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# Harsanyi support levels solutions

Manfred Besner\*

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## Abstract

We introduce a new class of values with transferable utility for level structures. In these hierarchical structures, each level corresponds to a partition of the player set, which becomes increasingly coarse from the trivial partition containing only singletons to the partition containing only the grand coalition. The new values, called Harsanyi support levels solutions, extend the Harsanyi solutions to level structures. As an important subset of these values, we present the class of weighted Shapley support levels values as a further result. The values from this class extend the weighted Shapley values to level structures and contain the Shapley levels value as a special case. Axiomatizations of the studied classes are provided.

**Keywords** Cooperative game · Level structure · (Weighted) Shapley (levels) value · Harsanyi set · Dividends

## 1 Introduction

Many institutions, companies, governments, and so on are organized in hierarchical structures. Typically, there is one unit at the top. In the following levels, each unit of the parent level is divided into two or more subordinate units, which usually have a lower rank than the higher ones. We can see a similar organizational structure in supply chains in some respects. Effectiveness increases by sharing or pooling physical objects, resources, and information. Queuing problems or electricity and other networks have a related background. A central characteristic of all these forms of organization is that a cooperating unit can itself be an actor to gain advantages of cooperation for the members of the unit. The question arises: how should we share the benefits and allocate the costs?

To distribute the profits of cooperating coalitions, the application of a cooperative game seems to be a natural approach. [Winter \(1989\)](#) defined a model for cooperative games with a hierarchical structure, called level structure, which comprises a sequence of coalition structures (the levels). At each level, the player set is partitioned into components where each higher level is coarser than the previous one (see also [Figure 1](#)). Winter's value ([Winter, 1989](#)) for such a model, we call it Shapley levels value, extends the Owen value

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(Owen, 1977), which is itself an extension of the Shapley value (Shapley, 1953b). Thus, this value satisfies adaptations of the symmetry axioms, satisfied by the Owen value, to level structures.

To treat symmetric players differently when there are exogenous weights for the players, Shapley (1953a) introduced the weighted Shapley values. Vidal-Puga (2012) established a value for coalition structures with weights given by the size of the coalitions. With a step by step top-down algorithm, Gómez-Rúa and Vidal-Puga (2011) extended it to level structures. Besner (2019) generalized this value to the class of the weighted Shapley hierarchy levels values for arbitrary, exogenously given, weights. These values satisfy an extension of the consistency property of the weighted Shapley values in Hart and Mas-Colell (1989). Interestingly, the weighted Shapley hierarchy levels values contain the Shapley levels value but do not satisfy the null player axiom in general.

The weighted values for coalition structures in Levy and McLean (1989) and McLean (1991) behave the other way round; they satisfy the null player property but do not correspond to a consistency property in the above sense. Levy and McLean (1989) examine several classes of weighted values for coalition structures which use the same weight system as the weighted Shapley values: either for the players within a component or the components themselves if the components act as players. The combined use of such a weight system, both for players and components, is only mentioned. This latter class of extensions of the weighted Shapley values and an extension of the class of random order values (Weber, 1988) to coalition structures is discussed in McLean (1991). Dragan (1992) called McLean's extensions of the weighted Shapley values McLean weighted coalition structure values. He presented a formula for them related to that of the Owen value. For a fixed coalition structure, these values coincide with a multiweighted Shapley value (Dragan, 1992).

Harsanyi (1959) introduced a new perspective on coalition functions. He used so-called (Harsanyi) dividends, assigned to all feasible coalitions of a player set according to the coalition function. Singletons receive the singleton worth as their dividend and the dividend of each larger coalition  $S$  amounts to the worth of  $S$  minus the sum of all dividends of the proper subcoalitions of  $S$ . The weighted Shapley values give the players as payoffs a share of the dividends from the coalitions in which they are members. Two players' shares of dividends from coalitions containing both players are always in the same ratio. In comparison, the Harsanyi solutions (Hammer, 1977; Vasil'ev, 1978), are more flexible. By these values, two players can receive dividend shares from coalitions, containing both, in various ratios for each coalition.

In this article, we introduce the Harsanyi support levels solutions. To the best of our knowledge, these are the first values for level structures (LS-values) that extend the Harsanyi solutions. We represent each value of the new class by a formula with dividends. The coefficients in the formulas constitute a dividend sharing system, i. e., all coefficients are non-negative and amount to one for each coalition. According to the definition of a Harsanyi solution, the LS-values from this class correspond to a Harsanyi solution for a fixed level structure. Therefore, they inherit all properties (adapted to level structures) of these values for a fixed level structure.

Within the framework of level structures, we can also consider the components of a level as players in so-called induced LS-games. Winter (1989) has shown that the Shapley levels value satisfies the level game property. This property means that the payoff to a component in an induced LS-game is equal to the sum of the payoffs to all players included in the component. We show that if a value for level structures satisfies the level game property and coincides for a fixed level structure with a Harsanyi solution, the value is a member

of the class of the Harsanyi support levels solutions. A corollary, our main result, extends an axiomatization of the class of the Harsanyi solutions by Vasil'ev (1981) and Derks et al. (2000) to a characterization of the class of the Harsanyi support levels solutions.

Since the Harsanyi solutions are no random order values in general, we cannot adopt the proof procedures for characterizations, e.g., in Winter (1989) or McLean (1991). We base our proofs on dividends where two new lemmas are a significant help.

As a further result, we present the class of the weighted Shapley support levels values as a proper subset of the class of the Harsanyi support levels solutions. The LS-values from this class coincide with the McLean weighted coalition structure values on a level structure with only three levels if we count the partition containing all singletons and the partition containing only the grand coalition as levels. To offer characterizations for this class, we adapt axiomatizations of the weighted Shapley values in Nowak and Radzik (1995).

In the concluding section, it is briefly explained that the values presented in this article can have an interesting solidarity characteristic: players who have formed a group support the other group members even in situations where they are not active and do not expect any direct reward for their support, as with the weighted Shapley Hierarchy levels values. In this way, they differ fundamentally from solidarity-based solution concepts such as the equal division value, the equal surplus division value (Driessen and Funaki, 1991), the egalitarian Shapley values (Joosten, 1996) or the solidarity value (Nowak and Radzik, 1994) and the Shapley-solidarity value (Calvo and Gutiérrez, 2013).

The outline of the paper is structured as follows. Section 2 contains preliminaries, as the main part, we introduce in Section 3 the Harsanyi support levels solutions with an appropriate class axiomatization. In Section 4, we offer the weighted Shapley support levels values as a further result. Examples in Section 5 illustrate the axioms of the characterizations and a numerical example compares different values. Section 6 discusses the results and concludes with a fundamental principle of group solidarity which is satisfied by our new values. The Appendix (Section 7) provides all the proofs, related properties and lemmas, and shows the logical independence of the axioms in our axiomatizations.

## 2 Preliminaries

### 2.1 TU-games

Let  $\mathbb{R}$  be the real numbers and  $\mathbb{R}_{++}$  the set of all positive real numbers. We fix a countably infinite set  $\mathfrak{U}$ , the universe of players, and denote by  $\mathcal{N}$  the set of all non-empty and finite subsets of  $\mathfrak{U}$ . A cooperative game with transferable utility (**TU-game**) is a pair  $(N, v)$  such that  $N \in \mathcal{N}$  is a set of players and  $v: 2^N \rightarrow \mathbb{R}$ ,  $v(\emptyset) = 0$ , a **coalition function** where  $2^N$  is the power set of  $N$ . The subsets  $S \subseteq N$  are called **coalitions**,  $v(S)$  is the **worth** of coalition  $S$ , the set of all nonempty subsets of  $S$  is denoted by  $\Omega^S$ , and the set of all TU-games with player set  $N$  is denoted by  $\mathbb{V}^N$ .

Let  $N \in \mathcal{N}$ ,  $(N, v) \in \mathbb{V}^N$  and  $S \subseteq N$ . The **dividends**  $\Delta_v(S)$  (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } S \in \Omega^N, \text{ and} \\ 0, & \text{if } S = \emptyset. \end{cases} \quad (1)$$

A TU-game  $(N, u_T) \in \mathbb{V}^N$ ,  $T \in \Omega^N$ , with  $u_T(S) := 1$  if  $T \subseteq S$  and  $u_T(S) := 0$  otherwise for all  $S \subseteq N$  is called an **unanimity game**. It is well-known that any coalition function  $v$  on

$N$  has a unique representation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \quad (2)$$

$(N, v)$  is called **totally positive** (Vasil'ev, 1975) if  $\Delta_v(T) \geq 0$  for all  $T \subseteq N$ . Player  $i \in N$  is called a **null player** in  $(N, v)$  if  $v(S \cup \{i\}) = v(S)$  for all  $S \subseteq N \setminus \{i\}$ ; players  $i, j \in N, i \neq j$ , are called (mutually) **dependent** (Nowak and Radzik, 1995) in  $(N, v)$  if  $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$  for all  $S \subseteq N \setminus \{i, j\}$ , which is equivalent to

$$\Delta_v(S \cup \{k\}) = 0, k \in \{i, j\}, \text{ for all } S \subseteq N \setminus \{i, j\}. \quad (3)$$

For all  $N \in \mathcal{N}$ , a **TU-value** or **solution**  $\phi$  is an operator that assigns to any  $(N, v) \in \mathbb{V}^N$  a payoff vector  $\phi(N, v) \in \mathbb{R}^N$ .

By  $W := \{f: \mathfrak{U} \rightarrow \mathbb{R}_{++}\}$ ,  $w_i := w(i)$  for all  $w \in W, i \in \mathfrak{U}$ , we define the collection of all positive **weight systems** on  $\mathfrak{U}$ .

