How and where satellite cities form around a large city: Sustain bifurcation mechanism of a long narrow economy

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How and where satellite cities form around a large city: Sustain bifurcation mechanism of a long narrow economy[1]

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Abstract

We investigate economic agglomeration in a long narrow economy, in which discrete locations are evenly spread over a line segment. We elucidate the mechanism how (new) satellite cities form around a central city by the bifurcation analysis of a monocentric city at the center. The validity and usefulness of this mechanism are ensured for various kinds of spatial economic models, namely Forslid & Ottaviano (J Econ Geo, 2003), Pflüger (Reg Sci Urban Econ, 2004), Pflüger and Südekum (J Urban Econ, 2008), and Murata and Thisse (J Urban Econ, 2005). Where satellite cities emerge is demonstrated to be dependent on the models and their economic parameters. For the first two models, for example, the larger the agglomeration forces, the farther away from the monocentric city satellite cities emerge. The transition of stable agglomeration patterns is observed and is put to use in the explanation of the history of city size distribution in the real world.

Keywords: Bifurcation; economic geography; long narrow economy; satellite city; sustain point.

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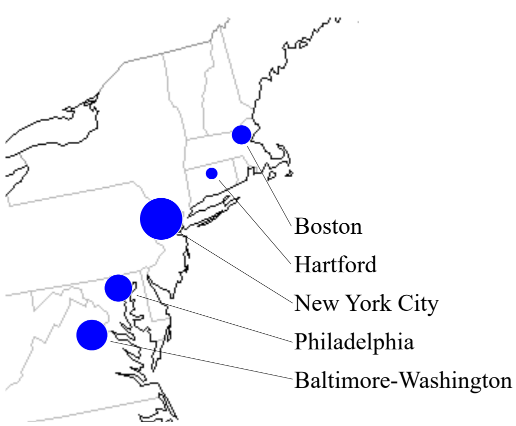


(a) East Atlantic

(b) Main Island in Japan

Figure 1: A chain of cities in the world

1. Introduction

A chain of cities prospers worldwide, e.g., on the Main Island of Japan and in a closed narrow corridor between the Atlantic Ocean and the Appalachian Mountains (see Fig. 1). A chain of cities can be found also at transnational scales, particularly in Europe (see Fig. 2), such as the Atlantic Axis (from Porto in Portugal to La Coruña in Spain), the STRING (from Hamburg in Germany to Oslo in Norway), and the so-called "blue, green and golden bananas".

The mechanism of the growth of a large city, such as New York City and Tokyo, among a chain of cities is of great interest in spatial economics. Geography, amongst other factors, plays a big role in characterizing chains of cities according to their locational and spatial topology. One such particular configuration is the line segment, whose analysis, despite its stylized geometry, is of great interest because: (i) it is both simple and generates asymmetries that confers advantages to some regions; and (ii) it is empirically relevant as it fits several real world examples of a chain of cities.

This paper aims to elucidate the agglomeration mechanism of a long narrow economy,

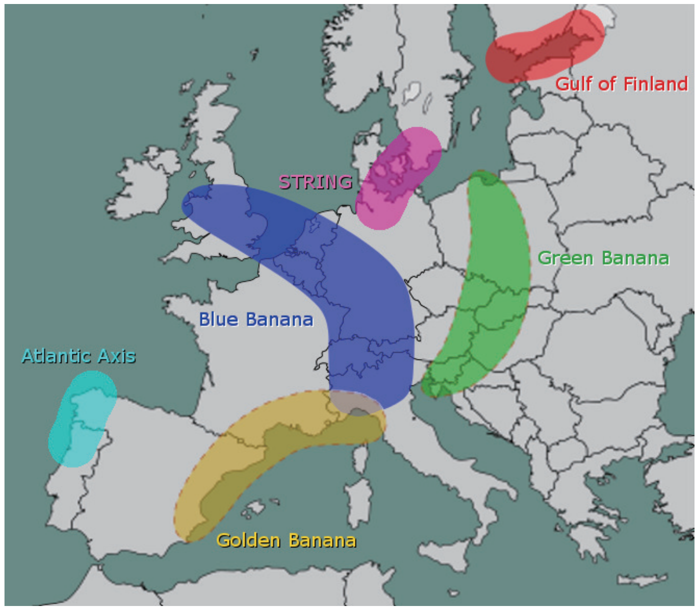


Figure 2: European transnational megalopolises (source: Maps on the Web: https://mapsontheweb.zoom-maps.com)

