Does the Labour Theory of Value Explain Economic Growth? A Modern Classical View

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Does the Labour Theory of Value Explain Economic Growth?

A Modern Classical View

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The labour theory of value (LTV) is the cornerstone of the classical and Marxian political economy for it explains the creation and valuation of wealth in capitalist societies and it remains the chief analytical tool in investigating economic phenomena. In this respect macroscopic phenomena, which include many distinct production processes evolving over long gestation periods, conceal the transformation of labour values into their monetary expression (prices). Consequently, the use of the LTV on a grand and dynamic scale is usually considered inapplicable in the construction of macroeconomic models. The classical/Marxian analysis is conducted through either multi-dimensional multi-sectoral models or the solution of the summation problem of heterogeneous commodities. However, many studies have corroborated the dynamic aspects of the LTV and probed for a reduction in the dimensionality of macroeconomic models. In this paper, on the one hand, we restate the dynamic aspects of the LTV over time and, on the other hand, ascertain its utility as a long-run macroeconomic tool. The way to proceed is to model the divergence of actual prices and quantities of commodities from their equilibria in a multi-sectoral economy and establish that the long-run behaviour of the system mirrors the long-run movement of the labour values.

***Keywords*:** labour theory of value, heterodox microeconomics, micro-founding of economic growth, dynamic input-output analysis

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1. Introduction

The LTV was proposed by Adam Smith as a way to explain the formation and evolution of relative prices of commodities by means of the labour time expended on them (*Wealth*, chs. 5-11). He argued that the relative prices of goods are determined by the ratios of the respective labour-times employed in their production, so long there is no capital and, therefore, profits; hence, all value added is created and earned by labour. However, the presence of capital in the production process gives rise to a redistribution of labour-created value added between labour and capital, that is, between wages and profits respectively. As a result, this redistribution gives rise to divergence between relative equilibrium prices of goods and respective relative labour times provided that the capital-intensities differ between industries and free competition establishes an average rate of profit. In short, the presence of capital and profits renders equilibrium prices to be partially determined by their respective relative labour times. Smith, in the presence of such deviations, sought for alternative explanations of the movement of relative equilibrium prices.

David Ricardo criticized Smith for essentially abandoning the LTV and through the use of disarmingly simple numerical examples, he argued that the LTV holds absolutely when there is no capital and is slightly modified in the presence of differences in capital intensities, turnover times, and changes in income distribution. Ricardo was keen to point out that the core statement of the LTV, that is, the relative prices of commodities depend on the relative labour times, holds to a large extent, figuratively speaking at 93 percent and, therefore, one does not abandon a theory with such high predictive capacity (*Principles*, ch. 1). More specifically, he argued that the LTV in the presence of capital holds fully so long as there is uniform capital intensity between industries, a case quite similar to a one commodity world. Furthermore, the differences in the times of completion of the production process, as in the case of wine or timber, not only do not modify the validity of the LTV in any empirically significant way but, moreover, both the sign and size of changes in relative prices can be further theorized (Tsoulfidis 2022). The same is true with the changes in income distribution for which Ricardo posited his 'fundamental principle of distribution', that is, the inverse and, therefore, competitive relationship between wages and profits. He further argued that the effects on relative prices are not only relatively small but predictable. By contrast, for Smith these changes in income distribution would affect the price level. In general, Ricardo's point of view was that capital goods (plant and equipment, tools, raw materials) were themselves commodities, hence they embody labour times from past production processes. Consequently, all produced commodities could be decomposed into their dated quantities of labour times spent on their production and, therefore, should explain the relative prices of commodities, apart from the afore-mentioned divergences. Despite this knowledge, since Ricardo's time, these divergences between relative prices and labour times, along the ideological implications, led many economists to criticise and gradually abandon the LTV, even by economists in the classical tradition.

Marx, unlike Smith and Ricardo, argued that the embodied labour time in commodities is what gives them value, whose monetary expression (*i.e.,* direct price) establishes a more accurate centre of gravitation for the ever-fluctuating market prices. At a more concrete level of analysis, there is a set of equilibrium prices, known as "prices of production" which constitute an even better centre of gravitation of market prices, provided that both centres are not only related but they are too close to each other. Finally, there is an even more concrete another set of prices and related to the other two centres of gravitation, the regulating prices of production, which embody the dominant or regulating technical conditions of production (Tsoulfidis 2015). From the above, it follows that the more developed capitalism and, therefore, the more generalized the process of commodity exchange, the higher expected to be the explanatory power of the LTV. Marx's perspective was that

- nature and human labour power are sources of use values, but labour power is the main, if not the only, source of exchange values
- all elements of fixed capital through their depreciation transfer to the product only a fraction of their exchange value and
- labour power is a non-reproducible commodity, whose price (the wage) is regulated by its production (reproduction of the workers), not by its product.

Consequently, the prices of commodities represent embodied labour time, while the sole source of obscuration is the unpaid part of labour which appears as a surplus product, and its monetary expressions are profits, interest, rents, and taxes. Marx demonstrated that prices of commodities are not recognised as the mirror image of the labour embodied in them, because the labour process alienates the worker from the object of his/her work and mystifies his/her surplus labour in the form of profits. It goes without saying that deviations in Ricardo's sense do exist but these not only are not distortive but rather they are signs of the full operation of the LTV under the conditions of generalised competition of capitals (*Capital* III: chs. 1 -12).

The what came to be known in the literature as the ‘transformation problem’ debate refers to the above issues and initiated a vivid discussion and debates initiated by Böhm-Bawerk (1898) and attracted economists such as Dmitriev (1904) and Bortkiewicz (1907). Some decades later, Seton (1957) and Samuelson (1971) argued that the ‘transformation problem’ remains unresolved and taking into account only the actual prices and quantities of commodities, labour values are deemed redundant and metaphysical; that is, they form a set of variables introduced by the old classical economists and Marx which have no place in modern economic analysis. In addition, authors in the Sraffian tradition considered the ‘transformation problem’ in similar vein and argued the impossibility of finding a consistent solution in its most general and realistic case. Steedman (1977), in particular, argued that the transformation of labour values to prices of production fails and the labour content alone cannot explain the magnitudes of prices, while it is insufficient as an analytic tool. By contrast, Morishima (1973) and Bródy (1974) showed the prerequisites for the LTV to be valid, while Morishima (1989) proved that the source of surplus in any economic system (Ricardian, Marxian, or Neoclassical) is the unpaid labour. Shaikh (1973, 1977 and 1984) showed how one can complete Marx’s iteration approach starting from labour values going to their monetary expressions (direct prices) and arrive at prices of production in a step-by-step and theoretically consistent way. In the meantime, Shaikh (1984 and 1998), Petrović (1987), Ochoa (1989), Tsoulfidis and Rieu (2006), Tsoulfidis (2008), Tsoulfidis and Paitaridis (2017), Tsoulfidis and Tsaliki (2019, ch. 4), among others, demonstrated the empirical strength of the LTV, by computing the deviations of prices of production from their respective labour values in several economies and time spans. Lastly, along a different line, Walker (1988) argued that the neoclassical (due to Seton and Samuelson) and neo-Ricardian (due to Steedman) treatment of the LTV was static and, therefore, inappropriate to deal with reality. He proposed a dynamic reformulation of the theory to account for the ability of the system to reproduce on an expanded scale.

In this study, we argue that the LTV is theoretically consistent and therefore it can be regarded as a valuable analytical tool, which after passing successfully a battery of empirical tests, the LTV can be formulated as a solid microscopic model to determine equilibrium prices. Consequently, the LTV can address questions related to macroscopic and long-run analysis. In fact, Ricardo and Marx utilized the LTV to analyze the long-run tendencies of the capitalist system.[1] In what follows, we argue that the theory of value (microeconomics) and output (macroeconomics) should constitute aspects of the same unified economic theory. Consequently, the 'transformation problem' can be viewed as a different version of the 'summation problem', that is, going from prices to outputs as a whole.

We should bear in mind that in the classical approach, the divide in micro- and macroeconomics does not have the same importance as in the neoclassical tradition and it can even be ambiguous if applied in any strict and absolute way. As a result, the often-cited efforts to micro-founding the macro-economy are not a problem within the classical approach, in which the two are one and the same theory addressing different questions. In fact, the LTV interconnects the two spheres; on a micro-economic scale, it explains the formation and motion of individual prices by means of their labour costs, while on a macro-economic scale, it explains the distribution of income and the movement of the economy as a whole. At the same time, evaluating the macro-economic variables in terms of labour values allows for the integration of different labour processes into an 'almost' one-sector economy and the modelling of long-run tendencies without taking into account the specific microscopic phenomena. In contrast, in (neoclassical) economics there is a schism between micro and macroeconomics that needs to be bridged; more importantly, in no science, other than economics, there is the need to connect its micro and macro aspects. In what follows, we attempt to construct a dynamic microscopic model of the dual dynamics of the prices and quantities of commodities in an n-sectoral economy, that mathematically supports Walker's claim. The long-run attractor of the system lends support to the view that the LTV can be used in macro-economic analysis both in the short- and long-run periods.

The structure of the rest of the paper is as follows: Section 2 presents the variables and the symbols used in the analysis. Section 3 summarizes the pertinent literature on static and dynamic

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[1] As Morishima (1989, p.18) notes, Marx's LTV purports to reveal the exploitative nature of capitalism to expropriate the product of labor, while Ricardo's LTV is more of an accounting system connecting the different labor processes resembling the neoclassical micro-foundations of macroeconomics.

multisectoral models, starting with von Neumann's model of general equilibrium ending with the seminal works by Flaschel and Semmler (1987 and 1990) on Classical cross-dual dynamics. Section 4 considers a purely classical model, where prices adjust to imbalances of quantities and *vice versa*. The analysis extends to include a Keynesian and a Sraffian kind of addenda about possible reactions of quantities and prices to imbalances of quantities and prices, respectively. Section 5 operationalizes the model in two stages; the first utilizes an autonomous system while the second proceeds by assuming that the parameters (technical coefficients, labour inputs, exogenous demand) change over time. Both cases are treated employing Li's theorems on the Lotka-Volterra system. Section 6 concludes the paper by making recommendations for future research efforts.

