf{y})$). Denote λ the normalized density of the income distribution, l, so that

$$\varrho_L(\mathbf{y}, z) = \frac{1}{n}\int_0^z \lambda(u)\,du.$$

Therefore,

$$\begin{aligned}
\iota_L(z_L) &= \int_{z_L}^1 \varrho_L(\mathbf{y}, z)\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz \\
&= \int_{z_L}^1 \left[\frac{1}{n}\int_0^z \lambda(u)\,du\right]\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz \\
&= \frac{1}{n}\int_{z_L}^1\int_0^z \lambda(u)\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dudz \\
&= \frac{1}{n}\int_0^z\int_{z_L}^1 \lambda(u)\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dzdu \\
&= \int_0^z \lambda(u)\left[\frac{1}{n}\int_{z_L}^1 \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz\right]du.
\end{aligned}$$

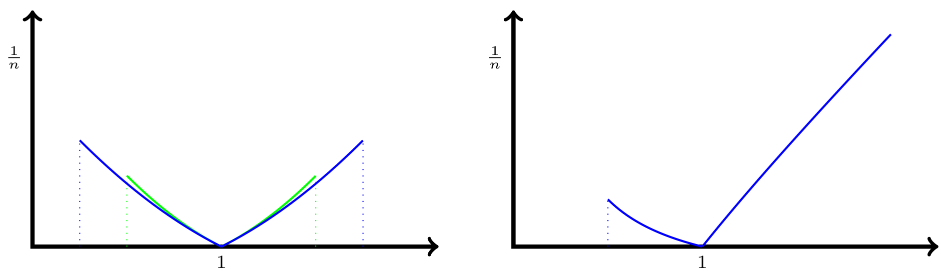
It follows that the contribution ot the inequality index of a household with income $y_L = \bar{y}z_L \leq \bar{y}$ is equal to

$$\eta(\mathbf{y}, z_L) = \frac{1}{n}\int_{z_L}^1 \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz. \tag{9}$$

Similarly, the individual contribution to the inequality measure of a household with income $y_H = \bar{y} z_H \geq \bar{y}$ writes as

$$\eta(\mathbf{y}, z_H) = \frac{1}{n} \int_1^{z_H} \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz. \tag{10}$$

□



(a) Uniform Distribution (b) Pareto Distribution

Figure 7: Individual contributions to the Gini index

Distribution-free inequality contribution Interestingly enough, if the functions α_L and α_H are independant from ϱ_L and ϱ_H, respectively, the individual contributions become distribution-free.

Proposition 3. *If $\alpha_L(z_L, \varrho_L, Y, n) \equiv \alpha_L(z_L, Y, n)$ and $\alpha_H(z_H, \varrho_H, Y, n) \equiv \alpha_H(z_H, Y, n)$, the contribution to the inequality measure of a single individual is independent of the distribution:*

$$\eta(z) = \begin{cases} \frac{1}{n} \int_z^1 \alpha_L(u, Y, n)\, du & \text{if } z \in [0, 1] \\ \frac{1}{n} \int_1^z \alpha_H(u, Y, n)\, du & \text{if } z \geq 1. \end{cases}$$

Proof. Follows immediately from Proposition 2. □

Figure 8 displays the individual contribution of a household for a given income level for various indices.

Example 2. *Variance index:* $\alpha_{var}(z_L, z_H, Y, n) \equiv 2\bar{y}^2(z_H - z_L)$ can be decomposed as the sum of $\alpha_{var}^L \equiv 2\bar{y}^2(1 - z_L)$ and $\alpha_{var}^H \equiv 2\bar{y}^2(z_H - 1)$. Hence,

$$\eta_{var}(z) = \frac{\bar{y}^2}{n}(1 - z)^2 \qquad \forall z \in \mathbb{R}_+$$

Pietra index: $\alpha_{Pietra} \equiv 1$ can be decomposed with $\alpha^L_{Pietra} \equiv \alpha^H_{Pietra} \equiv 1/2$. We thus obtain

$$\eta_{Pietra}(z) = \begin{cases} \frac{1}{2n}(1-z) & \text{if } z \in [0,1] \\ \frac{1}{2n}(z-1) & \text{if } z \geq 1 \end{cases}$$

Theil index: $\alpha_{Theil}(z_L, z_H) \equiv \ln(z_H/z_L)$ can be decomposed as the sum of $\alpha^L_{Theil} \equiv -\ln(z_L)$ and $\alpha^H_{Theil} \equiv \ln(z_H)$, so that

$$\eta_{Theil}(z) = \frac{1}{n}(1 + z\ln z - z) \qquad \forall z \in \mathbb{R}_+$$

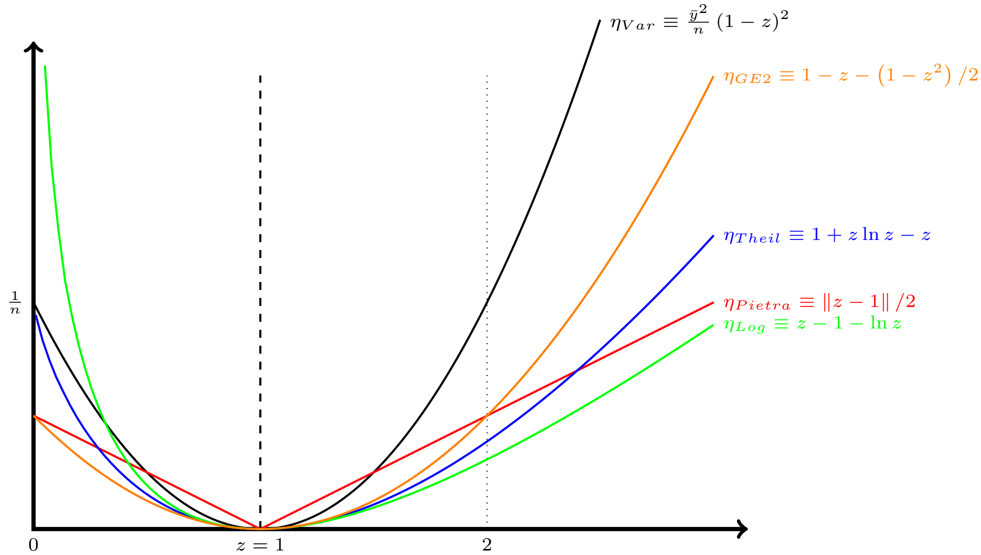


Figure 8: Individual contributions to inequality. All the indices pictured have a horizontal slope at $z = 1$ except for the Pietra index. This implies that for distributions that are very concentrated around the mean the Pietra index will display more inequality than the others. By contrast, the other indices generally assign more weight to large income spreads. Specifically, the log-normal index is the one that assigns the most weight to very-low-income households. It is therefore most sensitive to the existence of very poor households.

Mean-log deviation index: $\alpha_{log}(z_L, z_H) \equiv 1/z_L - 1/z_H$ can be decomposed as the sum of $\alpha^L_{log} \equiv 1/z_L - 1$ and $\alpha^H_{log} \equiv 1 - 1/z_H$. We thus obtain

$$\eta_{log}(z) = \frac{1}{n}[(z-1) - \ln z] \qquad \forall z \in \mathbb{R}_+$$

Generalized entropy index: $\alpha_{GE\varepsilon}(z_L, z_H) \equiv \left[(z_H)^{\varepsilon-1} - (z_L)^{\varepsilon-1}\right] / (\varepsilon - 1)$ can be decomposed as the sum of $\alpha^H_{GE\varepsilon} \equiv [1(z_H) - 1] / (\varepsilon - 1)$ and $\alpha^L_{GE\varepsilon} \equiv \left[1 - (z_L)^{\varepsilon-1}\right] / (\varepsilon - 1)$. We thus obtain

$$\eta_{GE\varepsilon}(z) = \frac{(1-z) - (1 - z^{\varepsilon})/\varepsilon}{n(\varepsilon - 1)} \qquad \forall z \in \mathbb{R}_+$$

Quantile contribution to the inequality measure

Finally, we make explicit how any given portion of the distribution contributes to the inequality measure. Practical applications include breaking down the inequality by quantiles.

Formally, define the income bracket $[y_1, y_2]$ associated to some quantile of interest, Q. There are three cases to consider, depending on whether the quantile lies entirely below the mean income level, entirely above, or straddles the mean income level.

Proposition 4. *The contribution of income bracket $Q = [y_1, y_2]$ to the overall inequality, $\iota(\mathbf{y})$, is equal to $\iota_Q(\mathbf{y}) =$*

$$\begin{cases} n(\varrho_L(\mathbf{y}, z_2) - \varrho_L(\mathbf{y}, z_1))\,\eta(z_2) + \int_{z_1}^{z_2} (\varrho_L(\mathbf{y}, z) - \varrho_L(\mathbf{y}, z_1))\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz & \text{if } z_2 \leq 1 \\ (\varrho_H(\mathbf{y}, z_1) - \varrho_H(\mathbf{y}, z_2))\,\eta(z_1) + \int_{z_1}^{z_2} (\varrho_H(\mathbf{y}, z) - \varrho_H(\mathbf{y}, z_2))\,\alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\,dz & \text{if } 1 \leq z_1 \\ \int_{z_1}^{1} (\varrho_L(\mathbf{y}, z) - \varrho_L(\mathbf{y}, z_1))\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\,dz & \\ + \int_{1}^{z_2} (\varrho_H(\mathbf{y}, z) - \varrho_H(\mathbf{y}, z_2))\,\alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\,dz & \text{if } z_1 \leq 1 \leq z_2 \end{cases}$$

where $z_1 = y_1/\bar{y}$ and $z_2 = y_2/\bar{y}$.

Proof. In Appendix A.3. □

4 Discussions

The formal representation of inequality indices by their corresponding α lends itself to useful interpretations and even yields policy implications.

4.1 The NET Principle as a minimal requirement for a useful inequality index

A positive value of α means that the inequality measure worsens when transfers from poorest to richest occur. Arguably, this is a desirable feature of any inequality measure. In fact, it would be problematic if an inequality index were unaffected—or, worse, decreased—as NETs are implemented. We shall call *NET principle* this property:

Definition 5. An inequality index, ι, satisfies the NET Principle if $\iota(\mathbf{y}') > \iota(\mathbf{y})$ whenever $\mathbf{y}'$ is the result of a NET applied to $\mathbf{y}$, for any income profile $\mathbf{y}$.