The following TU-values distribute the dividends of the coalitions proportionally to the weights of the players included in this coalitions. For all  $N \in \mathcal{N}$ ,  $(N, v) \in \mathbb{V}^N$ , and each  $w \in W$ , the (positively) **weighted Shapley value**  $Sh^w$  (Shapley, 1953a) is defined by

$$Sh_i^w(N, v) := \sum_{S \subseteq N, S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S) \text{ for all } i \in N. \quad (4)$$

As a special case of a weighted Shapley value, all weights are equal, the **Shapley value**  $Sh$  (Shapley, 1953b) is defined by

$$Sh_i(N, v) := \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N.$$

For all coalitions that include the same two players, for the weighted Shapley values, the dividend shares for these players are always in the same ratio. This need not be the case for the following TU-values, using sharing systems, where the ratio of the dividend shares of two players may also be different for two different coalitions containing both players. The collection  $\Lambda$  of all **sharing systems**  $\lambda \in \Lambda$  on  $\mathcal{N}$  is defined by

$$\Lambda := \left\{ \lambda = (\lambda_{S,i})_{S \in \mathcal{N}, i \in S} \mid \sum_{i \in S} \lambda_{S,i} = 1 \text{ and } \lambda_{S,i} \geq 0 \text{ for each } S \in \mathcal{N} \text{ and all } i \in S \right\}.$$

Hammer et al. (1977) and Vasil'ev (1978), independently of each other, defined a class of TU-values. While Hammer et al. (1977) and, e.g., Derks et al. (2000) this class call **selectope**, Vasil'ev (1978), Vasil'ev and van der Laan (2002), and, e.g., del Pozo et al. (2011) refer to this class as **Harsanyi set**, abbreviated to  $\mathcal{H}$ . For all  $N \in \mathcal{N}$ ,  $(N, v) \in \mathbb{V}^N$ , and each sharing system  $\lambda \in \Lambda$ , we have a **Harsanyi solution**  $H^\lambda \in \mathcal{H}$ , defined by

$$H_i^\lambda(N, v) := \sum_{T \subseteq N, T \ni i} \lambda_{T,i} \Delta_v(T) \text{ for all } i \in N.$$

By (4), it is immediate that the class of the weighted Shapley values is a proper subset of the Harsanyi set.

## 2.2 LS-games

In the following, we also look at games where coalitions of players of a player set  $N \in \mathcal{N}$  are regarded as players. Also in TU-games players can be coalitions or representatives of them respectively (see, e.g., Kalai and Samet (1987, Section 7)). Note, therefore, that for any

player set  $N \in \mathcal{N}$ , our universe  $\mathfrak{U}$  also contains all coalitions as players that the elements from the player set  $N$  can form.

We denote the subset of all individuals of  $\mathfrak{U}$  by  $\mathfrak{U}^I$ . An individual  $i \in \mathfrak{U}^I$  is called a **sub-member** of a player  $j \in \mathfrak{U}$  if  $i = j$  or  $j$  is a set which contains as elements “individuals”, or “sets of individuals”, or “individuals and sets of individuals”, or “sets of sets of individuals”, and so on, and  $i$  is an element of one of these possibly nested sets. A player set  $N \in \mathcal{N}$  is called **sub-member-disjoint** if no individual  $i \in \mathfrak{U}^I$  is a sub-member of any two players  $j, k \in N$ . To exclude from the outset inconsistencies in the worths or in the weighting of coalitions and associated coalitions with the same sub-members, we will only consider sub-member-disjoint player sets.<sup>1</sup> The subset of all sub-member-disjoint player sets  $N \in \mathcal{N}$  is denoted by  $\mathcal{N}^{dis}$ .

For any  $N \in \mathcal{N}^{dis}$ , a **coalition structure**  $\mathcal{B}$  on  $N$  is a partition of the player set  $N$ , i.e., a collection of nonempty, pairwise disjoint, and mutually exhaustive subsets of  $N$ . Each  $B \in \mathcal{B}$  is called a **component** and  $\mathcal{B}(i)$  denotes the component that contains a player  $i \in N$ . A **level structure** (Winter, 1989) on  $N$  is a finite sequence  $\underline{\mathcal{B}} := \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$  of coalition structures  $\mathcal{B}^r$ ,  $0 \leq r \leq h+1$ , on  $N$  such that (see also Figure 1):

- $\mathcal{B}^0 = \{\{i\}: i \in N\}$ ,
- $\mathcal{B}^{h+1} = \{N\}$ , and
- for each  $r$ ,  $0 \leq r \leq h$ ,  $\mathcal{B}^r$  is a refinement of  $\mathcal{B}^{r+1}$ , i. e.,  $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$  for all  $i \in N$ .

$\mathcal{B}^r$  is called the  $r$ th **level** of  $\underline{\mathcal{B}}$ ,  $\overline{\mathcal{B}}$  is the set of all components  $B \in \mathcal{B}^r$  of all levels  $\mathcal{B}^r \in \underline{\mathcal{B}}$ ,  $0 \leq r \leq h$ , and  $\mathbb{L}^N$  denotes the set of all level structures on the player set  $N$ .

For each  $B \in \mathcal{B}^\ell$ ,  $0 \leq \ell \leq r \leq h+1$ , we denote by  $\mathcal{B}^r(B)$  the component of the  $r$ th level containing  $B$  as a (not necessary proper) subset;  $\mathcal{B}^r(B)$  is called an **ancestor** of  $B$ ; if  $r = \ell + 1$ ,  $\mathcal{B}^r(B)$  is also called **parent** of  $B$ . We call all components with the same parent  $B \in \mathcal{B}^r$ ,  $1 \leq r \leq h+1$ , **children** of  $B$  and two different children of  $B$  are called **siblings** in  $\mathcal{B}^{r-1}$ . We explicitly point out that a component, for different levels, can be its own child or parent at the same time.

If we regard the components of the  $r$ th level,  $0 \leq r \leq h$ , as players and cancel all levels below the  $r$ th level, we get the **induced  $r$ th level structure**  $\underline{\mathcal{B}}^r := \{\mathcal{B}^r, \dots, \mathcal{B}^{h+1-r}\} \in \mathbb{L}^{\mathcal{B}^r}$  from  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ . We give an illustrating example.

**Example 2.1.** Let  $N = \{1, 2, 3\}$  and  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$  be given by  $\mathcal{B}^0 = \{\{1\}, \{2\}, \{3\}\}$ ,  $\mathcal{B}^1 = \{\{1, 2\}, \{3\}\}$ , and  $\mathcal{B}^2 = \{N\}$ . We regard, e.g., the components of the first level as players. Then, the induced first level structure  $\underline{\mathcal{B}}^1 = \{\mathcal{B}^1, \mathcal{B}^2\}$  from  $\underline{\mathcal{B}}$  is given by  $\mathcal{B}^{1^0} = \{\{\{1, 2\}\}, \{\{3\}\}\}$ , and  $\mathcal{B}^{1^1} = \{\{\{1, 2\}, \{3\}\}\}$ .

For each  $N \in \mathcal{N}^{dis}$ , a TU-game  $(N, v) \in \mathbb{V}^N$  together with a level structure  $\underline{\mathcal{B}} \in \mathbb{L}^N$  is an **LS-game** which is denoted by  $(N, v, \underline{\mathcal{B}})$ . We denote the set of all LS-games on  $N$  by  $\mathbb{VL}^N$ .

If we want to stress the property that a coalition  $T \in \Omega^N$ ,  $T = \bigcup_{B \subseteq T, B \in \mathcal{B}^r} B$ , is the union of components of the  $r$ th level from  $\underline{\mathcal{B}}$ , we denote  $T$  by  $T^r$ . Each such  $T^r$  is an **associated coalition** to a coalition of all players  $B \in \mathcal{B}^r$ ,  $B \subseteq T^r$ , in the induced  $r$ th level structure, denoted by  $\mathcal{T}^r := \{B \in \mathcal{B}^r : B \subseteq T^r\}$  and vice versa. This means that a coalition and its

<sup>1</sup>E.g., let  $N := \{i, j, \{i, j\}\}$  be a player set with three players where the third player is a coalition. In a TU-game  $(N, v) \in \mathbb{V}^N$ , it does not matter if the worth  $v(\{i, j\})$  of the coalition  $\{i, j\}$  differ from the worth  $v(\{\{i, j\}\})$  of the singleton  $\{\{i, j\}\}$ . For a game on level structures, we have an inconsistency in the induced first LS-game (see the definition below) if the coalition  $\{i, j\}$  is there a player and we require that the induced first LS-game is given by (5).

associated coalition have the same sub-members. Thus, for each  $N \in \mathcal{N}^{dis}$ , each LS-game  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ , and each induced  $r$ th level structure  $\underline{\mathcal{B}}^r$ ,  $0 \leq r \leq h$ , we have an associated LS-game  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r) \in \mathbb{V}\mathbb{L}^{\mathcal{B}^r}$ , called **induced  $r$ th LS-game**<sup>2</sup>, with a coalition function  $v^r$  which is given by

$$v^r(\mathcal{T}^r) := v(T^r) \text{ for all } \mathcal{T}^r \in \Omega^{\mathcal{B}^r}. \quad (5)$$

This means that a coalition in an induced LS-game that has the same sub-members as a coalition from the original game also has the same worth as this associated coalition. E.g., for the TU-game  $(N, v) \in \mathbb{V}\mathbb{L}^N$  and  $\underline{\mathcal{B}} \in \mathbb{L}^N$  in example 2.1, we have  $v^1(\{\{1, 2\}\}) = v(\{1, 2\})$ .

Also induced LS-games for induced LS-games and so on are possible. We will not go into this in detail. Note that, for all  $N \in \mathcal{N}^{dis}$ , each TU-game  $(N, v) \in \mathbb{V}^N$  corresponds to an LS-game  $(N, v, \underline{\mathcal{B}}_0) \in \mathbb{V}\mathbb{L}^N$  with a **trivial level structure**  $\underline{\mathcal{B}}_0 := \{\mathcal{B}^0, \mathcal{B}^1\}$  and we would like to point out that each LS-game  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  corresponds to a game with coalition structure (Aumann and Drèze, 1974; Owen, 1977) if  $\underline{\mathcal{B}} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$ .

For all  $N \in \mathcal{N}^{dis}$  and all  $\underline{\mathcal{B}} \in \mathbb{L}^N$ , an **LS-value**  $\varphi$  is an operator that assigns to any  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  a payoff vector  $\varphi(N, v, \underline{\mathcal{B}}) \in \mathbb{R}^N$ .