2. Preliminaries

Let us suppose an economy with n sectors producing n type of commodities or outputs which are represented by the column-vector of gross output, $\mathbf{x}$. By $\mathbf{y}$ we represent the column-vector of net output (net income), which is allocated into worker's consumption, $\mathbf{b}$, capitalists' consumption, $\mathbf{f}$, and savings plus hoarding, $\boldsymbol{\iota}$; the latter is treated as investment spending. Thus, we may write

$$\mathbf{y} = \mathbf{b} + \mathbf{f} + \boldsymbol{\iota}$$

Let us further suppose $\mathbf{p}^T$ to be the row-vector of prices and $\mathbf{l}^T$ the row-vector of employment coefficients. The money wage is given as

$$w = \mathbf{p}^T\mathbf{b} \tag{1}$$

We introduce the $n \times n$ matrix of technological coefficients, $\mathbf{A}$, so that the output of sector i is allocated to all other sectors and itself as intermediate input; thus, we may write

$$\mathbf{A}\,\mathbf{x} = \sum_j a_{ij}x_j$$

And the cost of sector j derived by its circulating capital inputs from all sectors is

$$\mathbf{p}^T\mathbf{A} = \sum_i p_i a_{ij}$$

$\mathbf{D}$ stands for the $n \times n$ matrix of depreciation coefficients, d_{ij}, so that the production of sector i which is distributed to all sectors as depreciation of the stock is

$$\mathbf{D}\,\mathbf{x} = \sum_j d_{ij} x_j$$

and the cost of sector j derived by itsfixed capital inputs from all sectors is

$$\mathbf{p}^T \mathbf{D} = \sum_i p_i d_{ij}$$

Furthermore, $\mathbf{K}$ stands for the $n \times n$ matrix of capital stock coefficients, k_{ij}, so that the total amount of the net income that is saved, S, may be written as

$$S = \mathbf{p}^T \boldsymbol{\iota} = \mathbf{p}^T \mathbf{K}\,\dot{\mathbf{x}} - \dot{\mathbf{p}}^T \mathbf{K}\,\mathbf{x} \tag{2}$$

The above relation also reflects the macroeconomic variable of investment and captures all withdrawals of money from the circuit of capital, be it in the form of actual investment expenditures, or in the form of hoarding. The amount going to investment is captured by the change in quantities (growth) while the amount hoarded is reflected as a change of prices (inflation)[2]. As usual, the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{b}$, $\mathbf{f}$, $\boldsymbol{\iota}$, $\mathbf{p}^T$ and $\mathbf{l}^T$ are non-negative, while matrices $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{K}$ are non-negative, invertible, (semi-) positive-definite and follow the Hawkins-Simon condition.

We apply the usual normalization of prices using as *numéraire* the gross output

$$\boldsymbol{\rho}^T = \mathbf{p}^T \left(\frac{\mathbf{e}^T \mathbf{x}}{\mathbf{p}^T \mathbf{x}} \right) \tag{3}$$

We apply a similar normalization condition on outputs with the difference being that in the numerator we have the sum of prices to derive what has been deemed ‘activity levels’ by von Neumann

$$\mathbf{q} = \mathbf{x} \left(\frac{\mathbf{p}^T \mathbf{e}}{\mathbf{p}^T \mathbf{x}} \right) \tag{4}$$

where $\mathbf{e}$ the nx1 summation vector. The advantage of this normalisation is that it maintains the form of the static Leontief-Sraffa system meaning that the total output is the clarifying *numéraire* for the system, as is the usual case. At the same time, it treats prices and quantities as shares of the total output, which is another way to say that $\boldsymbol{\rho}^T$ and $\mathbf{q}$ are relative prices and relative outputs, respectively; hence their values are restricted between 0 and 1. Furthermore, $\boldsymbol{\rho}^T$ and $\mathbf{q}$ are preferred

[2]Throughout the text, the dot over a variable indicates its time derivative.

over $\mathbf{p}^T$ and $\mathbf{x}$, respectively, in the sense that their change reflects the deviation of prices and outputs from the change in the total output. As a consequence, the vectors $\mathbf{b}$, $\mathbf{f}$, $\boldsymbol{\iota}$ and $\mathbf{l}^T$ are normalised accordingly.

3. Literature Review

Adam Smith's description of the dynamics of competition through the interaction of prices with quantities captured the attention of many economists of the 19th century. Starting with the proto-marginalists in the early 19th century (Cournot) and continuing to the neoclassical tradition (Walras) in the last quarter of the same century, there were efforts to provide a mathematical statement of the formation of prices in an economy. In the classical tradition, a mathematically rigorous price mechanism and its adjustment to market imbalances had to wait until von Neumann's (1945-1946) seminal economic growth model, which was a critique and, at the same time, an amendment to Walras's model. Von Neumann casting in mathematical terms the classical theory of value and distribution and assuming free competition managed to invigorate and, at the same time, operationalize both of them.

In what follows, we briefly review the various attempts made over the years to model the Classical/Marxian theory of value and distribution. These models are not fully comparable to each other. The reason is that some of them focus on the theory of production, some others are concerned with economic growth and still, others are dealing with free competition. On further consideration, however, we discover that there is a common thread connecting all these models. And that is, they are cast in input-output terms, a feature that makes them comparable to each other. More importantly, these models operationalize the classical theory of value and distribution and show how prices and outputs adjust to each other in the attainment of their long-run equilibrium. The salient features of the latter are the operation of free competition and economic growth.

3.1. Von Neumann's growth model, its problems and limitations

The Von Neumann growth model serves a dual purpose: First, it seeks to 'correct' Walras's general equilibrium from its static analytical framework by introducing immanent to capitalism dynamics featuring growth and expansion; second, the model serves as an application of Brouwer's Fixed Point Theorem in the case of a linear economic structure, proving that such a system of production has a viable equilibrium even in the dynamic case.[3] The main assumptions of his model are:

- Given the technology, as this is represented by the matrices of technological, depreciation and capital stock coefficients, $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{K}$, respectively
- Given the vectors of gross and net output $\mathbf{x}$ and $\mathbf{y}$, as well as the vectors of workers consumption $\mathbf{b}$, along with the vectors of prices of production $\mathbf{p}^T$ and labour times per unit of output, $\mathbf{l}^T$
- Workers, as a class, do not save and spend their total wage on consumption goods, and
- Capitalists, as a class, invest all of their profits.

Von Neumann assumed that for the system to be productive (as shown in system (5)) there must be a positive surplus; hence, the money value of output should be at most equal to the money sum of the costs marked-up by a given general interest (profit) rate. Also, for the system to be closed (as shown in the system (6)) the entire production must be consumed within a period; hence, the quantities entering in the production process should not exceed the quantities exiting the production process. With given the above, the model can be stated as a linear programming problem in the following form[4]

$$\begin{aligned} &\max \boldsymbol{\rho}^T \geqq 0 \ \text{ and } \ a > 0 \\ &\text{s.t. } \boldsymbol{\rho}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T - a\mathbf{K}) \leqq 0 \\ &\text{and to} \qquad \mathbf{A}, \mathbf{D}, \mathbf{K}, \mathbf{b}, \mathbf{l}^T \geqq 0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\max \mathbf{q} \geqq 0 \ \text{ and } \ b > 0 \\ &\text{s.t. } (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T - b\mathbf{K})\mathbf{q} \geqq 0 \\ &\text{and to} \qquad \mathbf{A}, \mathbf{D}, \mathbf{K}, \mathbf{b}, \mathbf{l}^T \geqq 0 \end{aligned} \tag{6}$$

where a and b are two positive constants, representing the rate of profit and the growth rate, respectively. The solution of the systems yields

[3] In von Neumann's formulation, the number of production processes is less to the number of commodities; hence, we have joint production and the input-output system is rectangular; as a consequence, the identity matrix is substituted by some matrix $\mathbf{B}$. As is well known, the joint production input-output systems may yield negative —thus, economically trivial— solutions for either prices or quantities. To circumvent this, von Neumann (1945-1946) assumed that the activity levels will turn zero any time that prices are to be negative. The idea is that no industry would function if its output would be sold at negative prices as profits would turn negative.

[4] The form in which von Neumann's model is presented here is derived from Bródy (1974) and it differs only slightly from the usual presentation of Morishima (1963).

$$\frac{1}{a} = \frac{1}{b} = \frac{\boldsymbol{\rho}^T \mathbf{K}\,\mathbf{q}}{\boldsymbol{\rho}^T (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T)\mathbf{q}}$$

Hence, a and b are the reciprocals of Perron-Frobenius eigenvalues of the matrices $\mathbf{K}(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T)^{-1}$ and $(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T)^{-1}\mathbf{K}$, respectively, that stand for the capital-surplus ratio, provided that the economic surplus is the sole output of the economy.[5] From an economic standpoint, the programming problem (5) reflects the costs of the production processes; so the optimum profit rate, π^*, can be given as

$$\pi^* = a = \frac{\boldsymbol{\rho}^T (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T)\mathbf{q}}{\boldsymbol{\rho}^T \mathbf{K}\mathbf{q}} \tag{7}$$

The programming problem (6) reflects the distribution of surplus for reproduction purposes; so the optimum growth rate, g^*, can be given as

$$g^* = b = \frac{\boldsymbol{\rho}^T (\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T)\mathbf{q}}{\boldsymbol{\rho}^T \mathbf{K}\mathbf{q}} \tag{8}$$

It is obvious from equations (7) and (8) that the optimum growth rate is equal to the optimum rate of profit; it is also clear that both of them are equal to the output-capital ratio, as one would expect from the relevant literature, either for the rate of profit or for the rate of growth (Harrod 1939). The prices and activity levels at equilibrium, $\boldsymbol{\rho}^{T*}$ and $\mathbf{q}^*$, are the left and right eigenvectors of the matrices $\mathbf{K}(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^{\mathrm{T}})^{-1}$ and $(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^{\mathrm{T}})^{-1}\mathbf{K}$, respectively. This state of equilibrium $(\pi^*, g^*, \boldsymbol{\rho}^{T*}, \mathbf{q}^*)$ is considered the optimum for an expanding capitalist economy and is known in the literature as the von Neumann's 'turnpike'. Interestingly, this state bears a striking resemblance to Harrod's 'warranted' rate of accumulation. In both models, the rate of profit is such that does not generate inflation (constant $\boldsymbol{\rho}^{T*}$) while the rate of growth is such that corresponds to full capacity (constant $\mathbf{q}^*$). In addition to this resemblance, the 'turnpike' bears similarity to the Harrodian instability, as it has been shown by Goodwin and Punzo (1987) and Goodwin (1990) to be asymptotically unstable. Consequently, von Neumann's model can be thought as a multi-sectoral generalisation of Harrod's growth model.

[5] The term capital-surplus ratio is based on von Neumann's concept of 'cost-surplus ratio', provided that he does not use capital stock and the cost of labour is included in the matrix $\mathbf{A}$, as a result, his "output" is in effect the surplus.

The initial von Neumann model is inflicted by a few weaknesses as we explicate below. First, there is no depreciation, though this can easily be counted by simply adding the matrix of depreciation coefficients along with the other flow matrices. Second, it is assumed that capitalists save all of their income; however, this assumption is not as restrictive as it appears at first sight, because it is used merely to underline the capitalists' motivation "to accumulate for accumulation's shake", and it may be moderated to include the case in which part of the surplus value is unproductively consumed.[6] Third, it is assumed that competition establishes one price for each commodity and a uniform rate of profit for each process; however, in reality, there are deferential profit and growth rates. Last but not least, the constancy of the parameters may be an oversimplification of the actual process of capital accumulation; for instance, in reality, there is technological change manifested in rising productivity or cyclical behaviour of employment. However, it is important to stress that significant technological change and its diffusion require the lapse of considerable time. In the meantime, the input-output and capital stock coefficients do not change in any significant way, and for all practical purposes, and in agreement with von Neumann and the classical tradition, we can treat them as a datum.

