Testing for the cointegration rank between Periodically Integrated processes

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Abstract

Cointegration between periodically integrated (PI) processes has been analyzed by many, including Bladen-Hovell, Chui, Osborn, and Smith (1989), Boswijk and Franses (1995), Franses and Paap (2004), Kleibergen and Franses (1999) and del Barrio Castro and Osborn (2008), to name a few. However, there is currently no published method that allows us to determine the cointegration rank between PI processes. The present paper fills this gap in the literature with a method for determining the cointegration rank between a set of PI processes based on the idea of pseudo-demodulation, as proposed in the context of seasonal cointegration by del Barrio Castro, Cubadda, and Osborn (2020). Once a pseudo-demodulated time series is obtained, the Johansen (1995) procedure can be applied to determine the cointegration rank. A Monte Carlo experiment shows that the proposed approach works satisfactorily for small samples.

Keywords: Reduced Rank Regression,Periodic Cointegration, Periodically Integrated Processes.

JEL codes: C32.

1 Introduction

There are two main ways of modeling non-stationary integration in seasonal time series: with seasonal integration and with periodic integration (see Ghysels and Osborn (2001) for details about the main characteristics and differences between seasonal and periodic integration). The latter may be seen as more attractive, as its non-stationary behavior is ruled by a common stochastic trend shared between the seasons present in the time series. Contrarily, in the case of seasonal integration, each of the time series seasons has its own stochastic trend (see Osborn (1993) and Ghysels and Osborn (2001) for details). Furthermore, periodic integration serves as a suitable data-generating process for seasonal time series when the preferences of economic agents vary along with the seasons of the year (see Hansen and Sargent (1993), Gersovitz and McKinnon (1978), and Osborn (1988)).

In terms of long-run relationships (cointegration) that can be established between seasonal non-stationary processes, we can also find seasonal and periodic cointegration. For seasonally integrated (SI) processes it is possible to define both, but in the case of periodically integrated (PI) processes, only full periodic cointegration can be established (see del Barrio Castro and Osborn (2008a) and Ghysels and Osborn (2001) for details). As for seasonal cointegration, methods for both single-equation and reduced-rank regressions have been proposed to test for the presence of cointegration and to determine the cointegration rank (see for example Hylleberg, Engle, Granger, and Yoo (1990); Engle, Granger, Hylleberg, and Lee (1993); Johansen and Schaumburg (1998); Cubadda (2000); and Ahn and Reinsel (1994)). Periodic cointegration was proposed by Birchenhall, Bladen-Hovell, Chui, Osborn, and Smith (1989). A single-equation method to test for the presence of periodic cointegration was proposed by Boswijk and Franses (1995). They claim that their method can be applied to both SI and PI processes, but del Barrio Castro and Osborn (2008a) have shown that the asymptotic distribution of the error-correction test for periodic cointegration that they derived does not apply to PI processes. del Barrio Castro and Osborn (2008) have also proposed a residual-based

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cointegration test for periodic cointegration between PI processes. But to the best of our knowledge, only the working paper by Kleibergen and Franses (1999) has tried to come up with a method for determining the cointegration rank between sets of PI processes, (see also Franses and Paap (2004) for details). The method proposed by Kleibergen and Franses (1999) relies on periodic vector autoregressive (VAR) models and implies the use of GMM and reduced-rank regression techniques. Finally, a full dynamic systems approach, in which equations are estimated jointly for observations relating to each season, can theoretically be applied (Ghysels and Osborn (2001) pp 171–176)–as was done in the application of Haldrup, Hylleberg, Pons, and Sansó (2007)–but the VAR becomes over-parameterized. Hence, this approach is feasible in practice, but only when data of a relatively high frequency is available.

In this paper, we propose a simple method for determining the cointegration rank between PI processes, inspired by the demodulation method suggested by del Barrio Castro, Cubadda and Osborn (2022) that merely requires the use of the procedure proposed by Johansen (1995) once the PI processes or time series are "filtered" or "demodulated."

The paper is organized as follows, in the next section, we describe and summarize the main characteristics of PI processes and the consequences of cointegration between them. After that, we present our reduced-rank approach for determining the cointegration rank between PI processes, followed by a Monte Carlo section where we show that our approach works well on small samples. Finally, the last section concludes.

It is useful to introduce some notation at this stage. Our analysis is concerned with seasonal processes that have S observations per year; for example, $S = 4$ for quarterly seasonal data. In the paper, the vector of seasons representation indicating a specific observation within the year is used, as is double subscript notation. It is important to appreciate that, in this vector notation, $x_{s\tau}$ indicates the s^{th} observation within the τ^{th} year. For example, with quarterly data, $x_{s\tau}$ is the s^{th} quarter of year τ in the available sample. Assuming that $t = 1$ represents the first period within a cycle, the identity $t = S(\tau - 1) + s$ provides a link between the usual time index and the vector notation. Finally it is understood that $x_{s-i,\tau} = x_{S-(s-i),\tau-1}$ for $s - i \leq 0$.

2 Periodic Integration and Cointegration between Periodically Integrated Processes

First, we will focus on the main characteristics of PI processes. One of these characteristics is going to be critical to the approach suggested in this paper, as it will allow us to determine the cointegration rank between PI processes. Secondly, we will consider possible cointegration between PI processes.

2.1 Periodic Integration

A periodic autoregressive process of order p (PAR(p)), is a generalization of an autoregressive process in which the parameters are allowed to vary with the season of the year, hence we have:

$$\begin{array}{c} y_{s\tau} = \phi_{1s} y_{s-1,\tau} + \phi_{2s} y_{s-2,\tau} + \cdots + \phi_{ps} y_{s-p,\tau} + \varepsilon_{s\tau} \\ s = 1, 2, \cdots, S \qquad \tau = 1, 2, \cdots, N \end{array} \tag{1}$$

where $\varepsilon_{s\tau}$ is the innovation of the process and we assume that $\varepsilon_{s\tau} \sim iid\left(0, \sigma_\varepsilon^2\right)$. PAR($p$) processes like (1) can be rewritten as a Vector Autoregressive model of order (P) (VAR(P)), also known as vector of seasons representation of a PAR process (see Franses and Paap (2004) and Ghysels and Osborn (2001) for more details), where the S seasons of the time series are stacked in an $S \times 1$ vector $Y_\tau = [y_{1\tau}, y_{2\tau}, \ldots, y_{S\tau}]'$ and

$$\mathbf{A}_0 Y_\tau = \mathbf{A}_1 Y_{\tau-1} + \mathbf{A}_2 Y_{\tau-2} + \cdots + \mathbf{A}_P Y_{\tau-P} + E_\tau \tag{2}$$

where, $E_\tau = [\varepsilon_{1\tau}, \varepsilon_{2\tau}, ..., \varepsilon_{S\tau}]'$, $\mathbf{A}_k$ for $k = 0, 1, \ldots, P$ are $S \times S$ matrices with generic elements

$$A_{0(i,j)} = \begin{cases} 1 & i = j, \\ -\phi_{i-j,i} & i < j, \\ 0 & i > j \end{cases}$$

$$A_{k(i,j)} = \phi_{i+Sk-j,i},$$

for $i = 1, 2, \ldots, S$, $j = 1, 2, \ldots, S$ and $P = 1 + [(p-1)/S]$, with $[a]$ denotes the integer part of a. The subscript (i,j) indicates the $(i,j)^{th}$ element of the respective matrix. The stationarity of (2)-(1) requieres that all the roots of $\left|\mathbf{A}_0 - \mathbf{A}_1 z - \mathbf{A}_2 z - \cdots - \mathbf{A}_P z^P\right| = 0$ lie outside of the unit circle. In order to understand the concept of periodic integration, let us focus on the PAR process of order one:

$$y_{s\tau} = \phi_s y_{s-1,\tau} + u_{s\tau}. \tag{3}$$

In (3) we assume that $u_{s\tau}$ is a stationary innovation, this assumption will help us later on to connect (3) with (1)[1]. The condition of periodic integration in (3) is $\prod_{s=1}^{S} \phi_s = 1$, and it implies that between the seasons of the time series we have $S-1$ cointegration relationships, or equivalently, that the seasons of the process share a common stochastic trend. This situation is clearly shown in previously mentioned vector of seasons representation of process (3):

$$\mathbf{A}_0 Y_\tau = \mathbf{A}_1 Y_{\tau-1} + U_\tau \tag{4}$$

where, $U_\tau = [u_{1\tau}, u_{2\tau}, ..., u_{S\tau}]'$, $\mathbf{A}_0$, and $\mathbf{A}_1$ are $S \times S$ matrices with generic elements

$$A_{0(h,j)} = \begin{cases} 1 & h = j,\ j = 1, ..., S \\ -\phi_h & h = j+1,\ j = 1, ..., S-1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$
$$A_{1(h,j)} = \begin{cases} \phi_1 & h = 1,\ j = S \\ 0 & \text{otherwise} \end{cases}.$$

Note that for (4) we have $|\mathbf{A}_0 - \mathbf{A}_1 z| = 1 - \phi_1\phi_2\phi_3\phi_4 z$, and we have a unit root when $\phi_1\phi_2\phi_3\phi_4 = 1$, that is the Periodic Integration condition. In the following lemma we summarize the stochastic behavior of Y_τ in (4).

Lemma 1 *For $Y_\tau = [y_{1\tau}, y_{2\tau}, y_{3\tau}, \ldots, y_{S\tau}]'$ with $y_{s\tau}$ $s = 1, 2, \cdots, S$ defined in (3-4) with $\prod_{s=1}^{S} \phi_s = 1$ and with $\left(1 - \psi_{1s} L - \cdots - \psi_{p-1,s} L^{p-1}\right) u_{s\tau} = \varepsilon_{s\tau}$ and $\varepsilon_{s\tau} \sim iid\left(0, \sigma^2\right)$, then*

$$Y_\tau = \mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \mathbf{A}_0^{-1} U_\tau + \mathbf{a}\mathbf{b}' \sum_{j=1}^{\tau-1} U_{\tau-j} \tag{6}$$

$$\begin{aligned} \frac{1}{\sqrt{T}} Y_{\lfloor Tr \rfloor} &\Rightarrow \sigma \mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}\Psi(1)^{-1} W(r) = \sigma \mathbf{a}\mathbf{b}'\Psi(1)^{-1} W(r) \\ &= \omega \mathbf{a} w(r) \end{aligned} \tag{7}$$

where $\mathbf{a}$ and $\mathbf{b}$ are defined in (47) in the appendix, $W(r)$ is an $S \times 1$ multivariate Brownian vector, $w(r)$ is a scalar Brownian motion, and the scalar ω is defined by (49) in the appendix. The definition of matrix $\Psi(1)$ can also be found in the appendix.

The fact that the stochastic behavior of the vector Y_τ is ruled by the scalar Brownian motion $w(r)$, clearly shows that there is a common stochastic trend shared by the seasons of the process $y_{s\tau}$ that is gathered in vector Y_τ and identified by the scalar Brownian motion $w(r)$ in (7). Or equivalently, we have $S-1$ cointegration relationships between the seasons of (4). If we rewrite (4) as:

$$\begin{aligned} Y_\tau &= \mathbf{A}_0^{-1}\mathbf{A}_1 Y_{\tau-1} + \mathbf{A}_0^{-1} U_\tau \\ Y_\tau - Y_{\tau-1} &= \left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right] Y_{\tau-1} + \mathbf{A}_0^{-1} U_\tau, \end{aligned} \tag{8}$$

[1] If we write (1) as:

$$\left(1 - \phi_{1s} L - \phi_{2s} L^2 - \cdots - \phi_{ps} L^p\right) y_{s\tau} = \varepsilon_{s\tau}$$

and factorize the polynomial $\left(1 - \phi_{1s} L - \phi_{2s} L^2 - \cdots - \phi_{ps} L^p\right)$ as

$$\left(1 - \phi_{1s} L - \phi_{2s} L^2 - \cdots - \phi_{ps} L^p\right) = \left(1 - \phi_s L\right)\left(1 - \psi_{1s} L - \cdots - \psi^*_{p-1,s} L^{p-1}\right)$$

(see Franses (1996), Boswijk and Franses (1996), and del Barrio Castro and Osborn (2008a)) then (1) is connected to (3) as $u_{s\tau}$ in (3) is defined as follows:

$$\left(1 - \psi_{1s} L - \cdots - \psi_{p-1,s} L^{p-1}\right) u_{s\tau} = \varepsilon_{s\tau},$$

hence as we assume that $u_{s\tau}$ is a stationary innovation in (3), $u_{s\tau}$ should follow a PAR$(p-1)$ stationary process. Hence for its vector of seasons representation with $U_\tau = [u_{1\tau}, u_{2\tau}, ..., u_{S\tau}]'$:

$$\mathbf{\Psi}_0 U_\tau = \mathbf{\Psi}_1 U_{\tau-1} + \mathbf{\Psi}_2 U_{\tau-2} + \cdots + \mathbf{\Psi}_K U_{\tau-K} + E_\tau$$

with $K = 1 + [(p-2)/S]$, all the roots of $\left|\mathbf{\Psi}_0 - \mathbf{\Psi}_1 z - \mathbf{\Psi}_2 z^2 - \cdots - \mathbf{\Psi}_K z^K\right| = 0$ should lie outside of the unit circle.

matrix $\left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right]$ has rank $S-1$. Clearly $\left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right] = \alpha\beta'$, where both α and β have dimension $S \times (S-1)$ and one possible choice for the columns of β are the last $S-1$ rows of $\mathbf{A}_0$.[2] Finally, it is clear that we have cointegration between the seasons of Y_τ. If we left-multiply expression (6) by β' we obtain:

$$\beta' Y_\tau = \beta' \mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \beta' \mathbf{A}_0^{-1} U_\tau + \beta' \mathbf{a}\mathbf{b}' \sum_{j=1}^{\tau-1} U_{\tau-j}.$$

With the definition of $\mathbf{a}$ in (47) and β' defined as the last $S-1$ rows of $\mathbf{A}_0$ (or as in footnote 2), it is evident that $\beta'\mathbf{a} = 0$. We clearly show that $\beta' Y_\tau \sim I(0)$ and that we have $S-1$ cointegration relationships between the S seasons of $y_{s\tau}$ (or Y_τ).

If we compare Lemma 1 expression (A2) in del Barrio Castro, Cubadda, and Osborn (2022) (BCCO hereafter) with Lemma 1 expression (7) in this paper, it is clear that the role played in our Lemma 1 by the $S \times 1$ vectors $\mathbf{a}$ and $\mathbf{b}$ is equivalent to the role played by the $S \times 1$ vectors $\mathbf{v}_j^-$ and $\mathbf{v}_j^+$ in BCCO. Note that, in BCCO, $\mathbf{v}_j^-$ collects the sequence of the S possible values of the complex demodulator operator $e^{-ti\omega_k} = e^{-[S(\tau-1)+s]i\omega_k}$, which is clearly a periodic function, as $\omega_k = 2\pi k/S$ with $k = 1, 2, \ldots, (S-1)/2$. The complex demodulator operator appears after recursive substitution in the complex-valued process integrated at frequency ω_k, $x_{s\tau}^- = e^{-i\omega_k} x_{s-1,\tau}^- + \varepsilon_{s\tau}$, which yields:

$$\begin{aligned} x_{s\tau}^- &= e^{-i\omega_k} x_{s-1,\tau}^- + \varepsilon_{s\tau} \\ x_{s\tau}^- &= e^{-[S(\tau-1)+s]i\omega_k}\left[x_0^- + \sum_{j=1}^{[S(\tau-1)+s]} e^{-[S(\tau-1)+s-j]i\omega_k}\varepsilon_j\right]. \end{aligned} \tag{9}$$

Hence, in (9) there are two parts: a complex-valued random walk integrated at the zero frequency $\left[x_0^- + \sum_{j=1}^{[S(\tau-1)+s]} e^{-[S(\tau-1)+s-j]i\omega_k}\varepsilon_j\right]$, and the demodulator operator $e^{-[S(\tau-1)+s]i\omega_k}$ that shifts the previous complex-valued random walk from the zero frequency to frequency ω_k. Thus, multiplying each observation of $x_{s\tau}^-$ by the complex conjugate of the demodulator operator $e^{-[S(\tau-1)+s]i\omega_k}$ (that is, $e^{[S(\tau-1)+s]i\omega_k}$) we obtain a complex value integrated at the zero frequency.

In this paper, there is an equivalent situation where in (7) the zero-frequency stochastic trend is associated with the scalar Brownian motion $w(r)$, and the $S \times 1$ vector $\mathbf{a}$ plays a role similar to the demodulator operator. But in the case of a PI process the periodic sequence of values collected in vector $\mathbf{a}$ causes spectral power at the zero frequency and at the seasonal frequencies. Figure 1 illustrates this situation, where

[2] Note that we have $S-1$ cointegration relationships between the seasons of (3) in the form $y_{s\tau} - \phi_s y_{s-1,\tau}$, which are clearly identified with the last $S-1$ rows of matrix $\mathbf{A}_0$, that is:

$$\beta' = \begin{bmatrix} -\phi_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi_3 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_S & 1 \end{bmatrix}.$$