Proposition 5. *An inequality index, ι, satisfies the NET Principle if and only if its underlying function α is always positive.*

Proof. The proof follows immediately from the representation in Theorem 2. □

The NET Principle is a direct reference to Dalton's Transfer Principle, which states that inequality should decrease whenever a transfer from a richer to a poorer person occurs that does not reverse their relative positions or, equivalently, that inequality should increase whenever a transfer from poorer to richer occurs. Clearly, by considering only transfers at the extremes of the income distribution, the NET Principle is a weakening of Dalton's Transfer Principle. It is a useful weakening, as some indices, like the Pietra index, satisfy the NET Principle, but not the Transfer Principle. In fact, in his seminal work, Dalton argued that the Pietra index was a poor metric because it did not satisfy the Transfer Principle.[8]Actually, we claim that the Pietra index is not such a poor metric, because it at least satisfies the NET Principle. By contrast, a bad index would be one that does not even satisfy the NET Principle.

4.2 Inequality concerns vs. poverty concerns

From a formal standpoint, inequality measurements should merely concern themselves with how spread out is the income distribution, not with whether the inequality occurs at the top or at the bottom. From an economic stanpoint, however, it is not clear that the issues of inequality and poverty should be considered separately.

We shall say that an inequality index is *symmetric* if it returns the same value for a distribution $\mathbf{y}$ and for distribution $\mathbf{y}'$, the symmetric of $\mathbf{y}$ with respect to the mean income. In other words, a symmetric index is insensitive to poverty issues. As it turns out, whether an index is symmetric depends intimately on the symmetry properties of its underlying α function:

[8]While Dalton considered the Transfer Principle to be a minimal requirement of a good inequality measure, empirical evidence suggests that a large fraction of the population rejects this principle (Amiel and Cowell, 1998, 1999; Gaertner and Schokkaert, 2012).

Proposition 6. *If an index, ι, is NET additive and if its underlying function α satisfies*

$$\alpha\left(z_L, z_H, \varrho_L, \varrho_H, Y, n\right) = \alpha\left(2 - z_H, 2 - z_L, \varrho_H, \varrho_L, Y, n\right) \tag{11}$$

for all $(z_L, z_H, \varrho_L, \varrho_H, Y, n) \in [0,1] \times [1,2] \times [0,1] \times [0,1] \times \mathbb{R}_+ \times \mathbb{N}$, then ι is symmetric.

Proof. In Appendix A.4. □

Symmetry of the weight function α is easily checked. In fact, as we have observed, the α functions of most commonly-used indices take on very simple expressions. For example, with $\alpha_{Gini} = 2 - \varrho_L - \varrho_H$, checking the symmetry of the Gini coefficient is immediate. Other representations of the Gini coefficient, like $\iota_{Gini}\left(\mathbf{y}\right) = \frac{2}{nY} \sum_{i=1}^{n} i y_i - \frac{n+1}{n}$ or $\iota_{Gini}\left(\mathbf{y}\right) = \frac{2}{nY} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|\right]$ are much less transparent.

4.3 The α function as guide to the most efficient way of reducing inequality

When considering α as an implicit function of z_L:

$$z_L \mapsto \alpha\left(z_L, z_H\left(\mathbf{y}, z_L\right), \varrho_L\left(\mathbf{y}, z_L\right), \varrho_H\left(\mathbf{y}, z_H\left(\mathbf{y}, z_L\right)\right), Y, n, \tilde{\mathbf{y}}\left(\mathbf{y}, z_L\right)\right), \tag{12}$$

we obtain a clear recommendation for reducing the inequality index efficiently (i.e., with the least amount of income transferred): if this function is decreasing in z_L the most impactful way to reduce the inequality index is to reverse the NET expansion $\tilde{\mathbf{y}}$.

Proposition 7. *When the function*

$$z_L \mapsto \alpha\left(z_L, z_H\left(\mathbf{y}, z_L\right), \varrho_L\left(\mathbf{y}, z_L\right), \varrho_H\left(\mathbf{y}, z_H\left(\mathbf{y}, z_L\right)\right), Y, n, \tilde{\mathbf{y}}\left(\mathbf{y}, z_L\right)\right)$$

is decreasing in z_L the redistribution policy that reduces the value of the inequality index the most consists in undoing the most extreme NET, that is, to take from the individuals with the highest income to give to those with the lowest income.

Proof. Immediate from the proof of Theorem 2. □

Clearly, whether the explicit function in (12) is decreasing in z_L depends on the income profile $\mathbf{y}$, so that general statements about a given index cannot typically be made. However, when the index is Inner-NET additive, one can make distribution-independent statements true for any income distribution: If the goal of the social planner is to operate budget-neutral transfers to reduce the value of the inequality index, the shape of the functions α_L and α_H offer a guide in doing so in the most efficient manner.

Proposition 8. *For any inner-NET additive index for which $\alpha_L\left(z_L, \varrho_L\left(\mathbf{y}, z_L\right), Y, n\right)$ is a decreasing function of z_L and $\alpha_H\left(z_H, \varrho_H\left(\mathbf{y}, z_H\right), Y, n\right)$ is an increasing function of z_H, the redistribution policy that reduces the value of the inequality index the most consists in undoing a NET, that is, to take from the individual(s) with the highest income to give to those with the lowest income.*

Proof. It follows from Theorem 2 and Expression (4) that the (marginal) impact of giving an extra dollar to an individual is equal to $-\alpha_L\left(z_L, \varrho_L\left(\mathbf{y}, z_L\right), Y, n\right)/n$ or $\alpha_H\left(z_H, \varrho_H\left(\mathbf{y}, z_H\right), Y, n\right)/n$, depending upon their income level. Monotonicity in z_L and z_H yield the result. $\square$

In particular, if the functions α_L and α_H are independant from ϱ_L and ϱ_H, respectively, Proposition 8 becomes distribution-free. Among the examples considered thus far, this is the case for the variance, Pietra, Theil, Mean-Log Deviation and Generalized Entropy indices seen in Example 2.

Hence, efficiently reducing inequality is very different from the usual progressivity of the income tax for redistribution purposes. Instead, it requires taking money from the richest so as to reduce their income to a common value (effectively an income cap) and transfer the funds to the poorest (effectively creating an income floor). In particular, it should be noted that mid-income households are not involved in this efficient redistribution scheme. This is true for all pairs of index and income profiles such that α is decreasing in z_L, which are quite common. Moreover, many commonly used indices share this feature independently of the distribution, as seen in Example 2.

5 Conclusion

We showed that introducing the concept of negative extremal transfers (NETs) provides a novel way of describing income distributions (Theorem 1). In addition, we established that the set of NETs provides a mathematical basis that naturally leads to a new representation of inequality indices, through the weight function α that they assign to all possible NETs (Theorem 2).

This NET-based mathematical structure sheds new light on the properties of inequality indices. In particular, it makes it easy to identify whether an index emphasizes incomes rankings over income spreads—as is the case of the Gini coefficient, contrary to most other indices considered here—or whether an index puts a premium on income spreads at the bottom of the distribution—as the Theil and lognormal indices do, but not the Pietra index nor the income variance. (Section 3.1)

Furthermore, practitioners who work on income inequalities may find an interest in being able to compute the contribution of each income quantile to the inequality (Proposition 4). In addition, being able to assess the contribution to the inequality measure of any single household (Propositions 2 and 3) provides guidance in how to efficiently operate transfers from rich to poor (Proposition 8).

Finally, although the analysis focused on income distributions and income inequality, the methods carried out could largely apply to, say, utility distributions and social welfare functions (SWFs)—instead of income distributions and inequality indices, respectively. As a result, one could represent an SWF by how much weight it places on extremal utility transfers and, as a result, tease out the impact of any segment of the population on the overall value of the SWF.

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A Proofs

A.1 Proof of Theorem 1

Let $\mathbf{y} \in \mathcal{I}(Y, n)$, and define the uniform distribution $\mathbf{y_0} = (\bar{y}, ..., \bar{y}) \in \mathbb{R}^n_+$. Suppose $\mathbf{y} \neq \mathbf{y_0}$, otherwise the result is trivial.

For any pair $(y_L, y_H) \in \mathbb{R}_+^2$ such that $y_L \leq \bar{y} \leq y_H$, define the profile $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$ such that

$$\begin{cases} \tilde{y}_i = y_L & \text{for all } i \text{ such that } y_i < y_L, \\ \tilde{y}_k = y_k & \text{for all } k \text{ such that } y_L \leq y_k \ \leq y_H, \\ \tilde{y}_j = y_H & \text{for all } j \text{ such that } y_j > y_H. \end{cases} \tag{13}$$

In words, the profile $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$ spreads the income distribution starting from $\mathbf{y_0}$ towards $\mathbf{y}$ while bounding incomes by y_L below and by y_H above.

Define the volumes of income transfers associated to the shift from distribution $\mathbf{y_0}$ to the distribution $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$:

$$T_L(\mathbf{y}, y_L) = \int_{y_L}^{\bar{y}} l(\mathbf{y}, y)\, dy, \tag{14}$$

$$T_H(\mathbf{y}, y_H) = \int_{\bar{y}}^{y_H} h(\mathbf{y}, y)\, dy. \tag{15}$$

By definition, T_L (resp. T_H) returns the total amount of income to be taken from individuals in $L(\mathbf{y}, \bar{y}) \setminus \{k | y_k = \bar{y}\}$ (resp. to be given to agents in $H(\mathbf{y}, \bar{y}) \setminus \{k | y_k = \bar{y}\}$) to obtain $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$ from $\mathbf{y}_0$.

Conversely, define on $\mathbb{R}_+$ the following two functions:

$$y_L(\mathbf{y}, \cdot) \equiv T_L^{-1}(\mathbf{y}, \cdot), \tag{16}$$

$$y_H(\mathbf{y}, \cdot) \equiv T_H^{-1}(\mathbf{y}, \cdot). \tag{17}$$

By definition, for $T \geq 0$, $y_L(\mathbf{y}, T)$ (resp. $y_H(\mathbf{y}, T)$) returns the extremal income levels of $\tilde{\mathbf{y}}$ after transfering an amount T from individuals in $L(\mathbf{y}, \bar{y}) \setminus \{k | y_k = \bar{y}\}$ to individuals in $H(\mathbf{y}, \bar{y}) \setminus \{k | y_k = \bar{y}\}$ according to the expansion path delimited by (13). For readibility, we abuse notations slightly and denote $\tilde{\mathbf{y}}(\mathbf{y}, T) \equiv \tilde{\mathbf{y}}(\mathbf{y}, y_L(\mathbf{y}, T), y_H(\mathbf{y}, T))$. By construction, $\tilde{\mathbf{y}}(\mathbf{y}, T)$ is the mean-preserving spread of the uniform distribution $\mathbf{y}_0$, which obtains from $\mathbf{y}_0$ by a sequence of NETs.