in which discrete locations are evenly spread over a line segment. We answer the question "How and where do (new) satellite cities form around a large city?" By satellite cities we refer to a pair of (potential) cities, placed on each side of a central city, forming a spatial pattern that resembles a megalopolis or *megaregion*. This is apparently a difficult mission as the associated agglomeration properties are dependent on spatial economic models and as well as on their economic parameters. To tackle this mission, we elucidate the bifurcation/agglomeration mechanism of this economy in the following two steps:

1. The bifurcation mechanism for a general spatial economy.
2. The bifurcation mechanism for well-known economic geography models.

The results for the first step are applicable to canonical spatial economics models. On the other hand, the results for the second step are more informative than those for the first step, although these results hinge on the particular assumptions of the models employed in our paper.

The literature reports several characteristic agglomeration patterns of this economy: the simplest core–satellite pattern for three places (Ago et al., 2006), a chain of spatially repeated core–periphery patterns *a la* Christaller and Lösch (e.g., Fujita and Mori, 1997), and a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities* (Mori, 1997). These patterns were numerically observed for a long narrow economy by changing agglomeration forces and transport costs (Ikeda et al., 2017). Yet such patterns were investigated somewhat fragmentarily and in an ad hoc manner up to now.

In the first step, as a novel theoretical contribution of the paper, we answer the question "How do (new) satellite cities form around a large city?" in a manner applicable to a general economic geography model with an arbitrary number of places. It is proved that a state of full agglomeration to a large single city at the center can encounter a bifurcation at a critical level[5] of transport costs (freeness of trade). Above (below) this level this state becomes economically unsustainable and leads to the emergence of satellite cities around the large central city. Nowadays it seems far more important to investigate the competition between central and satellite cities than to investigate the self-organization of cities in a flat land envisaged, e.g., by Central Place Theory (Christaller, 1933).

In the second step, we answer the question "Where do (new) satellite cities form around a large city?" As a pioneering work on this issue, Fujita and Krugman (1995) investigated the emergence of satellite places around a monocenter in an infinite continuous space, and studied the conditions under which a monocentric equilibrium is stable in the context of a general equilibrium model. However, their work is silent on the exact

[5]This critical level is called the sustain point (Fujita et al., 1999).

location of such places around the central city for a bounded corridor with finite length, i.e., it does not discuss *how far away* from the central city the satellite places emerge.

Recently, Allen and Arkolakis (2014) provided examples of equilibrium configurations and their stability along a continuous line segment under changes in trade costs and other parameters such as exogenous productivities. However, they do not discuss what patterns arise (and where) once full agglomeration becomes unsustainable.

In the study of a chain of cities in this paper, a line segment economy with discrete places would be more pertinent than the continuous space. In this paper, we answer our question based on the analysis of many-region versions of several spatial economic models: the footloose entrepreneur models by Forslid & Ottaviano (2003), by Pflüger (2004), and by Pflüger and Südekum (2008) (FE, PF and PFSU, respectively), and the MT model by Murata and Thisse (2005).

The choice of these models is not arbitrary. The FE, PF and PFSU models fall in the class of analytically solvable models and hence allow us to obtain some analytical proofs that would otherwise be unattainable. The MT model, although not analytically solvable, is remarkably simple for the case of full agglomeration (a single city). Moreover, our choice allows us to cover all three kinds of models according to the spatial scale of dispersion forces (see Akamatsu et al, 2021 for classification of models). Global dispersion forces occur when they depend on the proximity structure between regions (competition effects that extend over a certain distance). Local dispersion arises due to congestion inside each region. The FE and PF models belong to the class of models that exhibit only <u>global</u> forces. The MT model belongs to the class of models that exhibit only <u>local</u>

dispersion forces.[6] Finally, the PFSU model belongs to the class of models that contains both local and global dispersion forces.[7]

The location of the satellite places turns out to be dependent on the models and the values of their parameters. This location is demonstrated to be dependent on the agglomeration forces that are a consequence of: (i) the global size of the industrial sector relative to the traditional sector, (ii) the degree of scale economies in the industrial sector, and (iii) the size of congestion forces due to housing costs and/or commuting costs.

