Bond Prices-Implied Default Probability and Principal Recovery Rate Under Un-Recoverable Coupons

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Abstract

Assuming only bond principal receives recovery at default, it is possible to derive the implied forward curve of both the default probability and the recovery rate from the bond prices of the same corporate issuer if the issuer has at least two bonds outstanding at each maturity with different coupon rates.

Background

Yield to maturity is the most often used measure of bond value under a given maturity. Strictly speaking though, yield is not a fundamental value measure as bonds from the same issuer can have slightly different yields even for the same maturity when coupon differs. For issuers free of credit risk, the fundamental measure of bond value is the discount rate curve. But for issuers with credit risk, the fundamental measure is not the "risky" discount rate curve, or equivalently the credit spread curve. Rather, for credit risky issuers the fundamental measure is the forward curve of default probability and the recovery rate. Given these two curves, a defaultable bond of fixed coupon rate can be valued as the expected future payout while including default and recovery. Reversely however, given the bond prices of an issuer, it is not always possible to derive the two curves simultaneously. In practice, the recovery rates are often assumed to be some given value in order for the default curve to be derived.

There are however certain conditions under which both the default curve and the recovery curve can be derived simultaneously from bond prices[1]:

1. The recovery rate applies only to bond principal, and the coupon is un-recovered on default.
2. The issuer has two bonds outstanding at each maturity with different coupon rates.

If both conditions hold, the default probability curve and the recovery rate curve can be derived together from the prices of the outstanding bonds. This paper provides the formula for this case. In the real market, it is uncommon for issuers to meet strictly the condition 2. For large corporates who issue bonds regularly over relatively fixed schedule, condition 2 can be approximately satisfied.

Formula

Notations and conventions

i is the index of time taken in annual steps with $i = 0$ corresponding to the present time.

Period i is the period between time $i - 1$ and time i.

p_i is the conditional default probability within period i.

R_i is the (average) recovery rate within period i.

r_i is the risk-free discount rate for maturity at time i.

Δc_i is the difference in the coupon rates of the two bonds both with maturity i.

[1] Assuming the bonds are in the same seniority class and hence share the same recovery rate on default.

ΔV_i is the difference in the prices of the two bonds with different coupon rates and same maturity i.

By convention, $\Delta V_i \equiv \Delta c_i \equiv r_i \equiv 0$ for all $i \leq 0$.

$V_{i,1}$ and $c_{i,1}$ are the price and the coupon rate, respectively, of one of the two bonds at maturity i.

$d \geq 0$ is the delay time between default and receiving recovery.

All bond maturities are in annual increments. Interest rate and coupon rate are expressed in simple annual compounding.

Price implied default probability

$$1 - p_i = \frac{\left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)}{\left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)} \frac{(1 + r_i)^i}{(1 + r_{i-1})^{i-1}} \tag{1}$$

Price implied principal recovery rate

$$\frac{R_i}{(1 + r_{i+d})^{i+d}} = \frac{V_{i,1} - V_{i-1,1} - \frac{\Delta V_i}{\Delta c_i}(c_{i,1} + 1) + \frac{\Delta V_{i-1}}{\Delta c_{i-1}}(c_{i-1,1} + 2) - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}}{\left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)(1 + r_{i-1})^{i-1} - \left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)(1 + r_i)^i} \tag{2}$$

Implied constraints on bond prices

Because $\frac{\Delta V_N}{\Delta c_N} = \sum_{i=1}^{N} \frac{S_i}{(1+r_i)^i}$ (see section Derivation below), it follows that

$$\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}} \geq 0$$

As $1 - p_i \leq 1$, it follows that

$$\left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)(1 + r_i)^i \leq \left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)(1 + r_{i-1})^{i-1}$$

From $R_i \geq 0$ and the previous inequality,

$$\left(V_{i,1} - \frac{\Delta V_i}{\Delta c_i} c_{i,1}\right) - \left(V_{i-1,1} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}} c_{i-1,1}\right) \geq \left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right) - \left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)$$

From $R_i \leq 1$,

$$\left(V_{i,1} - \frac{\Delta V_i}{\Delta c_i} c_{i,1}\right) - \left(V_{i-1,1} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}} c_{i-1,1}\right)$$
$$\leq \left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)\left(1 - \frac{(1 + r_i)^i}{(1 + r_{i+d})^{i+d}}\right) - \left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)\left(1 - \frac{(1 + r_{i-1})^{i-1}}{(1 + r_{i+d})^{i+d}}\right)$$

In the case of no recovery delay, $d = 0$:

$$\left(V_{i,1} - \frac{\Delta V_i}{\Delta c_i} c_{i,1}\right) - \left(V_{i-1,1} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}} c_{i-1,1}\right) \leq -\left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)\left(1 - \frac{1}{1 + f_i}\right)$$

where $f_i \equiv \frac{(1+r_i)^i}{(1+r_{i-1})^{i-1}} - 1.$

Derivation

Define the following notations:

$S_i = (1-p_1)(1-p_2) *** (1-p_i)$ is the cumulative survival probability by time i.

$D_i = (1-p_1)(1-p_2) *** (1-p_{i-1})p_i$ is the net probability of defaulting exactly in the period i.