See Paap and Franses (1999) for more details. And for α (with $\left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right] = \alpha\beta'$) we will have:

$$\alpha = \begin{bmatrix} \phi_2^{-1} & (\phi_2\phi_3)^{-1} & \cdots & \left(\prod_{j=2}^{S}\phi_j\right)^{-1} \\ 0 & \phi_3^{-1} & \cdots & \left(\prod_{j=3}^{S}\phi_j\right)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_S^{-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Note also, that equivalently, we can use the normalized version of β'

$$\beta^{*\prime} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -\phi_1 \\ 0 & 1 & 0 & \cdots & 0 & -\phi_1\phi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\prod_{i=1}^{S-1}\phi_i \end{bmatrix}.$$

part (a) shows the average periodogram based on 10,000 replications of simulated PI process (3), in which $y_{s\tau} = \phi_s y_{s-1,\tau} + u_{s\tau}$, where $S = 4$, $\phi_1 = 0.8$, $\phi_2 = 1$, $\phi_3 = 0.5$, $\phi_4 = 1/(\phi_1\phi_2\phi_3)$, and $u_{s\tau} \sim Niid(0,1)$. In panel (b) of Figure 1 we present the average periodogram of $a_s^{-1} y_{s\tau}$, where a_s is the element with s^{th} position in vector $\mathbf{a}$. Clearly, part (a) shows spectral power at the zero, $\pi/2$, and π frequencies. Hence, $y_{s\tau}$ in (3) has zero frequency and seasonal behavior while the pseudo-demodulated process $a_s^{-1} y_{s\tau}$ has only zero-frequency spectral power, as seen in panel (b). This situation is explained by the misspecified constant parameter representation of the PI process (see Osborn (1991), Ghysels and Osborn (2001), and del Barrio Castro and Osborn (2008b)). As pointed out by del Barrio Castro and Osborn (2008b) "This representation provides the conventional non-periodic ARMA process that has autocovariance properties identical to those that result from analyzing a periodic process as a conventional non-periodic one." The misspecified constant-parameter representation of $y_{s\tau} = \phi_s y_{s-1,\tau} + u_{s\tau}$, with $S = 4$ and $\phi_1\phi_2\phi_3\phi_4 = 1$, is $y_{s\tau} - y_{s\tau-1} = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)\,\eta_{s\tau}$ (see section 2.2 in del Barrio Castro and Osborn (2008b) for details on how to obtain θ_1, θ_2, θ_3, and σ_η^2 for a given combination of values for ϕ_1, ϕ_2, $\phi_3, \phi_4 = 1/(\phi_1\phi_2\phi_3)$ and σ_u^2). Following section 2.2 in del Barrio Castro and Osborn (2008b) it is possible to see that the invertible constant-parameter representation associated with $y_{s\tau} = \phi_s y_{s-1,\tau} + u_{s\tau}$, where $\phi_1 = 0.8$; $\phi_2 = 1$; $\phi_3 = 0.5$; $\phi_4 = 1/(\phi_1\phi_2\phi_3)$; and $\sigma_u^2 = 1$, is $y_{s\tau} - y_{s\tau-1} = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)\,\eta_{s\tau}$, with[3] $\theta_1 = 0.8497$, $\theta_2 = 0.5976$, $\theta_3 = 0.4133$, and $\sigma_\eta^2 = 2.8139$. The moving average polynomial of order 3 $(1 + 0.8497L + 0.5976L^2 + 0.4133L^3)$ can be factorized as follows: $(1 + 0.8497L + 0.5976L^2 + 0.4133L^3) = (1 + 0.7703L)(1 + [0.0397 + 0.7314i]\,L)$ $(1 + [0.0397 - 0.7314i]\,L)$. On the other hand, the seasonal difference operator $(1 - L^4)$ can be factorized as $(1 - L^4) = (1 - L)(1 + L)(1 - iL)(1 + iL)$. The spectral power in Figure 1 part (a) at the zero frequency is higher than at frequencies $\pi/2$ and π. Clearly, in the MA(3) process with constant-parameter representation we do not have a factor associated with the zero frequency, and the spectral power at the Nyquist frequency in Figure 1 part (a) is lowered by the factor $(1 + 0.7703L)$. In the case of frequency $\pi/2$ it is lowered by the complex conjugate factors $(1 + [0.0397 \mp 0.7314i]\,L)$. Finally, note that expression (6) is very similar to (9). In particular, for a specific season s of vector Y_τ, say $y_{s\tau}$, we have:

$$
\begin{aligned}
y_{s\tau} &= a_s \left[\phi_1 y_{S,0} + \sum_{j=1}^{S} b_j \sum_{i=1}^{\tau-1} u_{j,\tau-i} \right] + u_{s\tau} + \sum_{j=1}^{s-1} \left(\prod_{i=s-j+1}^{s} \phi_i \right) u_{s-j,\tau} \\
&= a_s y_{s\tau}^{(0)} + Stationary \quad terms,
\end{aligned} \tag{10}
$$

where b_j is the element with j^{th} position in vector $\mathbf{b}$ of Lemma 1 (defined in (47)). The common stochastic trend shared by the seasons is $y_{s\tau}^{(0)} = \left[\phi_1 y_{S,0} + \sum_{j=1}^{S} b_j \sum_{i=1}^{\tau-1} u_{j,\tau-i} \right]$[4]. Finally note that from (10) it is also possible to see that $y_{s\tau}^{(0)}$ is an standard random walk behavior with initial condition $y_{S0}^{(0)}$ equal to $y_{S0}^{(0)} = \phi_1 y_{S,0}$ and periodic innovation $b_s u_{s\tau}$:

$$
\begin{aligned}
y_{s\tau}^{(0)} &= \left[\phi_1 y_{S,0} + \sum_{j=1}^{S} b_j \sum_{i=1}^{\tau-1} u_{j,\tau-i} \right] \\
y_{s\tau}^{(0)} &= y_{s-1\tau}^{(0)} + b_s u_{s\tau} \\
y_{S0}^{(0)} &= \phi_1 y_{S,0}.
\end{aligned} \tag{11}
$$

In this paper we propose the use of $a_s^{-1} y_{s\tau}$ to extract the zero-frequency stochastic trend $y_{s\tau}^{(0)}$. Hence, we use the previous pseudo-demodulation of PI processes to extract the common zero-frequency trend of each PI process, include these pseudo-demodulated times series in the standard Johansen (1996) procedure, and test for the cointegration rank between the pseudo-demodulated time series obtained from the PI processes. In the following section the possibilities of cointegration between PI processes are explored.

2.2 Cointegration between PI processes

Periodic cointegration was introduced by Birchenhall, Bladen-Hovell, Chui, Osborn, and Smith (1989), and it implies that long-run relationships are considered season by season. Hence, we have different cointegration

[3] Rounding to the fourth decimal place.

[4] Note that the term $u_{s\tau} + \sum_{j=1}^{s-1} \left(\prod_{i=s-j+1}^{s} \phi_i \right) u_{s-j,\tau}$ in (10) is a finite sums of $u_{s\tau}$ and its first $s-1$ lags and hence it is stationary.

vectors for each season. Periodic cointegration can be established for both seasonally integrated processes and periodically integrated processes. Boswijk and Franses (1995) distinguished between full and partial periodic cointegration. The latter applies when stationary linear combinations between seasonal non-stationary time series can be established for only some seasons $s = 1, 2, \ldots, S$. And full periodic cointegration implies that stationary linear combinations exit for all the seasons. Finally, full non-periodic cointegration implies that the same cointegration vectors are shared by all seasons.

Ghysels and Osborn (2001) and del Barrio Castro and Osborn (2008a) analyze cointegration between PI processes and show that the only possibilities are full periodic cointegration or full non-periodic cointegration.

In this paper, we follow the definition of periodic cointegration proposed by del Barrio Castro and Osborn (2008a) (see definition 1 in section 2.2), but we introduce an equivalent way of defining cointegration between PI processes that is more closely connected with the usual definition of cointegration at the zero frequency. First, we focus on the bivariate case, followed by an extension to the multivariate.

2.3 The bivariate case

In Ghysels and Osborn (2001), the following example is used to show that the only possibility of cointegration between two PI processes is fully periodic (Ghysels and Osborn (2001) page 169). Let us assume that we have two PI processes $y_{s\tau} = \phi_s^y y_{s-1,\tau} + \varepsilon_{s\tau}^y$ and $x_{s\tau} = \phi_s^x x_{s-1,\tau} + \varepsilon_{s\tau}^x$, with stationary innovations $\varepsilon_{s\tau}^y$ and $\varepsilon_{s\tau}^x$, and that the PI condition $\prod_{s=1}^{S} \phi_s^j = 1$ for $j = y$ and x holds. If we assume that there is cointegration between $y_{s\tau}$ and $x_{s\tau}$ in the last season, say S, the linear combination $y_{S\tau} - \beta x_{S\tau}$ should be stationary. Hence, by recursive substitution of $y_{s\tau}$ and $x_{s\tau}$ in $y_{S\tau} - \beta x_{S\tau}$, we find that:

$$
\begin{aligned}
& y_{S\tau} - \beta x_{S\tau} = \\
& y_{S-1\tau} - \beta \frac{\phi_S^x}{\phi_S^y} x_{S-1\tau} + \frac{\varepsilon_{S\tau}^y}{\phi_S^y} - \frac{\beta \varepsilon_{S-1,\tau}^x}{\phi_S^y} = \\
& y_{S-2\tau} - \beta \frac{\phi_S^x \phi_{S-1}^x}{\phi_S^y \phi_{S-1}^y} x_{S-2\tau} + \frac{\varepsilon_{S\tau}^y}{\phi_S^y \phi_{S-1}^y} - \frac{\beta \varepsilon_{S-1,\tau}^x}{\phi_S^y \phi_{S-1}^y} + \frac{\varepsilon_{S-1\tau}^y}{\phi_{S-1}^y} - \frac{\beta \phi_S^x \varepsilon_{S-1,\tau}^x}{\phi_S^y \phi_{S-1}^y} = \\
& y_{S-3\tau} - \beta \frac{\phi_S^x \phi_{S-1}^x \phi_{S-2}^x}{\phi_S^y \phi_{S-1}^y \phi_{S-2}^y} x_{S-3\tau} + Stationary \quad terms = \\
& \vdots \\
& y_{1\tau} - \beta \frac{\prod_{j=0}^{S-2} \phi_{S-j}^x}{\prod_{j=0}^{S-2} \phi_{S-j}^y} x_{1\tau} + Stationary \quad terms.
\end{aligned} \tag{12}
$$

From (12), we see that in order to have full non-periodic cointegration between $y_{s\tau}$ and $x_{s\tau}$, it must hold that $\phi_j^x = \phi_j^y$ for $j = 1, 2, \ldots, S$. In Lemma 1 in del Barrio Castro and Osborn (2008a) the result from (12) is extended to the general case of more than two variables, say n variables or n PI processes. They show that between a set of n PI processes the only possibilities are fully periodic cointegration and fully non-periodic cointegration. The intuition behind this result is that, as shown in Lemma 1 of the previous subsection, the S seasons of a PI process are driven by the same common stochastic trend. Hence, if we have cointegration between one of the seasons of a PI process, recursive substitution implies that cointegration will hold for the rest of the seasons, with a cointegration vector that will change for each season unless all the PI processes have the same coefficients associated with the PI condition $\prod_{s=1}^{S} \phi_s^k = 1$, that is, $\phi_s^k = \phi_s$ for $k = 1, 2, \ldots, n$ and $s = 1, 2, \ldots, S$. And precisely in this latter case, when all the PI processes share the same coefficients $\phi_s^k = \phi_s$ with the PI condition, we have full non-periodic cointegration. Finally, note that in (12), moving to the relationship between $y_{S,\tau-1}$ and $x_{S,\tau-1}$, by recursion in the last expression of (12), we have $y_{S,\tau-1} - \beta \left(\prod_{j=0}^{S-1} \phi_{S-j}^x \right) \left(\prod_{j=0}^{S-1} \phi_{S-j}^y \right)^{-1} x_{S,\tau-1} + Stationary \quad terms = y_{S,\tau-1} - \beta x_{S,\tau-1} + Stationary \quad terms$, and hence, the periodic sequence of values in the cointegration vector is completed.

The approach used in Ghysels and Osborn (2001), and in (12) above, is a little bit different from the usual approach to cointegration at the zero frequency. Following the lines of BCCO, here, we provide a different but equivalent approach to showing the possibility of cointegration between PI processes, one that is more related to the usual approach to zero-frequency cointegration.

First note, based on equations (6) and (7) from Lemma 1 and equation (10), that for a PI process $x_{s\tau} = \phi_s^x x_{s-1,\tau} + \varepsilon_{s\tau}^x$ with $\prod_{s=1}^{S} \phi_s^x = 1$, it is possible to write $x_{s\tau} = a_s^x x_{s\tau}^{(0)} + Stationary \quad terms$, where a_s^x is the element with the s^{th} position in the $S \times 1$ vector $\mathbf{a}^x$ defined in (47) but with ϕ_s replaced by ϕ_s^x for $s = 1, 2, \ldots, S$, and finally $x_{s\tau}^{(0)}$ is the common stochastic trend[5] shared by the seasons of the process PI process $x_{s\tau} = \phi_s^x x_{s-1,\tau} + \varepsilon_{s\tau}^x$. From this, we can define the zero-frequency cointegration relationship:

$$\begin{aligned} y_{s\tau}^{(0)} &= \beta^* x_{s\tau}^{(0)} + \varepsilon_{s\tau}^y \\ x_{s\tau}^{(0)} &= \left(a_s^x\right)^{-1} x_{s\tau} + Stationary \quad terms \\ x_{s\tau} &= \phi_s^x x_{s-1,\tau} + \varepsilon_{s\tau}^x \\ \prod_{s=1}^{S} \phi_s^x &= 1. \end{aligned} \tag{13}$$

System (13) is the usual cointegration system between two processes integrated at the zero frequency $y_{s\tau}^{(0)}$ and $x_{s\tau}^{(0)}$, with cointegration vector $[1, -\beta^*]$. Note that if we use the fact that $x_{s\tau}^{(0)} = \left(\mathbf{a}_s^x\right)^{-1} x_{s\tau} + Stationary \quad terms$ in $y_{s\tau}^{(0)} = \beta^* x_{s\tau}^{(0)} + \varepsilon_{s\tau}^y$ (13), we will obtain $y_{s\tau}^{(0)} = \beta^* \left(a_s^x\right)^{-1} x_{s\tau} + \varepsilon_{s\tau}^y + Stationary \quad terms$, that is, a periodic cointegration relationship between a PI process $x_{s\tau}$ and a standard $I(1)$ associated with the zero frequency $y_{s\tau}^{(0)}$:

$$\begin{aligned} y_{s\tau}^{(0)} &= \beta^* \left(a_s^x\right)^{-1} x_{s\tau} + Stationary \quad terms \\ x_{s\tau} &= \phi_s^x x_{s-1,\tau} + \varepsilon_{s\tau}^x \\ \prod_{s=1}^{S} \phi_s^x &= 1. \end{aligned} \tag{14}$$

Finally if we multiply (14) by a_s^y, the following system is obtained:

$$\begin{aligned} y_{s\tau} &= a_s^y \beta^* \left(a_s^x\right)^{-1} x_{s\tau} + Stationary \quad terms \\ x_{s\tau} &= \phi_s^x x_{s-1,\tau} + \varepsilon_{s\tau}^x \\ y_{s\tau} &= a_s^y y_{s\tau}^{(0)} + Stationary \quad terms. \end{aligned} \tag{15}$$

Hence, we move to a cointegrated system between two PI processes $y_{s\tau}$ and $x_{s\tau}$. The coefficients associated with the periodic integration condition in the case of $x_{s\tau}$ are gathered in $\mathbf{a}^x = \left[1, \phi_2^x, \phi_2^x \phi_3^x, \ldots, \prod_{s=2}^{S} \phi_s^x\right]'$, and a_s^x is the s^{th} element of the $S \times 1$ vector $\mathbf{a}^x$. In the case of $y_{s\tau}$, the coefficients are gathered in $\mathbf{a}^y = \left[1, \phi_2^y, \phi_2^y \phi_3^y, \ldots, \prod_{s=2}^{S} \phi_s^y\right]'$, and a_s^y is the s^{th} element of the $S \times 1$ vector $\mathbf{a}^y$. Clearly, in (15), the cointegration vector is periodic, as in (12). In the case of (12), it is possible to see that the periodic coefficients

[5] From (10) we will have $x_{s\tau}^{(0)} = \left[\phi_1^x x_{S,0} + \sum_{j=1}^{S} b_j^x \sum_{i=1}^{\tau-1} \varepsilon_{j,\tau-i}^x\right]$.

of the cointegration vector $[1, -\beta_s]$ evolve as follows:

$$
\begin{aligned}
\beta_S &= \beta = \beta \frac{\phi_1^y \phi_2^y \phi_3^y \cdots \phi_S^y}{\phi_1^x \phi_2^x \phi_3^x \cdots \phi_S^x} \\
\beta_{S-1} &= \beta \frac{\phi_S^x}{\phi_S^y} = \beta \frac{\phi_1^y \phi_2^y \phi_3^y \cdots \phi_{S-1}^y}{\phi_1^x \phi_2^x \phi_3^x \cdots \phi_{S-1}^x} \\
\beta_{S-2} &= \beta \frac{\phi_S^x \phi_{S-1}^x}{\phi_S^y \phi_{S-1}^y} = \beta \frac{\phi_1^y \phi_2^y \phi_3^y \cdots \phi_{S-2}^y}{\phi_1^x \phi_2^x \phi_3^x \cdots \phi_{S-2}^x} \\
\beta_{S-3} &= \beta \frac{\phi_S^x \phi_{S-1}^x \phi_{S-2}^x}{\phi_S^y \phi_{S-1}^y \phi_{S-2}^y} = \beta \frac{\phi_1^y \phi_2^y \phi_3^y \cdots \phi_{S-3}^y}{\phi_1^x \phi_2^x \phi_3^x \cdots \phi_{S-3}^x} \\
&\vdots \\
\beta_2 &= \beta \frac{\phi_S^x \phi_{S-1}^x \phi_{S-2}^x \cdots \phi_3^x}{\phi_S^y \phi_{S-1}^y \phi_{S-2}^y \cdots \phi_3^y} = \beta \frac{\phi_1^y \phi_2^y}{\phi_1^x \phi_2^x} \\
\beta_1 &= \beta \frac{\phi_S^x \phi_{S-1}^x \phi_{S-2}^x \cdots \phi_2^x}{\phi_S^y \phi_{S-1}^y \phi_{S-2}^y \cdots \phi_2^y} = \beta \frac{\phi_1^y}{\phi_1^x},
\end{aligned}
\tag{16}
$$

where we use the fact that $\prod_{i=1}^{S} \phi_i^j = 1$ for $j = y$ and $j = x$. And in the case of (15), the periodic coefficients of the cointegration vector $[1, -\beta_s^*]$ evolve as follows:

$$
\begin{aligned}
\beta_S^* &= \beta^* a_S^y \left(a_S^x\right)^{-1} = \beta^* \frac{\phi_2^y \phi_3^y \cdots \phi_S^y}{\phi_2^x \phi_3^x \cdots \phi_S^x} \\
\beta_{S-1}^* &= \beta^* a_{S-1}^y \left(a_{S-1}^x\right)^{-1} = \beta^* \frac{\phi_2^y \phi_3^y \cdots \phi_{S-1}^y}{\phi_2^x \phi_3^x \cdots \phi_{S-1}^x} \\
\beta_{S-2}^* &= \beta^* a_{S-2}^y \left(a_{S-2}^x\right)^{-1} = \beta^* \frac{\phi_2^y \phi_3^y \cdots \phi_{S-2}^y}{\phi_2^x \phi_3^x \cdots \phi_{S-2}^x} \\
\beta_{S-3}^* &= \beta^* a_{S-3}^y \left(a_{S-3}^x\right)^{-1} = \beta^* \frac{\phi_2^y \phi_3^y \cdots \phi_{S-3}^y}{\phi_2^x \phi_3^x \cdots \phi_{S-3}^x} \\
&\vdots \\
\beta_2^* &= \beta^* a_2^y \left(a_2^x\right)^{-1} = \beta^* \frac{\phi_2^y}{\phi_2^x} \\
\beta_1^* &= \beta^* a_1^y \left(a_1^x\right)^{-1} = \beta^*.
\end{aligned}
\tag{17}
$$

Hence, it is clear that $\beta^* = \beta\phi_1^y/\phi_2^x$, and that (12)&(16) and (15)&(17) are two alternative and equivalent ways of representing full periodic cointegration between two PI processes.

2.4 The multivariate case

Let us consider the $n \times 1$ vector process $Y_{s\tau}^{(n)} = \begin{bmatrix} y_{s\tau}^1 & y_{s\tau}^2 & \ldots & y_{s\tau}^n \end{bmatrix}' = \begin{bmatrix} Y_{s\tau}^{1'} & Y_{s\tau}^{2'} \end{bmatrix}'$ in which $Y_{s\tau}^1$ is $r \times 1$, that is, $Y_{s\tau}^1 = \begin{bmatrix} y_{s\tau}^1 & y_{s\tau}^2 & \ldots & y_{s\tau}^r \end{bmatrix}'$, and $Y_{s\tau}^2$ is $(n-r) \times 1$, that is, $Y_{s\tau}^2 = \begin{bmatrix} y_{s\tau}^{r+1} & y_{s\tau}^{r+2} & \ldots & y_{s\tau}^n \end{bmatrix}'$. Our objective is to define a triangular system for n PI processes with r cointegration relationships, or equivalently, $n-r$ common stochastic trends between the n PI processes. The elements of $Y_{s\tau}^2$ can be identified with the $n-r$ common stochastic trends of the triangular system. Hence, the elements of $Y_{s\tau}^2$ are such that:

$$
y_{s\tau}^k = \phi_s^k y_{s-1,\tau}^k + u_{s\tau}^k \quad \prod_{s=1}^{S} \phi_s^k = 1, \quad s = 1, 2, \ldots, S, \quad k = r+1, r+2, \ldots, n, \tag{18}
$$

where each $u_{s\tau}^k$ is a stationary periodic autoregressive process[6]:

$$
\left(1 - \psi_{1s}^k L - \cdots - \psi_{p-1,s}^k L^{p-1}\right) u_{s\tau}^k = \varepsilon_{s\tau}^k. \tag{19}
$$

[6] Then the for the vector of seasons representation of each $u_{s\tau}^k$, that is $U_\tau^k = \left[u_{1\tau}^k, u_{2\tau}^k, ..., u_{S\tau}^k\right]'$:

$$
\mathbf{\Psi}_0^j U_\tau^k = \mathbf{\Psi}_1^k U_{\tau-1}^k + \mathbf{\Psi}_2^k U_{\tau-2}^k + \cdots + \mathbf{\Psi}_G^k U_{\tau-G}^k + E_\tau^k
$$

with $G = 1 + [(p-2)/S]$, all the roots of all the roots of $\left|\mathbf{\Psi}_0^k - \mathbf{\Psi}_1^k z - \mathbf{\Psi}_2^k z^2 - \cdots - \mathbf{\Psi}_G^k z^G\right| = 0$ should lie outside of the unit circle.