Denote $\overline{T}(\mathbf{y}) = \int_0^{\bar{y}} l(\mathbf{y}, y)\, dy$. By construction, $\overline{T}(\mathbf{y})$ is the total amount of income taken from individuals in $L(\mathbf{y}, \bar{y}) \setminus \{k | y_k = \bar{y}\}$ and given to individuals

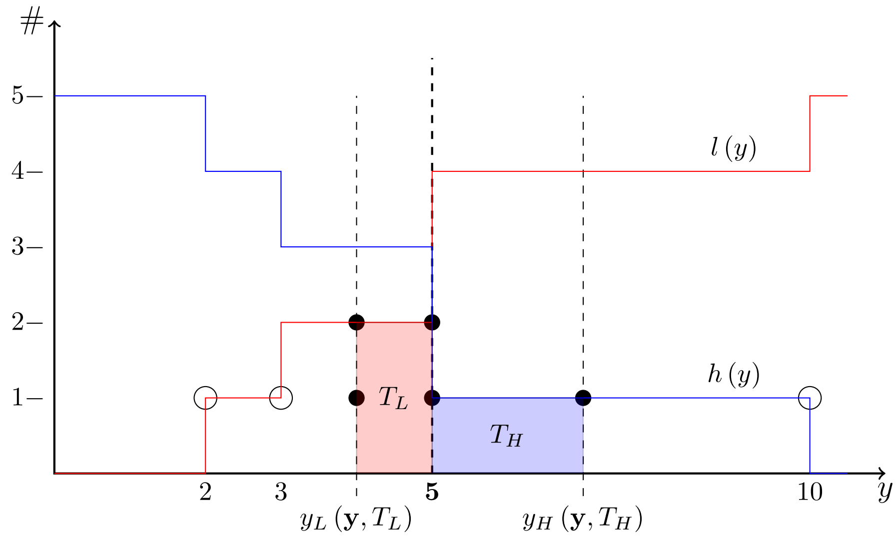

Figure 9: **Truncated profile**

in $H(\mathbf{y}, \bar{y}) \setminus \{k | y_k = \bar{y}\}$ to obtain profile $\mathbf{y}$ from the uniform profile $\mathbf{y}_0$:

$$\overline{T}(\mathbf{y}) = \int_0^{\bar{y}} l(\mathbf{y}, y)\, dy = \int_{\bar{y}}^{+\infty} h(\mathbf{y}, y)\, dy.$$

Equivalently,

$$\overline{T}(\mathbf{y}) = \sum_{i \in L(\mathbf{y}, \bar{y})} (\bar{y} - y_i) = \sum_{j \in H(\mathbf{y}, \bar{y})} (y_j - \bar{y}).$$

By construction, for any $0 \leq T \leq \overline{T}(\mathbf{y})$ the profile $\tilde{\mathbf{y}}(\mathbf{y}, T)$ is the result of a sequence of NETs from the uniform distribution $\tilde{\mathbf{y}}(\mathbf{y}, 0) = \mathbf{y}_0$. Furthermore, $\tilde{\mathbf{y}}\left(\mathbf{y}, \overline{T}(\mathbf{y})\right) = \mathbf{y}$, thus proving the first statement of the Theorem. The second statement immediately follows from noticing that the expansion $T \mapsto \tilde{\mathbf{y}}(\mathbf{y}, y_L(\mathbf{y}, T), y_H(\mathbf{y}, T))$ is unique to $\mathbf{y}$.

A.2 Proof of Theorem 2

This proof reuses the same notations in the proof of Theorem 1. Consider an index ι. Fix Y and n for now, and let $\mathbf{y} \in \mathcal{I}(Y, n)$. Consider the NET expansion of Theorem 1 from $\mathbf{y}_0$ to $\mathbf{y}$. By definition of an index, we have $\iota(\mathbf{y}_0) = 0$.

Define, for any $y_i \in \mathbb{R}_+$,

$$T(\mathbf{y}, y_i) = \begin{cases} T_L(\mathbf{y}, y_i) & \text{if } 0 \leq y_i \leq \bar{y}, \\ T_H(\mathbf{y}, y_i) & \text{if } y_i \geq \bar{y}, \end{cases}$$

the volume of transfer needed to reach income y_i by the means of the NET expansion $\tilde{\mathbf{y}}(\mathbf{y}, T)$ from the uniform profile, $\mathbf{y}_0$. Define $s : N \to N$ a permutation that reorders the agents in increasing order of $T(\mathbf{y}, y_i)$: for all $k \in N$,

$$T\left(\mathbf{y}, y_{s(k+1)}\right) \geq T\left(\mathbf{y}, y_{s(k)}\right).$$

By the anonymity of an index, such a reordering does not affect its value. Given s, one can define the profiles $\tilde{\mathbf{y}}^{(k)} \equiv \tilde{\mathbf{y}}\left(\mathbf{y}, T\left(\mathbf{y}, y_{s(k)}\right)\right)$, $k = 1, ..., n$. We abuse notation slightly to denote $\tilde{\mathbf{y}}^{(0)} \equiv \tilde{\mathbf{y}}(\mathbf{y}, 0) = \mathbf{y}_0$.

Observe that the permutation s is not uniquely defined. This is because it is the case that $T\left(\mathbf{y}, y_{s(k+1)}\right) = T\left(\mathbf{y}, y_{s(k)}\right)$ for some k. Note, in particular, that $T(\mathbf{y}, y_1) = T(\mathbf{y}, y_n) = \overline{T}(\mathbf{y})$, so that so that we can have $s(n) = 1$ or $s(n) = n$. However, the sequence of profile vectors $\tilde{\mathbf{y}}^{(k)}$ is unique.

For any $k \in N$, define the semi-open interval $R_k(\mathbf{y}) = \left(T\left(\mathbf{y}, y_{s(k-1)}\right), T\left(\mathbf{y}, y_{s(k)}\right)\right]$, where we extend notations to $T\left(\mathbf{y}, y_{s(0)}\right) \equiv 0$. By construction, in the interior of any interval $R_k(\mathbf{y})$, the extremal income values $y_L(\mathbf{y}, T)$ and $y_H(\mathbf{y}, T)$ do not cross any income levels y_i of the actual income distribution, $\mathbf{y}$. Hence, the sets of poorest and richest individuals of $\tilde{\mathbf{y}}(\mathbf{y}, T)$ are independent of the value of $T \in R_k(\mathbf{y})$. For any such $R_k(\mathbf{y})$, one can thus define the two sets $L_k \equiv L(\tilde{\mathbf{y}}(\mathbf{y}, T), y_L(\mathbf{y}, T)) = L(\mathbf{y}, y_L(\mathbf{y}, T))$ and $H_k \equiv H(\tilde{\mathbf{y}}(\mathbf{y}, T), y_H(\mathbf{y}, T)) = H(\mathbf{y}, y_H(\mathbf{y}, T))$ for any $T \in R_k(\mathbf{y})$ as well as their cardinality $l_k = \#L_k$ and $h_k = \#H_k$.

For any $T, T' \in R_k(\mathbf{y})$ such that $T' \geq T$. By construction, when applied to the profile $\tilde{\mathbf{y}}(\mathbf{y}, T')$,

$$l(\tilde{\mathbf{y}}(\mathbf{y}, T'), x) = l_k, \qquad \text{for all } y_L(\mathbf{y}, T') \leq x \leq y_L(\mathbf{y}, T), \tag{18}$$

$$h(\tilde{\mathbf{y}}(\mathbf{y}, T'), x) = h_k, \qquad \text{for all } y_H(\mathbf{y}, T) \leq x \leq y_H(\mathbf{y}, T'). \tag{19}$$

Thus for any T, T' in R_k,

$$T' - T = \int_{y_L(\mathbf{y},T')}^{y_L(\mathbf{y},T)} l\left(\tilde{\mathbf{y}}\left(\mathbf{y},T'\right),x\right)dx = \left[y_L\left(\mathbf{y},T\right) - y_L\left(\mathbf{y},T'\right)\right] l_k, \tag{20}$$

$$T' - T = \int_{y_H(\mathbf{y},T)}^{y_H\left(\mathbf{y},T'\right)} h\left(\tilde{\mathbf{y}}\left(\mathbf{y},T'\right),x\right)dx = \left[y_H\left(\mathbf{y},T'\right) - y_H\left(\mathbf{y},T\right)\right] h_k. \tag{21}$$

This makes it clear that $\tilde{\mathbf{y}}\left(\mathbf{y},T'\right)$ is obtained from $\tilde{\mathbf{y}}\left(\mathbf{y},T\right)$ by a NET:

$$\begin{cases} y_i' = y_i - \left(T' - T\right)/l_k & \text{for all } i \in L_k \\ y_i' = y_i + \left(T' - T\right)/h_k & \text{for all } i \in H_k \\ y_i' = y_i & \text{otherwise.} \end{cases} \tag{22}$$

Observe that the function $y_L\left(\mathbf{y},\cdot\right)$, as defined in (16) is strictly decreasing over $\left[0,\overline{T}\right]$. The partition of the interval $\left[0,\overline{T}\right]$ into the semi-open intervals $R_k\left(\mathbf{y}\right)$, $k \in N$, can therefore be associated to the partition of $\left[y_L\left(\mathbf{y},\bar{T}\right),\bar{y}\right]$ into the semi-open intervals of income levels

$$D_k\left(\mathbf{y}\right) = \left[y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k)}\right)\right), y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k-1)}\right)\right)\right)$$

where we again extend notations to $y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(0)}\right)\right) \equiv \bar{y}$. By construction $l\left(\mathbf{y},\cdot\right) = l_k$ over $D_k\left(\mathbf{y}\right)$.