3.2. Generalisations of von Neumann's model by Pasinetti, Goodwin and Punzo

The aforementioned limitations of the original von Neumann model generated quite interesting discussions starting with Morishima (1973) and Bródy (1974) among others. These authors, based on the von Neumann model and its resemblance to Harrod's, attempted to express Marx's theory of value and capital accumulation as an application and further elaboration of the aforesaid model. These efforts were manifestations of the need not only to extend the model so that to encompass important features of reality, but to provide a model that rigorously operationalizes the classical/Marxian theory of value and distribution. In this direction, we may include the works by Pasinetti (1983 and 1990), Goodwin (1983), Goodwin and Punzo (1987) and Punzo (1990).

Pasinetti (1983) considered the case of exogenously changing employment, $\mathbf{l}^T$, and final demand, $\mathbf{y}$, in a multi-sectoral model; he also considered an exogenous technological change affecting each

[6] This idea is hidden in von Neumann's assumption. Essentially, one could assume that capitalists receive a salary for their managing and entrepreneurial services, which constitutes their personal income; this salary may be treated as part of the cost and is directed to consumption, hence it can be added to the matrix $(\mathbf{A} + \mathbf{D} + \mathbf{b}\,\mathbf{l}^T)$.

sector differently. His scope was to explore the structural dynamics of an expanding economy and the implications surfacing from the interaction between different processes. His attempt is deemed highly successful, albeit his contribution with respect to analytic rigorousness of the von Neumann model has been limited. However, the new idea that Pasinetti brought in the relevant analysis is that of the vertically integrated sectors instead of the usual input-output relations; in so doing, he made the analysis much simpler and transparent in identifying the diverging technological change and differential growth rates.

Similarly, Goodwin (1983) attempted to extend the von Neumann model based on the fact that the optimum solution corresponds to the Perron-Frobenius eigenvalue. He assumed that each multi-sectoral system can be decomposed to its eigen-sectors, each of them corresponding to a specific eigenvalue, which is nothing more than the eigen-sectoral growth rate and eigen- sectoral rate of profit.[7] Through this decomposition, Goodwin and Punzo (1987) assessed the stability and instability properties of the von Neumann equilibrium and attempted to model the fluctuating evolution of the capitalist economies. Later, Pasinetti (1990) and Punzo (1990) showed the resemblance of the two approaches, while Steenge (1995) noted that the linear transformations leading to Pasinetti's vertically integrated sectors or to Goodwin's eigen-sectors are identical to the ones leading to Sraffa's standard commodity. Bródy (1989) reached to a similar model in an attempt to construct a unified disaggregated model of growth and cycles. He utilized a scalar objective function, which represents the accumulated gains from the imbalance between aggregate supply and demand; this is similar to Goodwin's use of a potential, but it extends the idea from a transitory to an intertemporal state. Unfortunately, Bródy did not proceed further with the analysis of his model and no continuation of this work is known to the authors.

3.3. Dynamics of quantities by Leontief and Lange

The input-output model originally proposed by Leontief did not deal with the dynamic dimensions of the system; however, later amendments by him, but mostly by Jorgenson (1961), Lange (1969) and Szyld (1985) *inter alia* introduced in the analysis the investment and the intertemporal change in quantities. The idea is simple and is founded by extending of Leontief's fundamental equation. Given that the gross output is decomposed to intermediate uses and final demands as

[7] In fact, the (eigen-)sectoral rate of profit is the eigenvalue divided by the (eigen-)sectoral rate of savings.

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Dx} + \mathbf{y} = \mathbf{Ax} + \mathbf{Dx} + \mathbf{b} + \mathbf{f} + \boldsymbol{\iota}$$

prices are considered to be constant in the long-run; hence, $\boldsymbol{\iota} = \mathbf{K}\,\dot{\mathbf{x}}$ in continuous time. As a result, the differential equations guiding the evolution of quantities are

$$\dot{\mathbf{x}} = \mathbf{K}^{-1}(\mathbf{I} - \mathbf{A} - \mathbf{D})\mathbf{x} - \mathbf{K}^{-1}(\mathbf{b} + \mathbf{f}) \tag{9}$$

Setting $\dot{\mathbf{x}} = \mathbf{0}$, we see that the fixed point of equation (9) is

$$\mathbf{x}^* = (\mathbf{I} - \mathbf{A} - \mathbf{D})^{-1}(\mathbf{b} + \mathbf{f})$$

which is Leontief's equilibrium when investment is zero. This fixed point is unstable, as the components of matrix $\mathbf{K}^{-1}(\mathbf{I} - \mathbf{A} - \mathbf{D})$ are non-negative; moreover, the character of instability (monotonic or oscillatory) is not guaranteed as it depends on the eigenvalues of the said matrix. However, the analysis is useful to identify or predict the short-run growth path of the economy, which is bounded by the maximal eigenvalue.

At this point is worth noting that this upper bound makes possible the further theorization of economic growth and its extension to the long-run. However, the latter, based on a multisectoral model, would require the consideration of technological and distributional changes, which cannot be derived endogenously from the model. Hence, the theoretical and empirical validity of such a model is restricted to more or less short-run analysis. Consequently, apart from some remarks on the stability of these models, little theoretical or empirical contributions were made.

3.4. Dynamics of prices by Nikaido and Kobayashi

In a similar fashion, the dynamics of prices have been considered as well and the idea behind this was not to assess growth, but to account for the adjustment mechanism of prices. Another point of interest was to account for the so-called Wicksell effect, which is the change in relative prices due to income distribution. This effect was originally discussed by Ricardo and Marx and further elaborated by Sraffa (1960) and it was implemented by neo-Ricardians to criticize the LTV. Nikaido and Kobayashi (1978) formulated their analysis around an adjustment mechanism for prices described by the following differential equations

$$\dot{\mathbf{p}}^T = \mathbf{p}^T\mathbf{A} + \mathbf{p}^T\mathbf{D} + w\mathbf{l}^T + \pi\mathbf{p}^T\mathbf{K} - \mathbf{p}^T \tag{10}$$

These differential equations were considered for a given rate of profit and were joined by an equation for the adjustment of wage aiming to formulate the proposed wage-prices spirals of the neo-Ricardian approach.[8] In equation (10), we may set $\dot{\mathbf{p}}^T = \mathbf{0}$ and compute the fixed point as

$$\mathbf{p}^{T*} = w\mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \pi\mathbf{K})^{-1}$$

which approximates the labour values, $\mathbf{v}^T = \mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D})^{-1}$, provided that π tends to zero or the capital intensities are no different between industries. In the purely circulating model studied by Nikaido and Kobayashi ($\mathbf{D} = 0$ and $\mathbf{K} = \mathbf{A}$), this fixed point is asymptotically stable under very loose conditions on the matrix $[\mathbf{I} - (1 + \pi)\mathbf{A}]^{-1}$ and very tight conditions on the wage; however, this result is expected to hold and in the case of a fixed capital model, since the matrix $(\mathbf{I} - \mathbf{A} - \mathbf{D} - \pi\mathbf{K})^{-1}$, we know from the extant empirical literature, is negative-definite (invertible?!) provided that the rate of profit does take extreme values. Furthermore, the nature of stability (monotonic or oscillatory) is not *a priori* specified, unless some further conventions are made about matrices $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{K}$.

3.5. Cross-dual dynamics by Flaschel and Semmler

It was again Nikaido (1983 and 1985) who discussed the interaction of prices and quantities in a combined system. His work questioned the stability of the classical equilibrium and suggested that the process of competition as described by the old classical economists and Marx does not lead to convergence. Flaschel and Semmler (1987 and 1990) attempted to dislodge his arguments by addressing the problem through a model of cross-dual interactions between the two sets of variables, prices and quantities. Essentially, they decomposed these dynamics into two effects:

(1) the Classical micro-dynamical adjustments, according to which prices respond to changes in quantities and *vice versa*, and

(2) the Keynesian micro-dynamical adjustments, where there is a disconnect between prices and outputs; that is prices respond to changes in prices and quantities respond to changes in quantities.

[8] Nikaido and Kobayashi (1978) work within Sraffa's own premises and they consider a purely circulating capital model with $\mathbf{D} = 0$ and $\mathbf{K} = \mathbf{A}$; in such models, the presence of stocks is usually presented by means of an output matrix $\mathbf{B}$, as in the von Neumann model.

Their investigation led to the conclusion that convergence to equilibrium is not guaranteed and further restrictions must be placed. In fact, there exist three different ways which allow to prove stability for a composite Classical/Keynesian system: namely the diagonal dominance of matrix $(\mathbf{I}-\mathbf{A}-\mathbf{D}-\pi\mathbf{K})^{-1}$, the quasi-negative definiteness of matrix $(\mathbf{I}-\mathbf{A}-\mathbf{D}-\pi\mathbf{K})^{-1}$ and a new two-level approach. In all cases, certain conditions for the strength of the Classical component must be assumed to obtain stable composite dynamics.[9]

4. Establishing a Cross-Dual Dynamic Model of Prices (Values) and Outputs

Our modelling of competition and its dynamics is different from Nikaido's (1983 and 1985) or Flaschel and Semmler's (1987 and 1990) in that we consider the normalised variables; hence, we model the sectoral deviations from the state of uniform growth. Furthermore, our emphasis is not limited to competition and the establishment of a short-run equilibrium; but extends to include the process of economic growth and the attainment of a long-run equilibrium. Starting with the normalised prices, we theorize their motion through their growth; if it is positive (negative), we have inflation (deflation), $\boldsymbol{\varpi}$, caused in the long-run by persisting imbalances in the market. Essentially, whenever the demand for a commodity increases (decreases) above (below) its supply, the price of that commodity is expected to rise (fall). The motion of the normalised prices follows that of the actual prices with one difference: dividing each price with the total product clears off the 'market effects'; hence the motion of prices does not reflect any short-run monetary phenomena. As a result, the growth in prices can be written as follows:

$$\frac{\mathrm{d}\boldsymbol{\rho}^T}{\mathrm{d}t} < \boldsymbol{\rho}^T >^{-1} \approx \mathbf{y}_{\boldsymbol{D}} - \mathbf{y}_{\boldsymbol{S}}$$

where $\mathbf{y}_{\boldsymbol{D}}$ is the vector of the normalised aggregate demand and $\mathbf{y}_{\boldsymbol{S}}$ is the vector of the normalised aggregate supply. However, prices may change because they adjust to other effects, such as changes in the income distribution (Wicksell effect). This adjustment is only temporary and of much smaller importance, which means that for all practical purposes can be omitted; however, we could always include it in the analysis at a later stage (Tsoulfidis and Tsaliki 2019, ch. 5). In addition, Nikaido and Kobayashi (1978) argued that this effect is due to the deviation of costs from

[9] For a discussion of the Classical and Keynesian components used by Flaschel and Semmler, see the presentation in Section 4, since our conventions are very similar to theirs.

prices. Hence, the above relation may be supplemented with the short-run Wicksell effect as follows

$$\frac{\mathrm{d}\boldsymbol{\rho}^T}{\mathrm{d}t} < \boldsymbol{\rho}^T >^{-1} \approx \underbrace{(\mathbf{y}_D - \mathbf{y}_S)}_{\substack{\text{long−run} \\ \text{(Classical) effect}}} + \underbrace{(\mathbf{c} - \mathbf{r})}_{\substack{\text{short−run} \\ \text{(Wicksell) effect}}}$$

where $\mathbf{c}$ is the vector of normalised costs and $\mathbf{r}$ is the 'normal' level of net revenue.