We start by defining the zero-frequency triangular system as follows:

$$\begin{aligned} Y_{s\tau}^{1(0)} &= \beta Y_{s\tau}^{2(0)} + U_{s\tau}^{1(0)} \\ Y_{s\tau}^{2(0)} &= \left(\mathbf{D}_s^2\right)^{-1} Y_{s\tau}^2 + Stationary \quad terms \end{aligned} \tag{20}$$

where $Y_{s\tau}^{1(0)}$ is an $r \times 1$ vector, β is an $r \times (n-r)$ matrix, and $U_{s\tau}^{1(0)}$ is an $r \times 1$ vector of innovations where each innovation follows a stationary PAR(p-1) process like in (19). Clearly, the cointegration vector in (20) is $[I_r - \beta]$. Finally, $\mathbf{D}_s^2$ is an $(n-r) \times (n-r)$ diagonal matrix such that:

$$\mathbf{D}_s^2 = diag \left[\begin{array}{ccccc} a_s^{r+1} & a_s^{r+2} & a_s^{r+2} & \cdots & a_s^n \end{array} \right], \tag{21}$$

where a_s^k, for $k = r+1, r+2, \ldots, n$, are the s^{th} elements of the $S \times 1$ vectors $\mathbf{a}^k$, for $k = r+1, r+2, \ldots, n$, associated with process (18), that is, $\mathbf{a}^k = \left[1, \phi_2^k, \phi_2^k \phi_3^k, \ldots, \prod_{s=2}^S \phi_s^k\right]'$ for $k = r+1,\ r+2, \ldots, n$. Note that (20) is the multivariate equivalent to (13) in the bivariate context. Finally, the triangular system for PI processes with n variables, r periodic cointegration relationships, or $n-r$ common stochastic trends between the seasons of the n PI processes, can be obtained by replacing $Y_{s\tau}^{2(0)} = \left(\mathbf{D}_s^2\right)^{-1} Y_{s\tau}^2$ with $Y_{s\tau}^{1(0)} = \beta Y_{s\tau}^{2(0)} + U_{s\tau}^{1(0)}$ and left-multiplying $Y_{s\tau}^{1(0)} = \beta Y_{s\tau}^{2(0)} + U_{s\tau}^{1(0)}$ by $\mathbf{D}_s^1$, a $r \times r$ diagonal matrix:

$$\mathbf{D}_s^1 = diag \left[\begin{array}{ccccc} a_s^1 & a_s^2 & a_s^2 & \cdots & a_s^r \end{array} \right], \tag{22}$$

where $\mathbf{a}_s^j$, for $j+1,\ 2, \ldots, r$, are the s^{th} elements of the $S \times 1$ vectors $\mathbf{a}^j$, for $j = 1,\ 2, \ldots, r$, defined as $\mathbf{a}^j = \left[1, \phi_2^j, \phi_2^j \phi_3^j, \ldots, \prod_{s=2}^S \phi_s^j\right]'$, such that $\prod_{s=1}^S \phi_s^j = 1$ for $j = 1,\ 2,\ \ldots,\ r$, in order to have PI processes:

$$\begin{aligned} Y_{s\tau}^1 &= \mathbf{D}_s^1 \beta \left(\mathbf{D}_s^2\right)^{-1} Y_{s\tau}^2 + \mathbf{D}_s^1 U_{s\tau}^{1(0)} \\ Y_{s\tau}^{2(0)} &= \left(\mathbf{D}_s^2\right)^{-1} Y_{s\tau}^2 + Stationary \quad terms \\ Y_{s\tau}^1 &= \mathbf{D}_s^1 Y_{s\tau}^{1(0)}. \end{aligned} \tag{23}$$

Definition 1 in del Barrio Castro and Osborn (2008a) establishes periodic cointegration for an $n \times 1$ vector, $Y_{s\tau}^{(n)}$, of PI processes if there exist $n \times r$ matrices, β_s, of rank r such that the linear combinations $\beta_s' Y_{s\tau}^{(n)}$ are (periodically) stationary for each season $s = 1, 2, \ldots, S$. In our case, we use the usual normalization for triangular systems (see Lütkepohl (2006)). Hence, we have $\beta_s' = \left[\begin{array}{cc} I_r & -\mathbf{D}_s^1 \beta \left(\mathbf{D}_s^2\right)^{-1} \end{array} \right]$. Boswijk and Franses (1995) define partial periodic cointegration when stationary linear combinations, $\beta_s' Y_{s\tau}^{(m)}$, exist in only some seasons, and full periodic cointegration when the linear combinations exist for all of the seasons. Full non-periodic cointegration is a particular case of full periodic cointegration in which the same $n \times r$ matrix, β, allows us to obtain stationary linear combinations for all of the seasons. Clearly, in order to have full non-periodic cointegration, we need all of the PI processes in the triangular system to have the same coefficients associated with the PI condition, that is, $\phi_s^j = \phi_s$ for $j = 1,\ 2,\ \ldots,\ n$ and $s = 1,\ 2,\ \ldots,\ S$.

Note that for (23) it is possible to write:

$$\begin{aligned} \left[\begin{array}{cc} I & -\mathbf{D}_s^1 \beta \left(\mathbf{D}_s^2\right)^{-1} \\ 0 & I \end{array} \right] \left[\begin{array}{c} Y_{s\tau}^1 \\ Y_{s\tau}^2 \end{array} \right] &= \left[\begin{array}{cc} 0 & 0 \\ 0 & \Phi_s^2 \end{array} \right] \left[\begin{array}{c} Y_{s-1,\tau}^1 \\ Y_{s-1,\tau}^2 \end{array} \right] + \left[\begin{array}{c} U_{s\tau}^1 \\ U_{s\tau}^2 \end{array} \right] \\ \Phi_s^2 &= diag \left[\begin{array}{cccc} \phi_s^{r+1} & \phi_s^{r+2} & \cdots & \phi_s^n \end{array} \right], \end{aligned} \tag{24}$$

using the fact that $\left(a_s^j\right)^{-1} = b_s^j$ and $b_s^j \phi_s^j = b_{s-1}^j$ it is possible to write $\left(a_s^j\right)^{-1} \phi_s^j = \left(a_{s-1}^j\right)^{-1}$, hence $\left(\mathbf{D}_s^2\right)^{-1} \Phi_s^2 = \left(\mathbf{D}_{s-1}^2\right)^{-1}$ and using results from the inverse of a partitioned matrices, we could re-write (24) as:

$$\left[\begin{array}{c} Y_{s\tau}^1 \\ Y_{s\tau}^2 \end{array} \right] = \left[\begin{array}{cc} 0 & \mathbf{D}_s^1 \beta \left(\mathbf{D}_{s-1}^2\right)^{-1} \\ 0 & \Phi_s^2 \end{array} \right] \left[\begin{array}{c} Y_{s-1,\tau}^1 \\ Y_{s-1,\tau}^2 \end{array} \right] + \left[\begin{array}{cc} I & \mathbf{D}_s^1 \beta \left(\mathbf{D}_s^2\right)^{-1} \\ 0 & I \end{array} \right] \left[\begin{array}{c} U_{s\tau}^1 \\ U_{s\tau}^2 \end{array} \right] \tag{25}$$

Note that (25) is quite similar to the Periodic Vector Autoregressive model of order 1 (PVAR(1)) (expression (2.1)) in Kleibergen and Franses (1999), that is $y_n = \varphi_s y_{n-1} + u_n$, in our case equivalently to φ_s, we also have a $n \times n$ matrix that varies across the seasons $s = 1,\ 2,\ \ldots,\ S$. But, it is clear from (25) that contrary to what it is stated for φ_s in (2.1) in Kleibergen and Franses (1999), our $n \times n$ matrix does not have full column rank. To have a full rank matrix, in a PVAR(1) like $Y_{s\tau}^{(n)} = \Phi_s^{(n)} Y_{s-1\tau}^{(n)} + U_{s\tau}^{(n)}$, cointegration between the PI

processes collected in $Y_{s\tau}^{(n)}$ must not hold, that is, for example with $\Phi_s^{(n)} = diag\left[\begin{array}{ccccc} \phi_s^1 & \phi_s^2 & \phi_s^3 & \cdots & \phi_s^n \end{array}\right]$, precisely the case of Lemma 3 in del Barrio Castro and Osborn (2008a) (for the quarterly case), where we have cointegration across the seasons of each PI process, but we do not cointegration between the seasons of differents PI processes in $Y_{s\tau}^{(n)}$.In what follows, Lemma 3 of del Barrio Castro and Osborn (2008a) will be extended from the particular case of $S = 4$, to the general case of S seasons per year[7].Hence, we have:

$$\begin{aligned} Y_{s\tau}^{(n)} &= \Phi_s^{(n)} Y_{s-1\tau}^{(n)} + U_{s\tau}^{(n)} \\ \Phi_s^{(n)} &= diag\left[\begin{array}{ccccc} \phi_s^1 & \phi_s^2 & \phi_s^3 & \cdots & \phi_s^n \end{array}\right] \\ U_{s\tau}^{(n)} &= \left[\begin{array}{ccccc} u_{s\tau}^1 & u_{s\tau}^2 & u_{s\tau}^3 & \cdots & u_{s\tau}^n \end{array}\right]' \\ Y_{s\tau}^{(n)} &= \left[\begin{array}{ccccc} y_{s\tau}^1 & y_{s\tau}^2 & y_{s\tau}^3 & \cdots & y_{s\tau}^n \end{array}\right]'. \end{aligned} \tag{26}$$

With $u_{s\tau}^k$ for $k = 1, 2, \ldots, n$ as in (19) (that is following stationary periodic PAR processes. see also footnote 6 above). The Vector of Seasons representation associated to (26) will be as follows:

$$\begin{aligned} \mathbf{A}_0^{(n)} Y_\tau^{(n)} &= \mathbf{A}_1^{(n)} Y_{\tau-1}^{(n)} + U_\tau^{(n)} \\ Y_\tau^{(n)} &= \left[\begin{array}{cccc} y_{1\tau}^1, \ldots, y_{S\tau}^1 & y_{1\tau}^2, \ldots, y_{S\tau}^2 & \cdots & y_{1\tau}^3, \ldots, y_{S\tau}^3 \end{array}\right]' \\ &= \left[\begin{array}{c} Y_\tau^1 \\ Y_\tau^2 \\ \vdots \\ Y_\tau^n \end{array}\right] \\ U_\tau^{(n)} &= \left[\begin{array}{cccc} u_{1\tau}^1, \ldots, u_{S\tau}^1 & u_{1\tau}^2, \ldots, u_{S\tau}^2 & \cdots & u_{1\tau}^3, \ldots, u_{S\tau}^3 \end{array}\right]' \\ &= \left[\begin{array}{c} U_\tau^1 \\ U_\tau^2 \\ \vdots \\ U_\tau^n \end{array}\right]. \end{aligned} \tag{27}$$

The matrices $\mathbf{A}_0^{(n)}$ and $\mathbf{A}_1^{(n)}$ are square block diagonal matrices of dimension $(n \times S) \times (n \times S)$ such that:

$$\begin{aligned} \mathbf{A}_0^{(n)} &= diag\left[\mathbf{A}_0^1, \mathbf{A}_0^2, \cdots, \mathbf{A}_0^n\right] \\ \mathbf{A}_1^{(n)} &= diag\left[\mathbf{A}_1^1, \mathbf{A}_1^2, \cdots, \mathbf{A}_1^n\right], \end{aligned} \tag{28}$$

with the following $S \times S$ submatrices:

$$\mathbf{A}_0^j = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\phi_2^j & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\phi_3^j & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\phi_4^j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_S^j & 1 \end{array}\right] \quad j = 1, 2, \ldots, n \tag{29}$$

$$\mathbf{A}_1^j = \left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & \phi_1^j \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \quad j = 1, 2, \ldots, n. \tag{30}$$

The stochastic behavior of the system is summarized in the following lemma.

Lemma 2 *For* $Y_\tau^{(n)} = \left[Y_\tau^{1\prime}, Y_\tau^{2\prime}, , , \ldots Y_\tau^3\right]'$ *defined in (27-28-29-30); with* $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right) u_{s\tau}^j = \varepsilon_{s\tau}^j$, *for* $j = 1, 2, \ldots$ n; *and* $E_{s\tau}^{(n)} = \left[\begin{array}{cccc} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \cdots & \varepsilon_{s\tau}^n \end{array}\right]'$ *is a white noise vector with the positive definite*

[7] As Lemma 1 extend Lemma 1 of Boswijk and Franses from $S = 4$ to a generic case of S seasons per year.

variance-covariance matrix $E\left[E^{(n)}_{s\tau}E^{(n)\prime}_{s\tau}\right]=\Sigma$*, then*

$$\frac{1}{\sqrt{T}}Y^{(n)}_{\lfloor Tr\rfloor}\Rightarrow\left(\mathbf{A}^{(n)}_0\right)^{-1}\mathbf{A}^{(n)}_1\left(\mathbf{A}^{(n)}_0\right)^{-1}\Psi^{(n)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(n)}(r) \tag{31}$$
$$=\begin{bmatrix}\mathbf{a}_1\mathbf{b}_1' & 0_{S\times S} & \cdots & 0_{S\times S}\\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & \cdots & 0_{S\times S}\\ \vdots & \vdots & \ddots & \vdots\\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n\mathbf{b}_n'\end{bmatrix}\Psi^{(n)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(n)}(r)$$
$$=\begin{bmatrix}\omega_1\mathbf{a}_1w_1(r)\\ \omega_2\mathbf{a}_2w_2(r)\\ \vdots\\ \omega_n\mathbf{a}_nw_n(r)\end{bmatrix},$$

∞*where* $\mathbf{a}_j$ *and* $\mathbf{b}_j$*, for* $j=1,2,\ldots,n$*, are defined in (53),* $W^{(n)}(r)$ *is a* $(n\times S)\times 1$ *multivariate Brownian Vector, and* $w_j(r)$*, for* $j=1,2,\ldots,n$*, are scalar Brownian motions defined in the appendix. Finally, the definition of matrix* $\Psi^{(n)}(1)$ *and the scalar terms* ω_j*, for* $j=1,2,\ldots,n$*, can also be found in the appendix, and* $\mathbf{P}$ *is a* $n\times n$ *matrix such that* $\Sigma=\mathbf{PP}'$*.*

It is clear from (31) that each PI process has is own stochastic trend identified with the scalar Brownian motions $w_j(r)$ for $j=1,2,\ldots,n$. Note also that for $Y^{(n)}_{s\tau}$ in (26), it is possible write (based on (31)):

$$\frac{1}{\sqrt{T}}Y^{(n)}_{s\lfloor Tr\rfloor}\Rightarrow\begin{bmatrix}\omega_1a^1_sw_1(r)\\ \omega_2a^2_sw_2(r)\\ \vdots\\ \omega_na^n_sw_n(r)\end{bmatrix}=\mathbf{D}^{(n)}_s\begin{bmatrix}\omega_1w_1(r)\\ \omega_2w_2(r)\\ \vdots\\ \omega_nw_n(r)\end{bmatrix} \tag{32}$$
$$\mathbf{D}^{(n)}_s=diag\begin{bmatrix}a^1_s & a^2_s & \cdots & a^n_s\end{bmatrix}.$$

In the case of cointegration between the PI processes collected in $Y^{(n)}_{s\tau}$ for (24) and (25) it is possible to write:

$$\begin{bmatrix}Y^1_{s\tau}\\ Y^2_{s\tau}\end{bmatrix}-\begin{bmatrix}\Phi^1_s & 0\\ 0 & \Phi^2_s\end{bmatrix}\begin{bmatrix}Y^1_{s-1,\tau}\\ Y^2_{s-1,\tau}\end{bmatrix}=\begin{bmatrix}-\Phi^1_s & \mathbf{D}^1_s\beta\left(\mathbf{D}^2_{s-1}\right)^{-1}\\ 0 & 0\end{bmatrix}\begin{bmatrix}Y^1_{s-1,\tau}\\ Y^2_{s-1,\tau}\end{bmatrix}+\begin{bmatrix}I & \mathbf{D}^1_s\beta\left(\mathbf{D}^2_s\right)^{-1}\\ 0 & I\end{bmatrix}\begin{bmatrix}U^1_{s\tau}\\ U^2_{s\tau}\end{bmatrix}.$$

Where we just subtract the same term in both sides of (25). Note also, that it is possible to write $\mathbf{D}^1_s=\Phi^1_s\mathbf{D}^1_{s-1}$, and hence obtain:

$$\begin{bmatrix}Y^1_{s\tau}\\ Y^2_{s\tau}\end{bmatrix}-\begin{bmatrix}\Phi^1_s & 0\\ 0 & \Phi^2_s\end{bmatrix}\begin{bmatrix}Y^1_{s-1,\tau}\\ Y^2_{s-1,\tau}\end{bmatrix}=\begin{bmatrix}-\Phi^1_s\\ 0\end{bmatrix}\begin{bmatrix}I & -\mathbf{D}^1_{s-1}\beta\left(\mathbf{D}^2_{s-1}\right)^{-1}\end{bmatrix}\begin{bmatrix}Y^1_{s-1,\tau}\\ Y^2_{s-1,\tau}\end{bmatrix}+\begin{bmatrix}I & \mathbf{D}^1_s\beta\left(\mathbf{D}^2_s\right)^{-1}\\ 0 & I\end{bmatrix}\begin{bmatrix}U^1_{s\tau}\\ U^2_{s\tau}\end{bmatrix}. \tag{33}$$

Hence (33) is a Periodic Vector Error Correction Model, expressed in terms of the periodic difference needed to achive stationarity for each of the elements of $Y^{(n)}_\tau=\begin{bmatrix}Y^{1\prime}_{s\tau} & Y^{2\prime}_{s\tau}\end{bmatrix}'$, say $y^k_{s\tau}-\phi^k_sy^k_{s-1,\tau}$ for $k=1,2,\ldots,n$. Furthermore, as the innovations $u^k_{s\tau}$ collected in both $U^1_{s\tau}$ and $U^2_{s\tau}$ follow stationary PAR$(p-1)$ processes, see (19), the Periodic Vector Error Correction Model (33) could to be augmented with $p-1$ lags of the periodic differences:

$$\begin{bmatrix}Y^1_{s\tau}\\ Y^2_{s\tau}\end{bmatrix}-\begin{bmatrix}\Phi^1_s & 0\\ 0 & \Phi^2_s\end{bmatrix}\begin{bmatrix}Y^1_{s-1,\tau}\\ Y^2_{s-1,\tau}\end{bmatrix}=\begin{bmatrix}-\Phi^1_s\\ 0\end{bmatrix}\begin{bmatrix}I & -\mathbf{D}^1_{s-1}\beta\left(\mathbf{D}^2_{s-1}\right)^{-1}\end{bmatrix}\begin{bmatrix}Y^1_{s-1,\tau}\\ Y^2_{s-1,\tau}\end{bmatrix}+\begin{bmatrix}I & \mathbf{D}^1_s\beta\left(\mathbf{D}^2_s\right)^{-1}\\ 0 & I\end{bmatrix}\times \tag{34}$$
$$\times\sum_{j=1}^{p-1}\left\{\begin{bmatrix}\Psi^1_{j,s} & 0\\ 0 & \Psi^2_{j,s}\end{bmatrix}\left(\begin{bmatrix}Y^1_{s-j,\tau}\\ Y^2_{s-j,\tau}\end{bmatrix}-\begin{bmatrix}\Phi^1_{s-j} & 0\\ 0 & \Phi^2_{s-j}\end{bmatrix}\begin{bmatrix}Y^1_{s-1-j,\tau}\\ Y^2_{s-1-j,\tau}\end{bmatrix}\right)\right\}$$
$$+\begin{bmatrix}E^1_{s,\tau}\\ E^2_{s,\tau}\end{bmatrix}.$$

With $\Psi^1_{j,s}$ and $\Psi^1_{j,s}$ been diagonal matrix defined as $\Psi^1_{j,s}=diag\begin{bmatrix}\psi^1_{js} & \psi^2_{js} & \psi^3_{js} & \cdots & \psi^r_{js}\end{bmatrix}$ and $\Psi^2_{j,s}=diag\begin{bmatrix}\psi^{r+1}_{js} & \psi^{r+2}_{js} & \psi^{r+3}_{js} & \cdots & \psi^n_{js}\end{bmatrix}$ with ψ^k_{js} been the coefficients in (19) for $k=1,2,\ldots,n$.