Let $t \in R_k$ and define the associated income level $\zeta = y_L\left(\mathbf{y},t\right) \in D_k$. Let $\varepsilon \in \left[0,T\left(\mathbf{y},y_{s(k)}\right) - t\right]$ and $t' = t + \varepsilon$. Define again the associated income level $\zeta' = y_L\left(\mathbf{y},t'\right)$. By construction $\zeta' \in D_k$ and is such that $\zeta' \leq \zeta$, so that it can be written as $\zeta' = \zeta - \delta$, with $\delta \in \left[0,\zeta - y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k)}\right)\right)\right]$. Define the two profiles

$$\mathbf{y}_\zeta = \tilde{\mathbf{y}}\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right) \quad \text{and} \quad \mathbf{y}_\zeta' \equiv \tilde{\mathbf{y}}\left(\mathbf{y},T\left(\mathbf{y},\zeta'\right)\right). \tag{23}$$

The latter distribution, $\mathbf{y}_\zeta'$, is obtained from $\mathbf{y}_\zeta$ by a transfer of $\varepsilon = T\left(\mathbf{y},\zeta - \delta\right) - T\left(\mathbf{y},\zeta\right)$ from the least wealthy to the most wealthy. Observe that this NET involves a total transfer of a share $l_k\delta/Y$ of total income.

Define now the impact, β, of the above NET on the inequality index ι. This impact depends on the characteristics of the NET itself—its starting income ζ, its ending income $y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right)$, the number of individuals involved on each side, $l\left(\mathbf{y},\zeta\right)$ and $h\left(\mathbf{y},y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right)\right)$, and the size of the transfer, $l_k\delta$—as well as potentially on the starting distribution $\mathbf{y}_\zeta$. Normalizing the impact of the NET per percent of total income transferred $(l_k\delta/Y)$, we obtain:

$$\iota\left(\mathbf{y}'_\zeta\right)-\iota\left(\mathbf{y}_\zeta\right)=\frac{l_k\delta}{Y}\beta\left(\zeta,y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right),l\left(\mathbf{y},\zeta\right),h\left(\mathbf{y},y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right)\right),l_k\delta,\mathbf{y}_\zeta\right). \tag{24}$$

Define $f:\zeta\mapsto\iota\left(\mathbf{y}_\zeta\right)$. Considering infinitesimal increments, define

$$f'\left(\zeta\right)=\lim_{\delta\to 0}\frac{f\left(\zeta-\delta\right)-f\left(\zeta\right)}{-\delta} \tag{25}$$

$$=-\lim_{\delta\to 0}\left[\frac{\iota\left(\mathbf{y}'_\zeta\right)-\iota\left(\mathbf{y}_\zeta\right)}{\delta}\right], \tag{26}$$

which exists by the fact that ι is continuously differentiable.

Define

$$\begin{aligned}\alpha\left(\zeta/\bar{y},y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right)/\bar{y},l\left(\mathbf{y},\zeta\right)/n,h\left(\mathbf{y},y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right)\right)/n,\mathbf{y}_\zeta\right)\\ \equiv\beta\left(\zeta,y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right),l\left(\mathbf{y},\zeta\right),h\left(\mathbf{y},y_H\left(\mathbf{y},T\left(\mathbf{y},\zeta\right)\right)\right),0,\mathbf{y}_\zeta\right),\end{aligned}$$

which we shall abreviate to $\alpha\left(\zeta/\bar{y},y_H/\bar{y},l/n,h/n,\mathbf{y}_\zeta\right)$ for readability. With this notation, Expression (24) becomes:

$$f'\left(\zeta\right)=-\frac{l\left(\mathbf{y},\zeta\right)}{Y}\alpha\left(\frac{\zeta}{\bar{y}},\frac{y_H}{\bar{y}},\frac{l}{n},\frac{h}{n},\mathbf{y}_\zeta\right). \tag{27}$$

Summing over the closure of D_k, $D_k\cup\left\{y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k-1)}\right)\right)\right\}$, and observing that $\lim_{\zeta\to y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k-1)}\right)\right)}\mathbf{y}_\zeta=\tilde{\mathbf{y}}^{(k-1)}$ and that $\mathbf{y}_\zeta=\tilde{\mathbf{y}}^{(k)}$ when $\zeta=y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k)}\right)\right)$, we obtain:

$$\iota\left(\tilde{\mathbf{y}}^{(k)}\right)-\iota\left(\tilde{\mathbf{y}}^{(k-1)}\right)=\int_{y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k)}\right)\right)}^{y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k-1)}\right)\right)}\frac{l\left(\mathbf{y},\zeta\right)}{Y}\alpha\left(\frac{\zeta}{\bar{y}},\frac{y_H}{\bar{y}},\frac{l}{n},\frac{h}{n},\mathbf{y}_\zeta\right)d\zeta. \tag{28}$$

Recall that $\tilde{\mathbf{y}}^{(0)}=\mathbf{y_0}$ is the egalitarian distribution, so that $\iota\left(\mathbf{y_0}\right)=0$, and that $\tilde{\mathbf{y}}^{(n)}\equiv\mathbf{y}$, so that $y_L\left(\mathbf{y},0\right)=\bar{y}$ and $y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(n)}\right)\right)=y_L\left(\mathbf{y},\overline{T}\right)$. Hence, we have, by summation of (28) over all k:

$$\iota(\mathbf{y}) = \iota(\mathbf{y_0}) + \sum_{k=1}^{n} \int_{y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k)}\right)\right)}^{y_L\left(\mathbf{y},T\left(\mathbf{y},y_{s(k-1)}\right)\right)} \left[\frac{l(\mathbf{y},\zeta)}{Y} \alpha\left(\frac{\zeta}{\bar{y}}, \frac{y_H}{\bar{y}}, \frac{l}{n}, \frac{h}{n}, \mathbf{y}_\zeta\right)\right] d\zeta$$

$$= \int_{y_L\left(\mathbf{y},\overline{T}\right)}^{\bar{y}} \frac{l(\mathbf{y},\zeta)}{Y} \alpha\left(\frac{\zeta}{\bar{y}}, \frac{y_H}{\bar{y}}, \frac{l}{n}, \frac{h}{n}, \mathbf{y}_\zeta\right) d\zeta. \quad (29)$$

Let $z_L = \zeta/\bar{y} \leq 1$ be the ratio of income ζ to average income. The previous expression can be directly rewritten as

$$\iota(\mathbf{y}) = \int_{y_L\left(\mathbf{y},\overline{T}\right)/\bar{y}}^{1} \left[\frac{l(\mathbf{y},\bar{y}z_L)}{n} \alpha\left(z_L, z_H, \frac{l}{n}, \frac{h}{n}, \mathbf{y}_{z_L}\right)\right] dz_L, \quad (30)$$

where $z_H = y_H/\bar{y} \geq 1$. We denote $\varrho_L(\mathbf{y}, z) = l(\mathbf{y}, \bar{y}z)/n$ the (normalized) cumulative distribution function associated with $\mathbf{y}$; it is increasing in z and belongs to $[0,1]$. Furthermore, for any $0 \leq z < y_L\left(\mathbf{y}, \overline{T}\right)/\bar{y}$, we have $\varrho_L(\mathbf{y}, z) = 0$. Hence,

$$\iota(\mathbf{y}) = \int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha\left(z_L, z_H(\mathbf{y}, z_L), \varrho_L(\mathbf{y}, z_L), \varrho_H(\mathbf{y}, z_H(\mathbf{y}, z_L)), \mathbf{y}_{z_L}\right) dz_L, \quad (31)$$

with $\varrho_H(\mathbf{y}, z_H) \equiv h(\mathbf{y}, \bar{y}z_H)/n$.

Finally, considering that the previous argument was made for given Y and given n, we must adapt the result to account for variations in these two dimensions. Indeed, the function α—which contains all the degrees of freedom in determining ι— potentially depends on both Y and n. Hence the result:

$$\iota(\mathbf{y}) = \int_0^1 \varrho_L(\mathbf{y}, z_L)\, \alpha\left(z_L, z_H(\mathbf{y}, z_L), \varrho_L(\mathbf{y}, z_L), \varrho_H(\mathbf{y}, z_H(\mathbf{y}, z_L)), \mathbf{y}_{z_L}, Y, n\right) dz_L.$$

A.3 Proof of Proposition 4

The proof covers three cases:

Case_1 $z_2 \leq 1$, meaning that the quantile lies wholly below the mean income.

Its contribution to inequality is given by

$$\iota_Q(\mathbf{y}) = \int_{z_1}^{z_2} n\varrho_L'(\mathbf{y}, z_L)\, \iota_L(z_L)\, dz_L$$

where

$$n\varrho_L'(\mathbf{y}, z_L) = n\frac{d\varrho_L(\mathbf{y}, z_L)}{dz_L}$$

is the (relative) income density function associated to the distribution of $z = y/\bar{y}$. Thus we have, through integration by parts:

$$\begin{aligned}
\iota_Q(\mathbf{y}) &= \int_{z_1}^{z_2} \varrho_L'(\mathbf{y}, z_L)\left(\int_{z_L}^{1} \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz\right) dz_L \\
&= \int_{z_1}^{z_2} \varrho_L'(\mathbf{y}, z_L)\left(\int_{z_2}^{1} \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz\right) dz_L \\
&+ \int_{z_1}^{z_2} \varrho_L'(\mathbf{y}, z_L)\left(\int_{z_L}^{z_2} \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz\right) dz_L \\
&= (\varrho_L(\mathbf{y}, z_2) - \varrho_L(\mathbf{y}, z_1))\left(\int_{z_2}^{1} \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz\right) \\
&+ \left[(\varrho_L(\mathbf{y}, z_L) - \varrho_L(\mathbf{y}, z_1))\int_{z_L}^{z_2} \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz\right]_{z_L=z_1}^{z_L=z_2} \\
&- \int_{z_1}^{z_2} (\varrho_L(\mathbf{y}, z_L) - \varrho_L(\mathbf{y}, z_1))(-\alpha_L(z_L, \varrho_L(\mathbf{y}, z_L), Y, n))\, dz_L \\
&= n(\varrho_L(\mathbf{y}, z_2) - \varrho_L(\mathbf{y}, z_1))\,\eta(z_2) \\
&+ \int_{z_1}^{z_2} (\varrho_L(\mathbf{y}, z) - \varrho_L(\mathbf{y}, z_1))\,\alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz.
\end{aligned}$$