In the FE model and the PF model, when agglomeration forces are very small, a large central place surrounded by two neighboring satellite places emerges for low enough trade costs, thus forming a hump-shaped megalopolis around the central city. When these forces are large, satellite cities appear far away from the primary city at the center. This would give an economic implication of *agglomeration shadow* (Arthur, 1990),[8] cast by cities with a large industry size over locations in the vicinity, in which little or no settlement takes place because competition among firms in neighboring regions is so intense that firms cannot gain profit there. In contrast, sufficiently separated satellite cities and the central region can share industry. The PFSU model displays a similar tendency, while two neighboring satellite places always emerge for the MT model.

The analysis on the FE model is put to use in the investigation of the population distribution in chains of cities presented above. The historical change of the population

[6]The seminal model by Helpman (1998) or the more recent model by Allen and Arkolakis (2014) also fall into this class and can actually be shown to be isomorphic under some conditions.

[7]For more information on this typology, its meaning, and consequences, we refer the interested reader to Akamatsu et al. (2021).

[8]See also Fujita et al. (1999), Ioannides and Overman (2004), and Fujita and Mori (2005).

distribution in a chain of cities in the mainland of Japan is successfully explained using the result of the analysis of an long narrow economy with five cities using the FE model, while the population distribution in the Atlantic Axis using that with seven cities. Thus, the analysis by economic geography models, in the framework proposed in this paper, would be of great assistance in the investigation of real data.

This paper is organized as follows: The bifurcation from the full agglomeration to a single city on a long narrow economy is described in Section 2. Economic geography models are introduced in Section 3. How and where satellite cities form for economic geography models is investigated in Section 4. Location of satellite cities is studied in Section 5. Real data is studied in Section 6. Section 7 concludes.

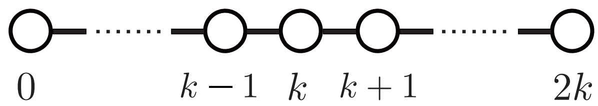


Figure 3: A long narrow economy

2. Full agglomeration to a single city on a long narrow economy

We employ a long narrow economy as a spatial platform for economic geography models. We would like to answer the question "How and where do (new) satellite cities form around a large city?" As a mechanism of the emergence of satellite cities around a large core city, we set forth the bifurcation at the state of the full agglomeration at the center. Note that this mechanism is applicable to diverse spatial economic models under the replicator dynamics so far as they have the state of the full agglomeration.

2.1. Modeling of the spatial economy

The long narrow economy has $K = 2k + 1$ ($k \in \mathbb{Z} : k \geq 1$) cities labeled $i \in N = \{0, ..., k, ..., 2k\}$, which are equally spread on a line segment (Fig. 3). The kth city is located at the center, and a city $i \neq k$ is said to be $\delta \equiv |i - k|$ steps away from the center.

There are inter-regionally mobile agents, the number of which at city $i \in N$ is denoted by λ_i under the constraint $\sum_{i \in N} \lambda_i = 1$. We introduce a spatial equilibrium according to which the mobile agents migrate among cities and choose to live in the city that offers them the highest utility. A customary way of defining such an equilibrium is to consider the following problem: Find $(\boldsymbol{\lambda}^*, \hat{v})$ satisfying

$$(v_i - \hat{v})\lambda_i^* = 0, \qquad v_i - \hat{v} \leq 0, \qquad \lambda_i^* \geq 0, \qquad \sum_{i \in N} \lambda_i^* = 1, \tag{1}$$

where $\hat{v}$ is the highest (indirect) utility of the solution to this problem.