As can be seen in subsection 2.1 and in lemma 2, the vector of seasons representation is a very convenient tool for representing PI processes. This representation allows us to clearly appreciate that the non-stationary stochastic behavior of the seasons of a PI process is ruled by a common stochastic trend. In the case of cointegration between the PI processes, we could also use the vector of seasons representation, in particular for (34) or (24) we a representation like in (27) but with the $(n \times S) \times (n \times S)$ matrices $\mathbf{A}_0^{(n)}$ and $\mathbf{A}_1^{(n)}$ defined as follows:

$$
\begin{aligned}
\mathbf{A}_0^{(n)} &= \begin{bmatrix} \mathbf{I}_{(r\times S)} & -\mathbf{D}^1\left(\beta \otimes \mathbf{I}_S\right)\left(\mathbf{D}^2\right)^{-1} \\ 0_{([n-r]\times S)\times(r\times S)} & \mathbf{A}_0^2 \end{bmatrix} \\
\mathbf{A}_0^{(n)} &= \begin{bmatrix} 0_{(r\times S)\times(r\times S)} & 0_{(r\times S)\times([n-r]\times S)} \\ 0_{([n-r]\times S)\times(r\times S)} & \mathbf{A}_1^2 \end{bmatrix} \\
\mathbf{A}_0^{(2)} &= diag\left[\mathbf{A}_0^{r+1}, \mathbf{A}_0^{r+2}, \cdots, \mathbf{A}_0^{n}\right] \\
\mathbf{A}_1^{(2)} &= diag\left[\mathbf{A}_1^{r+1}, \mathbf{A}_1^{r+2}, \cdots, \mathbf{A}_1^{n}\right] \\
\mathbf{D}^1 &= diag\left[diag(\mathbf{a}_1), diag(\mathbf{a}_2), \cdots, diag(\mathbf{a}_r)\right] \\
\mathbf{D}^2 &= diag\left[diag(\mathbf{a}_{r+1}), diag(\mathbf{a}_{r+2}), \cdots, diag(\mathbf{a}_n)\right].
\end{aligned}
\tag{35}
$$

Where $\mathbf{I}_{(r\times S)}$ and $\mathbf{I}_S$ are identity matrices of dimension $(r \times S)$ and S respectively. And $0_{([n-r]\times S)\times(r\times S)}$, $0_{(r\times S)\times(r\times S)}$ and $0_{(r\times S)\times([n-r]\times S)}$ are matrices of zeros with dimensions $([n-r] \times S) \times (r \times S)$, $(r \times S) \times (r \times S)$ and $(r \times S) \times ([n-r] \times S)$ respectively. Finally, the submatrices $\mathbf{A}_0^j$ and $\mathbf{A}_1^j$ for $j = r+1, r+2, \ldots, n$ are defined as in (29) and (30) but only for $j = r+1, r+2, \ldots, n$. The following lemma summarizes the stochastic behavior of the vector of seasons when we have r cointegration relationships between a set of n PI processes:

Lemma 3 *For* $Y_\tau^{(n)} = \left[Y_\tau^{1\prime}, Y_\tau^{2\prime}, , \ldots Y_\tau^{3}\right]'$ *defined in (27-35) associated to (34); with* $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right) u_{s\tau}^j = \varepsilon_{s\tau}^j$, *for* $j = 1, 2, \ldots n$; *and* $E_{s\tau}^{(n)} = \begin{bmatrix} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \cdots & \varepsilon_{s\tau}^n \end{bmatrix}'$ *is a white noise vector with the positive definite variance-covariance matrix* $E\left[E_{s\tau}^{(n)} E_{s\tau}^{(n)\prime}\right] = \Sigma$, *then*

$$
\frac{1}{\sqrt{T}} Y_{\lfloor Tr \rfloor}^{(n)} \Rightarrow \begin{bmatrix}
0_{S\times S} & \cdots & 0_{S\times S} & \beta_{11}\mathbf{a}_1\mathbf{b}_{r+1}' & \beta_{12}\mathbf{a}_1\mathbf{b}_{r+2}' & \cdots & \beta_{1n}\mathbf{a}_1\mathbf{b}_n' \\
0_{S\times S} & \cdots & 0_{S\times S} & \beta_{21}\mathbf{a}_2\mathbf{b}_{r+1}' & \beta_{22}\mathbf{a}_2\mathbf{b}_{r+2}' & \cdots & \beta_{2n}\mathbf{a}_2\mathbf{b}_n' \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0_{S\times S} & \cdots & 0_{S\times S} & \beta_{r1}\mathbf{a}_r\mathbf{b}_{r+1}' & \beta_{r2}\mathbf{a}_r\mathbf{b}_{r+2}' & \cdots & \beta_{rn}\mathbf{a}_r\mathbf{b}_n' \\
0_{S\times S} & \cdots & 0_{S\times S} & \mathbf{a}_{r+1}\mathbf{b}_{r+1}' & 0_{S\times S} & \cdots & 0_{S\times S} \\
0_{S\times S} & \cdots & 0_{S\times S} & 0_{S\times S} & \mathbf{a}_{r+2}\mathbf{b}_{r+2}' & \cdots & 0_{S\times S} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0_{S\times S} & \cdots & 0_{S\times S} & 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n\mathbf{b}_n'
\end{bmatrix} \Psi^{(n)}(1)^{-1}\left[\mathbf{P} \otimes \mathbf{I}_S\right] W^{(n)}(r)
$$

$$
= \begin{bmatrix}
\mathbf{a}_1 \sum_{j=1}^n \beta_{1j}\omega_{j+1} w_{j+1}(r) \\
\mathbf{a}_2 \sum_{j=1}^n \beta_{2j}\omega_{j+1} w_{j+1}(r) \\
\vdots \\
\mathbf{a}_r \sum_{j=1}^n \beta_{rj}\omega_{j+1} w_{j+1}(r) \\
\omega_{r+1}\mathbf{a}_{r+1} w_{r+1}(r) \\
\omega_{r+2}\mathbf{a}_{r+2} w_{r+2}(r) \\
\vdots \\
\omega_n \mathbf{a}_n w_n(r)
\end{bmatrix}. \tag{36}
$$

where $\mathbf{a}_j$ *for* $j = 1, 2, \ldots, n$ *and* $\mathbf{b}_j$, *for* $j = r+1, r+2, \ldots, n$, $W^{(n)}(r)$ *is a* $(n \times S) \times 1$, $w_j(r)$, *for* $j = r+1, r+2, \ldots, n$, *matrix* $\Psi^{(n)}(1)$ *and the scalar terms* ω_j, *for* $j = r+1, r+2, \ldots, n$ *are defined as in Lemma 2.*

As in Lemma 2 note that based on (36) it is possible to write for $Y_{s\tau}^{(n)}$ in (24):

$$\frac{1}{\sqrt{T}}Y_{s\lfloor Tr\rfloor}^{(n)} \Rightarrow \begin{bmatrix} a_s^1\sum_{j=1}^{n}\beta_{1j}\omega_{j+1}w_{j+1}(r) \\ \vdots \\ a_s^r\sum_{j=1}^{n}\beta_{rj}\omega_{j+1}w_{j+1}(r) \\ \omega_{r+1}a_s^{r+1}w_{r+1}(r) \\ \vdots \\ \omega_n a_s^n w_n(r) \end{bmatrix} = \mathbf{D}_s^{(n)} \begin{bmatrix} \sum_{j=1}^{n}\beta_{1j}\omega_{j+1}w_{j+1}(r) \\ \vdots \\ \sum_{j=1}^{n}\beta_{rj}\omega_{j+1}w_{j+1}(r) \\ \omega_{r+1}w_{r+1}(r) \\ \vdots \\ \omega_n w_n(r) \end{bmatrix} \tag{37}$$

$$\mathbf{D}_s^{(n)} = diag\begin{bmatrix} a_s^1 & \cdots & a_s^r & a_s^{r+1} & \cdots & a_s^n \end{bmatrix}.$$

Hence from (32) and (37) we could say that our approach of using $\tilde{y}_{s\tau}^j = \left(a_s^j\right)^{-1} y_{s\tau}^j$, as an input in the Johansen standard procedure will work without problems, if the parameters associated to the PI condition are known for each time series $y_{s\tau}^j$ $j = 1, 2, \ldots, n$. In the following section we present our proposal for determining the cointegration rank with reduced-rank regression techniques in systems of PI processes.

3 Econometric Methodology

As mentioned previously our proposal is based on the demodulation approach used in BCCO (2020). In the previous section, we clearly show that for a particular PI process we have $S-1$ cointegration relationships between the seasons, or equivalently, there is a common stochastic trend shared by the seasons of the PI process. In Lemmas 1 to 3, the common stochastic trends are identified with scalar Brownian motions that drive the long-run behavior of the seasons in each PI process in the systems. For example, in Lemma 2 we have n common stochastic trends identified with the scalar Brownian motions $w_j(r)$ $j = 1, 2, \ldots, n$. These stochastic trends are adjusted to each season in the PI process by the elements of the $S \times 1$ vectors a^j, for $j = 1, 2, \ldots, n$. Note that the elements of the vector are the coefficients associated with the restriction of being PI, that is, $\prod_{s=1}^{S}\phi_s^j = 1$, for $j = 1, 2, \ldots, n$. In Lemma 3, the stochastic non-stationary behavior of the seasons of the n PI processes is ruled by $n-r$ common stochastic trends identified with $w_j(r)$ for $j = r+1, r+2, \ldots, n$ adjusted to each season of the PI processes by the elements of the $S \times 1$ vectors $\mathbf{a}^j$, for $j = 1, 2, \ldots, n$.

Hence, our approach is based on the simple idea of demodulating each time series by multiplying each season by the reciprocal (or inverse) of the corresponding element of vector $\mathbf{a}^j = \begin{bmatrix} a_1^j & a_2^j & a_3^j & \cdots & a_S^j \end{bmatrix}' = \begin{bmatrix} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{bmatrix}'$, that is, we work with the new time series $\tilde{y}_{s\tau}^j = \left(a_s^j\right)^{-1} y_{s\tau}^j$. Clearly, our approach implies knowledge of the coefficients associated with the PI restriction $\prod_{s=1}^{S}\phi_s^j = 1$. This limitation can be easily resolved with a test for periodic integration, such as the likelihood ratio test proposed by Boswijk and Franses (1996) or the multivariate approach taken by Franses (1994).[8] In this paper, we use the Boswijk and Franses (1996) test rather than the one proposal by Franses (1994), as the latter has problems concerning over-parametrization (for quarterly data you need to run the Johansen procedure with four time series, i.e., each quarter is treated as a different time series). If we want to determine the cointegration rank between PI processes, a previous and necessary condition is to test (or be sure) that all the analyzed time series behave like PI processes. Furthermore, we can take advantage of this initial step and use it to obtain information about the values of the parameters associated with the PI condition (that is, $\prod_{s=1}^{S}\phi_s^j = 1$).

To summarize, our approach consists of the following steps:

- Testing for periodic integration using the Boswijk and Franses (1996) likelihood ratio test and retaining the values of the fitted coefficients associated with vector $\mathbf{a}^j = \begin{bmatrix} a_1^j & a_2^j & a_3^j & \cdots & a_S^j \end{bmatrix}' = \begin{bmatrix} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{bmatrix}'$,

[8] Although non-parametric tests of the null of periodic integration were proposed by del Barrio Castro and Osborn (2011, 2012), these tests are not valid here as they do not require an estimation of the coefficients associated with the restriction of being periodically integrated.

- Obtaining $\tilde{y}^j_{s\tau} = \left(a^j_s\right)^{-1} y^j_{s\tau}$ based on the estimation of the elements of $\mathbf{a}^j$ in the previous step, and finally,
- Including the demodulated time series $\tilde{y}^j_{s\tau}$ in the usual Johansen procedure and determining the cointegration rank.

Note that we can use the standard critical values of the Johansen procedure. Also, it is important to highlight that our approach has a clear advantage over the Boswijk and Franses (1995) and del Barrio Castro and Osborn (2008a) approaches, as these methods do not allow us to determine the cointegration rank between a set of PI time series. Finally, we do not need to use a periodic VAR framework or GMM jointly with reduced-rank regression techniques as in Kleibergen and Franses (1999).

The canonical correlation procedure by Johansen works with the demodulated time series $\tilde{y}^j_{s\tau} = \left(\mathbf{a}^j_s\right)^{-1} y^j_{s\tau}$, based on the true unknown parameters associated with the PI condition ($\prod_{s=1}^{S} \phi^j_s = 1$), collected in the $S \times 1$ vectors $\mathbf{a}^j$, for $j = 1, 2,$ and 3. But, in order to implement our approach, we use $\hat{\mathbf{a}}_j = \left[\begin{array}{cccccc} 1 & \hat{\phi}^j_2 & \hat{\phi}^j_2\hat{\phi}^j_3 & \cdots & \prod_{s=2}^{S}\hat{\phi}^j_s \end{array}\right]' = \left[\begin{array}{ccccc} \hat{a}^j_1 & \hat{a}^j_2 & \hat{a}^j_3 & \cdots & \hat{a}^j_S \end{array}\right]'$. From Boswijk and Franses (1996) and Boswijk, Franses, and Haldrup (1997) we know that the estimators of ϕ^j_s obtained from their test procedures are super-consistent. They show that $T\left(\hat{\phi}^j_s - \phi^j_s\right) = O_p(1)$, and hence, $\hat{\phi}^j_s = \phi^j_s + o_p(1)$.

In the quarterly case, for example, from Lemma 1 and Lemma 2, it is possible to write:

$$\begin{aligned}
T^{-1/2} y_{1,\lfloor Tr \rfloor} &\Rightarrow \sigma a^j_1 w_j(r) = \sigma w_j(r) \\
T^{-1/2} y_{2,\lfloor Tr \rfloor} &\Rightarrow \sigma a^j_2 w_j(r) = \phi^j_2 \sigma w_j(r) = \left(\phi^j_1\phi^j_3\phi^j_4\right)^{-1} \sigma w_j(r) \\
T^{-1/2} y_{3,\lfloor Tr \rfloor} &\Rightarrow \sigma a^j_3 w_j(r) = \phi^j_2\phi^j_3 \sigma w_j(r) = \left(\phi^j_1\phi^j_4\right)^{-1} \sigma w_j(r) \\
T^{-1/2} y_{4,\lfloor Tr \rfloor} &\Rightarrow \sigma a^j_4 w_j(r) = \phi^j_2\phi^j_3\phi^j_4 \sigma w_j(r) = \left(\phi^j_1\right)^{-1} \sigma w_j(r).
\end{aligned}$$

Hence, clearly $T^{-1/2}\left(a^j_s\right)^{-1} y_{s,\lfloor Tr \rfloor} \Rightarrow \sigma \left(a^j_s\right)^{-1} a^j_s w_j(r) = \sigma w_j(r)$. But what happens if we use $\left(\hat{a}^s_j\right)^{-1} y_{s,t}$? We can evaluate the effects of using $\hat{a}^s_j$ instead of the true values of a^s_j by paying attention to expression (6):

$$Y_\tau = \mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \mathbf{A}_0^{-1} U_\tau + \mathbf{ab}' \sum_{j=1}^{\tau-1} U_{\tau-j} = \mathbf{ab}' \sum_{j=1}^{\tau} U_j + O_p(1) \tag{38}$$

after premultiplying by $\hat{\mathbf{D}}^{-1} = diag\left(\hat{a}_1^{-1}, \hat{a}_2^{-1}, \cdots, \hat{a}_S^{-1}\right)$. Note that $\hat{\mathbf{D}}^{-1} Y_\tau = \hat{\mathbf{D}}^{-1}\mathbf{ab}' \sum_{j=1}^{\tau} U_j + O_p(1)$, and for example, in the quarterly case:

$$\begin{aligned}
\hat{\mathbf{D}}^{-1}\mathbf{a} &= \left[\begin{array}{cccc} 1 & \frac{\phi_2}{\hat{\phi}_2} & \frac{\phi_2\phi_3}{\hat{\phi}_2\hat{\phi}_3} & \frac{\phi_2\phi_3\phi_4}{\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4} \end{array}\right]' \\
as \quad \phi_1\phi_2\phi_3\phi_4 = 1 &\quad and \quad \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4 = 1 \\
\hat{\mathbf{D}}^{-1}\mathbf{a} &= \left[\begin{array}{cccc} 1 & \frac{\hat{\phi}_1\hat{\phi}_3\hat{\phi}_4}{\phi_1\phi_3\phi_4} & \frac{\hat{\phi}_1\hat{\phi}_4}{\phi_1\phi_4} & \frac{\hat{\phi}_1}{\phi_1} \end{array}\right]' \\
&= \left[\begin{array}{c} 1 \\ \frac{(\phi_1+o_p(1))(\phi_3+o_p(1))(\phi_4+o_p(1))}{\phi_1\phi_3\phi_4} \\ \frac{(\phi_1+o_p(1))(\phi_4+o_p(1))}{\phi_1\phi_4} \\ \frac{\phi_1+o_p(1)}{\phi_1} \end{array}\right] \\
&= \mathbf{1}_{4\times 1} + o_p(1).
\end{aligned} \tag{39}$$

Hence, we can conclude that $\hat{\mathbf{D}}^{-1} Y_\tau = \mathbf{1}_{S\times 1}\mathbf{b}' \sum_{j=1}^{\tau} U_j + O_p(1)$, and anticipate that the canonical correlation procedure by Johansen for determining the cointegration rank will provide similar results whether we work with the true values collected in $\mathbf{a}^j$ (that is, a^j_s for $s = 1, 2, \ldots, S$) or the fitted ones (that is, $\hat{a}^j_s$ for $s = 1, 2, \ldots, S$) obtained in the Boswijk and Franses (1996) test of periodic integration. In the following section, this claim is supported with a Monte Carlo experiment (see Tables 2.a to 2.d and Table 2.e). To finish with the issue of using fitted values of a_s^j for $s = 1, 2, \ldots, S$ rather than true ones, in our discussion above, we have follow the same kind of arguments used by Xiao and Phillips (1999) (see Remark 9) and show that from (32) and (37) we could obtain the same results from the true and fitted values as we show in (38) and (39) that with the fitted and true values the asymtotic behaviour of (32) and (37) will be exactly the same.

Another relevant issue is how to treat the deterministic part. In the case of periodic integration the usual two specifications for the deterministic part are either seasonal dummies or seasonal dummies and trends, see Boswijk and Franses (1996) and Paap and Franses (1999), the latter of which, in particular, show that other possible specifications for the deterministic part (like including a constant, a constant and a trend, or seasonal dummies and trend) are not relevant in the case of periodic integration as the addition of an intercept to (3) leads to a seasonally varying trend in $E[y_{s\tau}]$, and hence, an annual growth rate $\left(1 - L^S\right) y_{s\tau}$ that varies over seasons. Furthermore, excluding the special case of an $I(1)$ process, these authors show that a PI process with an intercept cannot have a trend that is common over the seasons, regardless of whether the intercept is constant over the seasons or varies. Additionally, as shown in Lee (1992), Lee and Siklos (1995), Johansen and Schaumburg (1998), and Cubadda (2001), when including seasonal dummies, we have a distribution of critical values like in the Johansen procedure when testing with a constant. Hence, in our case the relevant critical values with seasonal dummies are those from the standard Johansen trace test with a constant. And when dealing with seasonal dummies and trends we use the critical values of the Johansen procedure with a constant and a linear trend (see also Tables 2.a to 2.e).