Case_2 $z_1 \geq 1$, meaning that the quantile lies wholly above the mean income. Its contribution to inequality is given by

$$\iota_Q(\mathbf{y}) = -\int_{z_1}^{z_2} n\varrho_H'(\mathbf{y}, z_H)\,\iota_H(z_H)\, dz_H$$

where

$$-n\varrho_H'(\mathbf{y}, z_H) = -n\frac{d\varrho_H(\mathbf{y}, z_H)}{dz_H}$$

is the (relative) income density function associated to the distribution of

$z = y/\bar{y}$. Thus we have

$$
\begin{aligned}
\iota_Q(\mathbf{y}) &= -\int_{z_1}^{z_2} \varrho_H'(\mathbf{y}, z_H)\left(\int_1^{z_H} \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz\right) dz_H \\
&= -\int_{z_1}^{z_2} \varrho_H'(\mathbf{y}, z_H)\left(\int_1^{z_1} \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz\right) dz_H \\
&- \int_{z_1}^{z_2} \varrho_H'(\mathbf{y}, z_H)\left(\int_{z_1}^{z_H} \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz\right) dz_H \\
&= (\varrho_H(\mathbf{y}, z_1) - \varrho_H(\mathbf{y}, z_2))\left(\int_1^{z_1} \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz\right) dz_H \\
&- \left[(\varrho_H(\mathbf{y}, z_H) - \varrho_H(\mathbf{y}, z_2))\int_{z_1}^{z_H} \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz\right]_{z_H=z_1}^{z_H=z_2} \\
&+ \int_{z_1}^{z_2} (\varrho_H(\mathbf{y}, z_H) - \varrho_H(\mathbf{y}, z_2))(\alpha_H(z_H, \varrho_H(\mathbf{y}, z_H), Y, n))\, dz_H \\
&= (\varrho_H(\mathbf{y}, z_1) - \varrho_H(\mathbf{y}, z_2))\, \eta(z_1) \\
&+ \int_{z_1}^{z_2} (\varrho_H(\mathbf{y}, z) - \varrho_H(\mathbf{y}, z_2))\, \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz.
\end{aligned}
$$

Case_3 $z_1 \leq 1 \leq z_2$, meaning that the quantile straddles the mean income, we have

$$
\begin{aligned}
\iota_Q(\mathbf{y}) &= \int_{z_1}^{1} n\varrho_L'(\mathbf{y}, z_L)\, \iota_L(z_L)\, dz_L + \int_1^{z_2} n\varrho_H'(\mathbf{y}, z_H)\, \iota_H(z_H)\, dz_H \\
&= \int_{z_1}^{1} (\varrho_L(\mathbf{y}, z) - \varrho_L(\mathbf{y}, z_1))\, \alpha_L(z, \varrho_L(\mathbf{y}, z), Y, n)\, dz \\
&+ \int_1^{z_2} (\varrho_H(\mathbf{y}, z) - \varrho_H(\mathbf{y}, z_2))\, \alpha_H(z, \varrho_H(\mathbf{y}, z), Y, n)\, dz
\end{aligned}
$$

where the result comes from applying the expressions found in Case 1 and Case 2.

A.4 Proof of Proposition 6

Let ι be a NET-additive profile, so that it can be represented by a function $\alpha(z_L, z_H, \varrho_L, \varrho_H, Y, n)$, as per Proposition 3, and suppose α satisfies condition (11). Consider a profile, $\mathbf{y}$, whose support is included in the interval $[0, 2\bar{y}]$ and its symmetric with respect to the mean income, $\mathbf{y}'$. By construction,

$$\varrho_L(\mathbf{y}', z) \equiv \varrho_H(\mathbf{y}, 2-z) \qquad \text{and} \tag{32}$$

$$\varrho_H(\mathbf{y}', z) \equiv \varrho_L(\mathbf{y}, 2-z), \tag{33}$$

for all $z \in [0,2]$. We aim to show that $\iota(\mathbf{y}') = \iota(\mathbf{y})$.

From Theorem 2, we have

$$\iota(\mathbf{y}') = \int_0^1 \varrho_L(\mathbf{y}', z_L)\,\alpha\left(z_L, z_H(\mathbf{y}', z_L), \varrho_L(\mathbf{y}', z_L), \varrho_H(\mathbf{y}', z_H(\mathbf{y}', z_L)), Y, n\right) dz_L. \tag{34}$$

By definition of the NET expansion path we know that the function $z_L \mapsto z_H(\mathbf{y}', z_L)$ is such that

$$\varrho_L(\mathbf{y}', z_L)\, dz_L = -\varrho_H(\mathbf{y}', z_H(\mathbf{y}', z_L))\, dz_H$$

for all $z_L \in [0,1]$. Hence, a change of variable $z_H = z_H(\mathbf{y}', z_L)$ turns Expression (34) into:

$$\begin{aligned}\iota(\mathbf{y}') &= -\int_2^1 \varrho_H(\mathbf{y}', z_H)\,\alpha\left(z_L(\mathbf{y}', z_H), z_H, \varrho_L(\mathbf{y}', z_L(\mathbf{y}', z_H)), \varrho_H(\mathbf{y}', z_H), Y, n\right) dz_H \\ &= \int_0^1 \varrho_H(\mathbf{y}', 2-u)\,\alpha\left(z_L(\mathbf{y}', 2-u), 2-u, \varrho_L(\mathbf{y}', z_L(\mathbf{y}', 2-u)), \varrho_H(\mathbf{y}', 2-u), Y, n\right) du\end{aligned} \tag{35}$$

where the second line comes from another change of variable: $u = 2 - z_H$.

By construction of the NET expansion, we have

$$\int_1^{2-u} \varrho_H(\mathbf{y}', z)\, dz = \int_{z_L(\mathbf{y}', 2-u)}^1 \varrho_L(\mathbf{y}', z)\, dz.$$

Thus, by the symmetric relation between profiles $\mathbf{y}$ and $\mathbf{y}'$, (i.e., Expressions (32) and (33)) the above expression becomes

$$\begin{aligned}\int_1^{2-u} \varrho_L(\mathbf{y}, 2-z)\, dz &= \int_{z_L(\mathbf{y}', 2-u)}^1 \varrho_H(\mathbf{y}, 2-z)\, dz \\ \int_u^1 \varrho_L(\mathbf{y}, w)\, dw &= \int_1^{2-z_L(\mathbf{y}', 2-u)} \varrho_H(\mathbf{y}, w)\, dw\end{aligned} \tag{36}$$

where the last line follows from the change of variable $w = 2 - z$. Applying the definition of the function $z_H(\mathbf{y}, \cdot)$ we obtain

$$2 - z_L(\mathbf{y}', 2 - u) = z_H(\mathbf{y}, u) \tag{37}$$

Getting back to Expression (35) and replacing $z_L(\mathbf{y}', 2 - u)$ by $2 - z_H(\mathbf{y}, u)$ yields:

$$\begin{aligned}
\iota(\mathbf{y}') &= \int_0^1 \varrho_H(\mathbf{y}', 2-u)\,\alpha\left(2 - z_H(\mathbf{y}, u), 2-u, \varrho_L(\mathbf{y}', 2 - z_H(\mathbf{y}, u)), \varrho_H(\mathbf{y}', 2-u), Y, n\right) du \\
&= \int_0^1 \varrho_L(\mathbf{y}, u)\,\alpha\left(2 - z_H(\mathbf{y}, u), 2-u, \varrho_H(\mathbf{y}, z_H(\mathbf{y}, u)), \varrho_L(\mathbf{y}, u), Y, n\right) du \qquad (38) \\
&= \int_0^1 \varrho_L(\mathbf{y}, u)\,\alpha\left(u, z_H(\mathbf{y}, u), \varrho_L(\mathbf{y}, u), \varrho_H(\mathbf{y}, z_H(\mathbf{y}, u)), Y, n\right) du \qquad (39) \\
&= \iota(\mathbf{y}), \qquad (40)
\end{aligned}$$

where (38) follows from (32) and (33), and (39) follows from the assumption on α. Expression (40) is simply the expression of $\iota(\mathbf{y})$ via Theorem 2.

B A simple method for extracting α

Consider a mean-preserving expansion profile of Theorem 1, denoted $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$ using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H(\mathbf{y}, y_L) \geq \bar{y}$.

By Theorem 2, there exists a weighting function α such that

$$\begin{aligned}
\iota(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)) &= \int_0^1 \varrho_L(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H), \zeta)\,\alpha(\zeta, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}(\mathbf{y}, \zeta))\,d\zeta \\
&= \int_{z_L}^1 \varrho_L(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H), \zeta)\,\alpha(\zeta, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}(\mathbf{y}, \zeta))\,d\zeta
\end{aligned}$$

because $\varrho_L(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H), \zeta) = 0$ whenever $\zeta < z_L = y_L/\bar{y}$.

Hence,

$$\begin{aligned}\frac{d\iota\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\left(\mathbf{y},y_L\right)\right)\right)}{dy_L} &= -\frac{1}{\bar{y}}\varrho_L\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right),z_L\right)\alpha\left(z_L,z_H,\varrho_L,\varrho_H,Y,n,\tilde{\mathbf{y}}\left(\mathbf{y},z_L\right)\right)\\ &+\int_{z_L}^{1}\frac{d}{dy_L}\varrho_L\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right),\zeta\right)\alpha\left(\zeta,z_H,\varrho_L,\varrho_H,Y,n,\tilde{\mathbf{y}}\left(\mathbf{y},\zeta\right)\right)d\zeta\\ &= -\frac{1}{\bar{y}}\varrho_L\left(\tilde{\mathbf{y}}\left(\mathbf{y},\bar{y}z_L,\bar{y}z_H\right),z_L\right)\alpha\left(z_L,z_H,\varrho_L,\varrho_H,Y,n,\tilde{\mathbf{y}}\left(\mathbf{y},z_L\right)\right),\end{aligned}$$

since the second term is equal to zero because $\varrho_L\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right),\zeta\right)$ is independent of y_L for $\zeta\in(z_L,1]$.

It follows that α can be extracted by simply computing the derivative of ι along the NET expansion of Theorem 1.