The demand for a commodity consists of workers plus capitalists' consumption and is considered exogenously given, as it does not alter in any significant way in the short run and expands uniformly in the long run; in contrast, savings, investment and hoarding are variables more sensitive to both short- and long-run shocks. We also consider a level of 'normal' investment,$\bar{\boldsymbol{\iota}}$, as part of the normalized demand responsible for both the replacement of the worn-out capital and for proportional growth.[10] Hence, the normalised aggregate demand is defined as

$$\mathbf{y}_D = \frac{\mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}}}{\boldsymbol{\rho}^T \mathbf{K}\, \mathbf{q}}$$

In the same reasoning, the supply of a commodity depends solely on its production; consequently, the normalised aggregate supply is given by

$$\mathbf{y}_S = \frac{(\mathbf{I} - \mathbf{A} - \mathbf{D})\mathbf{q}}{\boldsymbol{\rho}^T \mathbf{K}\, \mathbf{q}}$$

The costs of production are defined from the side of production, taking into account prices of the means of production and labour costs; that is

$$\mathbf{c} = \frac{(\mathbf{A} + \mathbf{D})^T \boldsymbol{\rho} + w\mathbf{l}}{\boldsymbol{\rho}^T \mathbf{K}\mathbf{q}}$$

In addition, the net revenue consists of capitalists' income that is not consumed productively, hence that is not invested; the 'normal' level of it defines the balanced growth path produced in the Schemes of Expanded Reproduction. If the latter is to coincide with von Neuman's 'turnpike', where all surplus is capitalized, then the 'normal' level should be zero. Defining the investment

[10] Essentially, this 'normal' level of investment is identical to Marx's Schemes of Expanded Reproduction; on the one hand, it assures a rate of uniform growth under constant proportions (constant composition of capital), while on the other hand, it does not necessarily match to von Neumann's 'turnpike', as it may include a non-optimum case.

matrix as $\bar{\mathbf{E}} = \bar{\boldsymbol{\iota}} < \mathbf{q} >^{-1}$,[11] the normalized level or 'normal' net revenue, $\mathbf{r}$, that allows for the normal reproduction of the capitalist class, is given by

$$\mathbf{r} = \frac{(\mathbf{I} - \bar{\mathbf{E}})^T \boldsymbol{\rho}}{\boldsymbol{\rho}^T \mathbf{K}\, \mathbf{q}}$$

We also consider two sets of proportionality constants, $\varphi_{i(1)}$ and $\varphi_{i(2)}$. These coefficients determine the intensity of the two effects and bear no direct economic meaning; their main use is that of scaling the differential equations so that $\boldsymbol{\rho}^T$ and $\mathbf{q}$ are restricted to an economically meaningful range of values. As a result, the differential equations concerning the motion of normalised prices are written as

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{\rho}^T}{\mathrm{d}t} < \boldsymbol{\rho}^T >^{-1} = < \boldsymbol{\varphi}_1 > \frac{\mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}} - (\mathbf{I} - \mathbf{A} - \mathbf{D})\mathbf{q}}{\boldsymbol{\rho}^T \mathbf{K}\, \mathbf{q}} \\ - < \boldsymbol{\varphi}_2 > \frac{(\mathbf{I} - \mathbf{A} - \mathbf{D} - \bar{\mathbf{E}})^T\, \boldsymbol{\rho} - w\mathbf{l}^T}{\boldsymbol{\rho}^T \mathbf{K}\, \mathbf{q}} \end{aligned} \tag{11.a}$$

or

$$\begin{aligned} \frac{\dot{\rho}_i}{\rho_i} = -\varphi_{i(1)} \frac{q_i - \sum_j a_{ij} q_j - \sum_j d_{ij} q_j - b_i - f_i - \bar{\iota}_i}{\kappa(\rho_i, q_i)} \\ - \varphi_{i(2)} \frac{\rho_i - \sum_j a_{ji} \rho_j - \sum_j d_{ji} \rho_j - \sum_j \varepsilon_{ji} \rho_j - w\, l_i}{\kappa(\rho_i, q_i)} \end{aligned} \tag{11.b}$$

where $\kappa(\rho_i, q_i) = \boldsymbol{\rho}^T \mathbf{K}\, \mathbf{q}$ is the total cost of the stocks for the economy.

Moving now to the motion of activity levels, we observe that a change in quantities of commodities produced presents the expansion (or contraction) phase of the economy; thus, the rate of change of the quantities is nothing more but the growth rate of each sector. From the Classical/Marxian tradition and von Neumann's analysis, we know that the growth rate depends on the rate of profit; more specifically, the rate of growth is equal to capitalists' propensity to save times the rate of profit minus the rate of depreciation. However, the rate of change of the normalised quantities is not equal to the rate of profit *per se*, but it reflects the deviations of the sectoral rates of profit, $\boldsymbol{\pi}$, from the uniform rate of profit, π^*. Thus,

[11] It is worth mentioning that such investment or capital flows matrices are constructed from time to time. Their general structure is pretty much the same over the years. The idea is that many industries (such as those in consumer goods, services, government, and the like) do not produce investment goods, and their rows are filled with zeros. By contrast, in the investment goods industries, their coefficients, not too many compared to those of the matrix **A**, change so slowly that we can reasonably treat them as if they were constant.

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} < \mathbf{q} >^{-1} \approx \boldsymbol{\pi} - \pi^* < \mathbf{q} >^{-1}$$

As Flaschel and Semmler (1987 and 1990) argue, this interaction concerns the short-run implications of competition while post-Keynesian analyses by Bhaduri and Marglin (1990) argue that it should reflect the short-run implications of growth, as well. This critique can be dealt with if we add the rate of capacity utilisation in the determination of the growth rate; so, the rate of change of the normalised prices should be proportional to the deviation of the sectoral rates of profit from the uniform one, and the deviation of the sectoral capacity utilisation rates ($\mathbf{u}$) from the uniform one (u^*). Thus,

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} < \mathbf{q} >^{-1} \approx \underbrace{(\boldsymbol{\pi} - \pi^* < \mathbf{q} >^{-1})}_{\substack{\text{long-run} \\ \text{(profitability) effect}}} + \underbrace{(\mathbf{u} - u^* < \boldsymbol{\rho} >^{-1})}_{\substack{\text{short-run} \\ \text{(capacity) effect}}}$$

The sectoral rates of profit are determined as follows

$$\boldsymbol{\pi} = \frac{(\mathbf{I} - \mathbf{A} - \mathbf{D})^T \boldsymbol{\rho} - w\mathbf{l}}{\boldsymbol{\rho}^T \mathbf{K}\,\mathbf{q}}$$

while the uniform rate of profit corresponds to a modified version of equation (7), which incorporates the ‘normal’ level of investment; as von Neumann assumed, in equilibrium capitalists’ personal consumption is minimized and can always be assumed as a fraction of the costs (that is, the remuneration the capitalists earn for their entrepreneurial contributions to production). That is

$$\pi^* = \frac{\boldsymbol{\rho}^T \bar{\mathbf{E}}\,\mathbf{q}}{\boldsymbol{\rho}^T \mathbf{K}\,\mathbf{q}}$$

As usual, the sectoral rates of capacity utilisation are given as the ratio of actual output over potential output for each process. The actual output is of course measured by the outcome of the production processes, while the potential output is identified to the total expended capital. As a result,

$$\mathbf{u} = \frac{(\mathbf{I} - \mathbf{A} - \mathbf{D})\mathbf{q}}{\boldsymbol{\rho}^T \mathbf{K}\,\mathbf{q}}$$

while the uniform rate of capacity utilisation is given with respect to the net output measured from the side of consumption as

$$u^* = \frac{\boldsymbol{\rho}^T (\mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}})}{\boldsymbol{\rho}^T \mathbf{K}\mathbf{q}}$$

Again, we assume a set of proportionality constants, $\psi_{i(1)}$ and $\psi_{i(2)}$, that determine the intensity of the two effects and bare the same significance as $\varphi_{i(1)}$ and $\varphi_{i(2)}$. As a result, the differential equations concerning the motion of normalised quantities are given as

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} < \mathbf{q}^{-1} > = < \boldsymbol{\psi}_1 > \frac{\boldsymbol{\rho}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \bar{\mathbf{E}}) - \mathbf{w}\mathbf{l}^T}{\boldsymbol{\rho}^T\mathbf{K}\,\mathbf{q}} + < \boldsymbol{\psi}_2 > \frac{\mathbf{q}^T(\mathbf{I} - \mathbf{A} - \mathbf{D})^T - \mathbf{b}^T - \mathbf{f}^T - \bar{\boldsymbol{\iota}}^T}{\boldsymbol{\rho}^T\mathbf{K}\,\mathbf{q}} \tag{12.a}$$

or

$$\frac{\dot{q}_i}{q_i} = \psi_{i(1)} \frac{\rho_i - \sum_j \rho_j a_{ji} - \sum_j \rho_j d_{ji} - \sum_j \rho_j \bar{e}_{ji} - w l_i}{\kappa(\rho_i, q_i)} + \psi_{i(2)} \frac{q_i - \sum_j a_{ij} q_j - \sum_j d_{ij} q_j - b_i - f_i - \bar{\iota}_i}{\kappa(\rho_i, q_i)} \tag{12.b}$$

Equations (11.b) and (12.b) are a formulation of the fluctuations of normalized prices and quantities, respectively, from a ‘normal’ level. So far as the prices are concerned, this ‘normal’ level reflects the costs of production, while the ‘normal’ level of quantities represents the case of expanded reproduction under fixed proportion. Apparently, the latter is a restatement of Marx’s Schemes of Expanded Reproduction rendering the former a restatement of Marx’s labour values in an expanding economy. In the following section, it is shown that the economically meaningful fixed point of the system is identical to the state of an expanding economy where estimated prices are exactly proportional to the labour values.