Finally, note that for periodic autoregressive processes like (1), we can use periodic polynomials in the lag operator to obtain $\left(1 - \phi_{1s}L - \phi_{2s}L^2 - \cdots - \phi_{ps}L^p\right) y_{s\tau} = \varepsilon_{s\tau}$. And as in Note 1, we can use the following factorization: $\left(1 - \phi_{1s}L - \phi_{2s}L^2 - \cdots - \phi_{ps}L^p\right) = (1 - \phi_s L)\left(1 - \psi_{1s}L - \cdots - \psi^*_{p-1,s}L^{p-1}\right)$,where the coefficients ϕ_s, for $s = 1, 2, \ldots, S$, are those associated with the PI restriction $\prod_{s=1}^{S} \phi_s = 1$. The standard augmentation in the Johansen procedure satisfactorily handles non-periodic dynamic behavior and non-periodic stationarity, but in the presence of periodic stationary dynamics, it is convenient to use periodic augmentation in the canonical correlation procedure to test for the cointegration rank. Hence, with periodic augmentation the VAR model used when testing for cointegration is as follows:

$$Y_t^{(n)} = \left[\tilde{y}_t^1, \tilde{y}_t^2, \ldots, \tilde{y}_t^n\right]'$$
$$\Delta Y_t^{(n)} = \alpha\beta' Y_{t-1}^{(n)} + \sum_{j=1}^{p-1}\sum_{s=1}^{S} \Gamma_{sj} d_{st} \Delta Y_{t-j}^{(n)} + E_t,$$

where d_{st}, for $s = 1, 2, \ldots, S$, are the usual seasonal dummies. In the following Monte Carlo section we present the results of the performance of the canonical correlation procedure with periodic augmentation compared to standard augmentation, and we show that periodic augmentation clearly performs well.

4 Monte Carlo

For our Monte Carlo experiment, we take a three-variable approach and explore the three possible situations that we could face with three PI processes. That is, no cointegration between three PI processes, a situation with one common stochastic trend shared by three PI processes (that is, two periodic cointegration relationships between three PI processes), and a final situation with two common stochastic trends shared by three PI processes (that is, one periodic cointegration relationship between three PI processes).

As mentioned in the previous section, we compare the results obtained when using the Johansen cointegration rank test with the true parameters versus the fitted ones (based on the Boswijk and Franses (1996) test), in order to obtain the pseudo-demodulated time series. We also assess the adequacy of the critical values of the Johansen trace test in our case, in terms of the deterministic part (see Hamilton (1994) Table B.10 and Johansen (1995) Tables 15.1, 15.2, and 15.4). All of these issues will be present in the following subsection on the case of no cointegration.

4.1 No cointegration

We consider three PI processes with no cointegration, like in (26), that is:

$$\begin{aligned} y^1_{s\tau} &= \phi^1_s y^1_{s-1,\tau} + u^1_{s\tau} \\ y^2_{s\tau} &= \phi^2_s y^2_{s-1\tau} + u^2_{s\tau} \\ y^3_{s\tau} &= \phi^3_s y^3_{s-1,\tau} + u^3_{s\tau} \\ s &= 1,2,3 \; and \; 4 \\ \tau &= 1,2,\ldots,N \end{aligned} \tag{40}$$

with the following combinations of the parameters:

Table 1

	ϕ^1_1	ϕ^1_2	ϕ^1_3	ϕ^2_1	ϕ^2_2	ϕ^2_3	ϕ^3_1	ϕ^3_2	ϕ^3_3
i	1.05	1.1	0.9	1.05	0.9	1.1	0.9	1.05	1.1
ii	1.2	0.8	1	1.2	1	0.8	1	1.2	0.8

Note that in Table 1 we only provide the value of the first three parameters for each process. The unreported parameter, that is, ϕ^j_4, for $j = 1,2$, and 3, will be such that the PI condition holds. Hence we will have $\phi^j_4 = 1/\left(\phi^j_1\phi^j_2\phi^j_3\right)$. Also, for the innovations $u^j_{s\tau}$ we consider the following four possibilities:

$$\begin{aligned} &(1) \quad u^j_{s\tau} = \varepsilon^j_{s\tau} \quad \varepsilon^j_{s\tau} \sim Niid\,(0,1) \\ &(2) \quad u^j_{s\tau} = \varepsilon^j_{s\tau} - 0.5\varepsilon^j_{s-1,\tau} \\ &(3) \quad u^j_{s\tau} = \varphi u^j_{s-1,\tau} + \varepsilon^j_{s\tau} \quad \varphi = \{0.8, 0.95\} \\ &(4) \quad u^j_{s\tau} = \varphi_s u^j_{s-1,\tau} + \varepsilon^j_{s\tau} \quad \varphi_1 = 0.8,\; \varphi_2 = 1,\; \varphi_3 = 0.5 \\ &and \quad \varphi_4 = 0.8/\left(\varphi_1\varphi_2\varphi_3\right) \quad \varphi_4 = 0.95/\left(\varphi_1\varphi_2\varphi_3\right) \\ &\qquad j = 1,2,3, \end{aligned} \tag{41}$$

where $E^{(3)}_{s\tau} = \begin{bmatrix} \varepsilon^1_{s\tau} & \varepsilon^2_{s\tau} & \varepsilon^3_{s\tau} \end{bmatrix}'$ is a white noise vector with the positive definite variance-covariance matrix $E\left[E^{(3)}_{s\tau}E^{(3)\prime}_{s\tau}\right] = \Sigma$, with the following three possibilities for Σ:

$$\begin{aligned} (a) \quad \Sigma_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ (b) \quad \Sigma_2 &= \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.5 \\ 0.3 & 0.5 & 1 \end{bmatrix} \\ (b) \quad \Sigma_3 &= \begin{bmatrix} 1 & 0.8 & 0.95 \\ 0.8 & 1 & 0.8 \\ 0.95 & 0.8 & 1 \end{bmatrix}. \end{aligned} \tag{42}$$

We consider quarterly data, that is $S = 4$, and the following possibilities for the total number of years: $N = 50$, 100, and 250. Finally, all of the results are obtained with 10.000 replications.

In Tables 2.a to 2.d, we collect the quantiles of the Johansen trace test when applied to the pseudo-demodulated time series associated with the three PI processes of (40) with the combinations of parameters in Table 1 for a sample size of $N = 500$. In each table the results obtained with the true values associated with the PI processes and the fitted values obtained from the Boswijk and Franses test (1996) are reported. Table 2.a shows the results obtained with a constant, Table 2.b with seasonal dummies, Table 2.c with a constant and a trend, and finally, Table 2.d with seasonal dummies and trends. From Tables 2.a to 2.d it is possible to conclude that we obtain almost the same quantiles when the true fitted values of the coefficients associated with the PI restriction are used. Also, the quantiles in Tables 2.a and 2.b are very similar to each other, and to those in Table B.10 case 2 in Hamilton (1994) and Table 15.2 in Johansen (1995). Finally, the quantiles of Tables 2.c and 2.d are very similar to each other, and to those reported in Table 15.4 in Johansen (1995).

Additionally, in Table 2.e we report the empirical size for situation (40) with the combinations from Table 1 with seasonal dummies, using true and fitted values to obtain the pseudo-demodulation process, and with white noise innovations. These results confirm that we do not have important differences in the performance of the Johansen trace test when using true fitted values of the coefficients associated with the PI restriction.

The results concerning the size performance of the test are presented in Tables 3.a and 3.b. Table 3.a shows the results obtained with a white noise innovation, an AR(1) innovation with $\phi = 0.8$ and $\phi = 0.95$, and finally with an MA(1) innovation with $\theta = 0.5$. Table 2.b shows the results obtained from a PAR(1) innovation with $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ and $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ and with non-periodic and periodic augmentation. The columns labelled i and ii refer to the values of the coefficients ϕ_1^j, ϕ_2^j, and ϕ_3^j, for $j = 1, 2, 3$, shown in Table 1. Finally, the labels Σ_1, Σ_2, and Σ_3 refer to three options for the variance-covariance matrix $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right]$ (42) used in the Monte Carlo experiments.

As mentioned in the Econometric Methodology section, we first apply the Likelihood Ratio test by Boswijk and Franses (1996) to all the time series and retain the fitted values of $\hat{\phi}_1^j$, $\hat{\phi}_2^j$, $\hat{\phi}_3^j$, and $\hat{\phi}_4^j$, for $j = 1, 2, 3$, under the restriction $\prod_{s=1}^{4} \hat{\phi}_s^j = 1$. For case (1) in (41), to compute the Likelihood Ratio test we fit a restricted and unrestricted PAR(1). For case (2) in (41), the order of the PAR is 5, and finally, for cases (3) and (4) in (41), the PAR is of order 2. However, in the case of the VAR used to test the cointegration rank, the order is determined using the AIC criteria with a maximum order of augmentation of 9 lags. In the two remaining sections, the orders of the fitted PAR and VAR models are as defined here. Finally, all of the results are obtained including seasonal dummies.

Clearly, the results of Table 3.a, show that with the white noise innovation the Johansen method applied to the demodulated time series works adequately at detecting that we do not have cointegration between the three PI processes, and the results are very similar in the three scenarios about the variance-covariance matrix of $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$. In the the case of the AR(1) innovation $\varphi = 0.8$ and $\varphi = 0.95$, we observe an oversized Johansen test for $r_0 = 0$ compared to the white noise innovation. The oversizing tends to be resolved as the sample size increases. In the case of $\phi = 0.8$, the size $r_0 = 0$ moves from around 0.15 when $N = 50$ to 0.07 when $N = 250$, and in the case of $\varphi = 0.95$ the oversizing becomes more relevant, moving from around 0.40 when $N = 50$ to 0.12 when $N = 250$. The last case reported in Table 2.a is that of the MA(1) innovation with $\theta = 0.5$. Performance here in terms of size is very similar to what was observed with the AR(1) innovation with $\varphi = 0.8$; the oversizing is less pronounced. Finally, Table 3.b presents the results of a PAR(1) innovation with $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ and $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ and with non-periodic and periodic augmentation. As in the case of the AR innovations, here we also observe much more relevant oversizing than in the case of $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$, and the oversizing clearly tends to be reduced as the sample size increases. Note that with periodic augmentation the results improve a great deal and are in line with the results present with the AR(1) innovation. We have run a Monte Carlo experiment using standard integrated processes with the same innovations as in Tables 3.a and 3.b, and we can say that the results reported in Tables 3.a and 3.b are quite similar to those obtained with the standard integrated processes in the Johansen trace test. Hence we can say that, overall, the Johansen procedure applied to pseudo-demodulated time series does a good job of detecting the absence of cointegration between the three PI processes.

4.2 One periodic cointegration relationship

Compatible with (24), here we explore the situation with three PI processes with one periodic long-run relationship, or equivalently, a system of three PI processes ruled by two common stochastic trends, see Lemma 3. As in the previous subsection the values for ϕ_1^j, ϕ_2^j, and ϕ_3^j, for $j = 1, 2, 3$, and $\phi_4^j = 1/\left(\phi_1^j \phi_2^j \phi_3^j\right)$, for $j = 1, 2, 3$, are shown in Table 1. Hence, we have:

$$
\begin{aligned}
& y^1_{s\tau} = \beta_{1,s} y^2_{s\tau} + \beta_{2,s} y^3_{s\tau} + u^1_{s\tau} \\
& y^2_{s\tau} = \phi^3_s y^2_{s-1,\tau} + u^2_{s\tau} \\
& y^3_{s\tau} = \phi^3_s y^3_{s-1,\tau} + u^3_{s\tau} \\
& \quad s = 1, 2, \ldots, 4. \\
& \beta_{1,4} = 1 \qquad \beta_{2,4} = 1 \\
& \beta_{1,3} = \frac{\phi^2_4}{\phi^1_4} \qquad \beta_{2,3} = \frac{\phi^3_4}{\phi^1_4} \\
& \beta_{1,2} = \frac{\phi^2_4 \phi^2_4}{\phi 4 \phi^1_4} \qquad \beta_{2,S-2} = \beta_2 \frac{\phi^3_4 \phi^3_3}{\phi^1_4 \phi^1_3} \\
& \beta_{1,1} = \frac{\phi^2_4 \phi^2_3 \phi^2_2}{\phi^1_4 \phi^1_3 \phi^1_2} \qquad \beta_{2,S-2} = \beta_2 \frac{\phi^3_4 \phi^3_3 \phi^3_2}{\phi^1_4 \phi^1_3 \phi^1_2},
\end{aligned}
\tag{43}
$$

with $u^1_{s\tau}$, $u^2_{s\tau}$, and $u^3_{s\tau}$ as in (41) and also with the three cases considered in (42) for $E\left[E^{(3)}_{s\tau} E^{(3)\prime}_{s\tau}\right] = \Sigma$. Finally, we use the same sample sizes and replication numbers as in the previous subsection. The results are presented in Tables 4.a and 4.b, with the same organization in terms of the different schemes of serial correlation as the previous section. Overall we can say that the Johansen procedure does a good job of determining that the three PI processes share two common stochastic trends. In tables 3.a to 3.c, on the white noise innovation, we observe that the proportion of times that the null hypothesis of $r_0 = 0$ is rejected is always one, except in three cases when the sample size of $N = 50$, but it is very close to one. And in the case of $r_0 = 1$, the proportion of times that the null is rejected is very close to that which can be seen in Table 3.a. For the AR(1) innovation, the proportion of times that the null is rejected is lower than it is in the white noise innovation for the sample sizes of $N = 50$ and $N = 100$, but when $N = 250$, the proportion of times that the null is rejected is one when $\varphi = 0.8$ and very close to one when $\varphi = 0.95$. Hence, we can say that with the AR(1) innovation the power issue at $r_0 = 0$ tends to be resolved as the sample size increases. In the case of $r_0 = 1$ with an AR(1) innovation, we obtain proportions of rejection of the null that are in line with those seen with the white noise innovation. To finish, in Table 4.a, in the case of the MA(1) innovation, the proportion of times that the null is rejected for $r_0 = 0$ is always one, except in two cases with a sample size of $N = 50$. And, we observe a small oversizing effect when $r_0 = 1$, but it is resolved as the sample size increases. In Table 4.b the results for the PAR innovations with non-periodic and periodic augmentation are presented. Clearly, the results achieved with periodic augmentation help to largely resolve the problems observed in terms of power when $r_0 = 1$ with non-periodic augmentation.

4.3 Two periodic cointegration relationships

Finally, compatible with (24), here we explore the situation of three PI processes with two periodic long-run relationships, or equivalently, a system of three PI processes ruled by one common stochastic trend, see Lemma 3. As in the previous two cases, the values for ϕ^j_1, ϕ^j_2, and ϕ^j_3, for $j = 1, 2, 3$, and $\phi^j_4 = 1/\left(\phi^j_1 \phi^j_2 \phi^j_3\right)$, for $j = 1, 2, 3$, are shown in Table 1. Hence, we have the following:

$$
\begin{aligned}
& y^1_{s\tau} = \alpha_s y^3_{s\tau} + u^1_{s\tau} \\
& y^2_{s\tau} = \beta_s y^3_{s\tau} + u^2_{s\tau} \\
& y^3_{s\tau} = \phi^3_s y^3_{s-1,\tau} + u^3_{s\tau} \\
& \quad s = 1, 2, \ldots, 4. \\
& \alpha_4 = 1 \qquad \beta_4 = 1 \\
& \alpha_3 = \frac{\phi^3_4}{\phi^1_4} \qquad \beta_3 = \frac{\phi^3_4}{\phi^2_4} \\
& \alpha_2 = \frac{\phi^3_4 \phi^3_3}{\phi^1_4 \phi^1_3} \qquad \beta_2 = \frac{\phi^3_4 \phi^3_3}{\phi^2_4 \phi^2_3} \\
& \alpha_1 = \frac{\phi^3_4 \phi^3_3 \phi^3_2}{\phi^1_4 \phi^1_3 \phi^1_2} \qquad \beta_1 = \frac{\phi^3_4 \phi^3_3 \phi^3_2}{\phi^2_4 \phi^2_3 \phi^2_2}
\end{aligned}
\tag{44}
$$

We consider the same options for the innovations $u^1_{s\tau}$, $u^2_{s\tau}$, and $u^3_{s\tau}$, as well as the variance-covariance matrix $E\left[E^{(3)}_{s\tau}E^{(3)\prime}_{s\tau}\right] = \Sigma$, from the two previous subsections; we also use the same the sample sizes and number of replications. The results are shown in Tables 5.a and 5.b, following the same structure about serial correlation as in the sets of tables of the two previous subsections. In general, we can say that in Tables 5.a and 5.b the performance of the Johansen procedure with the pseudo-demodulated approach does a good job of determining the cointegration rank. Clearly, the Johansen procedure detects that there is a common stochastic trend shared by the three PI processes. Hence the procedure correctly detects that we have two periodic cointegration relationship between the three PI processes. The power problems observed in Tables 4.a and 4.b when $r_0 = 0$ are equivalent to those reported for $r_0 = 1$ in Tables 5.a and 3.b.

5 Conclusion

In this paper, we propose a easily implementable method for determining the cointegration rank between a set of PI processes. Our method relies on the use of pseudo-demodulated time series that can be obtained from an estimation of the parameters associated with the periodic integration restriction $\prod_{s=1}^{S}\phi^j_s = 1$ from the Likelihood Ratio test for periodic integration proposed by Boswijk and Franses (1996). Once we have these pseudo-demodulated time series, they can be introduced into Johansen's reduced-rank regression procedure. In the Monte Carlo section, we show that our approach to determining the cointegration rank between a set of periodically integrated processes performs adequately with small samples.

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Table 2.a Empirical Quantiles of Trace Test with Constant

mod	$\mathbf{a}_s$	rank	0,5	0,8	0,9	0,95	0,975	0,99
i	TRUE	$r_0 = 0$	2,4731	4,9076	6,5465	8,1347	9,8126	11,9604
i	TRUE	$r_0 = 1$	9,4428	13,4194	15,9128	18,1357	20,2987	23,2025
i	TRUE	$r_0 = 2$	20,5283	25,8922	29,0984	31,8603	34,6723	37,9846
i	FITTED	$r_0 = 0$	2,4710	4,9044	6,5420	8,1312	9,8093	11,9608
i	FITTED	$r_0 = 1$	9,4373	13,4186	15,9161	18,1309	20,2920	23,2116
i	FITTED	$r_0 = 2$	20,5241	25,8779	29,0954	31,8547	34,6680	38,0068
			0,5000	0,8000	0,9000	0,9500	0,9750	0,9900
ii	TRUE	$r_0 = 0$	2,5062	4,9281	6,6289	8,1455	9,8923	12,1150
ii	TRUE	$r_0 = 1$	9,4657	13,4303	15,8785	17,9800	20,2643	23,1774
ii	TRUE	$r_0 = 2$	20,5718	25,8057	29,0293	31,9314	34,6087	37,7776
ii	FITTED	$r_0 = 0$	2,5066	4,9247	6,6252	8,1430	9,9012	12,0929
ii	FITTED	$r_0 = 1$	9,4669	13,4277	15,8823	17,9727	20,2471	23,1829
ii	FITTED	$r_0 = 2$	20,5691	25,8037	29,0246	31,9100	34,6064	37,8231

Note: Based on 10.000 replication with $N = 500$and $S = 4$. Mod refers to the parameters values in Table 1. TRUE and FITTED to the results obtained with the true coefficents or the fitted one obtained from the Boswijk and Franses (1996) test. The DGPs are defined in (40) with the innovations as (1) in (41). And $r_0$is the number of cointegrating vectors under the null hypothesis.