Moreover, upon observing that

$$\varrho_L\left(\tilde{\mathbf{y}}\left(\mathbf{y},\bar{y}z_L,\bar{y}z_H\right),z_L\right)=\varrho_L\left(\mathbf{y},z_L\right),$$

we have that

$$\alpha\left(z_L,z_H,\varrho_L,\varrho_H,Y,n,\mathbf{y}_{z_L}\right)=-\frac{\bar{y}}{\varrho_L\left(\mathbf{y},z_L\right)}\frac{d\iota\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\left(\mathbf{y},y_L\right)\right)\right)}{dy_L}\tag{41}$$

whenever $\varrho_L\left(\mathbf{y},z_L\right)\neq 0$.

C Mathematical derivations of examples/Not intended for publication

This section is devoted to computing the NET representation of well-known inequality measures.

C.1 The Variance Index

Consider the variance of the distribution as the inequality measure:

$$\iota_{var}\left(\mathbf{y}\right)=\frac{1}{n}\sum_{i=1,\ldots,n}\left(y_i-\bar{y}\right)^2.$$

Consider a mean-preserving expansion profile of Lemma 1, denoted $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$ using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H(\mathbf{y}, y_L) \geq \bar{y}$:

$$\iota_{var}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)) = \frac{1}{n}\left\{ l(y_L)(y_L - \bar{y})^2 + \sum_{\{i|y_i \in [y_L, y_H]\}} (y_i - \bar{y})^2 + h(y_H)(y_H - \bar{y})^2 \right\}$$

where we abuse notations slightly and write $l(y_L) = l(\mathbf{y}, y_L)$ and $h(y_H) = h(\mathbf{y}, y_H)$. Consider the following partial derivatives on neighborhoods where $l(y_L)$ and $h(y_H)$ are constant:[9]

$$\begin{aligned} \frac{\partial\iota(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{\partial y_L} &= \frac{2}{n} l(y_L)(y_L - \bar{y}) \\ \frac{\partial\iota(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{\partial y_H} &= \frac{2}{n} h(y_H)(y_H - \bar{y}) \end{aligned}$$

Mean-preserving expansions require $dy_H/dy_L = -l(y_L)/h(y_H)$, so that:

$$\begin{aligned} \frac{d\iota_{var}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} &= \frac{2}{n}\left[l(y_L)(y_L - \bar{y}) - \frac{l(y_L)}{h(y_H)} h(y_H)(y_H - \bar{y}) \right] \\ &= -\frac{2}{n} l(y_L)(y_H - y_L) \\ &= -2\bar{y}\frac{l(y_L)}{n}\left(\frac{y_H}{\bar{y}} - \frac{y_L}{\bar{y}}\right) \\ &= -2\bar{y}\varrho_L(\mathbf{y}, y_L)(z_H - z_L) \end{aligned}$$

with $z_L = y_L/\bar{y}$ et $z_H = y_H/\bar{y}$ et $\varrho_L = l/n$. Integrating over the full range of z_L, we get:

$$\int_0^1 \frac{d\iota_{var}(\tilde{\mathbf{y}}(\mathbf{y}, \bar{y}z_L, \bar{y}z_H))}{d\bar{y}z_L}\bar{y}dz_L = \iota_{var}(\tilde{\mathbf{y}}(\mathbf{y}, \bar{y}, \bar{y})) - \iota_{var}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, 0, \max_i y_i\right)\right) = 0 - \iota_{var}(\mathbf{y}).$$

Hence,

$$\begin{aligned} \iota_{var}(\mathbf{y}) &= -\int_0^1 -2\bar{y}\varrho_L(\mathbf{y}, y_L)(z_H - z_L)\bar{y}dz_L \\ &= 2\bar{y}^2\int_0^1 \varrho_L(\mathbf{y}, y_L)(z_H - z_L)dz_L. \end{aligned}$$

[9] The set of pairs (y_L, y_H) where $l(\mathbf{y}, y_L)$ and $h(\mathbf{y}, y_H)$ are not constant is of measure zero and will not affect the integral to be computed later.

By comparison with the expression of Theorem 1, this yields:

$$\alpha_{var}\left(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}\left(\mathbf{y}, z_L\right)\right) \equiv 2\bar{y}^2\left(z_H - z_L\right).$$

C.2 The Gini Coefficient

Consider a profile $\mathbf{y}$ where agents are ordered: $y_i \leq y_{i+1}$:

$$\iota_{Gini}\left(\mathbf{y}\right) = \frac{2}{nY}\sum_{i=1}^{n} i y_i - \frac{n+1}{n}.$$

Consider a mean-preserving expansion profile of Lemma 1, denoted $\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)$ using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H\left(\mathbf{y}, y_L\right) \geq \bar{y}$:

$$\begin{aligned}
\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)\right) &= \frac{2}{nY}\left\{\frac{l\left(\mathbf{y}, y_L\right)\left(1 + l\left(\mathbf{y}, y_L\right)\right)}{2} y_L + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} i y_i\right. \\
&\quad \left. + \left(\frac{n\left(n+1\right)}{2} - \frac{\left(n - h\left(\mathbf{y}, y_H\right)\right)\left(n - h\left(\mathbf{y}, y_H\right) + 1\right)}{2}\right) y_H\right\} - \frac{n+1}{n} \\
&= \frac{2}{nY}\left\{\frac{l\left(\mathbf{y}, y_L\right)\left(1 + l\left(\mathbf{y}, y_L\right)\right)}{2} y_L + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} i y_i\right. \\
&\quad \left. + \frac{1}{2}\left(n^2 + n - \left(n - h\left(\mathbf{y}, y_H\right)\right) - \left(n - h\left(\mathbf{y}, y_H\right)\right)^2\right) y_H\right\} - \frac{n+1}{n} \\
&= \frac{2}{nY}\left\{\frac{l\left(\mathbf{y}, y_L\right)\left(1 + l\left(\mathbf{y}, y_L\right)\right)}{2} y_L + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} i y_i\right. \\
&\quad \left. \frac{1}{2}\left(h\left(\mathbf{y}, y_H\right) + h\left(\mathbf{y}, y_H\right) * \left(2n - h\left(\mathbf{y}, y_H\right)\right)\right) y_H\right\} - \frac{n+1}{n}
\end{aligned}$$

Rearranging further we obtain:

$$\begin{aligned}
\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)\right) &= \frac{2}{nY}\left\{\frac{l\left(\mathbf{y}, y_L\right)\left(1 + l\left(\mathbf{y}, y_L\right)\right)}{2} y_L + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} i y_i\right. \\
&\quad \left. + \frac{h\left(\mathbf{y}, y_H\right)}{2}\left(1 + 2n - h\left(\mathbf{y}, y_H\right)\right) y_H\right\} - \frac{n+1}{n} \\
&= \frac{1}{nY}\left\{\left(1 + l\left(\mathbf{y}, y_L\right)\right) l\left(\mathbf{y}, y_L\right) y_L + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} 2 i y_i\right. \\
&\quad \left. + \left(1 + 2n - h\left(\mathbf{y}, y_H\right)\right) h\left(\mathbf{y}, y_H\right) y_H\right\} - \frac{n+1}{n}
\end{aligned}$$

Differentiating with respect to y_L yields :

$$\frac{d\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right)\right)}{dy_L}=\frac{\partial\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right)\right)}{\partial y_L}+\frac{dy_H}{dy_L}\frac{\partial\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right)\right)}{\partial y_H}.$$

Consider the following partial derivatives on neighborhoods where $l\left(\mathbf{y},y_L\right)$ and $h\left(\mathbf{y},y_H\right)$ are constant:[10]

$$\begin{aligned}\frac{\partial\iota_{Gini}}{\partial y_L}&=\frac{1}{nY}\left(1+l\left(\mathbf{y},y_L\right)\right)l\left(\mathbf{y},y_L\right)\\ \frac{\partial\iota_{Gini}}{\partial y_H}&=\frac{1}{nY}\left(1+2n-h\left(\mathbf{y},y_H\right)\right)h\left(\mathbf{y},y_H\right)\end{aligned}$$

Using the fact that

$$\frac{dy_H}{dy_L}=-\frac{l\left(\mathbf{y},y_L\right)}{h\left(\mathbf{y},y_H\right)},$$

we obtain

$$\begin{aligned}\frac{d\iota_{Gini}\left(\tilde{\mathbf{y}}\left(y_L\right)\right)}{dy_L}&=\frac{\partial\iota_{Gini}}{\partial y_L}+\frac{dy_H}{dy_L}\frac{\partial\iota_{Gini}}{\partial y_H}\\ &=\frac{1}{nY}\left\{\left(1+l\left(\mathbf{y},y_L\right)\right)l\left(\mathbf{y},y_L\right)-\frac{l\left(\mathbf{y},y_L\right)}{h\left(\mathbf{y},y_H\right)}\left(1+2n-h\left(\mathbf{y},y_H\right)\right)h\left(\mathbf{y},y_H\right)\right\}\\ &=\frac{l\left(\mathbf{y},y_L\right)}{nY}\left\{\left(1+l\left(\mathbf{y},y_L\right)\right)-\left(1+2n-h\left(\mathbf{y},y_H\right)\right)\right\}\\ &=-\frac{l\left(\mathbf{y},y_L\right)}{nY}\left[2n-l\left(\mathbf{y},y_L\right)-h\left(\mathbf{y},y_H\right)\right].\end{aligned}$$

Using the normalized notations $\varrho_L=l/n$, $\varrho_H=1-\varrho_L=h/n$, $z_L=y_L/\bar{y}$ and $z_H=y_H/\bar{y}$, and integrating over the full range of z_L, we get:

$$\int_0^1\frac{d\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y},\bar{y}z_L,\bar{y}z_H\right)\right)}{d\bar{y}z_L}\bar{y}dz_L=\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y},\bar{y},\bar{y}\right)\right)-\iota_{Gini}\left(\tilde{\mathbf{y}}\left(\mathbf{y},0,\max_i y_i\right)\right)=0-\iota_{Gini}\left(\mathbf{y}\right).$$

Hence,

$$\begin{aligned}\iota_{Gini}\left(\mathbf{y}\right)&=-\int_0^1-\frac{l\left(\mathbf{y},y_L\right)}{nY}\left[2n-l\left(\mathbf{y},y_L\right)-h\left(\mathbf{y},y_H\right)\right]\bar{y}dz_L\\ &=\int_0^1\varrho_L\left(\mathbf{y},y_L\right)\frac{\bar{y}}{Y}\left[2n-l\left(\mathbf{y},y_L\right)-h\left(\mathbf{y},y_H\right)\right]dz_L\\ &=\int_0^1\varrho_L\left(\mathbf{y},y_L\right)\left[2-\varrho_L\left(\mathbf{y},y_L\right)-\varrho_H\left(\mathbf{y},y_H\right)\right]dz_L\end{aligned}$$

[10]The set of pairs (y_L,y_H) where $l\left(\mathbf{y},y_L\right)$ and $h\left(\mathbf{y},y_H\right)$ are not constant is of measure zero and will not affect the integral to be computed later.