We consider the replicator dynamics (Taylor and Jonker, 1978): $\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \phi)$, where $\boldsymbol{\lambda} = (\lambda_i \mid i \in N)$, $\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = (F_i(\boldsymbol{\lambda}, \phi) \mid i \in N)$, and:

$$F_i(\boldsymbol{\lambda}, \phi) = (v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi))\lambda_i, \quad i \in N. \tag{2}$$

Here, $\bar{v} = \sum_{i \in N} \lambda_i v_i$ represents the weighted average utility and $\phi \in (0, 1)$ is the trade freeness, which is an inverse measure of transportation costs. We choose the freeness of trade as the bifurcation parameter in order to capture the historical tendency of falling/increasing transport costs, as is customary in geographical economics.[9] We do *not*, however, disregard the important role of other costs in determining the size and distribution of cities, such as congestion or commuting costs, as we shall see, e.g., for the PFSU and MT models (Section 3).

Stationary points $(\boldsymbol{\lambda}, \phi)$ are defined as solutions of the static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \mathbf{0}. \tag{3}$$

There are several features about stationary points:

(1) Parameter dependency: Stationary points form equilibrium paths $(\boldsymbol{\lambda}(\phi), \phi)$. The pattern $\boldsymbol{\lambda}$, in general, varies with the parameter ϕ.

(2) Stability: A stable spatial equilibrium is obtained as a stable stationary point. This point is linearly stable if every eigenvalue of the Jacobian matrix of $\boldsymbol{F}(\boldsymbol{\lambda}, \phi)$ has a negative real part, and is linearly unstable if at least one eigenvalue has a positive one.

(3) Sustainability: A corner solution with zero population at one or more locations

[9]For a generic spatial economic model, we thus think of ϕ as an index that captures integration between regions in the broadest sense possible, i.e., it may reflect export hurdles due to trade tariffs, the quality of transportation infrastructures, or any kind of institutional barrier.

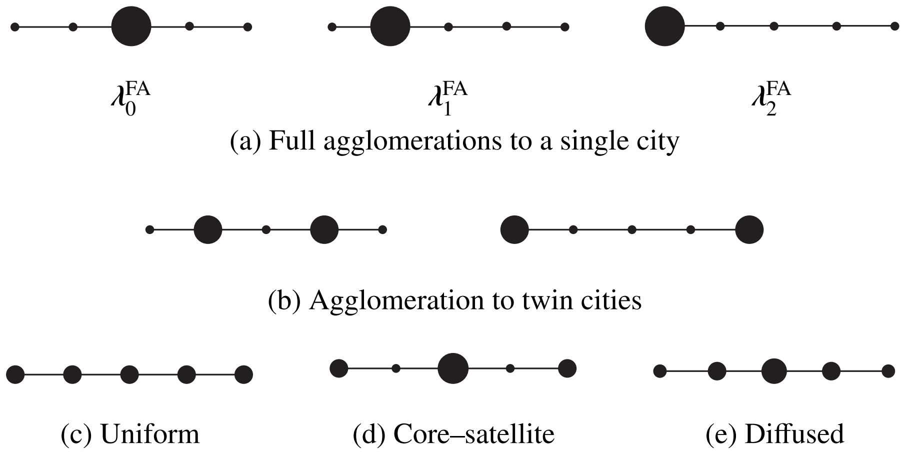


Figure 4: Agglomeration patterns in a long narrow economy for $K = 5$ cities

($\lambda_i = 0$) is called sustainable if $v_i - \bar{v} \leq 0$ is satisfied and is called unsustainable otherwise. Unsustainable corner solution are unstable.

There are several patterns $\boldsymbol{\lambda}$ of interest. The full agglomeration (FA) to a single place $i = k - \delta$ located δ steps away from the center (Fig. 4(a)) is defined as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\rm FA} \ \text{ with } \ \begin{cases} \lambda_i = 1 \ \text{ for } \ i = k - \delta, \\ \lambda_i = 0 \ \text{ for } \ i \neq k - \delta, \end{cases} \qquad 0 \leq \delta \leq k.$$

(We assume $1 \leq i \leq k$ for simplicity, while a similar discussion holds for $i > k$ by bilateral symmetry of the long narrow economy.) Twin cities, uniform, core–satellite, and diffused patterns, respectively, in Figs. 4(b)–(e) play an important role in the agglomeration analysis of economic geography models (Section 4.3).