5. The Qualitative Behaviour of the Model

Equations (11.b) and (12.b) have a form widely known in the theory of dynamical systems. Foremost, we may notice that the signs on the right-hand side are opposite; this is a probe to the duality between the variables, an idea not novel in the literature. By formulating the von Neumann model by two parallel linear programming problems, as in equations (5) and (6), we can see that the two equations are dual to each other making the two sets of variables also dual to each other. This duality property is an indication for the system to bear a Hamiltonian structure. Systems sharing this structure may take the form

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial x_i} \\ \dfrac{\partial \mathcal{H}}{\partial y_i} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial y_i} \\ -\dfrac{\partial \mathcal{H}}{\partial x_i} \end{bmatrix}$$

where $\mathcal{H}(x_i, y_i)$ is the Hamiltonian function and x_i and y_i are the dual variables.[12] This Hamiltonian function acts as an 'integral' and as a possible Lyapunov function of the system; using this alone, we can discuss the asymptotic and global stability of the system. Essentially, it is impossible for a solution in a Hamiltonian system to be asymptotically stable; it can either be a 'saddle' hence unstable, or a 'centre' hence globally and structurally stable, but asymptotically unstable. Consequently, such a system may 'stabilize' in oscillations but not converge in a fixed point.

On further consideration we find that the above system is not entirely Hamiltonian, but it retains a pseudo-Hamiltonian structure, as it can be written in the following form

$$\begin{bmatrix} \dot{q}_i \\ \dot{\rho}_i \end{bmatrix} = \frac{1}{\kappa(\rho_i, q_i)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_{i(1)} q_i \rho_i & \varphi_{i(2)} {\rho_i}^2 \\ \psi_{i(2)} {q_i}^2 & \psi_{i(1)} q_i \rho_i \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathcal{L}}{\partial q_i} \\ \dfrac{\partial \mathcal{L}}{\partial \rho_i} \end{bmatrix} \tag{13}$$

where

$$\mathcal{L}(\rho_i, q_i) = (\mathbf{I} - \mathbf{A} - \mathbf{D} - \bar{\mathbf{E}})\boldsymbol{\rho} - w\mathbf{l}^T \ln \boldsymbol{\rho} + \mathbf{q}^T(\mathbf{I} - \mathbf{A} - \mathbf{D}) - \ln \mathbf{q}^T\,(\mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}}) \tag{14}$$

the Lyapunov function of the system. This pseudo-Hamiltonian form and this particular Lyapunov function are familiar, as they refer to the generalised Lotka-Volterra system, which describes the interaction of $2n$ species that are either preys, predators or competitors towards each other. Within our framework, the normalized quantities behave like predators towards normalized prices, while both quantities and prices behave competitively towards each other. Interestingly, as with the original Lotka-Volterra system, the system becomes equivalent to a Hamiltonian one proposing that $\varphi_{i(2)} = \psi_{i(1)} = 0$ in the case where the short-run effects are negligible.

[12] The dual variables are the positions and *momenta* of a mechanical system, while the Hamiltonian function is equal to the total mechanical energy. Goodwin and Punzo (1987) attempted to bring the same reasoning to input-output systems by introducing the idea of an 'economic potential' equal to the net output of the economy $\mathbf{p}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \mathbf{b}\,\mathbf{l}^T)\mathbf{x}$, an idea which was not met with success.

The behaviour and global stability of the generalized Lotka-Volterra systems have been studied *inter alia* by Li (1994) and Takeuchi (1996). In the following, we utilize their results in order to comprehend the behaviour of the system. We consider two cases: the autonomous one, where the parameters of the model $\mathbf{A}$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{l}^T$, $\mathbf{b}$, $\mathbf{f}$, $\bar{\boldsymbol{\iota}}$ and w are constant, which may be valid in the short run, and the non-autonomous case, where they change over time, which is true in the long run mirroring the real function of an economy.

5.1. The autonomous case

Setting $\dot{q}_i = \dot{\rho}_i = 0$ in equations (11.b) and (12.b), it is relatively easy to compute a fixed point (A)

$$\boldsymbol{\rho}_A{}^{T^*} = \mathbf{0}^T \qquad \text{and} \qquad \mathbf{q}_A{}^* = \mathbf{0}$$

which is trivial and does not bear any economic meaning. However, there is another fixed point (B) that we may arrive at, which is non-trivial and is of particular interest, that is

$$\boldsymbol{\rho}_B{}^{T^*} = w\mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \bar{\mathbf{E}})^{-1}$$

and

$$\mathbf{q}_B{}^* = (\mathbf{I} - \mathbf{A} - \mathbf{D})^{-1}(\mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}})$$

We observe that the normalized prices are proportional to a quantity quite similar to the labour values $\mathbf{v}^T = \mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D})^{-1}$, augmented accordingly so that they take into account the 'normal' level of capitalists' consumption and the 'normal' level of reproduction. These nonstandard labour values have been referred by Walker (1988), in an attempt to extend labour values beyond the case of static equilibrium, and by Abraham-Frois and Berrebi (1997, ch. 6) in their demonstration of the 'transformation problem' and its solution;[13] these nonstandard labour values are defined as

$$\tilde{\mathbf{v}}^T = \mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \bar{\mathbf{E}})^{-1}$$

so

[13] Abraham-Frois and Berrebi (1997, p. 151) give the following result: Since $\omega = \mathbf{v}^T\mathbf{b}$ is the wage rate, the rate of surplus value is expressed as $e = (1 - \mathbf{v}^T\mathbf{b})(\mathbf{v}^T\mathbf{b})^{-1}$; hence $(1 + e)\mathbf{v}^T\mathbf{b} = 1$. As a result, the labour values can be written as

$$\mathbf{v}^T = \mathbf{v}^T(\mathbf{A} + \mathbf{D}) + \mathbf{l}^T = \mathbf{v}^T(\mathbf{A} + \mathbf{D}) + \mathbf{l}^T(1 + e)\mathbf{v}^T\mathbf{b}$$

and solved as

$$\mathbf{v}^T = \omega\, \mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - e\mathbf{b}\mathbf{l}^T)^{-1}$$

where $e\mathbf{b}\mathbf{l}^T$ is the surplus value at equilibrium. This is in fact $\bar{\mathbf{E}}$, namely the surplus value at equilibrium is tending to the 'normal' level of investment that is the constant-composition-of-capital investment found in the Schemes of Expanded Reproduction. Walker (1988) argued that introducing this part of surplus value in the accounting of labour values is correct in the dynamic case, as the labour values should now represent not only the ability of the system to produce, but also to be reproduced.

$$\boldsymbol{\rho}_B{}^{T*} = w\,\tilde{\mathbf{v}}^T = w\,\mathbf{l}^T(\mathbf{I} - \mathbf{A} - \mathbf{D} - \bar{\mathbf{E}})^{-1} \tag{15}$$

In this sense, the exact proportionality between prices and labour values is established as an equilibrium solution for the system.

Similarly, the activity levels in equilibrium are nothing more but the equation of static equilibrium in Leontief's input-output model, where intermediate demand also contains the capital stock coefficients and the exogenous demand contains both consumption and investment. Specifically, the activity level is written as

$$\mathbf{q}_B{}^{*} = (\mathbf{I} - \mathbf{A} - \mathbf{D})^{-1}(\mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}}) \tag{16}$$

This condition is described by Marx in his Schemes of Expanded Reproduction as a 'warranted' growth path for the capitalist system (Chatzarakis *et al.* 2022).

According to equations (3) and (4) for normalized prices and quantities, the above equilibrium values of $\boldsymbol{\rho}^T$ and $\mathbf{q}$ correspond neither to loss of growth nor to an absence of inflation. In fact, the equilibrium is the case where each quantity x_i grows proportionally to the growth of the total product, $\mathbf{p}^T\mathbf{x}$; this proportionality is regulated by Leontief's equilibrium relation in equation (16) and, essentially, it mirrors Marx's Schemes of Expanded Reproduction.

From the first theorem provided by Li (1994) in the case of constant coefficients, the fixed point A is asymptotically unstable and more specifically a 'saddle'. As for the fixed point B, a structural stability is ensured so long as

(1) the coefficients a_{ji}, d_{ij}, k_{ij}, $\bar{e}_{ji}$, w, l_i, b_i, f_i and $\bar{\iota}_i$ are all positive, and

(2) the relation $a_{ji} + d_{ij} + \bar{e}_{ji} < 1$ is true for any i and j.

Hence, the solutions of the system are not expected to diverge from fixed point B. Further elaboration of the Lyapunov function, using Li's criterion, shows that the attainment of equilibrium depends on the magnitude of the two effects. In the case of the Classical effects alone ($\boldsymbol{\varphi}_2 = \boldsymbol{\psi}_2 = \mathbf{0}$), the eigenvalues of the system in the vicinity of the fixed point B are imaginary, hence no asymptotic stability is established; the fixed point is a 'centre' and the solutions keep oscillating around it.[14] Introducing the post-Keynesian and Sraffian effects ($\boldsymbol{\varphi}_2 \neq \mathbf{0}$ and $\boldsymbol{\psi}_2 \neq \mathbf{0}$), the same eigenvalues turn complex with negative real parts and asymptotic stability is also achieved; hence, the fixed point is an 'attracting focus' and the solutions converge to it via oscillations.

In Figure 1, a symbolic representation of the phase space is given, in which all normalized prices, ρ_i, are compressed on the vertical axis, while all normalized quantities, q_i, on the horizontal one. On the left-hand side panel, the case of Classical effects alone is presented, where the trajectories oscillate around the equilibrium (black dot) but never attain it; on the right-hand side panel, the post-Keynesian and Sraffian features are introduced, so the trajectories attain the equilibrium.

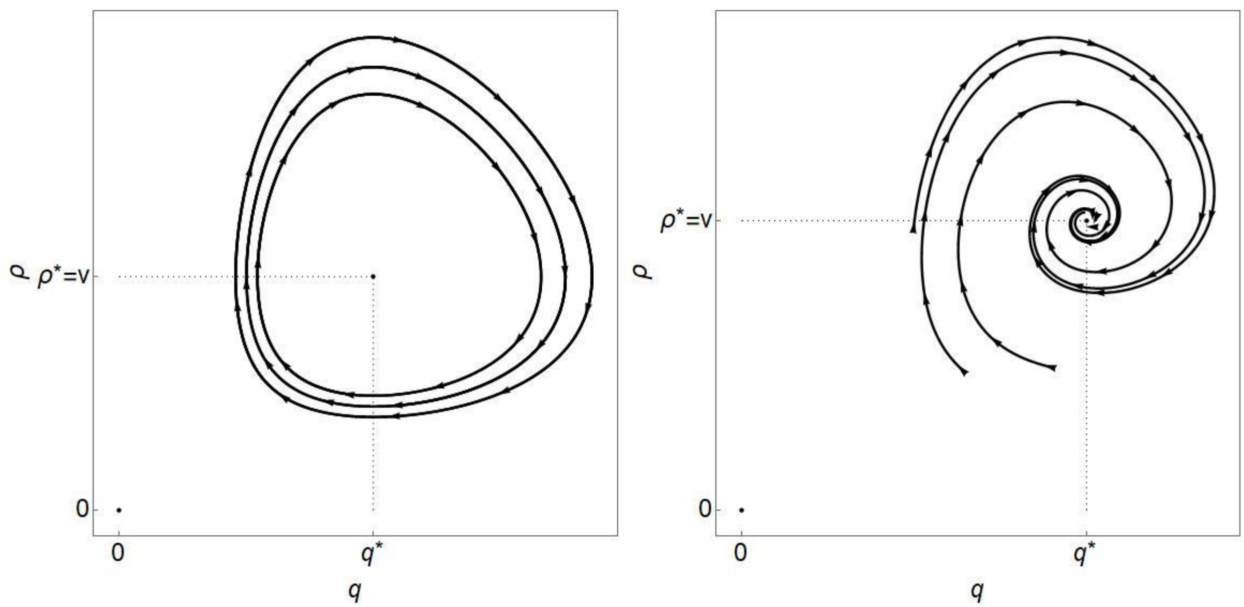

Figure 1: A symbolic phase space of the system of equations (13) in the short-run case.