By  $W^{dis} \subseteq W$ , we define the collection of all positive weight systems on  $\mathfrak{U}$  where, for each  $w \in W^{dis}$ , we have  $w_i = w_j$  for all  $i, j \in \mathfrak{U}$  if  $i$  has the same sub-members as  $j$ . This means that, e.g., for an LS-game  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $N \in \mathcal{N}^{dis}$ , we have that the weight  $w_i$  of a player  $i \in N$  is equal to the weight  $w_{\{i\}}$  of the singleton  $\{i\} \in \mathcal{B}^0$  in the zeroth LS-game.

We also want a corresponding relationship for sharing systems. The collection  $\Lambda^{dis} \subseteq \Lambda$  of all sharing systems  $\lambda \in \Lambda^{dis}$  on  $\mathcal{N}^{dis}$  is defined by

$$\Lambda^{dis} := \left\{ \lambda = (\lambda_{S,i})_{S \in \mathcal{N}^{dis}, i \in S} \mid \sum_{i \in S} \lambda_{S,i} = 1 \text{ and } \lambda_{S,i} \geq 0 \text{ for each } S \in \mathcal{N}^{dis} \text{ and all } i \in S \right. \\ \left. \text{such that for all } S' \in \mathcal{N}^{dis} \text{ with } |S| = |S'| \text{ and for each } i \in S \text{ we have a } j \in S' \right. \\ \left. \text{which has the same sub-members as } i, \text{ we have } \lambda_{S,i} = \lambda_{S',j} \right\}.$$

This means that if we have two coalitions with the same number of players, and for every player in the first coalition we have a player in the second coalition with the same sub-members, then the related players should also have the same weight for their respective coalitions.

### 2.3 Axioms

We refer to the following axioms for LS-values  $\varphi$  which are mostly simple adaptations of standard axioms for TU-values. All axioms below apply for all  $N \in \mathcal{N}^{dis}$ . We do not repeat this phrase at the beginning of each axiom.

**Efficiency, E.** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ , we have  $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$ .

Efficiency requires that the sum of the payoffs should result in the total worth of the game. The next property means that a player who contributes nothing to any coalition should obtain no payoff.

**Null player, N.** For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  and  $i \in N$  such that  $i$  is a null player in  $(N, v)$ , we have  $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$ .

<sup>2</sup>In Owen (1977), a corresponding game, for the special case  $r = 1$  and a level structure  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$ , is called a quotient game.

**Additivity, A.** For all  $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ , we have  $\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}})$ .

Additivity states that a player's payoff from the sum of two games is the sum of the player's payoff for the two games. By the following axiom, all players receive a non-negative payoff if all coalitions have non-negative dividends.

**Positivity, Pos** (Vasil'ev, 1975). For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  such that  $(N, v)$  is totally positive, we have  $\varphi_i(N, v, \underline{\mathcal{B}}) \geq 0$  for all  $i \in N$ .

**Strict monotonicity, SMon** (Megiddo, 1974). For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  and  $\alpha \in \mathbb{R}_{++}$ , we have  $\varphi_i(N, v + \alpha \cdot u_N, \underline{\mathcal{B}}) > \varphi_i(N, v, \underline{\mathcal{B}})$  for all  $i \in N$ .

This property means that if the grand coalition improves its performance and the performance of all other coalitions remains the same, all players should receive a higher payoff. In our investigations, the next axiom plays an important role. It claims that the sum of all players' payoffs of a component coincides with this component's payoff in an induced LS-game where the component is regarded as a player.

**Level game property, LG** (Winter, 1989). For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $B \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r). \quad (6)$$

The following axiom says that if two siblings are symmetric in the induced  $r$ th LS-game, the total payoff to all players of the first sibling is equal to the total payoff to all players of the second sibling.

**Symmetry between components, S<sup>3</sup>** (Winter, 1989). For all  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ , two siblings  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_k, B_\ell$  are symmetric in  $(\mathcal{B}^r, v^r) \in \mathbb{V}\mathcal{B}^r$ , we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

## 2.4 The Shapley levels value

The following formula is presented in Calvo et al. (1996, Eq. (1)).

For all  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ , and  $T \in \Omega^N$ ,  $T \ni i$ , define

$$K_{\underline{\mathcal{B}}, T}(i) := \prod_{r=0}^h K_{\underline{\mathcal{B}}, T}^r(i), \quad \text{where } K_{\underline{\mathcal{B}}, T}^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i), B \cap T \neq \emptyset\}|} \quad (7)$$

be the reciprocal cardinality of the set that contains all children of the component of the  $(r+1)$ th level containing player  $i$ , which contain at least one player from coalition  $T$ . Then, the **Shapley levels value**<sup>4</sup> (Winter, 1989)  $Sh^L$  is given by

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_{\underline{\mathcal{B}}, T}(i) \Delta_v(T) \quad \text{for all } i \in N. \quad (8)$$

<sup>3</sup>This axiom is called coalitional symmetry in Winter (1989).

<sup>4</sup>The value is also known as level(s) structure value or Winter's (Shapley type) value. Our designation is used, e. g., in Álvarez-Mozos et al. (2017).

It is easy to see that  $Sh^L$  coincides with  $Sh$  if  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$ . Winter (1989) used the Owen value (Owen, 1977) as a starting point for his LS-value. Therefore, Winter has extended the efficiency, null player, symmetry and additivity axioms to axioms for level structures. He splits symmetry into symmetry between components and an individual symmetry axiom. If we define a level structure as above, i. e., the singletons are the elements of the lowest level, Winter (1989, remark 1.6) pointed out that we can omit the individual symmetry axiom. In this sense, we present Winter's first axiomatization of the Shapley levels value<sup>5</sup>.

**Theorem 2.2.** (Winter, 1989)  $Sh^L$  is the unique LS-value that satisfies **E**, **N**, **S**, and **A**.

It should be noted that there exist some further axiomatizations of the Shapley levels value by Calvo et al. (1996), Khmel'nitskaya and Yanovskaya (2007), Casajus (2010), Alvarez-Mozos and Tejada (2011), and Besner (2019).

### 3 Harsanyi support levels solutions

With the Shapley levels value, symmetric siblings always get the same payoff. But sometimes there are compelling reasons, not included in the coalition function itself, to treat symmetric siblings differently. We can have fixed, but for each coalition possibly different, metrics for each player, representing, e. g., the effort, commitment, qualification, or power of a player. Sometimes a player's influence on other partners is stronger when another player is not on the team. Or a player can only exert its influence if specific other players have joined the coalition to support this player. The following values give the possibility to deal with such situations.

Before we give a formal definition, we describe the payoff algorithm: Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ , and  $\lambda \in \Lambda^{dis}$ . By (2),  $v$  can be expressed as a linear combination of unanimity games. This means that we have a game  $v_T$  for each coalition  $T \in \Omega^N$  such that  $\Delta_{v_T}(T) = \Delta_v(T)$ ,  $\Delta_{v_T}(S) = 0$  for all  $S \subseteq N$ ,  $S \neq T$ , and  $v = \sum_{T \in \Omega^N} v_T$ .

In the game  $v_T$ , each component  $B^h$  of the  $h$ th level that contains at least one player  $i \in T$  receives a share of  $\Delta_v(T)$ , proportional to its sharing weight  $\lambda_{\mathcal{C}_T^h, B^h}$ , where  $\mathcal{C}_T^h$  is the coalition that contains all such components  $B^h$  as elements. Then the share of each component  $B^h$  involved is distributed among all its children  $B^{h-1} \subseteq B^h$  of the  $(h-1)$ th level, containing at least one player  $i \in T$ , proportional to their sharing weights  $\lambda_{\mathcal{C}_T^{h-1}, B^{h-1}}$ , where  $\mathcal{C}_T^{h-1}$  is the coalition that contains all such siblings  $B_{h-1}$  as elements, and so on for all levels. In the end, each player  $i \in T$  gets its share, "supported" by the sharing weights of all its ancestors. To receive the total payoff of a player, these payoffs are finally added up over all games  $v_T$  where the player is not a null player.

**Definition 3.1.** For all  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $\lambda \in \Lambda^{dis}$ , and  $T \in \Omega^N$ ,  $T \ni i$ ,  $0 \leq r \leq h$ , let

$$\mathcal{C}_T^r(i) := \{B : B \in \mathcal{B}^r, B \subseteq \mathcal{B}^{r+1}(i), B \cap T \neq \emptyset\} \quad (9)$$

be the set that contains all children of the component of the  $(r+1)$ th level containing player  $i$ , which contain at least one player from coalition  $T$ . Define

<sup>5</sup>Winter (1989) introduced his value axiomatically and used this axiomatization as a definition.

$$K_{\underline{\mathcal{B}}, T}^\lambda(i) := \prod_{r=0}^h \lambda_{\mathcal{C}_T^r(i), \mathcal{B}^r(i)}. \quad (10)$$

Then, for each  $\lambda \in \Lambda^{dis}$ , the **Harsanyi support levels solution**  $H^{\lambda SL}$  is given by

$$H_i^{\lambda SL}(N, v, \underline{\mathcal{B}}) = \sum_{T \subseteq N, T \ni i} K_{\underline{\mathcal{B}}, T}^\lambda(i) \Delta_v(T) \text{ for all } i \in N. \quad (11)$$

The class of all Harsanyi support levels solutions is called **Harsanyi support levels set** and is denoted by  $\mathcal{H}^{SL}$ .

**Remark 3.2.** We see that the Shapley levels value is a Harsanyi support levels solution with a  $\lambda \in \Lambda^{dis}$  such that we have for each  $\lambda_{\mathcal{C}_T^r(i), \mathcal{B}^r(i)}$  in (10) a  $K_{\underline{\mathcal{B}}, T}^r(i)$  in (7) with  $\lambda_{\mathcal{C}_T^r(i), \mathcal{B}^r(i)} = K_{\underline{\mathcal{B}}, T}^r(i)$ .