2.2. Full agglomerations to a single city

The state of the full agglomeration is studied, e.g., in Fujita and Krugman (1995) and is of great economic interest. This state has several special features:

(1) Parameter independency: The full agglomeration $\boldsymbol{\lambda}_\delta^{\rm FA}$ has a special feature (called invariant patterns in Ikeda et al., 2012, 2018, and Aizawa et al., 2020):

Proposition 1. *The full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\rm FA}$ ($0 \leq \delta \leq k$) is a stationary point of the replicator dynamics for any value of ϕ (and any value of any other parameter).*

Proof. See Appendix A.1 for the proof. □

(2) Stability: The stability of $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\delta}^{\text{FA}}$ is described by the following proposition

Proposition 2. *The state of the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\delta}^{\text{FA}}$ at the place $i = k - \delta$ is stable if the utility $v_{k-\delta}$ is strictly larger than the utility v_i elsewhere ($v_i < v_{k-\delta}$ for any $i \neq k - \delta$).*

Proof. See Appendix A.2 for the proof. □

(3) Sustainability: The *local sustain point* $\phi = \phi_i^{\text{s}}$ for an individual place i and the *global sustain point* $\phi = \phi^{\text{s}}$ for the whole system are distinguished as below.

Definition 1. *(i)* **Local sustainability.** *A region i ($\neq k-\delta$) is* locally sustainable *if $v_i - \bar{v} \leq 0$ holds. A* local sustain point *for this region is defined by $\phi = \phi_i^{\text{s}}$ satisfying $v_i - \bar{v} = 0$.*
(ii) **Global sustainability.** *A global sustain point $\phi = \phi^{\text{s}}$ of $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\delta}^{\text{FA}}$ is defined as a point where its stability changes, i.e., $v_{i_{\text{sat}}} - \bar{v} = 0$ ($\bar{v} = v_{k-\delta}$) for some $i = i_{\text{sat}}$ and $v_j - \bar{v} < 0$ for all $j \neq i_{\text{sat}},\ k - \delta$. The place $i = i_{\text{sat}}$ is $\delta_{\text{sat}}(\equiv k - i_{\text{sat}})$ steps away from the center.*

2.3. Bifurcation from the full agglomeration at the center: Emergence of satellite cities

To answer the major question of this paper, "how do (new) satellite cities form around a large city?", we elucidate the bifurcation mechanism of the state $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^{\text{FA}}$ of the full agglomeration at the center. This state is chosen herein as it has superior stability among full agglomerations in various locations (Section 4.1).

The full agglomeration $\boldsymbol{\lambda}_0^{\text{FA}}$ at the center has a sustain point $(\boldsymbol{\lambda}_0^{\text{FA}}, \phi_{\delta}^{\text{c}})$ (for some $\delta \in N_{\delta}$) where $v_{k-\delta} - v_k = v_{k+\delta} - v_k = 0$ is satisfied.[10] This is a bifurcation point, from which emerge one or two satellite cities, δ steps away from the central region, as stated in the following proposition (see Fig. 5 for the bifurcation of $K = 5$ cities).

[10]Since the state has the bilateral symmetry about the center, cities $i = k \pm \delta$ has the same indirect utility.