It is worth noting that the two conditions of structural stability are very simple and realistic and constitute a prerequisite for the employment of input-output analysis. Subsequently, the case for structural stability is not an incidental one in a real economy, but it should prevail if no exogenous forces are in effect. In Appendix A, simulations are provided using input-output tables of the US economy that prove the validity of our argument for the autonomous (short-run) case.

[14] In this case, the system is indeed exhibiting a Hamiltonian behaviour.

5.2. The non-autonomous case

In the case of time-varying parameters, the system is no longer autonomous; thus, the traditional techniques of locating and characterizing fixed points are no longer in effect. To understand the behaviour of the solutions, we should know the precise form of the coefficients with respect to time which is not possible, as there are no clear laws (theoretical or empirical) describing the precise motion of $\mathbf{A}$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{b}$, $\mathbf{f}$ and $\bar{\boldsymbol{\iota}}$. An obvious (technical) hypothesis that we could introduce into the analysis about their evolution is that it is very slow over time. This is a realistic hypothesis and is supported by the empirical findings of the way in which the above matrices $\mathbf{A}$, $\mathbf{D}$, $\mathbf{K}$ and vectors $\mathbf{b}$, $\mathbf{f}$ and $\bar{\boldsymbol{\iota}}$ change over time (see Carter 1970 and Tsoulfidis and Tsaliki 2019, ch. 5). These slow changes which are absolutely consistent with the views of the old classical economists and Marx. We know that technological change is a slow moving process and if we want to study it, we start with a benchmark year and estimate direct or equilibrium prices in constant monetary units and then we make another estimate some (say five or ten) years later and see how prices have changed and on the basis of this change characterize the technological change in both quantitative and qualitative terms. In this case, we may use Melnikov's perturbation theory (Wiggins 2003, ch. 29) and Kolmogorov-Arnold-Moser theorem (Verhulst 2000, ch. 15) to treat this slow evolution as a perturbation over time.[15] Hence, the fixed point of the system would remain the same and the behaviour of the system resembles that of the autonomous system; the deviation from this behaviour would be defined by the magnitude of the perturbations over time.

However, a fundamental problem arises. The fixed point of the autonomous system (equations (15) and (16)) is dependent on the parameters of the system. Hence, it cannot remain fixed; it 'moves' in the phase space depending on the evolution of the parameters and it may no longer be an invariant solution of the system. However, the relatively simple form of the system and its pseudo-Hamiltonian structure whose trajectories are defined by equations

$$\bar{\boldsymbol{\rho}}^{T} = w(t)\tilde{\mathbf{v}}^{T}(t) = w(t)\mathbf{l}^{T}(t)\big(\mathbf{I} - \mathbf{A}(\mathbf{t}) - \mathbf{D}(\mathbf{t}) - \bar{\mathbf{E}}(\boldsymbol{t})\big)^{-1} \tag{17}$$

[15] In short, Melnikov's perturbation theory is used to measure the distance between stable and unstable asymptotes in the phase space; as for the Kolmogorov-Arnold-Moser theorem, it is used to measure the persistence of periodic and quasiperiodic trajectories in the phase space. Their use is important when a system is led to chaos through small perturbations and they are particularly useful in the case of Hamiltonian systems. Hence, the smaller the effect of the post-Keynesian and Sraffian effects, essentially the closer the equations (11.b) and (12.b) are to a Hamiltonian system, the more successful these perturbation methods would be, in our case.

$$\overline{\mathbf{q}} = \left(\mathbf{I} - \mathbf{A}(\mathbf{t}) - \mathbf{D}(\mathbf{t})\right)^{-1}\left(\mathbf{b}(\mathbf{t}) + \mathbf{f}(\mathbf{t}) + \bar{\boldsymbol{\iota}}(\boldsymbol{t})\right) \tag{18}$$

act indeed as an invariant solution; furthermore, it maintains the economic meaning of the system.

As for the local and global stability of the system, we cannot perform the usual techniques, but we may employ Li's (1994) second theorem that refers to an 'attractor' of a non-autonomous dynamical system as in equations (11.b) and (12.b). According to this theorem, the solutions of the system are bounded around the trajectory defined by equations (17) and (18) if and only if:

1. The coefficients $\mathbf{Z}(t) = \mathbf{I} - \mathbf{A}(t) - \mathbf{D}(t) - \bar{\mathbf{E}}(t)$ and $\mathbf{U}(t) = \mathbf{I} - \mathbf{A}(t) - \mathbf{D}(t)$ are uniformly bounded over time; thus, they cannot rise above some real number (smaller than unity) as time tends to infinity. Indeed, by their economic meaning, the components of these matrices lie always below one, so that the system is closed.
2. The *infinima* (lowest final values) of each component of the matrices $\mathbf{Z}(t)$ and $\mathbf{U}(t)$ must be greater than zero. This condition may not hold at all times, but the less-strict condition of non-negativity of matrices $\mathbf{Z}(t)$ and $\mathbf{U}(t)$ is a prerequisite for the system to be productive.
3. Given $\mathbf{Z}(t) = \mathbf{I} - \mathbf{A}(t) - \mathbf{D}(t) - \bar{\mathbf{E}}(t)$ and $\mathbf{U}(t) = \mathbf{I} - \mathbf{A}(t) - \mathbf{D}(t)$, then
   a) the relations
   $$z_{ii}(t)z_{jj}(t) > (n-1)z_{ij}(t)z_{ji}(t) \quad \text{and} \quad u_{ii}(t)u_{jj}(t) > (n-1)u_{ij}(t)u_{ji}(t)$$
   for every i and j such that $i \neq j$, imply monotonic boundness around the 'attractor' defined by equations (17) and (18), while
   b) the relations
   $$z_{ij}(t)z_{ji}(t) \neq 0 \qquad \text{and} \qquad u_{ij}(t)u_{ji}(t) \neq 0$$
   for every i and j such that $i \neq j$, imply oscillatory boundness around the 'attractor' defined by equations (17) and (18).

Eventually, the solutions of the system are not necessarily attracted by the 'equilibrium trajectory' of equations (17) and (18), but they definitely 'follow' it; essentially, the solutions are attracted by the 'equilibrium trajectory' in the manner of the gravitational convergence, either monotonically or oscillatory. This gravitational convergence reflects, on the one hand the non-stabilization of the system in the short-run, and on the other hand its tendential behaviour in the long-run; as demonstrated *inter alia* by Flaschel (2010) and Tsoulfidis and Tsaliki (2019), this is the standard representation of the tendencies observed in the operation of the capitalist mode of production as theorized by the classical political economists and Marx.

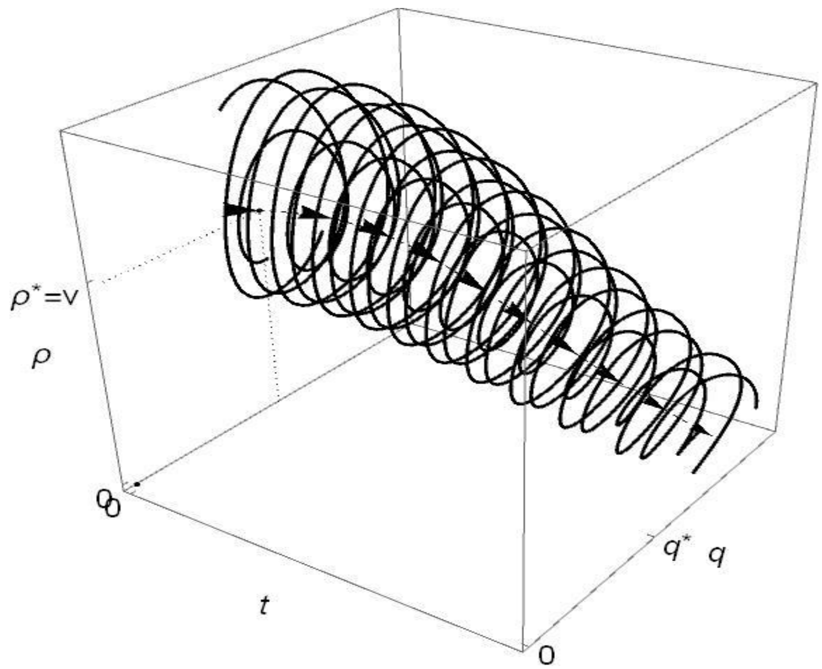

Figure 2: A symbolic phase space of the system of equations (13) in the long-run case.

Figure 2 depicts the symbolic phase space of equations (11.b) and (12.b), in which the two axes denote the normalized prices, ρ_i, and quantities, q_i, while the third axis denotes time, t, as the system is non-autonomous and time is itself a dynamic variable. What we may observe is that the solutions oscillate around the 'equilibrium trajectory' (curve of black arrows), which is not constant over time and for $t \to 0$ tends to the equilibrium point of the autonomous system. This trajectory signifies the condition of balanced growth, where again prices are exactly proportionate to labour values; this is an extension of Marx's Expanded Reproduction, where technological and distributional changes are accounted (Chatzarakis *et al.* 2022). The actual economy (the $\boldsymbol{\rho}^T$ and q) gravitates around this trajectory, suggesting that the labour values are an attractor for prices and the LTV serves as a microfoundation for macroeconomic theory. In Appendix B, simulations are provided using input-output tables of the US economy and a convention for their intertemporal change that prove the validity of our argument for the non-autonomous (long-run) case.

6. Conclusions

In an overview of the literature of classical and Marxian theory of political economy and its modern interpretations, we realise that the LTV served in many and different ways: first, as a keystone for a naturalist and empiricist explanation of economic phenomena, as it relates observed quantities (prices) with the source of the production (labour); second, as an accounting system that allowed for the treatment on the macroscopic scale of all different processes; finally, as an analytical tool capable of revealing the hidden exploitative nature of the capitalist system, as it relates the process of creating wealth through the production process by exploiting wage labour.