**Remark 3.3.** For each  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\lambda \in \Lambda^{dis}$ , and for all  $T \in \Omega^N$ ,  $T \ni i$ , we have  $\sum_{i \in T} K_{\underline{\mathcal{B}}, T}^\lambda(i) = 1$  and  $K_{\underline{\mathcal{B}}, T}^\lambda(i) \geq 0$ . Therefore, for fixed  $N$  and  $\underline{\mathcal{B}} \in \mathbb{L}^N$ , each  $H^{\lambda SL} \in \mathcal{H}^{SL}$  on  $(N, v, \underline{\mathcal{B}})$  coincides with a  $H^{\lambda'} \in \mathcal{H}$ ,  $\lambda' \in \Lambda$ , on  $(N, v) \in \mathbb{V}^N$ , where  $\lambda'_{T, i} = K_{\underline{\mathcal{B}}, T}^\lambda(i)$  for each  $T \in \Omega^N$  and all  $i \in T$  and the level structure is disregarded.

To prepare our main result, we first assume that a sharing system  $\lambda \in \Lambda^{dis}$  is exogenously given. We use a more or less technical property for unanimity games with some factor.

**$\lambda$ -balanced sharing between components in unanimity games** (with some factor), **SCU $^\lambda$** . For each  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\lambda \in \Lambda^{dis}$ , and for all  $\alpha \in \mathbb{R}$  and  $T \in \Omega^N$ , two siblings  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ ,  $B_k, B_\ell \cap T \neq \emptyset$ , and  $\mathcal{C}_T^r(B_k) := \{B : B \in \mathcal{B}^r, B \subseteq \mathcal{B}^{r+1}(B_k), B \cap T \neq \emptyset\}$ , we have

$$\lambda_{\mathcal{C}_T^r(B_k), B_\ell} \sum_{i \in B_k} \varphi_i(N, \alpha \cdot u_T, \underline{\mathcal{B}}) = \lambda_{\mathcal{C}_T^r(B_k), B_k} \sum_{i \in B_\ell} \varphi_i(N, \alpha \cdot u_T, \underline{\mathcal{B}}). \quad (12)$$

This property says that for two siblings which contain both at least one player from  $T$  the ratio of the payoff totals to all players of the two siblings in the unanimity game is equal to the ratio of the sharing weights of both siblings. The sharing weights refer to the partition that has as elements all siblings containing players from  $T$ .

The following proposition lists some axioms that are satisfied by a Harsanyi support levels solution.

**Proposition 3.4.** Let  $\lambda \in \Lambda^{dis}$ .  $H^{\lambda SL}$  satisfies **E**, **N**, **A**, **Pos**, **SCU $^\lambda$** , and **LG**.

We present a first axiomatization that replaces **S** in Theorem 2.2 by **SCU $^\lambda$** .

**Proposition 3.5.** Let  $\lambda \in \Lambda^{dis}$ .  $H^{\lambda SL}$  is the unique LS-value that satisfies **E**, **N**, **SCU $^\lambda$** , and **A**.

It follows that the Harsanyi support levels solutions have an exceptional status among extensions of Harsanyi solutions to LS-values.

**Theorem 3.6.** An LS-value  $\varphi$  coincides for fixed  $N$  and  $\underline{\mathcal{B}} \in \mathbb{L}^N$  with a Harsanyi solution and satisfies **LG** if and only if  $\varphi \in \mathcal{H}^{SL}$ .

By Theorem 3.6 and the probably most famous characterization of the Harsanyi set in Vasil'ev (1981) and Derks et al. (2000), the following corollary is immediate.

**Corollary 3.7.** An LS-value  $\varphi$  satisfies **E**, **N**, **Pos**, **A**, and **LG** if and only if  $\varphi \in \mathcal{H}^{SL}$ .

## 4 Further results

For applications, the class of weighted Shapley values is an important subset of the Harsanyi set. Therefore, we would like to highlight the following subset of the Harsanyi support levels set that extends the class of weighted Shapley values for level structures. Each possible coalition is assigned a positive weight. In contrary to the weighted Shapley hierarchy levels values, these values satisfy the null player property in general.

Again, we briefly describe how the dividends are distributed by the following LS-values. Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_h$ , and  $w \in W^{dis}$ . The dividends of each coalition  $T \in \Omega^N$  are divided as follows: Each component  $B^h$  of the  $h$ th level that contains at least one player  $i \in T$  receives a share of  $\Delta_v(T)$ , proportional to its weight  $w_{B^h}$ . Then the share of each component  $B^h$  involved is distributed among all its children  $B^{h-1} \subseteq B^h$  containing at least one player  $i \in T$ , proportional to their weights  $w_{B^{h-1}}$ , and so on for all levels. In the end, each player  $i \in T$  gets its share, “supported” by the weights of all its ancestors.

**Definition 4.1.** For all  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $w \in W^{dis}$ , and for all  $T \in \Omega^N$ ,  $T \ni i$ , define

$$K_{\underline{\mathcal{B}}, T}^w(i) := \prod_{r=0}^h \frac{w_{\mathcal{B}^r(i)}}{\sum_{\substack{B \in \mathcal{B}^r: B \subseteq \mathcal{B}^{r+1}(i), \\ B \cap T \neq \emptyset}} w_B}. \quad (13)$$

Then, for each  $w \in W^{dis}$ , the **weighted Shapley support levels value**  $Sh^{wSL}$  is given by

$$Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) = \sum_{T \subseteq N, T \ni i} K_{\underline{\mathcal{B}}, T}^w(i) \Delta_v(T) \text{ for all } i \in N. \quad (14)$$

The class of all weighted Shapley support levels values is denoted by  $\mathcal{WS}^{SL}$ .

**Remark 4.2.** If the weights for all coalitions are equal, (13) coincides with (7). Therefore, the Shapley levels value is also a member of the class of the weighted Shapley support levels values.

**Remark 4.3.** For all  $N \in \mathcal{N}^{dis}$  and level structures  $\underline{\mathcal{B}} \in \mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ , a weighted Shapley levels value  $Sh^{wSL}$  coincides with a Harsanyi support levels solution  $H^{\lambda SL}$  where we have

$$\lambda_{C_T^r(i), \mathcal{B}^r(i)} = \frac{w_{\mathcal{B}^r(i)}}{\sum_{\substack{B \in \mathcal{B}^r: B \subseteq \mathcal{B}^{r+1}(i), \\ B \cap T \neq \emptyset}} w_B} \text{ for all } T \in \Omega^N.$$

$Sh^{wSL}$  coincides with  $Sh^w$  if  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$  and, if  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$ , the  $K_{\underline{\mathcal{B}}, T}^w(i)$  coincide with the “ $\lambda_i^S$ ” given in Dragan (1992, Sec. 2(e)). Therefore, in this case, the  $Sh^{wSL}$  coincide with the McLean weighted coalition structure values (Levy and McLean, 1989; McLean, 1991; Dragan, 1992).

Dependent players are always symmetric but not vice versa. Therefore, our next axiom can be seen as a weighted weakening of **S**. The TU-version of this axiom comes from Nowak and Radzik (1995, Axiom A4 (w-Mutual Dependence)).

**w-weighted dependence between components, Dep<sup>w</sup>.** For all  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $w \in W^{dis}$ , two siblings  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_k, B_\ell$

are dependent in  $(\mathcal{B}^r, v^r) \in \mathbb{V}^{\mathcal{B}^r}$ , we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

The meaning of this axiom is that if two siblings are dependent in the  $r$ th LS-game the ratio of the sum of all players' payoffs from the two siblings is equal to the ratio of the weights of both siblings. It follows an extension of the mutual dependence axiom in [Nowak and Radzik \(1995, Axiom 7\)](#).

**Mutual dependence between components, MDep.** For all  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}})$ ,  $(N, v', \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ , two siblings  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_k, B_\ell$  are dependent in  $(\mathcal{B}^r, v^r) \in \mathbb{V}^{\mathcal{B}^r}$  and  $(\mathcal{B}^r, v'^r) \in \mathbb{V}^{\mathcal{B}^r}$ , we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) \sum_{i \in B_\ell} \varphi_i(N, v', \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}) \sum_{i \in B_k} \varphi_i(N, v', \underline{\mathcal{B}}).$$

This axiom considers games where two siblings are dependent in two induced  $r$ th LS-games. Then this axiom states that the ratio of the payoff totals to all players of the siblings remains invariant for such games if none of the sums is zero. The following proposition lists some axioms that are satisfied by a weighted Shapley support levels value.

**Proposition 4.4.** *Let  $w \in W^{dis}$ .  $Sh^{wSL}$  satisfies **E**, **N**, **A**, **LG**, **SMon**, **Dep<sup>w</sup>**, and **MDep**.*

The following proposition can be seen as an extension of characterizations of the weighted Shapley values with exogenously given weights in [Nowak and Radzik \(1995\)](#).

**Proposition 4.5.** *Let  $w \in W^{dis}$ .  $Sh^{wSL}$  is the unique LS-value that satisfies **E**, **N**, **Dep<sup>w</sup>**, and **A**.*

Our last theorem axiomatizes the class of weighted Shapley support levels values in general. It is closely related to an axiomatization of the class of weighted Shapley values in [Nowak and Radzik \(1995, Theorem 2.4, Remark 2.3\)](#).

**Theorem 4.6.** *An LS-value  $\varphi$  satisfies **E**, **N**, **SMon**, **A**, and **MDep** if and only if  $\varphi \in \mathcal{WS}^{SL}$ .*

## 5 Examples

Laura is the proprietor of an independent owner-managed and medium-sized software company. Her company is active as an IT specialist at two locations in Europe and one in East Asia. She places her trust in innovative software engineering, and almost all orders are carried out as agile projects. That is why each location has several agile teams, each made up of all-rounders and highly specialized software developers, supported by local lean management. Since projects have different scope and specialization requirements, the individual teams do not only work together within their locations. Especially for larger projects, the teams from Europe and East Asia complement each other by the different time zones and create more releases in less time. Sometimes certain specialists from one team work together with other (parts of) teams, and in small projects also only a few employees can work together.

Laura’s business has been going well for years. After deduction of all costs such as office expenses, buildings, equipment acquisition and maintenance, software licenses, employees’ fixed salaries, marketing, taxes, management bonuses, and own profit share, a positive share of total turnover, let’s say five percent, remains as a year’s profit. To motivate and to build loyalty among her highly qualified IT staff, Laura wants to pay this amount to her developers.