The value of a defaultable bond, with fixed coupon rate c and maturity N, is defined as

$$V \;= NPV(\text{undefaulted coupon}) + NPV(\text{undefaulted principal}) + NPV(\text{default recovery})$$

$$= c\sum_{i=1}^{N}\frac{S_i}{(1+r_i)^i} + \frac{S_N}{(1+r_N)^N} + (1+\delta c)\sum_{i=1}^{N}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$$

δ is a binary function whose value is 0 if coupon is not recovered and 1 if coupon is recovered under the same recovery rate as the principal.

If coupon is not recovered ($\delta = 0$), then $\frac{\Delta V}{\Delta c}$ with the same maturity is independent of the recovery rate:

$$\frac{\Delta V_N}{\Delta c_N} = \sum_{i=1}^{N}\frac{S_i}{(1+r_i)^i}$$

Using proof by recursion, it can be shown that

$$S_i = \left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)(1+r_i)^i$$

Next,

$$\frac{\Delta V_N}{\Delta c_N} = \sum_{i=1}^{N-1}\frac{S_i}{(1+r_i)^i} + \frac{S_{N-1}(1-p_N)}{(1+r_N)^N} = \frac{\Delta V_{N-1}}{\Delta c_{N-1}} + \frac{S_{N-1}(1-p_N)}{(1+r_N)^N}$$

The formula (1) then follows from the expression of S_{N-1}.

To derive formula (2) for the recovery rate,

$$D_i = \prod_{k=1}^{i-1}(1-p_k)p_i = S_{i-1}p_i$$

$$= \left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)(1+r_{i-1})^{i-1}\left(1 - \frac{\left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)}{\left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)}\frac{(1+r_i)^i}{(1+r_{i-1})^{i-1}}\right)$$

$$= \left(\frac{\Delta V_{i-1}}{\Delta c_{i-1}} - \frac{\Delta V_{i-2}}{\Delta c_{i-2}}\right)(1+r_{i-1})^{i-1} - \left(\frac{\Delta V_i}{\Delta c_i} - \frac{\Delta V_{i-1}}{\Delta c_{i-1}}\right)(1+r_i)^i$$

Using the price expression for either one of the two bonds,

$$V_{N1} = c_{N1}\sum_{i=1}^{N}\frac{S_i}{(1+r_i)^i} + \frac{S_N}{(1+r_N)^N} + \sum_{i=1}^{N}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$$

$$= c_{N1}\frac{\Delta V_N}{\Delta c_N} + \left(\frac{\Delta V_N}{\Delta c_N} - \frac{\Delta V_{N-1}}{\Delta c_{N-1}}\right) + \sum_{i=1}^{N}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$$

Hence,

$$\frac{D_N R_N}{(1+r_{N+d})^{N+d}} = V_{N1} - \left(c_{N1}\frac{\Delta V_N}{\Delta c_N} + \left(\frac{\Delta V_N}{\Delta c_N} - \frac{\Delta V_{N-1}}{\Delta c_{N-1}}\right)\right) - \sum_{i=1}^{N-1}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$$

$$= V_{N1} - \left(c_{N1}\frac{\Delta V_N}{\Delta c_N} + \left(\frac{\Delta V_N}{\Delta c_N} - \frac{\Delta V_{N-1}}{\Delta c_{N-1}}\right)\right) - \frac{D_{N-1} R_{N-1}}{(1+r_{N-1+d})^{N-1+d}} - \sum_{i=1}^{N-2}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$$

Reducing $\frac{D_{N-1} R_{N-1}}{(1+r_{N-1+d})^{N-1+d}}$ the same way,

$$\frac{D_N R_N}{(1+r_{N+d})^{N+d}}$$

$$= V_{N1} - \left(c_{N1}\frac{\Delta V_N}{\Delta c_N} + \left(\frac{\Delta V_N}{\Delta c_N} - \frac{\Delta V_{N-1}}{\Delta c_{N-1}}\right)\right)$$

$$- \left(V_{N-1,1} - \left(c_{N-1,1}\frac{\Delta V_{N-1}}{\Delta c_{N-1}} + \left(\frac{\Delta V_{N-1}}{\Delta c_{N-1}} - \frac{\Delta V_{N-2}}{\Delta c_{N-2}}\right)\right) - \sum_{i=1}^{N-2}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}\right) - \sum_{i=1}^{N-2}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$$

The $\sum_{i=1}^{N-2}\frac{D_i R_i}{(1+r_{i+d})^{i+d}}$ term is cancelled out, so

$$\frac{D_N R_N}{(1+r_{N+d})^{N+d}}$$

$$= V_{N1} - \left(c_{N1}\frac{\Delta V_N}{\Delta c_N} + \left(\frac{\Delta V_N}{\Delta c_N} - \frac{\Delta V_{N-1}}{\Delta c_{N-1}}\right)\right) - \left(V_{N-1,1} - \left(c_{N-1,1}\frac{\Delta V_{N-1}}{\Delta c_{N-1}} + \left(\frac{\Delta V_{N-1}}{\Delta c_{N-1}} - \frac{\Delta V_{N-2}}{\Delta c_{N-2}}\right)\right)\right)$$

$$= V_{N1} - V_{N-1,1} - \frac{\Delta V_N}{\Delta c_N}(c_{N1}+1) + \frac{\Delta V_{N-1}}{\Delta c_{N-1}}(c_{N-1,1}+2) - \frac{\Delta V_{N-2}}{\Delta c_{N-2}}$$