Over the years, many research efforts sought to cast in mathematical terms and, at the same time, prove the logical consistency and great explanatory content of the LTV. Morishima (1973), Bródy (1974), Okishio (1974) and Shaikh (1973 and 1977) were among the first who showed the consistency of the LTV within a static framework; however, little was done to extend the analysis to a dynamic domain. There were some efforts inspired by Leontief's input-output system (*e.g.*, Jorgenson 1961; Lange 1969) but were restricted to the evolution of quantities, while some other efforts inspired mainly by Sraffa's analysis of income distribution (Nikaido and Kobayashi 1978) were restricted to price reactions. Yet, it is well known since von Neumann's (1945-1946) model that quantities and prices are dual variables and their motion should be modelled together. Nikaido (1983 and 1985) and Flaschel and Semmler (1987 and 1990) proceeded to develop such cross-dual models, but their results were met with only partial success; in particular, the former's research concluded with negative results for the stability properties of the equilibrium, while in the latter's modelling the introduction of Keynesian elements and the restrictions on the workings of competition were decisive for the attainment of the stability of equilibrium.

In our model, we attempted to permeate a long-run and macroscopic flavour into the analysis. The reaction of prices to the adjustment of quantities is extracted by the Classical/Marxian perception on inflation, while the reaction of quantities to the adjustment of prices is derived by the Classical/Marxian perception of growth. In addition, we took into consideration the price (Wicksell) effects as discussed by Ricardo, Marx and Sraffa, and the capacity effect as discussed by the post-Keynesians. Through this, we arrived at a dynamic model that extends the von Neumann model of growth beyond the case of an optimum equilibrium and, at the same time, resembles Flaschel and Semmler's (1987 and 1990) modelling of Classical competition. Its qualitative behaviour is studied by means of Li's (1994) two theorems on the generalised Lotka-Volterra system. Through these theorems, we proved that the system is globally and structurally stable, so long as its parameters are within economically meaningful limits. Avoiding the strict assumptions of Flaschel and Semmler (1987 and 1990), we prove that labour values are long-run attractors of the system and the gravitational convergence confirms the non-stabilization of the system in the short-run and, at the same time, its tendential towards equilibrium behaviour in the long-run.

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Appendices

Appendix A: Simulations for the Autonomous (Short-Run) Case

We consider a simplified example of a realistic 5-sectoral economy to subject to empirical testing our model using realistic parameter values. The data used are derived by the WIOD 2014 input-output table of 54 industries for the US economy, as was aggregated by Tsoulfidis (2021: ch. 7) into five meaningfully constructed sectors. Following Tsoulfidis (2021: ch. 7) calculations, we consider the matrix of technical coefficients to be

$$\mathbf{A} = \begin{pmatrix} 0.1388 & 0.1312 & 0.0321 & 0.0026 & 0.0026 \\ 0.1209 & 0.3254 & 0.1875 & 0.0833 & 0.0574 \\ 0.0152 & 0.0145 & 0.0116 & 0.0084 & 0.0084 \\ 0.0667 & 0.0935 & 0.1167 & 0.1284 & 0.0648 \\ 0.0762 & 0.0876 & 0.0725 & 0.1849 & 0.2166 \end{pmatrix}$$

and the matrix of capital stock coefficients, derived by multiplying the column vector of investments shares by the row vector of capital-output ratios, to be

$$\mathbf{K} = \begin{pmatrix} 0.0503 \\ 0.2905 \\ 0.2684 \\ 0.2109 \\ 0.1799 \end{pmatrix} \begin{pmatrix} 2.105 & 0.584 & 1.513 & 0.838 & 2.63 \end{pmatrix}$$

$$= \begin{pmatrix} 0.106 & 0.0294 & 0.0761 & 0.0422 & 0.1325 \\ 0.6118 & 0.1697 & 0.4394 & 0.0833 & 0.7648 \\ 0.5654 & 0.1568 & 0.406 & 0.2248 & 0.7067 \\ 0.4443 & 0.1233 & 0.3191 & 0.1767 & 0.5554 \\ 0.3789 & 0.1051 & 0.2721 & 0.1507 & 0.4737 \end{pmatrix}$$

Then, the column vector of normalised exogenous demand $\mathbf{y} = \mathbf{b} + \mathbf{f} + \bar{\boldsymbol{\iota}}$ is

$$\mathbf{y} = \begin{pmatrix} 0.00282 \\ 0.06100 \\ 0.00876 \\ 0.10812 \\ 0.20357 \end{pmatrix}$$

and the normalised row vector of labour hours is

$$w\mathbf{l}^T = (0.0049 \quad 0.0319 \quad 0.0169 \quad 0.0715 \quad 0.1737)$$

The initial conditions used are

$$\mathbf{q}(0) = \begin{pmatrix} 0.040 \\ 0.161 \\ 0.045 \\ 0.329 \\ 0.438 \end{pmatrix} \text{ and } \boldsymbol{\rho}^T(0) = (0.287 \quad 0.16 \quad 0.37 \quad 0.163 \quad 0.272)$$

In the case of a Classical system ($\varphi_1 = 1$, $\varphi_2 = 0$, $\psi_1 = 1$ and $\psi_2 = 0$), the simulations produce the results given in Figure A1, where the normalized prices are in the vertical axis and the normalized quantities are in the horizontal axis, the blue curves are the solution of equations (13) for this case, and the black dots represent the fixed points; each subplot is a slice of the actual 10-dimensional phase space that corresponds to a particular sector.

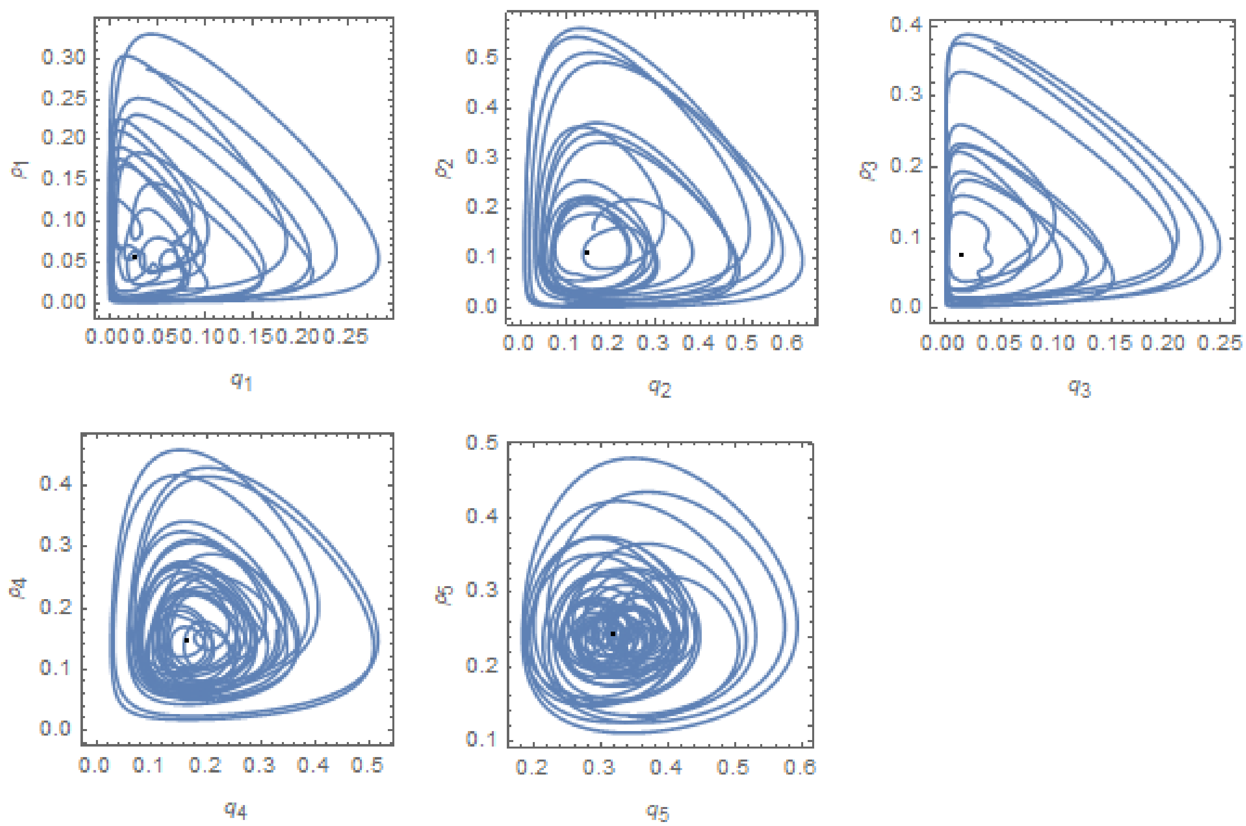

Figure A1: The phase space of the system of equations (13) in the short-run Classical case by employing an aggregated input-output table of the US economy.

In the case of a composite Classical-Keynesian-Sraffian system ($\varphi_1 = 1$, $\varphi_2 = 0.2$, $\psi_1 = 1$ and $\psi_2 = 0.2$), the simulations produce the results given in Figure A2, following the same notation.