Laura is an enthusiastic cooperative game theorist. She defines a coalition function in which each possible coalition of developers has a worth of five percent of the sum of all revenues from all projects in which only members of that coalition were involved as developers. To give all developers a “fair” share, Laura wants to include certain conditions (axioms) in the payoff calculation. The worth of the grand coalition (that is exactly the amount she wants to pay out), nothing more and nothing less, should be fully shared (**E**). If a developer has been in training or has been ill all year and so has not taken part in a single project, he/she should not receive any bonus (**N**). Under no circumstances should any employee receive deductions from his or her salary. Hence, **Pos** must be satisfied, since the game constructed here is totally positive. Laura wants to make sure that it doesn’t matter to the developers whether it’s a monthly or annual billing as long as the staff doesn’t change. Furthermore, the total amount paid to an individual developer should not depend on whether a project is calculated for this year or next (**A**).

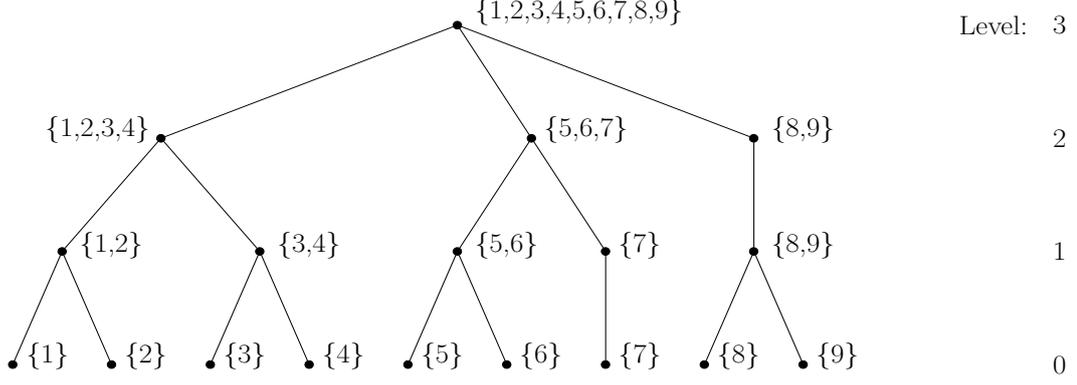
The development teams and the locations each have a high degree of self-responsibility, for example, in the distribution of work or the recruitment of new employees. Laura wants to take this into account by considering these units (components) to be largely independent within this framework. Therefore, the payoff to the locations/teams in a modeled game where the locations/teams are the players (induced level game) should be equal to the sum of the payoffs to all players of the respective locations/teams (**LG**). Thus all axioms of Corollary 3.7 must be satisfied, and Laura should use a value of the Harsanyi support levels set for the payoff of the bonuses.

If Laura has key figures for the players/teams/locations and wants to check them or set them first,  $\mathbf{SCU}^\lambda$  can be useful. For example, she could conclude that due to possible higher costs of living, the weights for the European locations should be higher as for the East Asian locations. However, if all three locations are involved in the same project, the workload of the Asian location can increase. In this case, the weights for all three locations could converge.

In Laura’s view, if the revenues for projects in which all her employees are involved increase and the revenues from all other projects remain the same, then all employees should have more in their pockets than before (**SMon**). Let two teams at the same location or members of the same team always jointly work on projects (they are dependent in  $(N, v)$ ). Provided the total workforce does not change, Laura should choose a weighted Shapley support levels value if she believes that the total payoff to both teams or both team members should always be in the same ratio (**MDep**).

We give a numerical example to compare the distributions for different values in such situations. Let  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $N := \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ , and  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3\}$ , with  $\mathcal{B}^1 := \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}, \{8, 9\}\}$ ,  $\mathcal{B}^2 := \{\{1, 2, 3, 4\}, \{5, 6, 7\}, \{8, 9\}\}$  (see Figure 1). The coalition function  $v$  is given by the following dividends:

$$\begin{aligned} \Delta_v(\{3\}) &= 1, & \Delta_v(\{1, 2\}) &= 2, & \Delta_v(\{1, 2, 3\}) &= 6, \\ \Delta_v(\{5, 6, 7\}) &= 6, & \Delta_v(\{1, 2, 5, 7\}) &= 2, & \Delta_v(\{1, 2, 3, 8, 9\}) &= 5, \\ \Delta_v(\{5, 6, 7, 8, 9\}) &= 6, & \Delta_v(\{1, 2, 3, 5, 6, 7, 8, 9\}) &= 12, \end{aligned}$$



**Figure 1:** Structure of the components in different levels

and all other coalitions have zero dividends in  $v$ .

For reasons of simplicity, we choose a weight system  $w \in W^{dis}$  such that  $w_S := |S|$  for all  $S \in \Omega^N$ .

Furthermore, we denote by  $i^\#$  the number of all sub-members of a player  $i \in \mathfrak{U}$ . We use a level sharing system  $\lambda \in \Lambda^{dis}$ , given by  $\lambda_{\mathcal{B}^2, \{1,2,3,4\}} := 0.4$ ,  $\lambda_{\mathcal{B}^2, \{5,6,7\}} := 0.3$ ,  $\lambda_{\mathcal{B}^2, \{8,9\}} := 0.3$ , and, for all other  $S \in \mathcal{N}^{dis}$ ,  $S \neq S'$  such that  $|S'| = |\mathcal{B}^2|$  and for each  $i \in S'$  we have a  $j \in \mathcal{B}^2$  which has the same sub-members as  $i$ , we have

$$\lambda_{S,i} := \frac{i^\#}{\sum_{j \in S} j^\#} \text{ for all } i \in S.$$

Therefore our numerical example mimics the situations in Laura's company described above, very simplified of course. It should be mentioned that here the weighted Shapley hierarchy levels value  $Sh^{wHL}$  coincides with the LS-value presented in [Gómez-Rúa and Vidal-Puga \(2011\)](#). We obtain Table 1. Note that player 4 is a null player, players 1 and 2 and players 8 and 9 respectively are dependent in  $(N, v)$ , and components  $\{5, 6\}$  and  $\{7\}$  are dependent in  $(\mathcal{B}^1, v^1)$ .

**Table 1:** Comparison of different values

Value	Payoff to the players (rounded)								
	1	2	3	4	5	6	7	8	9
$Sh(N, v)$	6	6	5.5	0	5.2	4.7	5.2	3.7	3.7
$Sh^L(N, v, \underline{\mathcal{B}})$	4.625	4.625	7.25	0	3.75	3.25	7	4.75	4.75
$Sh^{wHL}(N, v, \underline{\mathcal{B}})$	5.1524	5.1524	7.1690	1.3357	4.9847	4.6524	4.8190	3.3667	3.3667
$Sh^{wSL}(N, v, \underline{\mathcal{B}})$	5.2381	5.2381	8.3333	0	5.1048	4.5333	4.8190	3.3667	3.3667
$H^{\lambda SL}(N, v, \underline{\mathcal{B}})$	5.1048	5.1048	8.0667	0	4.9714	4.4	4.6857	3.8333	3.8333

The payoffs to the dependent players in  $(N, v)$  are equal, they have the same weights. The dependent components in  $(\mathcal{B}^1, v^1)$  have different weights. Thus, the payoffs to players 5 and 6 do not sum up to the payoff to player 7 for  $Sh^{wHL}$ ,  $Sh^{wSL}$  or  $H^{\lambda SL}$ . We see that the Shapley levels value prefers player 7 who has not joined a team with other players in the first level. The payoffs to players 8 and 9 are higher by  $H^{\lambda SL}$  than by  $Sh^{wHL}$  or  $Sh^{wSL}$  since  $H^{\lambda SL}$  takes into account the higher weight of the component  $\{8, 9\}$  within  $\mathcal{B}^2$ .

## 6 Conclusion and discussion

The rapidly increasing volume of collected data and global networking make it possible and necessary to share benefits between cooperating actors, often hierarchically structured. According to the above examinations, for the distribution of the generated surpluses, the presented new classes of LS-values provide an alternative to the Shapley levels value and the weighted Shapley hierarchy levels values. A close examination of the Shapley levels value definition, given by (8), shows that in unanimity games, it is not advantageous for the individual player to merge into components: all components which are siblings have the same weight, even if a sibling is a singleton (see also the payoff to player 7 in Table 1).

The same need not apply to our new classes of LS-values and the weighted Shapley hierarchy levels values: the greater the weight of a component, the higher the payoff in an unanimity game. By the weighted Shapley hierarchy levels values, null players also receive a positive payoff if they contribute to the weight of the components involved (see Table 1).

On the contrary, the LS-values of the Harsanyi support levels set (and, thus, also the weighted Shapley support levels values) always leave null players without benefits. Again, it can be a great competitive advantage for the players to join forces. First, for those who form a cooperating subgroup within the carrier of an unanimity game. And then for all players who belong to coalitions with positive dividends within the whole coalition function. Here, the players of a component can “support” each other in changing unanimity games, even if they do not belong to the carrier. Nevertheless, null players do not receive any payoff, even if they contribute to the total weight of components. However, we can also interpret this in the spirit of cooperative game theory, which is in complete contrast to non-cooperative game theory:

“As a null player in this game, I support you with all my power, even if I have nothing of it at first. But I know that you as a null player in another game will support me with all your power, even if you have nothing of it there.”

This “support” is a fundamental principle of solidarity in all societies. E.g., it forms the business model of all associations and insurance companies. It can be disadvantageous, to kick a null or dummy player out of a group where everyone can rely on each other! This also means that an adaption to LS-games of the well-known null player out property in [Derks and Haller \(1999\)](#) is violated, which states that removing a null player from the game does not change the payoff to the other players. An exception is the Shapley levels value. Although it belongs to both the weighted Shapley support levels values and the weighted Shapley hierarchy levels values, null players are not paid and cannot support the corresponding group with their power (weight). Here the groups are powerless in the sense that each group, regardless of size and importance, has the same weight. Despite the fact that additivity is usually considered to be of little economic importance (see, e.g., [Casajus and Huettner \(2014\)](#)), it takes on a new quality in the context of group formation, group solidarity, and the Harsanyi support levels solutions.