Table 2.b Empirical Quantiles of Trace Test with Seasonal Dummies

mod	$\mathbf{a}_s$	rank	0,5	0,8	0,9	0,95	0,975	0,99
i	TRUE	$r_0 = 0$	2,4727	4,9064	6,5433	8,1319	9,8102	11,9671
i	TRUE	$r_0 = 1$	9,4410	13,4136	15,9168	18,1430	20,3155	23,2044
i	TRUE	$r_0 = 2$	20,5270	25,8886	29,0798	31,8660	34,6736	38,0473
i	FITTED	$r_0 = 0$	2,4724	4,9022	6,5419	8,1302	9,8053	11,9600
i	FITTED	$r_0 = 1$	9,4401	13,4168	15,9183	18,1371	20,2866	23,2089
i	FITTED	$r_0 = 2$	20,5207	25,8754	29,0933	31,8508	34,6695	38,0166
			0,5000	0,8000	0,9000	0,9500	0,9750	0,9900
ii	TRUE	$r_0 = 0$	2,5069	4,9258	6,6216	8,1426	9,8814	12,1116
ii	TRUE	$r_0 = 1$	9,4665	13,4352	15,8805	17,9773	20,2531	23,2049
ii	TRUE	$r_0 = 2$	20,5704	25,7971	29,0176	31,9143	34,6061	37,7697
ii	FITTED	$r_0 = 0$	2,5077	4,9250	6,6213	8,1399	9,8988	12,0945
ii	FITTED	$r_0 = 1$	9,4646	13,4293	15,8734	17,9770	20,2431	23,1891
ii	FITTED	$r_0 = 2$	20,5648	25,8026	29,0229	31,9129	34,6084	37,8095

Note: See the note of table 2.a.

Table 2.c Empirical Quantiles of Trace Test withconstant and Trend

mod	$\mathbf{a}_s$	rank	0,5	0,8	0,9	0,95	0,975	0,99
i	TRUE	$r_0 = 0$	4,7457	7,7617	9,7202	11,7634	13,5321	15,8662
i	TRUE	$r_0 = 1$	13,9358	18,4514	21,2791	23,6911	26,1444	28,8353
i	TRUE	$r_0 = 2$	27,0797	33,2513	36,7054	39,9997	42,8724	46,7600
i	FITTED	$r_0 = 0$	4,7419	7,7565	9,7171	11,7579	13,5260	15,8549
i	FITTED	$r_0 = 1$	13,9367	18,4424	21,2782	23,6898	26,1489	28,8410
i	FITTED	$r_0 = 2$	27,0699	33,2314	36,7108	39,9864	42,8834	46,7630
			0,5000	0,8000	0,9000	0,9500	0,9750	0,9900
ii	TRUE	$r_0 = 0$	4,8091	7,7974	9,7992	11,6816	13,6039	15,6681
ii	TRUE	$r_0 = 1$	13,9003	18,6648	21,4928	23,9217	26,1132	28,9832
ii	TRUE	$r_0 = 2$	27,0189	33,2001	36,7969	40,0044	42,8102	46,4758
ii	FITTED	$r_0 = 0$	4,8056	7,7927	9,7992	11,6722	13,5869	15,6694
ii	FITTED	$r_0 = 1$	13,8900	18,6671	21,4765	23,9346	26,0939	28,9880
ii	FITTED	$r_0 = 2$	27,0008	33,1834	36,8049	39,9706	42,7710	46,4735

Note: See the note of table 2.a.

Table 2.d Empirical Quantiles of Trace Test with Seasonal Dummies and Trends

mod	$\mathbf{a}_s$	rank	0,5	0,8	0,9	0,95	0,975	0,99
i	TRUE	$r_0 = 0$	4,7432	7,7518	9,7177	11,7490	13,5247	15,8137
i	TRUE	$r_0 = 1$	13,9308	18,4375	21,2623	23,6894	26,1400	28,8139
i	TRUE	$r_0 = 2$	27,0665	33,2482	36,6841	39,9596	42,8916	46,7446
i	FITTED	$r_0 = 0$	4,7439	7,7511	9,7209	11,7482	13,5231	15,8321
i	FITTED	$r_0 = 1$	13,9283	18,4335	21,2587	23,6818	26,1453	28,8059
i	FITTED	$r_0 = 2$	27,0635	33,2485	36,6881	39,9759	42,8657	46,7518
			0,5000	0,8000	0,9000	0,9500	0,9750	0,9900
ii	TRUE	$r_0 = 0$	4,8071	7,7889	9,7934	11,6648	13,5801	15,6767
ii	TRUE	$r_0 = 1$	13,8872	18,6679	21,4583	23,9272	26,1060	28,9560
ii	TRUE	$r_0 = 2$	27,0079	33,1957	36,7767	39,9534	42,7393	46,4956
ii	FITTED	$r_0 = 0$	4,8056	7,7879	9,7943	11,6641	13,5833	15,6805
ii	FITTED	$r_0 = 1$	13,8816	18,6629	21,4543	23,9444	26,0890	28,9466
ii	FITTED	$r_0 = 2$	27,0034	33,1852	36,7901	39,9601	42,7511	46,4838

Note: See the note of table 2.a.

Table 2.e Size

						Σ_1
	PI		FITTED	FITTED	TRUE	TRUE
Variables	Rank	N	i	ii	i	ii
3	$r_0 = 0$	50	0,0698	0,0671	0,0702	0,0665
3	$r_0 = 0$	100	0,0624	0,0631	0,0627	0,0627
3	$r_0 = 0$	250	0,0561	0,0582	0,0564	0,0581
3	$r_0 = 1$	50	0,0065	0,0037	0,0062	0,0039
3	$r_0 = 1$	100	0,0045	0,0042	0,0044	0,0041
3	$r_0 = 1$	250	0,0037	0,0032	0,0037	0,0033
3	$r_0 = 2$	50	0,0010	0,0002	0,0010	0,0002
3	$r_0 = 2$	100	0,0010	0,0004	0,0010	0,0005
3	$r_0 = 2$	250	0,0009	0,0002	0,0009	0,0002
2	$r_0 = 0$	50	0,0602	0,0566	0,0607	0,0571
2	$r_0 = 0$	100	0,0563	0,0559	0,0562	0,0557
2	$r_0 = 0$	250	0,0541	0,0524	0,0541	0,0527
2	$r_0 = 1$	50	0,0055	0,0048	0,0056	0,0049
2	$r_0 = 1$	100	0,0045	0,0051	0,0047	0,0050
2	$r_0 = 1$	250	0,0034	0,0034	0,0034	0,0034
1	$r_0 = 0$	50	0,0536	0,0524	0,0545	0,0528
1	$r_0 = 0$	100	0,0546	0,0473	0,0548	0,0474
1	$r_0 = 0$	250	0,0540	0,0496	0,0542	0,0497

Note: Based on 10.000 replication with $S = 4$, $i$and ii refers to the parameters values in Table 1. TRUE and FITTED to the results obtained with the true coefficents or the fitted one obtained from the Boswijk and Franses (1996) test. The DGPs are defined in (40) with the innovations as (1) in (41). $r_0$is the number of cointegrating vectors under the null hypothesis. The Trace test is conducted at a nominal 5% level of significance. Finally Σ_1 refers to (42).

Table 3.a No Cointegration

rank		Σ_1		Σ_2		Σ_3	
	N	i	ii	i	ii	i	ii
	White Noise (1) in (41).						
$r_0 = 0$	50	0.0799	0.0786	0.0767	0.0769	0.0783	0.0721
$r_0 = 0$	100	0.0641	0.0671	0.0647	0.0662	0.0633	0.0676
$r_0 = 0$	250	0.0611	0.0577	0.0623	0.0643	0.0615	0.0613
$r_0 = 1$	50	0.0052	0.0059	0.0054	0.0052	0.0062	0.0058
$r_0 = 1$	100	0.0046	0.0062	0.0055	0.0045	0.0041	0.0053
$r_0 = 1$	250	0.0041	0.0041	0.0044	0.0039	0.0046	0.0055
$r_0 = 2$	50	0.0003	0.0008	0.0005	0.0007	0.0009	0.0012
$r_0 = 2$	100	0.0008	0.0009	0.0008	0.0010	0.0005	0.0012
$r_0 = 2$	250	0.0004	0.0005	0.0009	0.0005	0.0002	0.0007
	$AR(1) \quad \varphi = 0.8$ (3) in (41).						
$r_0 = 0$	50	0,1620	0,1602	0,1551	0,1654	0,1569	0,1621
$r_0 = 0$	100	0,1015	0,1032	0,0993	0,1081	0,1001	0,0936
$r_0 = 0$	250	0,0753	0,0708	0,0767	0,0721	0,0700	0,0757
$r_0 = 1$	50	0,0168	0,0167	0,0147	0,0165	0,0175	0,0177
$r_0 = 1$	100	0,0092	0,0094	0,0081	0,0094	0,0095	0,0086
$r_0 = 1$	250	0,0052	0,0060	0,0059	0,0067	0,0048	0,0074
$r_0 = 2$	50	0,0021	0,0020	0,0022	0,0015	0,0022	0,0017
$r_0 = 2$	100	0,0013	0,0010	0,0015	0,0012	0,0014	0,0013
$r_0 = 2$	250	0,0005	0,0006	0,0007	0,0007	0,0005	0,0011
	$AR(1) \quad \varphi = 0.95$ (3) in (41).						
$r_0 = 0$	50	0.4071	0.4066	0.4113	0.4083	0.6811	0.6089
$r_0 = 0$	100	0.2323	0.2442	0.2474	0.2464	0.3981	0.3754
$r_0 = 0$	250	0.1233	0.1168	0.1285	0.1121	0.1818	0.1825
$r_0 = 1$	50	0.0842	0.0817	0.0811	0.0817	0.2161	0.1785
$r_0 = 1$	100	0.0291	0.0319	0.0333	0.0344	0.0801	0.0744
$r_0 = 1$	250	0.0124	0.0124	0.0106	0.0101	0.0224	0.0245
$r_0 = 2$	50	0.0173	0.0146	0.0120	0.0138	0.0463	0.0360
$r_0 = 2$	100	0.0047	0.0053	0.0054	0.0060	0.0131	0.0135
$r_0 = 2$	250	0.0017	0.0013	0.0009	0.0011	0.0030	0.0037
	$MA(1) \quad \theta = 0.5$ (2) in (41).						
$r_0 = 0$	50	0.1050	0.1012	0.1018	0.1035	0.1037	0.1015
$r_0 = 0$	100	0.0769	0.0754	0.0792	0.0788	0.0779	0.0761
$r_0 = 0$	250	0.0658	0.0695	0.0737	0.0711	0.0721	0.0686
$r_0 = 1$	50	0.0073	0.0088	0.0081	0.0093	0.0076	0.0065
$r_0 = 1$	100	0.0057	0.0048	0.0045	0.0069	0.0040	0.0057
$r_0 = 1$	250	0.0058	0.0049	0.0045	0.0056	0.0052	0.0042
$r_0 = 2$	50	0.0010	0.0009	0.0006	0.0016	0.0007	0.0006
$r_0 = 2$	100	0.0010	0.0004	0.0006	0.0010	0.0011	0.0011
$r_0 = 2$	250	0.0005	0.0009	0.0007	0.0005	0.0010	0.0011

Note: Based on 10.000 replication with $S = 4$, $i$and ii refers to the parameters values in Table 1. The DGPs are defined in (40) with the innovations deffined in (41). $r_0$is the number of cointegrating vectors under the null hypothesis. The Trace test is conducted at a nominal 5% level of significance. Finally Σ_1, Σ_2 and Σ_3 refers to (42).

Table 3.b No Cointegration

rank		Σ_1		Σ_2		Σ_3	
	N	i	ii	i	ii	i	ii
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ (4) in (41).						
$r_0 = 0$	50	0.6025	0.4876	0.5645	0.4849	0.5706	0.4693
$r_0 = 0$	100	0.3486	0.2254	0.2710	0.2268	0.2799	0.2194
$r_0 = 0$	250	0.1030	0.1055	0.1024	0.1093	0.1072	0.1130
$r_0 = 1$	50	0.1555	0.1100	0.1318	0.1015	0.1345	0.1012
$r_0 = 1$	100	0.0566	0.0327	0.0395	0.0295	0.0413	0.0288
$r_0 = 1$	250	0.0092	0.0104	0.0110	0.0107	0.0121	0.0105
$r_0 = 2$	50	0.0268	0.0185	0.0229	0.0192	0.0233	0.0185
$r_0 = 2$	100	0.0077	0.0047	0.0060	0.0055	0.0057	0.0038
$r_0 = 2$	250	0.0014	0.0019	0.0015	0.0017	0.0011	0.0019
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ (4) in (41). Periodic Augmentation						
$r_0 = 0$	50	0.3582	0.3782	0.3597	0.3762	0.3554	0.3663
$r_0 = 0$	100	0.1837	0.1954	0.1799	0.1983	0.1852	0.1936
$r_0 = 0$	250	0.0805	0.0883	0.0871	0.0971	0.0874	0.0931
$r_0 = 1$	50	0.0642	0.0719	0.0609	0.0712	0.0607	0.0682
$r_0 = 1$	100	0.0195	0.0261	0.0204	0.0211	0.0205	0.0216
$r_0 = 1$	250	0.0061	0.0074	0.0074	0.0081	0.0075	0.0088
$r_0 = 2$	50	0.0112	0.0107	0.0111	0.0122	0.0096	0.0100
$r_0 = 2$	100	0.0031	0.0037	0.0040	0.0031	0.0023	0.0035
$r_0 = 2$	250	0.0010	0.0010	0.0006	0.0009	0.0012	0.0011
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ in (41).						
$r_0 = 0$	50	0.7380	0.6211	0.6775	0.6109	0.6811	0.6089
$r_0 = 0$	100	0.5065	0.3707	0.4009	0.3665	0.3981	0.3754
$r_0 = 0$	250	0.1886	0.1854	0.1898	0.1766	0.1818	0.1825
$r_0 = 1$	50	0.2599	0.1797	0.2148	0.1743	0.2161	0.1785
$r_0 = 1$	100	0.1310	0.0728	0.0829	0.0696	0.0801	0.0744
$r_0 = 1$	250	0.0268	0.0225	0.0209	0.0213	0.0224	0.0245
$r_0 = 2$	50	0.0536	0.0395	0.0439	0.0408	0.0463	0.0360
$r_0 = 2$	100	0.0237	0.0121	0.0144	0.0125	0.0131	0.0135
$r_0 = 2$	250	0.0046	0.0036	0.0024	0.0027	0.0030	0.0037
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ in (41). Periodic Augmentation						
$r_0 = 0$	50	0.4401	0.4660	0.4489	0.4852	0.4342	0.4721
$r_0 = 0$	100	0.2751	0.3051	0.3197	0.3321	0.2734	0.2999
$r_0 = 0$	250	0.1219	0.1409	0.1606	0.1685	0.1295	0.1428
$r_0 = 1$	50	0.0992	0.1040	0.0951	0.1064	0.0946	0.1069
$r_0 = 1$	100	0.0420	0.0494	0.0511	0.0598	0.0385	0.0496
$r_0 = 1$	250	0.0125	0.0139	0.0168	0.0188	0.0124	0.0155
$r_0 = 2$	50	0.0175	0.0178	0.0205	0.0192	0.0168	0.0184
$r_0 = 2$	100	0.0070	0.0080	0.0065	0.0102	0.0058	0.0074
$r_0 = 2$	250	0.0019	0.0020	0.0028	0.0024	0.0020	0.0017

Note: See the note of table 3.a.

Table 4.a One Periodic Cointegration Relationship

rank		Σ_1		Σ_2		Σ_3	
	N	i	ii	i	ii	i	ii
	White Noise (1) in (41).						
$r_0 = 0$	50	1.0000	0.9987	1.0000	0.9992	1.0000	0.9947
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0620	0.0636	0.0623	0.0649	0.0651	0.0713
$r_0 = 1$	100	0.0591	0.0593	0.0559	0.0611	0.0581	0.0581
$r_0 = 1$	250	0.0557	0.0584	0.0556	0.0594	0.0558	0.0538
$r_0 = 2$	50	0.0058	0.0045	0.0050	0.0055	0.0055	0.0060
$r_0 = 2$	100	0.0047	0.0040	0.0051	0.0043	0.0042	0.0054
$r_0 = 2$	250	0.0044	0.0044	0.0046	0.0038	0.0040	0.0040
	$AR(1)$ $\varphi = 0.8$ (3) in (41).						
$r_0 = 0$	50	0.5043	0.8068	0.6357	0.7834	0.5462	0.6018
$r_0 = 0$	100	0.9551	0.9958	0.9834	0.9971	0.9957	0.9767
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0560	0.0849	0.0688	0.0826	0.0631	0.0748
$r_0 = 1$	100	0.0681	0.0702	0.0654	0.0698	0.0689	0.0698
$r_0 = 1$	250	0.0581	0.0676	0.0610	0.0611	0.0601	0.0652
$r_0 = 2$	50	0.0054	0.0088	0.0052	0.0085	0.0052	0.0075
$r_0 = 2$	100	0.0049	0.0055	0.0047	0.0052	0.0054	0.0054
$r_0 = 2$	250	0.0050	0.0040	0.0040	0.0041	0.0037	0.0048
	$AR(1)$ $\varphi = 0.95$ (3) in (41).						
$r_0 = 0$	50	0.3633	0.6894	0.3657	0.6041	0.5017	0.5594
$r_0 = 0$	100	0.5737	0.8214	0.6027	0.7685	0.6408	0.6887
$r_0 = 0$	250	0.9817	0.9835	0.9902	0.9885	0.9911	0.9715
$r_0 = 1$	50	0.0673	0.1180	0.0675	0.1039	0.0915	0.0995
$r_0 = 1$	100	0.0713	0.0912	0.0739	0.0815	0.0741	0.0800
$r_0 = 1$	250	0.0668	0.0693	0.0690	0.0719	0.0647	0.0710
$r_0 = 2$	50	0.0122	0.0158	0.0094	0.0133	0.0156	0.0141
$r_0 = 2$	100	0.0083	0.0097	0.0075	0.0091	0.0087	0.0084
$r_0 = 2$	250	0.0051	0.0062	0.0059	0.0057	0.0064	0.0058
	$MA(1)$ $\theta = 0.5$ (2) in (41).						
$r_0 = 0$	50	1.0000	1.0000	1.0000	0.9999	1.0000	0.9994
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0881	0.0857	0.0904	0.0817	0.0814	0.0789
$r_0 = 1$	100	0.0710	0.0687	0.0662	0.0609	0.0720	0.0661
$r_0 = 1$	250	0.0633	0.0678	0.0611	0.0629	0.0658	0.0587
$r_0 = 2$	50	0.0084	0.0071	0.0069	0.0056	0.0086	0.0090
$r_0 = 2$	100	0.0062	0.0059	0.0047	0.0052	0.0061	0.0043
$r_0 = 2$	250	0.0045	0.0052	0.0067	0.0037	0.0051	0.0035

Note: See the note of table 3.a, but with DPGs defined in (43).