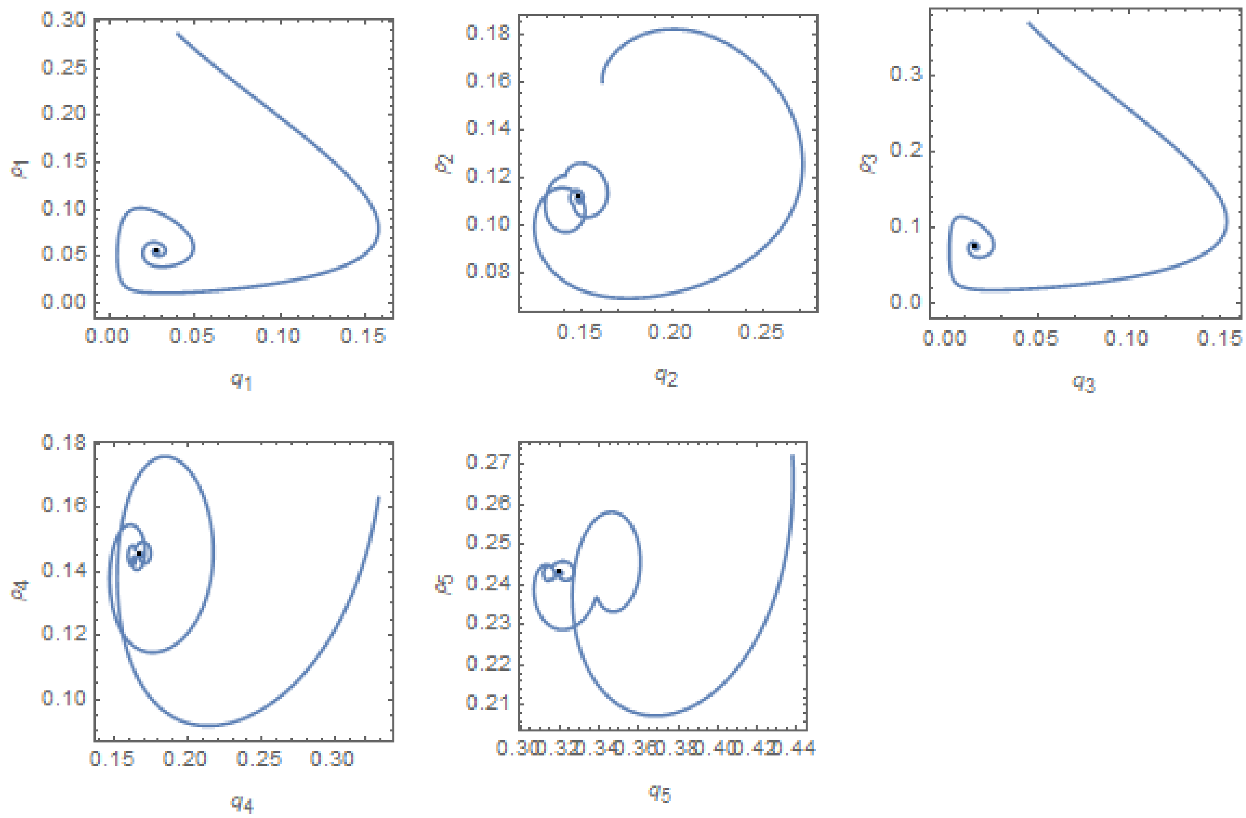

Figure A2: The phase space of the system of equations (13) in the short-run Classical-Keynesian-Sraffian case by employing an aggregated input-output table of the US economy.

We observe that the simulations produce a more complicated phase space than its symbolic representations in Figure A2; however, the general behaviour of the two cases is strikingly similar. We should note that the fixed points (black dots) coincide with the labour values calculated by Tsoulfidis (2021, ch. 7) and the prices are proved to oscillate around them.

Appendix B: Simulations for the Non-Autonomous (Long-Run) Case

Now, we may proceed by allowing matrices **A** and **K** and vectors **y** and **l** to change over time. We choose simple logistic relations, so that the change is slow and smooth, while at the same time respecting Li's criteria of stability (the sum of the entries of **A** along either the columns or the rows do not exceed unity); furthermore, we consider the 'stylized facts' that **A** and **l** decrease over time, as more efficient techniques of production are chosen, and **K** and **y** increase over time, as capital stock accumulates and demand grows in an expanding economy (Chatzarakis *et al* 2022).

Utilizing the same initial conditions, the Classical system ($\varphi_1 = 1$, $\varphi_2 = 0$, $\psi_1 = 1$ and $\psi_2 = 0$), behaves according to Figure B1, where the normalized prices are in the vertical axis and the normalized quantities are in the horizontal axis, the blue curves are the solution of equations (13) for this case, and the black curves depict the 'equilibrium' trajectory of labour values and balanced growth.

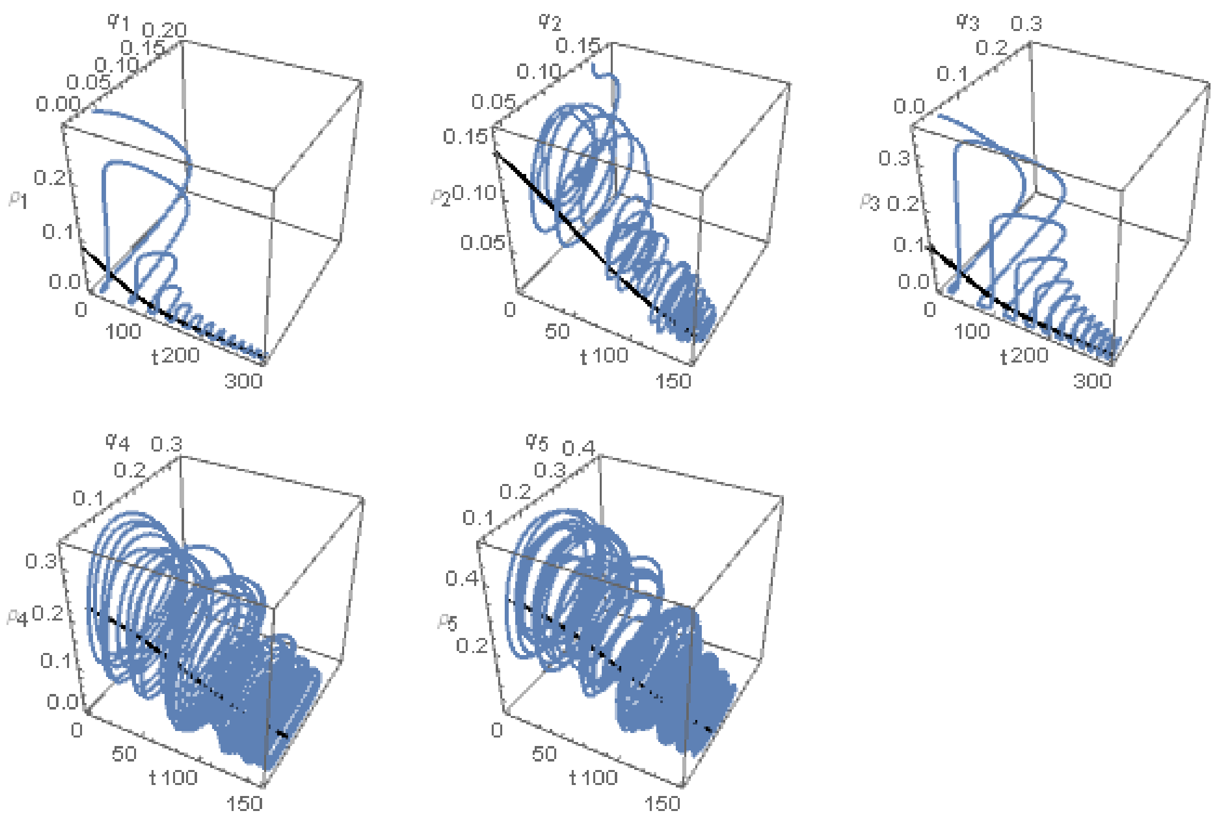

Figure B1: The phase space of the system of equations (13) in the long-run Classical case by employing and aggregated input-output table of the US economy

The main difference between Figures A1 and B1 is that the "centre of gravity" does move in such a manner that the labour values decrease –reflecting the rise in productivity.

In the case of a composite Classical-Keynesian-Sraffian system ($\varphi_1 = 1$, $\varphi_2 = 0.2$, $\psi_1 = 1$ and $\psi_2 = 0.2$), the simulations produce the results given in Figure B2, following the same notation.

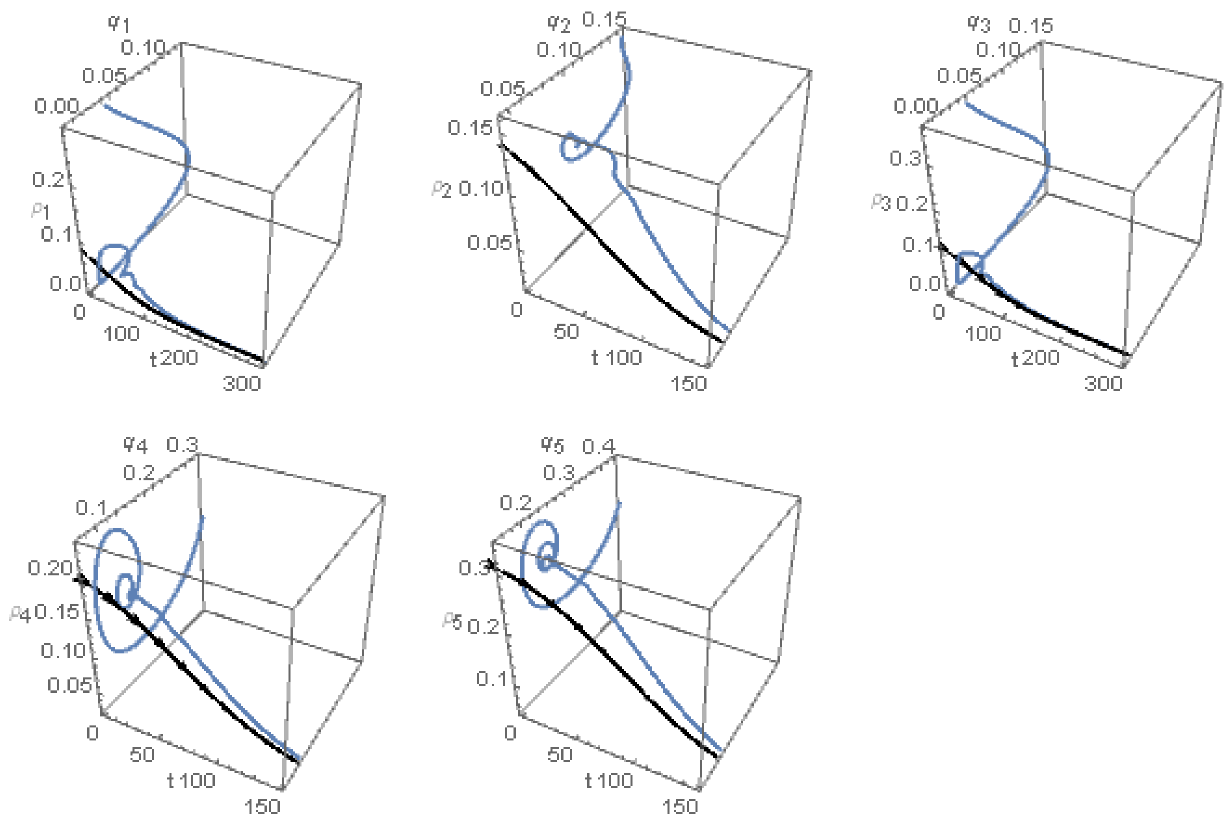

Figure B2: The phase space of the system of equations (13) in the long-run Classical-Keynesian-Sraffian case by employing and aggregated input-output table of the US economy.

In conclusion, in the case of a solely Classical system, normalised prices and quantities oscillate around the fixed point of labour values and 'normal growth' in a manner resembling the process of gravitational convergence. In the case of a composite Classical-Keynesian-Sraffian system, they are attracted towards the equilibrium point. It is worth noting that the simulations were repeated for different initial conditions and different values of the reaction coefficients, as well as for multiple logistic curves in the non-autonomous case, yielding similar results.

In this case, the equilibrium trajectories correspond to evolving labour values, whose initial point are the labour values calculated by Tsoulfidis (2021, ch. 7). Again, the prices are proved to oscillate around them or converge on them.