## 7 Appendix

### 7.1 Logical independence

**Remark 7.1.** *The axioms in Corollary 3.7 are logically independent.*

*Proof.* • **E, N, Pos, A:** The axiomatization must also be valid for  $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$ . In this case, the axioms in Corollary 3.7 coincide with the corresponding axioms for TU-values (and an always satisfied axiom **LG**). It is well-known that these axioms for TU-values are logically independent and, therefore, none of the corresponding axioms can be redundant.

- **LG:** For all  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $w \in W^{dis}$ , and for all  $T \in \Omega^N$ ,  $T \ni i$ , define

$$A_{\underline{\mathcal{B}}, T}^w(i) := \prod_{r=0}^h \frac{w_{\mathcal{B}^r(i) \cap T}}{\sum_{\substack{B \in \mathcal{B}^r: B \subseteq \mathcal{B}^{r+1}(i), \\ B \cap T \neq \emptyset}} w_{B \cap T}}.$$

Then, the LS-values  $Sh^{wAL}$ , given by

$$Sh_i^{wAL}(N, v, \underline{\mathcal{B}}) = \sum_{T \subseteq N, T \ni i} A_{\underline{\mathcal{B}}, T}^w(i) \Delta_v(T) \text{ for all } i \in N,$$

satisfy obviously **E, N, Pos**, and **A**, but not **LG** in general<sup>6</sup>. □

**Remark 7.2.** *The axioms in Proposition 4.5 and Theorem 4.6 are logically independent:*

*Proof.* The proof is omitted since it is analogous to the proof of Remark 7.1. □

## 7.2 Additional axioms and lemmas, used in the proofs

The first axiom is defined on TU-games, coincides with **SCU** <sup>$\lambda$</sup>  in the case of a trivial level structure, and is used in the proof of Theorem 3.6.

**$\lambda$ -balanced sharing in unanimity games** (with some factor), **SU** <sup>$\lambda$</sup> <sub>TU</sub>. For all  $N \in \mathcal{N}$ ,  $(N, v) \in \mathbb{V}^N$ ,  $\alpha \in \mathbb{R}$ ,  $\lambda \in \Lambda$ ,  $T \in \Omega^N$ , and  $i, j \in T$ , we have

$$\lambda_{T, j} \phi_i(N, \alpha \cdot u_T) = \lambda_{T, i} \phi_j(N, \alpha \cdot u_T).$$

The following axiom is a weakening of **SCU** <sup>$\lambda$</sup>  and is also used in the proof of Theorem 3.6. Property (12) has to be satisfied only for singletons.

**$\lambda$ -balanced sharing between singletons in unanimity games** (with some factor), **SSU** <sup>$\lambda$</sup> . For each  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\lambda \in \Lambda^{dis}$ , and for all  $\alpha \in \mathbb{R}$  and  $T \in \Omega^N$ ,  $i, j \in T$  such that  $j \in \mathcal{B}^1(i)$ , and  $\mathcal{C}_T^0(\{i\}) := \{B : B \in \mathcal{B}^0, B \subseteq \mathcal{B}^1(i), B \cap T \neq \emptyset\}$ , we have

$$\lambda_{\mathcal{C}_T^0(\{i\}), \{j\}} \varphi_i(N, \alpha \cdot u_T, \underline{\mathcal{B}}) = \lambda_{\mathcal{C}_T^0(\{i\}), \{i\}} \varphi_j(N, \alpha \cdot u_T, \underline{\mathcal{B}}).$$

<sup>6</sup>Let  $(N, u_S, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $w \in W^{dis}$  such that the weights are given as in Table 2,  $N := \{1, 2, 3, 4, 5\}$ , and  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3\}$  with  $\mathcal{B}^1 := \{\{1, 2\}, \{3, 4\}, \{5\}\}$ ,  $\mathcal{B}^2 := \{\{1, 2, 3, 4\}, \{5\}\}$ , and  $u_S$  be the unanimity game with carrier  $S := \{1, 2, 3, 5\}$ .

**Table 2:** Weights of the coalitions

$S$	{1}	{2}	{3}	{4}	{5}	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
$w_S$	1	1	1	1	5	3	3	3	3	3	3	5	5	5	5	7

It follows,  $Sh_{\{1,2,3,4\}}^{wAL}(\mathcal{B}^2, v^2, \underline{\mathcal{B}}^2) = \frac{7}{12} \neq \frac{1}{2} = \sum_{i \in \{1,2,3,4\}} Sh_i^{wAL}(N, v, \underline{\mathcal{B}})$ .

Our first lemma states that each non-empty coalition  $S$  for each level is a subset of only one coalition that is a union of components from this level which have a non-empty intersection with  $S$ .

**Lemma 7.3.** *Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $\mathcal{B}^r \in \underline{\mathcal{B}}$ ,  $0 \leq r \leq h$ . Each  $S \in \Omega^N$  is a subset of exactly one coalition  $T^r \in \Omega^N$ ,  $T^r = \bigcup_{\substack{B \subseteq T^r, B \in \mathcal{B}^r \\ B \cap S \neq \emptyset}} B$ . Thus, we can also uniquely designate each  $S \in \Omega^N$  as  $S_{T^r}$ .*

*Proof.* Each coalition  $T^r \in \Omega^N$  is a union of components  $B \in \mathcal{B}^r$ .  $\mathcal{B}^r$  is a partition, and so each player  $i \in S$ ,  $S \in \Omega^N$ , is contained in only one component  $B \in \mathcal{B}^r$ . Therefore, for each coalition  $S \in \Omega^N$  there is exactly one coalition  $T^r \in \Omega^N$  which is a union of all components  $B \in \mathcal{B}^r$  containing at least one player  $i \in S$ .  $\square$

The second lemma shows that for each coalition  $\mathcal{T}$  in an induced level structure the dividend in the induced LS-game is equal to the sum of the dividends in the original game from all coalitions  $S$  of the original level structure which are subsets of a coalition  $T$  associated to  $\mathcal{T}$  and have the property of the previous lemma with respect to coalition  $T$ .

**Lemma 7.4.** *Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $\mathcal{B}^r \in \underline{\mathcal{B}}$ ,  $0 \leq r \leq h$ , and  $S_{T^r}$  be the coalitions from lemma 7.3 with associated coalition  $T^r$ . Then, we have in the  $r$ th LS-game  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)$  for each  $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$ , associated to a  $T^r \in \Omega^N$ ,*

$$\Delta_{v^r}(\mathcal{T}^r) = \sum_{S_{T^r} \subseteq \mathcal{T}^r} \Delta_v(S_{T^r}). \quad (15)$$

*Proof.* Let  $t = |\{B \in \mathcal{B}^r : B \subseteq T^r\}|$  the number of components  $B \in \mathcal{B}^r$  which are subsets from a coalition  $T^r \in \Omega^N$  with associated  $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$ . We use induction on the size  $t$ ,  $1 \leq t \leq |\mathcal{B}^r|$ .

*Initialization:* Let  $t = 1$ .  $T^r$  is a component  $B \in \mathcal{B}^r$ . We have

$$\Delta_{v^r}(\mathcal{T}^r) \stackrel{(1)}{=} v^r(\mathcal{T}^r) \stackrel{(5)}{=} v(T^r) \stackrel{(1)}{=} \sum_{S \subseteq T^r} \Delta_v(S) \stackrel{\text{Lem. 7.3}}{=} \sum_{S_{T^r} \subseteq \mathcal{T}^r} \Delta_v(S_{T^r}).$$

*Induction step:* Assume that (15) holds for an arbitrary  $\hat{t} \geq 1$  (IH). Let now  $\hat{\mathcal{T}}^r \in \Omega^{\mathcal{B}^r}$  with associated  $\hat{T}^r \in \Omega^N$ ,  $\hat{t} = |\{B \in \mathcal{B}^r : B \subseteq \hat{T}^r\}|$  and  $T^r = \hat{T}^r \cup \hat{B}$ ,  $\hat{B} \in \mathcal{B}^r$ ,  $\hat{B} \not\subseteq \hat{T}^r$ . We have  $t = \hat{t} + 1$  and it follows

$$\begin{aligned} \Delta_{v^r}(\mathcal{T}^r) &\stackrel{(1)}{=} \sum_{\mathcal{Q}^r \subseteq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \stackrel{(1)}{=} v(T^r) - \sum_{\mathcal{Q}^r \subsetneq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \\ &\stackrel{(1)}{=} \Delta_v(T^r) + \sum_{S \subsetneq T^r} \Delta_v(S) - \sum_{\substack{\mathcal{Q}^r \subsetneq \mathcal{T}^r \\ \mathcal{Q}^r \subsetneq \mathcal{B}^r}} \sum_{S_{\mathcal{Q}^r} \subseteq \mathcal{Q}^r} \Delta_v(S_{\mathcal{Q}^r}) \\ &\stackrel{\text{Lem. 7.3}}{=} \Delta_v(T^r) + \sum_{S \subsetneq T^r} \Delta_v(S) - \sum_{\substack{S \subsetneq T^r \\ S \neq S_{T^r}}} \Delta_v(S) \\ &= \Delta_v(T^r) + \sum_{S_{T^r} \subsetneq \mathcal{T}^r} \Delta_v(S_{T^r}) = \sum_{S_{T^r} \subseteq \mathcal{T}^r} \Delta_v(S_{T^r}). \end{aligned}$$

$\square$

## 7.3 Proofs

### 7.3.1 Proof of Proposition 3.4

Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $\lambda \in \Lambda^{dis}$ , and  $K_{\underline{\mathcal{B}}, T}^\lambda$  be the expressions according to Def. 3.1.

• **E, N, A, Pos:** It is well-known that all  $H^\lambda \in \mathcal{H}$ ,  $\lambda \in \Lambda$ , satisfy the mentioned axioms for TU-values. Thus, the claim follows by Remark 3.3.

• **LG:** Let  $B \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ . If  $r = 0$ , (6) trivially is satisfied because the zeroth LS-game corresponds to the original LS-game.