Table 4.b One Periodic Cointegration Relationship

rank		Σ_1		Σ_2		Σ_3	
	N	i	ii	i	ii	i	ii
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ (4) in (41).						
$r_0 = 0$	50	0.4095	0.4903	0.4099	0.5486	0.5030	0.4548
$r_0 = 0$	100	0.5318	0.5913	0.5346	0.7208	0.6804	0.6029
$r_0 = 0$	250	0.9599	0.9446	0.9872	0.9792	0.9959	0.9285
$r_0 = 1$	50	0.0694	0.0744	0.0676	0.0848	0.0809	0.0710
$r_0 = 1$	100	0.0559	0.0646	0.0628	0.0711	0.0674	0.0615
$r_0 = 1$	250	0.0614	0.0630	0.0600	0.0644	0.0585	0.0639
$r_0 = 2$	50	0.0114	0.0093	0.0111	0.0123	0.0126	0.0088
$r_0 = 2$	100	0.0073	0.0081	0.0075	0.0052	0.0078	0.0052
$r_0 = 2$	250	0.0070	0.0058	0.0053	0.0055	0.0057	0.0067
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ (4) in (41).						
	Periodic Augmentation						
$r_0 = 0$	50	0.5849	0.7801	0.5301	0.5921	0.4668	0.4909
$r_0 = 0$	100	0.9018	0.9727	0.8157	0.9199	0.8135	0.8637
$r_0 = 0$	250	0.9999	0.9985	0.9994	0.9987	0.9995	0.9683
$r_0 = 1$	50	0.0446	0.0202	0.0367	0.0164	0.0227	0.0138
$r_0 = 1$	100	0.0326	0.0075	0.0278	0.0089	0.0243	0.0096
$r_0 = 1$	250	0.0195	0.0032	0.0251	0.0043	0.0183	0.0031
$r_0 = 2$	50	0.0034	0.0003	0.0030	0.0005	0.0008	0.0003
$r_0 = 2$	100	0.0013	0.0002	0.0013	0.0000	0.0009	0.0003
$r_0 = 2$	250	0.0007	0.0001	0.0005	0.0002	0.0003	0.0000
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ (4) in (41).						
$r_0 = 0$	50	0.4727	0.5637	0.4699	0.6244	0.5025	0.5459
$r_0 = 0$	100	0.4113	0.5424	0.4289	0.6542	0.4902	0.6506
$r_0 = 0$	250	0.4077	0.5646	0.4892	0.7793	0.7821	0.8555
$r_0 = 1$	50	0.0987	0.1094	0.0961	0.1244	0.0917	0.1072
$r_0 = 1$	100	0.0795	0.0880	0.0791	0.1097	0.0698	0.0803
$r_0 = 1$	250	0.0574	0.0586	0.0591	0.0811	0.0648	0.0676
$r_0 = 2$	50	0.0178	0.0196	0.0205	0.0223	0.0147	0.0150
$r_0 = 2$	100	0.0102	0.0141	0.0137	0.0174	0.0086	0.0094
$r_0 = 2$	250	0.0089	0.0060	0.0074	0.0090	0.0067	0.0067
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ (4) in (41).						
	Periodic Augmentation						
$r_0 = 0$	50	0.9399	0.9455	0.9088	0.8877	0.7976	0.7227
$r_0 = 0$	100	0.9997	0.9983	0.9993	0.9922	0.9931	0.9103
$r_0 = 0$	250	0.9998	0.9996	0.9999	0.9965	0.9999	0.9564
$r_0 = 1$	50	0.0992	0.0318	0.0993	0.0330	0.0510	0.0247
$r_0 = 1$	100	0.1240	0.0191	0.1375	0.0242	0.0478	0.0113
$r_0 = 1$	250	0.0482	0.0059	0.0822	0.0104	0.0276	0.0048
$r_0 = 2$	50	0.0037	0.0003	0.0056	0.0010	0.0021	0.0004
$r_0 = 2$	100	0.0018	0.0005	0.0023	0.0003	0.0012	0.0006
$r_0 = 2$	250	0.0015	0.0002	0.0008	0.0001	0.0004	0.0000

Note: See the note of table 3.a, but with DPGs defined in (43).

Table 5.a Two Periodic Cointegration Relationship

rank		Σ_1		Σ_2		Σ_3	
	N	i	ii	i	ii	i	ii
	White Noise (1) in (41).						
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0536	0.0492	0.0548	0.0541	0.0559	0.0529
$r_0 = 2$	100	0.0556	0.0532	0.0530	0.0507	0.0572	0.0521
$r_0 = 2$	250	0.0519	0.0579	0.0525	0.0535	0.0528	0.0487
	$AR(1)$ $\varphi = 0.8$ (3) in (41).						
$r_0 = 0$	50	0.9776	0.9712	0.9817	0.9813	0.9843	0.9829
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.6379	0.6212	0.6401	0.6296	0.6563	0.6540
$r_0 = 1$	100	0.9991	0.9986	0.9987	0.9974	0.9991	0.9991
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0464	0.0458	0.0478	0.0459	0.0482	0.0473
$r_0 = 2$	100	0.0481	0.0540	0.0521	0.0515	0.0492	0.0493
$r_0 = 2$	250	0.0542	0.0529	0.0520	0.0520	0.0559	0.0515
	$AR(1)$ $\varphi = 0.95$ (3) in (41).						
$r_0 = 0$	50	0.3342	0.2926	0.3089	0.2713	0.3004	0.2916
$r_0 = 0$	100	0.6592	0.6445	0.6374	0.6378	0.6161	0.6701
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0555	0.0546	0.0516	0.0456	0.0501	0.0447
$r_0 = 1$	100	0.1892	0.1888	0.1785	0.1817	0.1719	0.2006
$r_0 = 1$	250	0.9682	0.9804	0.9702	0.9798	0.9691	0.9858
$r_0 = 2$	50	0.0086	0.0095	0.0087	0.0061	0.0088	0.0073
$r_0 = 2$	100	0.0271	0.0272	0.0283	0.0266	0.0249	0.0270
$r_0 = 2$	250	0.0525	0.0550	0.0518	0.0530	0.0521	0.0535
	$MA(1)$ $\theta = 0.5$ (2) in (41).						
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0827	0.0808	0.0727	0.0700	0.0687	0.0659
$r_0 = 2$	100	0.0659	0.0638	0.0643	0.0684	0.0673	0.0596
$r_0 = 2$	250	0.0580	0.0593	0.0526	0.0575	0.0594	0.0581

Note: See the note of table 3.a, but with DPGs defined in (44).

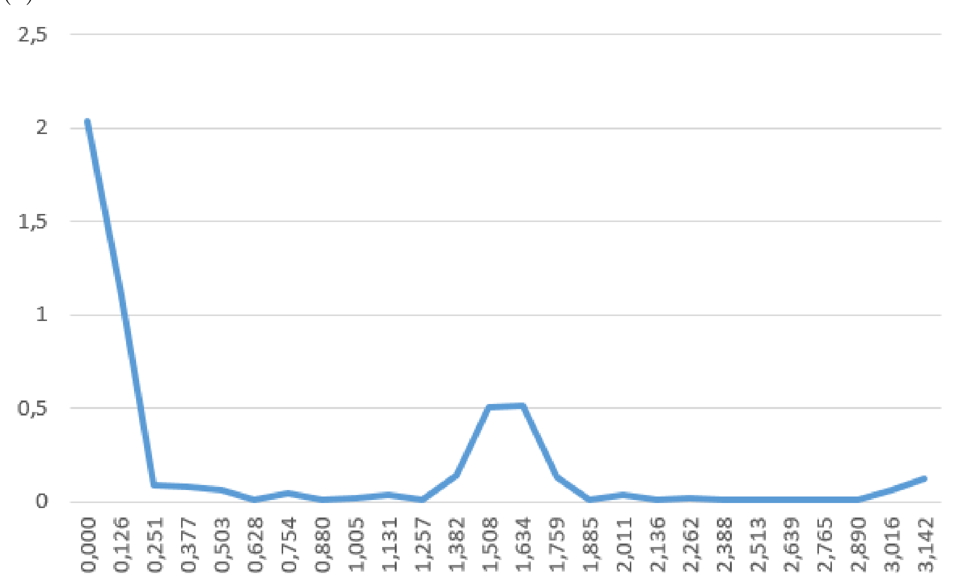
Table 5.b Two Periodic Cointegration Relationship

rank		Σ_1		Σ_2		Σ_3	
	N	i	ii	i	ii	i	ii
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ (4) in (41).						
$r_0 = 0$	50	0.3542	0.3415	0.4092	0.3662	0.4685	0.3903
$r_0 = 0$	100	0.7496	0.7373	0.7983	0.7979	0.8622	0.8176
$r_0 = 0$	250	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0666	0.0622	0.0754	0.0681	0.0889	0.0756
$r_0 = 1$	100	0.2423	0.2543	0.2810	0.2848	0.3354	0.3114
$r_0 = 1$	250	0.9257	0.8715	0.9424	0.9970	0.9893	0.9973
$r_0 = 2$	50	0.0114	0.0096	0.0109	0.0114	0.0132	0.0118
$r_0 = 2$	100	0.0224	0.0268	0.0280	0.0306	0.0312	0.0323
$r_0 = 2$	250	0.0344	0.0485	0.0381	0.0431	0.0434	0.0488
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.8$ (4) in (41).						
	Periodic Augmentation						
$r_0 = 0$	50	0.9419	0.9087	0.9048	0.8653	0.8343	0.8325
$r_0 = 0$	100	0.9999	0.9996	0.9993	0.9981	0.9965	0.9974
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
$r_0 = 1$	50	0.4855	0.3939	0.3816	0.3065	0.2501	0.2624
$r_0 = 1$	100	0.9561	0.9166	0.8827	0.8490	0.7359	0.8073
$r_0 = 1$	250	0.9999	1.0000	0.9994	0.9999	0.9979	0.9998
$r_0 = 2$	50	0.0301	0.0285	0.0173	0.0233	0.0137	0.0224
$r_0 = 2$	100	0.0402	0.0483	0.0334	0.0471	0.0337	0.0432
$r_0 = 2$	250	0.0328	0.0450	0.0237	0.0480	0.0374	0.0537
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ (4) in (41).						
$r_0 = 0$	50	0.2795	0.2578	0.2951	0.2584	0.3339	0.2819
$r_0 = 0$	100	0.4036	0.3853	0.4049	0.4011	0.4689	0.4214
$r_0 = 0$	250	0.9261	0.9166	0.9093	0.9728	0.9688	0.9759
$r_0 = 1$	50	0.0394	0.0383	0.0447	0.0373	0.0544	0.0443
$r_0 = 1$	100	0.0716	0.0683	0.0706	0.0775	0.0902	0.0805
$r_0 = 1$	250	0.4760	0.4952	0.3908	0.5861	0.6127	0.6054
$r_0 = 2$	50	0.0060	0.0060	0.0068	0.0061	0.0082	0.0083
$r_0 = 2$	100	0.0093	0.0100	0.0113	0.0099	0.0099	0.0118
$r_0 = 2$	250	0.0261	0.0304	0.0303	0.0326	0.0310	0.0374
	$PAR(1)$ $\varphi_1\varphi_2\varphi_3\varphi_4 = 0.95$ (4) in (41).						
	Periodic Augmentation						
$r_0 = 0$	50	0.9989	0.9976	0.9973	0.9953	0.9930	0.9941
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.8775	0.8420	0.8007	0.7839	0.6998	0.7473
$r_0 = 1$	100	0.9999	0.9995	0.9990	0.9978	0.9927	0.9983
$r_0 = 1$	250	0.9999	0.9999	0.9996	1.0000	0.9999	1.0000
$r_0 = 2$	50	0.0613	0.0608	0.0444	0.0603	0.0441	0.0598
$r_0 = 2$	100	0.0556	0.0672	0.0429	0.0712	0.0587	0.0697
$r_0 = 2$	250	0.0420	0.0484	0.0259	0.0580	0.0510	0.0599

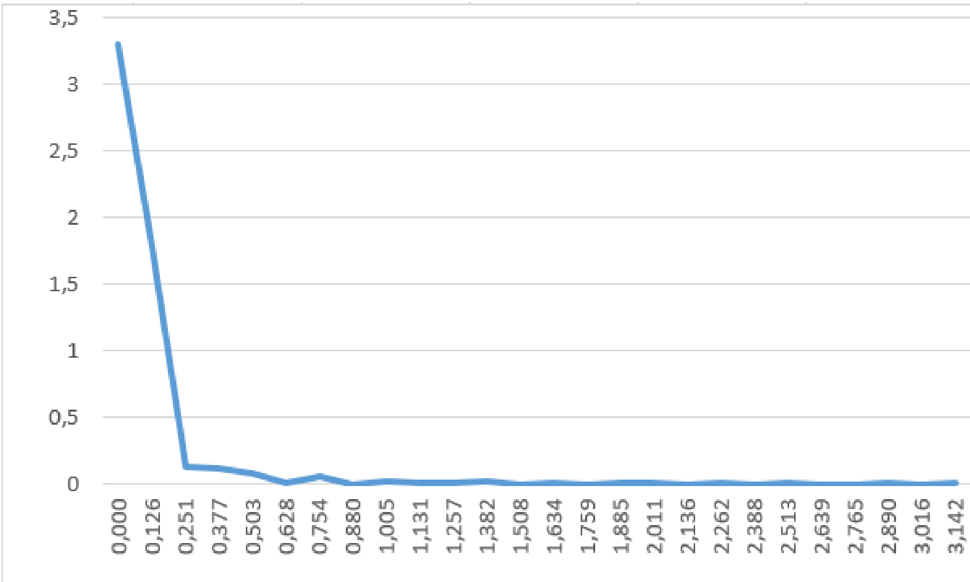
Note: See the note of table 3.a, but with DPGs defined in (44).

Figure 1

(a)



(b)



Part (a) Average periodogram of a PI process $y_{s\tau} = \phi_s y_{s-1,\tau} + u_{s\tau}$with $S = 4$,$T = 100$,$\phi_1 = 0.8$, $\phi_2 = 1$, $\phi_3 = 0.5$and $\phi_4 = 1/(\phi_1\phi_2\phi_3)$and $u_{s\tau} \sim Niid(0,1)$. Part (b) Average periodogram of $a_s^{-1}y_{s\tau}$with $a_s$been the s^{th} element of $\mathbf{a}$ defined in (47). Based in 5000 replications.

7 Appendix

Proof of Lemma 1:

First note that, as in the quarterly case studied by Paap and Franses (1999), successively substituting in (4) yields

$$\begin{aligned} Y_\tau &= [\mathbf{A}_0^{-1}\mathbf{A}_1]^\tau Y_0 + \mathbf{A}_0^{-1}U_\tau + \sum_{j=1}^{\tau-1}[\mathbf{A}_0^{-1}\mathbf{A}_1]^j \mathbf{A}_0^{-1}U_{\tau-j} \\ &= \mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \mathbf{A}_0^{-1}U_\tau + \mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j}. \end{aligned} \tag{45}$$

This result follows because matrix $\mathbf{A}_0^{-1}\mathbf{A}_1$ is idempotent. First, note that the matrix $\mathbf{A}_0$ (see chapter 2 pp 45-48 of Pollock (1999)) is an $S \times S$ lower-triangular Toeplitz matrix associated with the polynomial $(1-\phi_s L)$. Hence the matrix $\mathbf{A}_0^{-1}$ collects the coefficients of the expansion of the inverse polynomial associated with $(1-\phi_s L)$[9]. Based on the form of the matrices $\mathbf{A}_0^{-1}$ and $\mathbf{A}_1$, it is clear that the resulting matrix $\mathbf{A}_0^{-1}\mathbf{A}_1$ is an $S \times S$ matrix with the first $S-1$ columns having elements equal to zero and the last column equal to the column vector $\mathbf{v} = \begin{bmatrix} \phi_1 & \phi_1\phi_2 & \phi_1\phi_2\phi_3 & \cdots & \prod_{s=1}^{S}\phi_s \end{bmatrix}'$. Finally note that the last element of $\mathbf{v}$, that is, $\prod_{s=1}^{S}\phi_s$, is equal to 1, as we have Periodic Integration. Also, as the first $S-1$ columns of $\mathbf{A}_0^{-1}\mathbf{A}_1$ are equal to zero and the lower left element of this matrix is equal to one, implies that $[\mathbf{A}_0^{-1}\mathbf{A}_1]^j = \mathbf{A}_0^{-1}\mathbf{A}_1$ for $j = 2, 3, ...$ Clearly, (45) provides a representation of (3), where the matrix $\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}$ gives the effect of the accumulated vector of shocks $\sum_{j=1}^{\tau-1} U_{\tau-j}$ (see for example Boswijk and Franses (1996), Paap and Franses (1999) and del Barrio Castro and Osborn (2008a)). The matrix $\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}$ has rank one and hence can be written as

$$\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1} = \mathbf{a}\mathbf{b}' \tag{46}$$

where, for (46),

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} 1 & \phi_2 & \phi_2\phi_3 & \cdots & \prod_{s=2}^{S}\phi_s \end{bmatrix}' \\ \mathbf{b} &= \begin{bmatrix} 1 & \phi_1\prod_{s=3}^{S}\phi_s & \phi_1\prod_{s=4}^{S}\phi_s & \cdots & \phi_1 \end{bmatrix}'. \end{aligned} \tag{47}$$

Hence using (45) and (46) it is clear that () holds.

Now if we focus on U_τ, this is the $S \times 1$ vector that collects the stacked observations of $u_{s\tau}$ that we assume that follows a stationary PAR of order $P-1$. That is, $\left(1 - \psi_{1s}L - \cdots - \psi_{p-1,s}L^{p-1}\right) u_{s\tau} = \varepsilon_{s\tau}$ with $\varepsilon_{s\tau} \sim iid\left(0, \sigma_\varepsilon^2\right)$. It is possible to write for U_τ follows VAR of order $\mathcal{P} = int\left[(P+S-2)/S\right]$, that is:

$$\begin{aligned} \Psi_0 U_\tau - \Psi_1 U_{\tau-1} - \cdots - \Psi_{\mathcal{P}} U_{\tau-\mathcal{P}} &= E_\tau \\ \Psi(B)\,U_\tau &= E_\tau \\ \left(\Psi_0 I - \Psi_1 B - \cdots - \Psi_{\mathcal{P}} B^{\mathcal{P}}\right) &= \Psi(B). \end{aligned}$$

With B here playing the role of the lag operator for the $S \times 1$ vectors Y_τ, U_τ and E_τ. For the cumulate sum $\sum_{j=1}^{\tau} E_j$ it is possible to write $\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} E_j \Rightarrow \sigma W(r)$. With $W(r)$ been a $S \times 1$ standard vector Brownian

[9] That is:

$$\mathbf{A}_0^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 \\ \phi_2 & 1 & 0 & 0 & \cdots & 0 \\ \phi_2\phi_3 & \phi_3 & 1 & 0 & \cdots & 0 \\ \phi_2\phi_3\phi_4 & \phi_3\phi_4 & \phi_4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{j=2}^{S}\phi_j & \prod_{j=3}^{S}\phi_j & \prod_{j=4}^{S}\phi_j & \prod_{j=5}^{S}\phi_j & \cdots & 1 \end{bmatrix}.$$

motion. Hence for the cumulate sum $\sum_{j=1}^{\tau-1} U_{\tau-j}$ in (45), it is possible to write:

$$\frac{1}{\sqrt{T}} \sum_{j=1}^{\lfloor Tr \rfloor} U_j \Rightarrow \sigma \Psi(1)^{-1} W(r). \tag{48}$$

With $\Psi(\mathbf{1})^{-1}$, being the inverse of the polynomial matrix $\Psi(L)$ evaluated at $L = \mathbf{I}$. Result (7) is obtained straightforwardly using (48), (46) and (45). Finally note that ω and $w(r)$ are defined as follows:

$$\begin{aligned} w(r) &= \omega^{-1}\sigma \mathbf{b}'\Psi(1)^{-1} W(r) \\ \omega &= \sigma\left(\mathbf{b}'\Psi(1)^{-1}\Psi(1)^{-1\prime}\mathbf{b}\right)^{1/2}. \end{aligned} \tag{49}$$

■

Proof of Lemma 2:

First note that in model (27), by recursive substitution, we can have:

$$Y_{\tau}^{(n)} = \left[\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\right]^{\tau} Y_0^{(n)} + \left(\mathbf{A}_0^{(n)}\right)^{-1} U_{\tau}^{(n)} + \sum_{j=1}^{\tau-1}\left[\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\right]^{j}\left(\mathbf{A}_0^{(n)}\right)^{-1} U_{\tau-j}^{(n)}, \tag{50}$$

and that the inverse matrix $\left(\mathbf{A}_0^{(n)}\right)^{-1}$ will be also block diagonal, such that:

$$\left(\mathbf{A}_0^{(n)}\right)^{-1} = diag\left[\left(\mathbf{A}_0^{1}\right)^{-1}, \left(\mathbf{A}_0^{2}\right)^{-1}, \dots, \left(\mathbf{A}_0^{n}\right)^{-1}\right]$$

$$with:$$

$$\left(\mathbf{A}_0^{j}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi_2^j & 1 & 0 & 0 & \cdots & 0 \\ \phi_2^j\phi_3^j & \phi_3^j & 1 & 0 & \cdots & 0 \\ \phi_2^j\phi_3^j\phi_4^j & \phi_3^j\phi_4^j & \phi_4^j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=2}^{S}\phi_k^j & \prod_{k=3}^{S}\phi_k^j & \prod_{k=4}^{S}\phi_k^j & \prod_{k=5}^{S}\phi_k^j & \cdots & 1 \end{bmatrix} \quad j = 1, 2, \dots, n. \tag{51}$$

The product $\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}$ is also block diagonal, with the following form:

$$\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)} = diag\left[\left(\mathbf{A}_0^{1}\right)^{-1}\mathbf{A}_1^{1}, \left(\mathbf{A}_0^{2}\right)^{-1}\mathbf{A}_1^{2}, \dots, \left(\mathbf{A}_0^{n}\right)^{-1}\mathbf{A}_1^{n}\right]$$

$$with:$$

$$\left(\mathbf{A}_0^{j}\right)^{-1}\mathbf{A}_1^{j} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \phi_1^j \\ 0 & 0 & 0 & \cdots & 0 & \phi_1^j\phi_2^j \\ 0 & 0 & 0 & \cdots & 0 & \phi_1^j\phi_2^j\phi_3^j \\ 0 & 0 & 0 & \cdots & 0 & \phi_1^j\phi_2^j\phi_3^j\phi_4^j \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \prod_{k=1}^{S}\phi_k^j \end{bmatrix} \quad j = 1, 2, \dots, n. \tag{52}$$

Clearly, as we have PI processes the lower right element of the sub-matrices $\left(\mathbf{A}_0^{j}\right)^{-1}\mathbf{A}_1^{j}$ are equal to $\prod_{k=1}^{S}\phi_k^j = 1$. Hence, it is easy to check that matrix $\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}$ is idempotent. Then it is possible to write for (50):

$$Y_\tau^{(n)} = \left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}Y_0^{(n)} + \left(\mathbf{A}_0^{(n)}\right)^{-1}U_\tau^{(n)} + \left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\left(\mathbf{A}_0^{(n)}\right)^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j}^{(n)} \tag{53}$$

$$\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\left(\mathbf{A}_0^{(n)}\right)^{-1} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1' & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n\mathbf{b}_n' \end{bmatrix}$$

$$\mathbf{a}_j = \begin{bmatrix} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{bmatrix}'$$

$$\mathbf{b}_j = \begin{bmatrix} 1 & \phi_1^j\prod_{s=2}^{S}\phi_s^j & \phi_1^j\prod_{s=3}^{S}\phi_s^j & \cdots & \phi_1^j \end{bmatrix}'.$$

Note that, from (53), each of the n PI processes collected in the vector $Y_\tau^{(n)}$ has his own stochastic trend, that is $\mathbf{b}_j'\sum_{k=1}^{\tau-1}U_{\tau-k}^j$ for $j = 1, 2, \ldots, n$. And also we have cointegration between the seasons of each PI process. In (53) we have the cumulate sum $\sum_{j=1}^{\tau-1}U_{\tau-j}^{(n)}$ and that we can write:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}U_j^{(n)} = \begin{bmatrix} \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}U_\tau^1 \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}U_\tau^2 \\ \vdots \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}U_\tau^n \end{bmatrix} \Rightarrow \Psi^{(n)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(3)}(r). \tag{54}$$

In order to prove (54) first note that the connection between $u_{s\tau}^j$ and $\varepsilon_{s\tau}^j$ for $j = 1, 2, \ldots, n$ is the following $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right)u_{s\tau}^j = \varepsilon_{s\tau}^j$. Also we assume that $E_{s\tau}^{(n)} = \begin{bmatrix} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \cdots & \varepsilon_{s\tau}^n \end{bmatrix}'$ is a white noise vector with the positive definite variance-covariance matrix $E\left[E_{s\tau}^{(n)}E_{s\tau}^{(n)\prime}\right] = \Sigma$ then for the $(n\times S)\times 1$ vector $E_\tau^{(n)} = \left[E_\tau^{1\prime}, E_\tau^{2\prime}, \ldots, E_\tau^{n\prime}\right]' = \left[\varepsilon_{1\tau}^1, \ldots, \varepsilon_{S\tau}^1, \varepsilon_{1\tau}^2, \ldots, \varepsilon_{S\tau}^2, \ldots, \varepsilon_{1\tau}^n, \ldots, \varepsilon_{S\tau}^n\right]'$ we will have:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}E_j^{(n)} \Rightarrow \left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(n)}(r).$$

Where $W^{(n)}(r)$ is a $(n\times S)\times 1$ multivariate Vector Brownian motion with variance covariance matrix $\mathbf{I}_{(n\times S)\times(n\times S)}$ and $\mathbf{P}$ is a lower triangular matrix of order $n\times n$ such that such that $\Sigma = \mathbf{PP}'$. Hence $\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(n)}(r)$ will have a variance covariance matrix $\Sigma\otimes\mathbf{I}_S$. Now note that ,as $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right)u_{s\tau}^j = \varepsilon_{s\tau}^j$, there will be a vector of season representation for each $u_{s\tau}^j$ $j = 1, 2, \ldots, n$, that is, a VAR representation of order $P = \lfloor(p-2)/S\rfloor + 1$ as follows:

$$\left(\Psi_0^j - \Psi_1^j L - \cdots - \Psi_P^j L^P\right)U_\tau^j = E_\tau^j$$
$$\left(\Psi_0^j - \Psi_1^j L - \cdots - \Psi_P^j L^P\right) = \Psi^j(L).$$

And in the case of the $n-$variate vector $U_\tau^{(n)} = \left[U_\tau^{1\prime}, U_\tau^{2\prime}, \ldots, U_\tau^{n\prime}\right]'$ we will have:

$$\left(\Psi_0^{(n)} - \Psi_1^{(n)}L - \cdots - \Psi_P^{(n)}L^P\right)U_\tau^{(n)} = E_\tau^{(n)}$$
$$\left(\Psi_0^{(n)} - \Psi_1^{(n)}L - \cdots - \Psi_P^{(n)}L^P\right) = \Psi^{(n)}(L)$$

such that $\Psi_i^{(n)}$ $i = 0,1,\ldots,P$ are block diagonal matrices with diagonal elements Ψ_i^j $j = 1,2,\ldots,n$ for $i = 0,1,\ldots,P$. Hence we have $U_\tau^{(n)} = \Psi^{(n)}(L)^{-1} E_\tau^{(n)}$ and it will be possible to write:

$$\frac{1}{\sqrt{T}} \sum_{j=1}^{\lfloor Tr \rfloor} U_j^{(n)} = \frac{1}{\sqrt{T}} \sum_{j=1}^{\lfloor Tr \rfloor} \Psi^{(n)}(1)^{-1} E_j^{(n)} + o_p(1).$$

Hence (54) will come naturally. Next from (53) we have:

$$\begin{aligned} \frac{1}{\sqrt{T}} Y_{\lfloor Tr \rfloor}^{(n)} &\Rightarrow \left(\mathbf{A}_0^{(n)}\right)^{-1} \mathbf{A}_1^{(n)} \left(\mathbf{A}_0^{(n)}\right)^{-1} \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \\ &= \begin{bmatrix} \mathbf{a}_1 \mathbf{b}_1' & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_2 \mathbf{b}_2' & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n \mathbf{b}_n' \end{bmatrix} \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r). \end{aligned} \tag{55}$$

In order to define the n scalar Brownian motions of lemma 2, that is, $w_j(r)$, for $j = 1,2,\ldots,n$, we call $\mathbf{r}_j$ the j^{th} row of an identity matrix of order n (that is $\mathbf{I}_n$) hence, we could write (55) as:

$$\begin{aligned} \frac{1}{\sqrt{T}} Y_{\lfloor Tr \rfloor}^{(n)} &\Rightarrow \begin{bmatrix} \mathbf{a}_1 \mathbf{b}_1' & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_2 \mathbf{b}_2' & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n \mathbf{b}_n' \end{bmatrix} \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \\ &= \begin{bmatrix} \mathbf{a}_1 \left(\mathbf{r}_1 \otimes \mathbf{b}_1'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \\ \mathbf{a}_2 \left(\mathbf{r}_2 \otimes \mathbf{b}_2'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \\ \vdots \\ \mathbf{a}_j \left(\mathbf{r}_j \otimes \mathbf{b}_j'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \\ \vdots \\ \mathbf{a}_n \left(\mathbf{r}_n \otimes \mathbf{b}_n'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \end{bmatrix}. \end{aligned} \tag{56}$$

Note that the generic element $\left(\mathbf{r}_j \otimes \mathbf{b}_j'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r)$ for $j = 1,2,\ldots,n$, are scalar Brownian Motions defined as linear combinations of the $(n \times S)$ elements of the Vector Brownian Motion $W^{(n)}(r)$. Note that the variance of $\left(\mathbf{r}_j \otimes \mathbf{b}_j'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r)$ is $\left(\mathbf{r}_j \otimes \mathbf{b}_j'\right) \Psi^{(n)}(1)^{-1} [\Sigma \otimes \mathbf{I}_S] \Psi^{(n)}(1)'^{-1} \left(\mathbf{r}_j \otimes \mathbf{b}_j'\right)'$. Hence we define $w_j(r)$ and ω_j as:

$$\begin{aligned} w_j(r) &= \omega_j^{-1} \left(\mathbf{r}_j \otimes \mathbf{b}_j'\right) \Psi^{(n)}(1)^{-1} [\mathbf{P} \otimes \mathbf{I}_S] W^{(n)}(r) \\ & j = 1,2,\ldots,n \end{aligned} \tag{57}$$

with:

$$\begin{aligned} \omega_j &= \left[\left(\mathbf{r}_j \otimes \mathbf{b}_j'\right) \Psi^{(n)}(1)^{-1} [\Sigma \otimes \mathbf{I}_S] \Psi^{(n)}(1)'^{-1} \left(\mathbf{r}_j \otimes \mathbf{b}_j'\right)'\right]^{1/2} \\ & j = 1,2,\ldots,n. \end{aligned} \tag{58}$$

■

Proof of Lemma 3:

As in the proof of Lemma 2 it is possible by recursive substitution in (27) with $\mathbf{A}_0^{(n)}$ and $\mathbf{A}_1^{(n)}$ defined as in (35) to arrive to expression (50), but in this case the inverse matrix $\left(\mathbf{A}_0^{(n)}\right)^{-1}$ will be as follows (note that we use results for the inverse of partioned matrices:

$$\left(\mathbf{A}_0^{(n)}\right)^{-1} = \begin{bmatrix} \mathbf{I}_{(r\times S)} & \mathbf{D}^1 (\beta \otimes \mathbf{I}_S) \left(\mathbf{D}^2\right)^{-1} \left(\mathbf{A}_0^2\right)^{-1} \\ 0_{([n-r]\times S)\times(r\times S)} & \left(\mathbf{A}_0^2\right)^{-1} \end{bmatrix}. \tag{59}$$

Note that $\mathbf{A}_0^{(2)} = diag\left[\mathbf{A}_0^{r+1}, \mathbf{A}_0^{r+2}, \cdots, \mathbf{A}_0^n\right]$, hence we have that $\left(\mathbf{A}_0^{(2)}\right)^{-1} = diag\left[\left(\mathbf{A}_0^{r+1}\right)^{-1}, \left(\mathbf{A}_0^{r+2}\right)^{-1}, \cdots, \left(\mathbf{A}_0^n\right)^{-1}\right]$ and we know, that the specific expresson for $\left(\mathbf{A}_0^j\right)^{-1}$ will as reported in (51) for $j = r+1, r+2, \ldots, n$. Note that, for $\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}$ we will have:

$$\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)} = \left[\begin{array}{cc} \mathbf{I}_{(r\times S)} & \mathbf{D}^1\left(\beta\otimes\mathbf{I}_S\right)\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1} \\ 0_{([n-r]\times S)\times(r\times S)} & \left(\mathbf{A}_0^2\right)^{-1} \end{array}\right]$$
$$\times\left[\begin{array}{cc} 0_{(r\times S)\times(r\times S)} & 0_{(r\times S)\times([n-r]\times S)} \\ 0_{([n-r]\times S)\times(r\times S)} & \mathbf{A}_1^2 \end{array}\right]$$
$$= \left[\begin{array}{cc} 0_{(r\times S)\times(r\times S)} & \mathbf{D}^1\left(\beta\otimes\mathbf{I}_S\right)\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2 \\ 0_{([n-r]\times S)\times(r\times S)} & \left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2 \end{array}\right]. \quad (60)$$

Note that for $\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2$ it is possible to see, that as is Lemma 2 above we will have $\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2 = diag\left[\left(\mathbf{A}_0^{r+1}\right)^{-1}\mathbf{A}_1^{r+1}, \left(\mathbf{A}_0^{r+2}\right)^{-1}\mathbf{A}_1^{r+2},\ldots,\left(\mathbf{A}_0^{n}\right)^{-1}\mathbf{A}_1^{n}\right]$ and the specific expression for the $\left(\mathbf{A}_0^{j}\right)^{-1}\mathbf{A}_1^{j}$ for $j = r+1, r+2,\ldots,n$ could be found in (52). Hence it is possible to see that $\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}$ (60) is idempotent. Hence, we will also have here:

$$Y_\tau^{(n)} = \left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}Y_0^{(n)} + \left(\mathbf{A}_0^{(n)}\right)^{-1}U_\tau^{(n)} + \left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\left(\mathbf{A}_0^{(n)}\right)^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j}^{(n)}.$$

With $\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}$ defined as (60) and with $\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\left(\mathbf{A}_0^{(n)}\right)^{-1}$ with the following expression:

$$\left(\mathbf{A}_0^{(n)}\right)^{-1}\mathbf{A}_1^{(n)}\left(\mathbf{A}_0^{(n)}\right)^{-1} = \left[\begin{array}{cc} 0_{(r\times S)\times(r\times S)} & \mathbf{D}^1\left(\beta\otimes\mathbf{I}_S\right)\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2 \\ 0_{([n-r]\times S)\times(r\times S)} & \left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2 \end{array}\right] \quad (61)$$
$$\times\left[\begin{array}{cc} \mathbf{I}_{(r\times S)} & \mathbf{D}^1\left(\beta\otimes\mathbf{I}_S\right)\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1} \\ 0_{([n-r]\times S)\times(r\times S)} & \left(\mathbf{A}_0^2\right)^{-1} \end{array}\right]$$
$$= \left[\begin{array}{cc} 0_{(r\times S)\times(r\times S)} & \mathbf{D}^1\left(\beta\otimes\mathbf{I}_S\right)\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2\left(\mathbf{A}_0^2\right)^{-1} \\ 0_{([n-r]\times S)\times(r\times S)} & \left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2\left(\mathbf{A}_0^2\right)^{-1} \end{array}\right].$$

In the case of the sub-matrix $\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2\left(\mathbf{A}_0^2\right)^{-1}$ that is present in the two non-zero blocks of (61), it is possible to see ,following the lines of Lemma 2 (see (53)), that we have:

$$\left(\mathbf{A}_0^{(2)}\right)^{-1}\mathbf{A}_1^{(2)}\left(\mathbf{A}_0^{(2)}\right)^{-1} = \left[\begin{array}{cccc} \mathbf{a}_{r+1}\mathbf{b}'_{r+1} & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_{r+2}\mathbf{b}'_{r+2} & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n\mathbf{b}'_n \end{array}\right]$$
$$\mathbf{a}_j = \left[\begin{array}{cccccc} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{array}\right]'$$
$$\mathbf{b}_j = \left[\begin{array}{cccccc} 1 & \phi_1^j\prod_{s=2}^{S}\phi_s^j & \phi_1^j\prod_{s=3}^{S}\phi_s^j & \cdots & \phi_1^j \end{array}\right]'$$
$$For \qquad j = r+1, r+2,\ldots,n.$$

Finally in the case of $\mathbf{D}^1\left(\beta\otimes I_S\right)\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2\left(\mathbf{A}_0^2\right)^{-1}$, first note that for $\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2\left(\mathbf{A}_0^2\right)^{-1}$ it is possible to write:

$$\left(\mathbf{D}^2\right)^{-1}\left(\mathbf{A}_0^2\right)^{-1}\mathbf{A}_1^2\left(\mathbf{A}_0^2\right)^{-1} = \left[\begin{array}{cccc} diag\left(\mathbf{a}_{r+1}\right)^{-1} & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & diag\left(\mathbf{a}_{r+2}\right)^{-1} & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & diag\left(\mathbf{a}_n\right)^{-1} \end{array}\right]\left[\begin{array}{cccc} \mathbf{a}_{r+1}\mathbf{b}'_{r+1} & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_{r+2}\mathbf{b}'_{r+2} & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{a}_n\mathbf{b}'_n \end{array}\right]$$
$$= \left[\begin{array}{cccc} \mathbf{1}_S\mathbf{b}'_{r+1} & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{1}_S\mathbf{b}'_{r+2} & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{1}_S\mathbf{b}'_n \end{array}\right],$$

where $\mathbf{1}_S$ is a $S \times 1$ vector of ones. And hence for $\mathbf{D}^1 \left(\beta \otimes I_S\right) \left(\mathbf{D}^2\right)^{-1} \left(\mathbf{A}_0^2\right)^{-1} \mathbf{A}_1^2 \left(\mathbf{A}_0^2\right)^{-1}$, we could write:

$$\begin{aligned}
&\mathbf{D}^1 \left(\beta \otimes I_S\right) \left(\mathbf{D}^2\right)^{-1} \left(\mathbf{A}_0^2\right)^{-1} \mathbf{A}_1^2 \left(\mathbf{A}_0^2\right)^{-1} \\
&= \begin{bmatrix} diag\left(\mathbf{a}_1\right) & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & diag\left(\mathbf{a}_2\right) & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & diag\left(\mathbf{a}_r\right) \end{bmatrix} \times \\
&\times \begin{bmatrix} \beta_{11}\mathbf{I}_S & \beta_{12}\mathbf{I}_S & \cdots & \beta_{1n}\mathbf{I}_S \\ \beta_{21}\mathbf{I}_S & \beta_{22}\mathbf{I}_S & \cdots & \beta_{2n}\mathbf{I}_S \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{r1}\mathbf{I}_S & \beta_{r2}\mathbf{I}_S & \cdots & \beta_{rn}\mathbf{I}_S \end{bmatrix} \times \begin{bmatrix} \mathbf{1}_S\mathbf{b}'_{r+1} & 0_{S\times S} & \cdots & 0_{S\times S} \\ 0_{S\times S} & \mathbf{1}_S\mathbf{b}'_{r+2} & \cdots & 0_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{S\times S} & 0_{S\times S} & \cdots & \mathbf{1}_S\mathbf{b}'_n \end{bmatrix} \\
&= \begin{bmatrix} \beta_{11}\mathbf{a}_1\mathbf{b}'_{r+1} & \beta_{12}\mathbf{a}_1\mathbf{b}'_{r+2} & \cdots & \beta_{1n}\mathbf{a}_1\mathbf{b}'_n \\ \beta_{21}\mathbf{a}_2\mathbf{b}'_{r+1} & \beta_{22}\mathbf{a}_2\mathbf{b}'_{r+2} & \cdots & \beta_{2n}\mathbf{a}_2\mathbf{b}'_n \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{r1}\mathbf{a}_r\mathbf{b}'_{r+1} & \beta_{r2}\mathbf{a}_r\mathbf{b}'_{r+2} & \cdots & \beta_{rn}\mathbf{a}_r\mathbf{b}'_n \end{bmatrix}.
\end{aligned}$$

To complete the proof of Lemma 3 we only need to proceed as in Lemma 2 from (54) onwards.■