By comparison with the expression of Theorem 1, this yields:

$$\alpha_{Gini}\left(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}\left(\mathbf{y}, z_L\right)\right) \equiv 2 - \varrho_L - \varrho_H.$$

Note that $1 - \varrho_H$ is the share of the population whose income is below $z_H \bar{y}$, so that $\alpha_{Gini} = 1 + (1 - \varrho_H - \varrho_L)$ equals 1 plus the share of the population whose incomes are strictly between $z_L \bar{y}$ and $z_H \bar{y}$.

C.3 The Pietra Index

The Pietra index writes:

$$\iota_{Pietra}\left(\mathbf{y}\right) = \frac{\sum_{\{i|y_i \leq \bar{y}\}} \left(\bar{y} - y_i\right)}{Y}$$

Consider a mean-preserving expansion profile of Lemma 1, denoted $\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)$ using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H\left(\mathbf{y}, y_L\right) \geq \bar{y}$:

$$\iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)\right) = \frac{1}{Y}\left\{ l\left(y_L\right)\left(\bar{y} - y_L\right) + \sum_{\{i|y_i \leq \bar{y}\}} \left(\bar{y} - y_i\right)\right\}$$

Consider the following partial derivatives on neighborhoods where $l\left(\mathbf{y}, y_L\right)$ and $h\left(\mathbf{y}, y_H\right)$ are constant:[11]

$$\frac{\partial \iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)\right)}{\partial y_L} = -\frac{l\left(\mathbf{y}, y_L\right)}{Y}$$
$$\frac{\partial \iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)\right)}{\partial y_H} = 0$$

Using the fact that

$$\frac{dy_H}{dy_L} = -\frac{l\left(\mathbf{y}, y_L\right)}{h\left(\mathbf{y}, y_H\right)},$$

[11] The set of pairs (y_L, y_H) where $l\left(\mathbf{y}, y_L\right)$ and $h\left(\mathbf{y}, y_H\right)$ are not constant is of measure zero and will not affect the integral to be computed later.

we obtain

$$\begin{aligned}\frac{d\iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right)\right)}{dy_L} &= \frac{\partial\iota_{Pietra}}{\partial y_L}+\frac{\partial\iota_{Pietra}}{\partial y_H}\frac{dy_H}{dy_L}\\ &= -\frac{l\left(\mathbf{y},y_L\right)}{Y}\\ &= -\frac{\varrho_L\left(\mathbf{y},y_L\right)}{\bar{y}}.\end{aligned}$$

Using the normalized notations $\varrho_L = l/n$, $\varrho_H = 1-\varrho_L = h/n$, $z_L = y_L/\bar{y}$ and $z_H = y_H/\bar{y}$, and integrating over the full range of z_L, we get:

$$\int_0^1 \frac{d\iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y},\bar{y}z_L,\bar{y}z_H\right)\right)}{d\bar{y}z_L}\bar{y}dz_L = \iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y},\bar{y},\bar{y}\right)\right)-\iota_{Pietra}\left(\tilde{\mathbf{y}}\left(\mathbf{y},0,\max_i y_i\right)\right) = 0-\iota_{Pietra}\left(\mathbf{y}\right).$$

Hence,

$$\begin{aligned}\iota_{Pietra}\left(\mathbf{y}\right) &= -\int_0^1 -\frac{\varrho_L\left(\mathbf{y},y_L\right)}{\bar{y}}\bar{y}dz_L\\ &= \int_0^1 \varrho_L\left(\mathbf{y},y_L\right)dz_L\end{aligned}$$

By comparison with the expression of Theorem 1, this yields:

$$\alpha_{Pietra}\left(z_L,z_H,\varrho_L,\varrho_H,Y,n,\tilde{\mathbf{y}}\left(\mathbf{y},z_L\right)\right)\equiv 1.$$

C.4 The Theil Index

The Theil index writes:

$$\iota_{Theil}\left(\mathbf{y}\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{\bar{y}}\ln\left(\frac{y_i}{\bar{y}}\right).$$

Consider a mean-preserving expansion profile of Lemma 1, denoted $\tilde{\mathbf{y}}\left(\mathbf{y},y_L,y_H\right)$ using the notations of Expression (13), where $y_L \le \bar{y}$ and $y_H\left(\mathbf{y},y_L\right)\ge\bar{y}$:

$$\iota_{Theil}\left(\mathbf{y}\right) = \frac{1}{n\bar{y}}\left\{l\left(\mathbf{y},y_L\right)y_L\ln\left(\frac{y_L}{\bar{y}}\right)+\sum_{i=l(\mathbf{y},y_L)+1}^{n-h(\mathbf{y},y_H)}y_i\ln\left(\frac{y_i}{\bar{y}}\right)+h\left(\mathbf{y},y_H\right)y_H\ln\left(\frac{y_H}{\bar{y}}\right)\right\}.$$

Consider the following partial derivatives on neighborhoods where $l\left(\mathbf{y},y_L\right)$

and $h(\mathbf{y}, y_H)$ are constant:[12]

$$\begin{aligned}
\frac{\partial \iota_{Theil}}{\partial y_L} &= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \left[\ln\left(\frac{y_L}{\bar{y}}\right) + \frac{y_L}{\bar{y}} \left(\frac{y_L}{\bar{y}}\right)^{-1} \right] \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \left[\ln\left(\frac{y_L}{\bar{y}}\right) + 1 \right] \\
\frac{\partial \iota_{Theil}}{\partial y_H} &= \frac{1}{n\bar{y}} h(\mathbf{y}, y_H) \left[\ln\left(\frac{y_H}{\bar{y}}\right) + 1 \right].
\end{aligned}$$

Using the fact that

$$\frac{dy_H}{dy_L} = -\frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)},$$

we obtain

$$\begin{aligned}
\frac{d\iota_{Theil}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} &= \frac{\partial \iota_{Theil}}{\partial y_L} + \frac{\partial \iota_{Theil}}{\partial y_H}\frac{dy_H}{dy_L} \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \left[\ln\left(\frac{y_L}{\bar{y}}\right) + 1 \right] - \frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)} \frac{1}{n\bar{y}} h(\mathbf{y}, y_H) \left[\ln\left(\frac{y_H}{\bar{y}}\right) + 1 \right] \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \left[\ln\left(\frac{y_L}{\bar{y}}\right) - \ln\left(\frac{y_H}{\bar{y}}\right) \right] \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \left[\ln\left(\frac{y_L}{y_H}\right) \right]
\end{aligned}$$

Following Expression (41), we obtain:

$$\begin{aligned}
\alpha_{Theil}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}(\mathbf{y}, z_L)) &= -\frac{\bar{y}}{\varrho_L(\mathbf{y}, y_L)} \frac{d\iota_{Theil}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} \\
&= \ln\left(\frac{z_H}{z_L}\right).
\end{aligned}$$

C.5 The Mean Log Deviation

The mean log deviation index writes:

$$\iota_{log}(\mathbf{y}) = \frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{y_i}{\bar{y}}\right).$$

Consider a mean-preserving expansion profile of Lemma 1, denoted $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$

[12]The set of pairs (y_L, y_H) where $l(\mathbf{y}, y_L)$ and $h(\mathbf{y}, y_H)$ are not constant is of measure zero and will not affect the integral to be computed later.

using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H(\mathbf{y}, y_L) \geq \bar{y}$:

$$\iota_{log}(\mathbf{y}) = \frac{1}{n}\left\{ l(\mathbf{y}, y_L) \ln\left(\frac{y_L}{\bar{y}}\right) + \sum_{i=l(\mathbf{y},y_L)+1}^{n-h(\mathbf{y},y_H)} \ln\left(\frac{y_i}{\bar{y}}\right) + h(\mathbf{y}, y_H) \ln\left(\frac{y_H}{\bar{y}}\right) \right\}.$$

Consider the following partial derivatives on neighborhoods where $l(\mathbf{y}, y_L)$ and $h(\mathbf{y}, y_H)$ are constant:[13]

$$\begin{aligned}
\frac{\partial \iota_{log}}{\partial y_L} &= \frac{1}{n} l(\mathbf{y}, y_L) \left[\frac{1}{\bar{y}} \left(\frac{y_L}{\bar{y}}\right)^{-1} \right] \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \frac{\bar{y}}{y_L} \\
\frac{\partial \iota_{log}}{\partial y_H} &= \frac{1}{n\bar{y}} h(\mathbf{y}, y_H) \frac{\bar{y}}{y_H}.
\end{aligned}$$

Using the fact that

$$\frac{dy_H}{dy_L} = -\frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)},$$

we obtain

$$\begin{aligned}
\frac{d\iota_{log}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} &= \frac{\partial \iota_{log}}{\partial y_L} + \frac{\partial \iota_{log}}{\partial y_H} \frac{dy_H}{dy_L} \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \frac{\bar{y}}{y_L} - \frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)} \frac{1}{n\bar{y}} h(\mathbf{y}, y_H) \frac{\bar{y}}{y_H} \\
&= \frac{1}{n\bar{y}} l(\mathbf{y}, y_L) \left[\frac{\bar{y}}{y_L} - \frac{\bar{y}}{y_H} \right]
\end{aligned}$$

Following Expression (41), we obtain:

$$\begin{aligned}
\alpha_{log}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}(\mathbf{y}, z_L)) &= -\frac{\bar{y}}{\varrho_L(\mathbf{y}, y_L)} \frac{d\iota_{Theil}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} \\
&= \frac{\bar{y}}{y_L} - \frac{\bar{y}}{y_H} \\
&= \frac{1}{z_L} - \frac{1}{z_H}.
\end{aligned}$$

[13] The set of pairs (y_L, y_H) where $l(\mathbf{y}, y_L)$ and $h(\mathbf{y}, y_H)$ are not constant is of measure zero and will not affect the integral to be computed later.