Let now  $1 \leq r \leq h$ . Obviously, by (9), we have for all  $S \subseteq N$ ,  $S \cap B \neq \emptyset$ ,

$$\sum_{j \in B, j \in S} \prod_{\ell=0}^{r-1} \lambda_{\mathcal{C}_S^\ell(j), \mathcal{B}^\ell(j)} = 1. \quad (16)$$

In the game  $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)$ , we have for all  $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$ ,  $\mathcal{T}^r \ni B$ , associated  $T^r$ , and an arbitrary  $i \in B$ ,

$$K_{\underline{\mathcal{B}}^r, \mathcal{T}^r}^\lambda(B) \stackrel{(9)}{=} \prod_{\ell=r}^h \lambda_{\mathcal{C}_{T^r}^\ell(i), \mathcal{B}^\ell(i)}. \quad (17)$$

Let  $i \in B$  be fixed and  $S_{T^r} \in \Omega^N$  the coalitions from Lemma 7.3 with related coalitions  $T^r$ . Note, if  $i \in S_{T^r}$ , we have  $B \subseteq T^r$ . For all  $S_{T^r}$ ,  $S_{T^r} \ni i$ ,  $r \leq \ell \leq h$ , we have

$$\lambda_{\mathcal{C}_{S_{T^r}}^\ell(i), \mathcal{B}^\ell(i)} \stackrel{(9)}{=} \lambda_{\mathcal{C}_{T^r}^\ell(i), \mathcal{B}^\ell(i)}. \quad (18)$$

Lem.7.3

It applies,  $\mathcal{B}^\ell(i) = \mathcal{B}^\ell(j)$  for all  $j \in B$  and  $r \leq \ell \leq h$ . For all  $S_{T^r} \ni i$ , it follows,

$$\begin{aligned} \sum_{\substack{j \in B, \\ j \in S_{T^r}}} K_{\underline{\mathcal{B}}, S_{T^r}}^\lambda(j) &\stackrel{(10)}{=} \sum_{\substack{j \in B, \\ j \in S_{T^r}}} \prod_{\ell=0}^h \lambda_{\mathcal{C}_{S_{T^r}}^\ell(j), \mathcal{B}^\ell(j)} \stackrel{(18)}{=} \sum_{\substack{j \in B, \\ j \in S_{T^r}}} \prod_{\ell=0}^{r-1} \lambda_{\mathcal{C}_{S_{T^r}}^\ell(j), \mathcal{B}^\ell(j)} \prod_{\ell=r}^h \lambda_{\mathcal{C}_{T^r}^\ell(i), \mathcal{B}^\ell(i)} \\ &\stackrel{(16)}{=} \prod_{\ell=r}^h \lambda_{\mathcal{C}_{T^r}^\ell(i), \mathcal{B}^\ell(i)} \stackrel{(17)}{=} K_{\underline{\mathcal{B}}^r, \mathcal{T}^r}^\lambda(B). \end{aligned} \quad (19)$$

Finally, we get the following:

$$\begin{aligned} \sum_{j \in B} H_j^{\lambda SL}(N, v, \underline{\mathcal{B}}) &\stackrel{(11)}{=} \sum_{j \in B} \sum_{\substack{S \subseteq N, \\ S \ni j}} K_{\underline{\mathcal{B}}, S}^\lambda(j) \Delta_v(S) \stackrel{\text{Lem. 7.3}}{=} \sum_{j \in B} \sum_{\substack{S_{T^r} \subseteq N, \\ S_{T^r} \ni j}} K_{\underline{\mathcal{B}}, S_{T^r}}^\lambda(j) \Delta_v(S_{T^r}) \\ &= \sum_{S_{T^r} \subseteq N} \sum_{\substack{j \in B, \\ j \in S_{T^r}}} K_{\underline{\mathcal{B}}, S_{T^r}}^\lambda(j) \Delta_v(S_{T^r}) \stackrel{(19)}{=} \sum_{\substack{S_{T^r} \subseteq N, \\ \mathcal{T}^r \ni B}} K_{\underline{\mathcal{B}}^r, \mathcal{T}^r}^\lambda(B) \Delta_v(S_{T^r}) \\ &\stackrel{\text{Lem. 7.3}}{=} \sum_{\mathcal{T}^r \subseteq \mathcal{B}^r, \mathcal{T}^r \ni B} K_{\underline{\mathcal{B}}^r, \mathcal{T}^r}^\lambda(B) \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}) \\ &\stackrel{\text{Lem. 7.4}}{=} \sum_{\mathcal{T}^r \subseteq \mathcal{B}^r, \mathcal{T}^r \ni B} K_{\underline{\mathcal{B}}^r, \mathcal{T}^r}^\lambda(B) \Delta_{v^r}(\mathcal{T}^r) \stackrel{(11)}{=} Sh_B^{\lambda SL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r). \end{aligned}$$

• **SCU<sup>λ</sup>:** Let  $\alpha \in \mathbb{R}$ ,  $T \in \Omega^N$ ,  $k, \ell \in N$ ,  $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ ,  $0 \leq r \leq h$ ,  $\mathcal{B}^r(k), \mathcal{B}^r(\ell) \cap T \neq \emptyset$ , and  $\mathcal{C}_T^r(\mathcal{B}^r(k)) := \{B : B \in \mathcal{B}^r, B \subseteq \mathcal{B}^{r+1}(\mathcal{B}^r(k)), B \cap T \neq \emptyset\}$ . If  $\lambda_{\mathcal{C}_T^r(\mathcal{B}^r(k)), \mathcal{B}^r(k)} = 0$  or

$\lambda_{\mathcal{C}_T^r(\mathcal{B}^r(k)), \mathcal{B}^r(\ell)} = 0$ , (12) is satisfied by (10) and (11). Otherwise, if  $r = 0$ , we have,

$$\frac{H_k^{\lambda SL}(N, \alpha \cdot u_T, \underline{\mathcal{B}})}{\lambda_{\mathcal{C}_T^0(\{k\}), \{k\}}} \stackrel{(11)}{=} \frac{K_{\underline{\mathcal{B}}, T}^\lambda(k)}{\lambda_{\mathcal{C}_T^0(\{k\}), \{k\}}} \alpha \stackrel{(9)}{=} \frac{K_{\underline{\mathcal{B}}, T}^\lambda(\ell)}{\lambda_{\mathcal{C}_T^0(\{k\}), \{\ell\}}} \alpha \stackrel{(10)}{=} \frac{H_\ell^{\lambda SL}(N, \alpha \cdot u_T, \underline{\mathcal{B}})}{\lambda_{\mathcal{C}_T^0(\{k\}), \{\ell\}}}.$$

Analogously, in the  $r$ th LS-game,  $0 \leq r \leq h$ , we have

$$\frac{H_{\mathcal{B}^r(k)}^{\lambda SL}(\mathcal{B}^r, (\alpha \cdot u_T)^r, \underline{\mathcal{B}}^r)}{\lambda_{\mathcal{C}_T^r(\mathcal{B}^r(k)), \mathcal{B}^r(k)}} \stackrel{\text{Lem. 7.3}}{=} \frac{H_{\mathcal{B}^r(\ell)}^{\lambda SL}(\mathcal{B}^r, (\alpha \cdot u_T)^r, \underline{\mathcal{B}}^r)}{\lambda_{\mathcal{C}_T^r(\mathcal{B}^r(k)), \mathcal{B}^r(\ell)}} \stackrel{\text{Lem. 7.4}}{=}$$

and the claim follows by **LG**.  $\square$

### 7.3.2 Proof of Proposition 3.5

Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $\lambda \in \Lambda^{dis}$ ,  $S \in \Omega^N$ , and  $\varphi$  be an LS-value that satisfies all axioms of Proposition 3.5. Due to Proposition 3.4, (2), and **A**, it is sufficient to show that  $\varphi$  is uniquely defined on the game  $v_S := \Delta_v(S) \cdot u_S$ .

If  $B \in \mathcal{B}^r$ ,  $B \cap S = \emptyset$ , we have, by **N**,

$$\sum_{i \in B} \varphi_i(N, v_S, \underline{\mathcal{B}}) = 0 \text{ for all } r, 0 \leq r \leq h. \quad (20)$$

We use induction on the size  $m$ ,  $0 \leq m \leq h$ , for all levels  $r$ ,  $0 \leq r \leq h$ , with  $m := h - r$ .

*Initialization:* Let  $m = 0$  and so  $r = h$ . It follows for all  $i \in S$  with  $\prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(i)} > 0$ , a such  $i$  always exists, and all  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , and  $B \subseteq \mathcal{B}^{r+1}(i)$ ,

$$\begin{aligned} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &\stackrel{\text{SCU}^\lambda}{=} \frac{\lambda_{\mathcal{C}_S^r(i), B}}{\lambda_{\mathcal{C}_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\ \Rightarrow \sum_{\substack{B \in \mathcal{B}^r, \\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \sum_{\substack{B \in \mathcal{B}^r, \\ B \cap S \neq \emptyset}} \frac{\lambda_{\mathcal{C}_S^r(i), B}}{\lambda_{\mathcal{C}_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \stackrel{\text{(20)}}{=} \Delta_v(S) \\ \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(i)} \Delta_v(S). \end{aligned} \quad (21)$$

By **SCU** $^\lambda$  and (21), we have for all  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , and  $B \subseteq \mathcal{B}^{r+1}(i)$  with  $\prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(B)} = 0$ ,

$$\sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) = 0 = \prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(B)} \Delta_v(S). \quad (22)$$

*Induction step:* Assume that (21) and (22) hold to  $\varphi$  with an arbitrary  $m-1$ ,  $0 \leq m-1 \leq$

$h - 1$  (IH). It follows for all  $i \in S$  with  $\prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(i)} > 0$ ,

$$\begin{aligned} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) & \stackrel{\text{SCU}^\lambda}{=} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{\lambda_{\mathcal{C}_S^r(i), B}}{\lambda_{\mathcal{C}_S^r(i), \mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\ & \stackrel{\substack{(20) \\ (IH)}}{=} \prod_{\ell=r+1}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(i)} \Delta_v(S) \\ \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) & = \prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(i)} \Delta_v(S). \end{aligned}$$

Analogously, for all  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , with  $\prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(B)} = 0$ , we have

$$\sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) = 0 = \prod_{\ell=r}^h \lambda_{\mathcal{C}_S^\ell(i), \mathcal{B}^\ell(B)} \Delta_v(S).$$

Therefore,  $\varphi$  is uniquely defined on  $v_S$  (take  $m = h$  and so  $r = 0$ ).  $\square$

### 7.3.3 Proof of Theorem 3.6

Let  $N \in \mathcal{N}^{dis}$  and  $\underline{\mathcal{B}} \in \mathbb{L}^N$  be fixed.

$\Rightarrow$ : By Remark 3.3 and Proposition 3.4, each  $\varphi \in \mathcal{H}^{SL}$  on  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$  coincides with a Harsanyi solution  $\phi \in \mathcal{H}$  on  $(N, v) \in \mathbb{V}^N$  and satisfies **LG**.

$\Leftarrow$ : Obviously, if we use the coinciding axioms for TU-values (with the exception of **LG**) in Proposition 3.4, we have characterizations of Harsanyi solutions with exogenously given sharing systems  $\lambda \in \Lambda$ . Note that these axiomatizations hold for all  $N \in \mathcal{N}$  and, especially, for games which are defined on the same player sets as used for induced LS-games of the original LS-game on a fixed  $N \in \mathcal{N}^{dis}$ ,  $\underline{\mathcal{B}} \in \mathbb{L}^N$ . Thus, each  $\phi \in \mathcal{H}$  satisfies the standard axioms **E**<sub>TU</sub>, **N**<sub>TU</sub>, **A**<sub>TU</sub> for TU-values, and **SU**<sub>TU</sub> <sup>$\lambda$</sup>  for some  $\lambda \in \Lambda$ . Any LS-value  $\varphi$  that coincides with a Harsanyi solution on  $N \in \mathcal{N}^{dis}$ ,  $\underline{\mathcal{B}} \in \mathbb{L}^N$ , must also coincide with a Harsanyi solution in the induced LS-games. Therefore, any LS-value  $\varphi$  that coincides with a Harsanyi solution must satisfy the simply transferred LS-versions of these axioms: **E**, **N**, **A**, and **SSU** <sup>$\lambda'$</sup>  for some  $\lambda' \in \Lambda^{dis}$  and also in corresponding induced LS-games. Note that **SSU** <sup>$\lambda'$</sup>  is implied by **SU** <sup>$\lambda$</sup> , and **SCU** <sup>$\lambda$</sup>  is implied by **SSU** <sup>$\lambda'$</sup>  and **LG** for some  $\lambda, \lambda' \in \Lambda^{dis}$  for a fixed  $N \in \mathcal{N}^{dis}$ ,  $\underline{\mathcal{B}} \in \mathbb{L}^N$ . Thus, all the axioms of Proposition 3.5 must be satisfied and we have  $\varphi \in \mathcal{H}^{SL}$ .  $\square$

### 7.3.4 Proof of Corollary 3.7

The claim follows immediately due to the axiomatization of the Harsanyi set (Vasil'ev, 1981) by the TU-versions of **E**, **N**, **Pos**, and **A** and Theorem 3.6.  $\square$

### 7.3.5 Proof of Proposition 4.4

Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $w \in W^{dis}$ , and  $K_{\underline{\mathcal{B}}, T}^w$  be the expressions according to Def. 4.1.

- **E**, **N**, **A**, **LG**: The claim follows directly by Remark 4.3 and Proposition 3.4.
- **SMon**: The claim follows immediately by (14).
- **Dep** <sup>$w$</sup> : Let  $k, \ell \in N$ ,  $0 \leq r \leq h$ ,  $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$  and  $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$  be dependent in

$(\mathcal{B}^r, v^r) \in \mathbb{V}^{\mathcal{B}^r}$ . If  $r = 0$ , then  $k, \ell$  are dependent in  $(N, v) \in \mathbb{V}^N$  and we get

$$\begin{aligned} \frac{Sh_k^{wSL}(N, v, \underline{\mathcal{B}})}{w_{\{k\}}} &\stackrel{(14)}{=} \sum_{T \subseteq N, T \ni k} \frac{K_{\underline{\mathcal{B}}, T}^w(k)}{w_{\{k\}}} \Delta_v(T) \stackrel{(3)}{=} \sum_{T \subseteq N, \{k, \ell\} \subseteq T} \frac{K_{\underline{\mathcal{B}}, T}^w(k)}{w_{\{k\}}} \Delta_v(T) \\ &\stackrel{\substack{Def. \\ 4.1}}{=} \sum_{T \subseteq N, \{k, \ell\} \subseteq T} \frac{K_{\underline{\mathcal{B}}, T}^w(\ell)}{w_{\{\ell\}}} \Delta_v(T) = \frac{Sh_\ell^{wSL}(N, v, \underline{\mathcal{B}})}{w_{\{\ell\}}}. \end{aligned}$$

Thus, we have also in the  $r$ th LS-game,  $0 \leq r \leq h$ ,

$$\frac{Sh_{\mathcal{B}^r(k)}^{wSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)}{w_{\mathcal{B}^r(k)}} = \frac{Sh_{\mathcal{B}^r(\ell)}^{wSL}(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)}{w_{\mathcal{B}^r(\ell)}}$$

and the claim follows by **LG**.

- **MDep**: The claim follows immediately by **Dep**<sup>w</sup>. □

**Convention 7.5.** In order to avoid cumbersome case distinctions in the proof of Proposition 4.5, if we consider only one single component isolated as a player, we define the component dependent on itself. Then **Dep**<sup>w</sup> is trivially satisfied.

### 7.3.6 Proof of Proposition 4.5

Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ ,  $w \in W^{dis}$ ,  $S \in \Omega^N$ , and  $\varphi$  be an LS-value that satisfies all axioms of Theorem 4.5. Due to Proposition 4.4, (2), and **A**, it is sufficient to show that  $\varphi$  is uniquely defined on the game  $v_S := \Delta_v(S) \cdot u_S$ .

By Lemma 7.3, for each level  $r$ ,  $0 \leq r \leq h$ , exists exactly one coalition  $T_S^r$  with associated  $\mathcal{T}_S^r \subseteq \mathcal{B}^r$ , which is the smallest coalition of all  $R^r$ ,  $R^r \supseteq S$ , with associated  $\mathcal{R}^r \subseteq \mathcal{B}^r$  and so in each game  $(\mathcal{B}^r, v_S^r, \underline{\mathcal{B}}^r) \in \mathbb{VL}^{\mathcal{B}^r}$ , we have  $\Delta_{v_S^r}(\mathcal{T}_S^r) = \Delta_v(S)$  and  $\Delta_{v_S^r}(\mathcal{R}^r) = 0$  for  $\mathcal{R}^r \subseteq \mathcal{B}^r$ ,  $\mathcal{R}^r \neq \mathcal{T}_S^r$ . Therefore, by (3), possibly using Conv. 7.5, all components  $B \in \mathcal{B}^r$ ,  $B \cap S \neq \emptyset$ , are dependent in  $(\mathcal{B}^r, v_S^r)$ . If  $B \in \mathcal{B}^r$ ,  $B \cap S = \emptyset$ , we have, by **N**,  $\sum_{i \in B} \varphi_i(N, v_S, \underline{\mathcal{B}}) = 0$ .

We use induction on the size  $m$ ,  $0 \leq m \leq h$ , for all levels  $r$ ,  $0 \leq r \leq h$ , with  $m := h - r$ .

*Initialization:* Let  $m = 0$  and so  $r = h$ . For an arbitrary  $i \in S$ , we get

$$\begin{aligned} \sum_{\substack{B \in \mathcal{B}^h, \\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &\stackrel{\text{Dep}^w}{=} \sum_{\substack{B \in \mathcal{B}^h, \\ B \cap S \neq \emptyset}} \frac{w_B}{w_{\mathcal{B}^h(i)}} \sum_{j \in \mathcal{B}^h(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \stackrel{\mathbf{E}}{=} \Delta_v(S) \\ \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \prod_{k=h-m}^h \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k: B \subseteq \mathcal{B}^{k+1}(i), \\ B \cap S \neq \emptyset}} w_B} \Delta_v(S). \end{aligned} \quad (23)$$

*Induction step:* Assume that (23) holds to  $\varphi$  with an arbitrary  $m - 1$ ,  $0 \leq m - 1 \leq h - 1$  (*IH*). It follows, for an arbitrary  $i \in S$ ,

$$\begin{aligned} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &\stackrel{\text{Dep}^w}{=} \sum_{\substack{B \in \mathcal{B}^r, B \cap S \neq \emptyset, \\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{w_B}{w_{\mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\ &\stackrel{(IH)}{=} \prod_{k=h-m+1}^h \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k: B \subseteq \mathcal{B}^{k+1}(i), \\ B \cap S \neq \emptyset}} w_B} \Delta_v(S) \\ \Rightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \prod_{k=h-m}^h \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k: B \subseteq \mathcal{B}^{k+1}(i), \\ B \cap S \neq \emptyset}} w_B} \Delta_v(S). \end{aligned}$$

Therefore,  $\varphi$  is uniquely defined on  $v_S$  (take  $m = h$  and so  $r = 0$ ).  $\square$

### 7.3.7 Proof of Theorem 4.6

By Proposition 4.4, we only have to show the way back.

Let  $N \in \mathcal{N}^{dis}$ ,  $(N, v, \underline{\mathcal{B}}) \in \mathbb{V}\mathbb{L}^N$ ,  $\underline{\mathcal{B}} = \{\mathcal{B}^0, \dots, \mathcal{B}^{h+1}\}$ , and  $\varphi$  be an LS-value that satisfies **E**, **N**, **SMon**, **A**, and **MDep**. By **SMon** and **N**, we have  $\varphi_i(N, u_N, \underline{\mathcal{B}}) > 0$  for all  $i \in N$ . Take a  $w \in W^{dis}$  such that  $w_B := \sum_{i \in B} \varphi_i(N, u_N, \underline{\mathcal{B}})$  for all  $B \in \underline{\mathcal{B}}$ . By **MDep**, we have for all  $B_k, B_\ell \in \mathcal{B}^r$ ,  $0 \leq r \leq h$ , such that  $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ , and  $B_k, B_\ell$  are dependent in  $(\mathcal{B}^r, v^r) \in \mathbb{V}^{\mathcal{B}^r}$ ,

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}$$

and **Dep**<sup>w</sup> is satisfied. The claim follows by Proposition 4.5.  $\square$

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