C.6 Generalized Entropy Indices

Generalized entropy indices are parametrized by a coefficent $\varepsilon \in \mathbb{R}_+ \setminus \{0,1\}$:

$$\iota_{GE\varepsilon}(\mathbf{y}) = \frac{1}{\varepsilon(\varepsilon-1)} \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{y_i}{\bar{y}} \right)^{\varepsilon} - 1 \right].$$

Consider a mean-preserving expansion profile of Lemma 1, denoted $\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H)$ using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H(\mathbf{y}, y_L) \geq \bar{y}$:

$$\iota_{GE\varepsilon}(\mathbf{y}) = \frac{1}{\varepsilon(\varepsilon-1)} \frac{1}{n} \sum_{i=1}^{n} \left\{ l(\mathbf{y}, y_L) \left[\left(\frac{y_L}{\bar{y}} \right)^{\varepsilon} - 1 \right] + \sum_{i=l(\mathbf{y},y_L)+1}^{n-h(\mathbf{y},y_H)} \left[\left(\frac{y_i}{\bar{y}} \right)^{\varepsilon} - 1 \right] + h(\mathbf{y}, y_H) \left[\left(\frac{y_H}{\bar{y}} \right)^{\varepsilon} - 1 \right] \right\}.$$

Consider the following partial derivatives on neighborhoods where $l(\mathbf{y}, y_L)$ and $h(\mathbf{y}, y_H)$ are constant:[14]

$$\begin{aligned}
\frac{\partial \iota_{GE\varepsilon}}{\partial y_L} &= \frac{1}{\varepsilon(\varepsilon-1)} \frac{1}{n} l(\mathbf{y}, y_L) \left[\frac{\varepsilon}{\bar{y}} \left(\frac{y_L}{\bar{y}} \right)^{\varepsilon-1} \right] \\
&= \frac{1}{(\varepsilon-1)} \frac{1}{\bar{y}n} l(\mathbf{y}, y_L) \left(\frac{y_L}{\bar{y}} \right)^{\varepsilon-1} \\
\frac{\partial \iota_{GE\varepsilon}}{\partial y_H} &= \frac{1}{(\varepsilon-1)} \frac{1}{\bar{y}n} h(\mathbf{y}, y_H) \left(\frac{y_H}{\bar{y}} \right)^{\varepsilon-1}.
\end{aligned}$$

Using the fact that

$$\frac{dy_H}{dy_L} = -\frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)},$$

we obtain

$$\begin{aligned}
\frac{d\iota_{GE\varepsilon}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} &= \frac{\partial \iota_{GE\varepsilon}}{\partial y_L} + \frac{\partial \iota_{GE\varepsilon}}{\partial y_H} \frac{dy_H}{dy_L} \\
&= \frac{1}{(\varepsilon-1)} \frac{1}{\bar{y}n} \left[l(\mathbf{y}, y_L) \left(\frac{y_L}{\bar{y}} \right)^{\varepsilon-1} - \frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)} h(\mathbf{y}, y_H) \left(\frac{y_H}{\bar{y}} \right)^{\varepsilon-1} \right] \\
&= \frac{1}{(\varepsilon-1)} \frac{1}{\bar{y}n} l(\mathbf{y}, y_L) \left[\left(\frac{y_L}{\bar{y}} \right)^{\varepsilon-1} - \left(\frac{y_H}{\bar{y}} \right)^{\varepsilon-1} \right]
\end{aligned}$$

[14] The set of pairs (y_L, y_H) where $l(\mathbf{y}, y_L)$ and $h(\mathbf{y}, y_H)$ are not constant is of measure zero and will not affect the integral to be computed later.

Following Expression (41), we obtain:

$$\begin{aligned}\alpha_{GE\varepsilon}\left(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}\left(\mathbf{y}, z_L\right)\right) &= -\frac{\bar{y}}{\varrho_L\left(\mathbf{y}, y_L\right)} \frac{d\iota_{GE\varepsilon}\left(\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)\right)}{dy_L} \\ &= \frac{1}{(\varepsilon-1)}\left[\left(z_H\right)^{\varepsilon-1}-\left(z_L\right)^{\varepsilon-1}\right].\end{aligned}$$

C.7 The Atkinson Indices

The Atkinson index is defined relative to a social welfare function. It is defined as the normalized ratio of the egalitarian equivalent income—the individual income which, if received by all individuals, would yield the same social welfare as the current income distribution—over the mean income.

The underlying utility function generally considered in the literature is $u(y) = \left(y^{1-\varepsilon}-1\right)/(1-\varepsilon)$ for $\varepsilon \in \mathbb{R}_+ \setminus \{1\}$. When given an empirical distribution, the Atkinson index using parameter ε writes:

$$\iota_{Atk\varepsilon}(\mathbf{y}) = 1 - \frac{1}{\bar{y}}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}.$$

Consider a mean-preserving expansion profile of Theorem 1, denoted $\tilde{\mathbf{y}}\left(\mathbf{y}, y_L, y_H\right)$ using the notations of Expression (13), where $y_L \leq \bar{y}$ and $y_H\left(\mathbf{y}, y_L\right) \geq \bar{y}$:

$$\iota_{Atk\varepsilon}(\mathbf{y}) = 1 - \frac{1}{\bar{y}}\left(\frac{1}{n}\right)^{\frac{1}{1-\varepsilon}}\left\{l\left(\mathbf{y}, y_L\right) y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} y_i^{1-\varepsilon} + h\left(\mathbf{y}, y_H\right) y_H^{1-\varepsilon}\right\}^{\frac{1}{1-\varepsilon}}.$$

Consider the following partial derivatives on neighborhoods where $l\left(\mathbf{y}, y_L\right)$ and $h\left(\mathbf{y}, y_H\right)$ are constant:[15]

$$\begin{aligned}\frac{\partial \iota_{Atk\varepsilon}}{\partial y_L} &= -\frac{1}{\bar{y}}\left(\frac{1}{n}\right)^{\frac{1}{1-\varepsilon}} \frac{1}{1-\varepsilon} l\left(\mathbf{y}, y_L\right)(1-\varepsilon) y_L^{-\varepsilon}\left\{l\left(\mathbf{y}, y_L\right) y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} y_i^{1-\varepsilon} + h\left(\mathbf{y}, y_H\right) y_H^{1-\varepsilon}\right\}^{\frac{\varepsilon}{1-\varepsilon}} \\ &= -\frac{1}{\bar{y}}\frac{l\left(\mathbf{y}, y_L\right)}{n}\left(\frac{1}{n}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_L^{-\varepsilon}\left\{l\left(\mathbf{y}, y_L\right) y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} y_i^{1-\varepsilon} + h\left(\mathbf{y}, y_H\right) y_H^{1-\varepsilon}\right\}^{\frac{\varepsilon}{1-\varepsilon}} \\ \frac{\partial \iota_{Atk\varepsilon}}{\partial y_H} &= -\frac{1}{\bar{y}}\frac{h\left(\mathbf{y}, y_H\right)}{n}\left(\frac{1}{n}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_H^{-\varepsilon}\left\{l\left(\mathbf{y}, y_L\right) y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y}, y_L)+1}^{n-h(\mathbf{y}, y_H)} y_i^{1-\varepsilon} + h\left(\mathbf{y}, y_H\right) y_H^{1-\varepsilon}\right\}^{\frac{\varepsilon}{1-\varepsilon}}.\end{aligned}$$

[15] The set of pairs (y_L, y_H) where $l\left(\mathbf{y}, y_L\right)$ and $h\left(\mathbf{y}, y_H\right)$ are not constant is of measure zero and will not affect the integral to be computed later.

Using the fact that

$$\frac{dy_H}{dy_L} = -\frac{l(\mathbf{y}, y_L)}{h(\mathbf{y}, y_H)},$$

we obtain

$$\begin{aligned}
\frac{d\iota_{Atk\varepsilon}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} &= \frac{\partial \iota_{Atk\varepsilon}}{\partial y_L} + \frac{\partial \iota_{Atk\varepsilon}}{\partial y_H}\frac{dy_H}{dy_L} \\
&= -\frac{1}{\bar{y}}\frac{l(\mathbf{y}, y_L)}{n}\left(\frac{1}{n}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left\{l(\mathbf{y}, y_L)\, y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y},y_L)+1}^{n-h(\mathbf{y},y_H)} y_i^{1-\varepsilon} + h(\mathbf{y}, y_H)\, y_H^{1-\varepsilon}\right\}^{\frac{\varepsilon}{1-\varepsilon}}\left[y_L^{-\varepsilon} - y_H^{-\varepsilon}\right]
\end{aligned}$$

Following Expression (41), we obtain:

$$\begin{aligned}
\alpha_{Atk\varepsilon}(z_L, z_H, \varrho_L, \varrho_H, Y, n, \tilde{\mathbf{y}}(\mathbf{y}, z_L)) &= -\frac{\bar{y}}{\varrho_L(\mathbf{y}, y_L)}\frac{d\iota_{Atk\varepsilon}(\tilde{\mathbf{y}}(\mathbf{y}, y_L, y_H))}{dy_L} \\
&= \left(\frac{1}{n}\right)^{\frac{\varepsilon}{1-\varepsilon}}\left[y_L^{-\varepsilon} - y_H^{-\varepsilon}\right]\left\{l(\mathbf{y}, y_L)\, y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y},y_L)+1}^{n-h(\mathbf{y},y_H)} y_i^{1-\varepsilon} + h(\mathbf{y}, y_H)\, y_H^{1-\varepsilon}\right\}^{\frac{\varepsilon}{1-\varepsilon}} \\
&= \bar{y}^{\varepsilon}\left[y_L^{-\varepsilon} - y_H^{-\varepsilon}\right]\left(\frac{1}{\bar{y}}\right)^{\varepsilon}\left\{\frac{1}{n}\left[l(\mathbf{y}, y_L)\, y_L^{1-\varepsilon} + \sum_{i=l(\mathbf{y},y_L)+1}^{n-h(\mathbf{y},y_H)} y_i^{1-\varepsilon} + h(\mathbf{y}, y_H)\, y_H^{1-\varepsilon}\right]\right\}^{\frac{\varepsilon}{1-\varepsilon}} \\
&= \left[z_L^{-\varepsilon} - z_H^{-\varepsilon}\right](1 - \iota_{Atk\varepsilon}(\tilde{\mathbf{y}}(\mathbf{y}, z_L)))^{\varepsilon}.
\end{aligned}$$