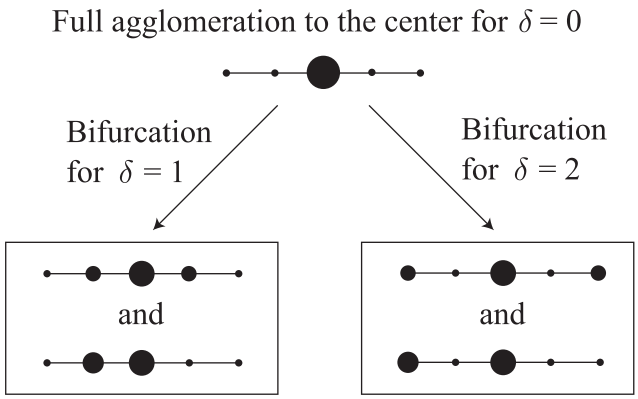


Figure 5: Possible bifurcations for $K = 5$ cities

Proposition 3. *The sustain point* $(\boldsymbol{\lambda}_0^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$ *has three bifurcating paths: a bifurcating path with two satellite cities at* $i = k \pm \delta$*, that with a satellite city at* $i = k - \delta$*, and that with a satellite city at* $i = k + \delta$*.*

Proof. See Lemma 2 in Appendix A.3 for the proof. □

The path with a satellite city at $i = k - \delta$ and that at $i = k + \delta$ are identified hereafter since the spatial pattern for the former and that for the latter are bilaterally symmetric.

The stability of the bifurcating paths is described by the following proposition.

Proposition 4. *The global sustain point has zero or one stable bifurcating path just after bifurcation. Bifurcating curves of local sustain points, other than the global sustain point, are all unstable just after bifurcation.*

Proof. See Appendix A.4 for the proof. □

Thus the global sustain point can possibly engender a stable bifurcating path accommodating satellite cities at $i_{\mathrm{sat}} = k \pm \delta_{\mathrm{sat}}$. When the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^{\mathrm{FA}}$ is stable, v_k for the central place is largest among all v_i ($i \in N$) by Proposition 2. When ϕ crosses the global sustain point ϕ^{s}, the full agglomeration becomes unstable and $v_{i_{\mathrm{sat}}}$ becomes largest among all v_i ($i \in N$). In this sense, the place i_{sat} is the most plausible location of satellite cities.

How a stable bifurcating path branches at this sustain (bifurcation) point is described by the following proposition.

Proposition 5. *(i) A stable bifurcating path from the full agglomeration λ_0^{FA}, if it exists, branches in the opposite direction from the stable full agglomeration.*

(ii) A bifurcating path is unstable, when this path snaps back, i.e., the path and the stable full agglomeration reside in the same direction from the global sustain point.

Proof. See Appendix A.5 for the proof. □

Thus a stable bifurcating path, when it exists, realizes a continuation of stable paths that reside on both sides of the sustain point. In the literature, such continuation is observed for the PF model (Pflüger, 2004). In this paper, such continuation in a long narrow economy is actually observed for the FE, PF, and PFSU models (Section 4.3).

We determine "where do (new) satellite cities form" by the following two steps:

Step 1: Obtain the global sustain point and the associated location δ_{sat}.

Step 2: Find a stable bifurcating path from this global sustain point.

When a stable bifurcating path exists, δ_{sat} obtained in this manner gives the location of satellite cities that can emerge through the transition of stable equilibria. When it dose not exist, a dynamical jump occurs, while δ_{sat} obtained still remains a candidate of the location of agglomeration as it has the largest indirect utility among all places.

3. Economic geography models

We shall present four economic geography models, namely the FE model (Forslid and Ottaviano, 2003), the PF model (Pflüger, 2004), the MT model (Murata and Thisse, 2005) and the PFSU model (Pflüger and Südekum, 2008). The choice of these models is far from arbitrary. First, we aim to encompass diverse modeling, that is, our choice allows us to cover all three kinds of models according to a characterization based on the source and spatial scale of net agglomeration/dispersion forces.[11] The distinction between global and local dispersion forces is that the former acts between regions and are dependent on the distance structure, whereas the latter acts within regions and are independent of the distance structure (Akamatsu et al., 2021). The FE and PF models, for instance, belong to the class of models that exhibit only global forces. The MT model belongs to the class of models that exhibit only local dispersion forces. Finally, the PFSU model belongs to the class of models that containing both local and global dispersion forces.

Second, the FE, PF and PFSU models fall in the class of the so-called footloose entrepreneur models (Baldwin et al., 2003), which are analytically solvable and hence are more tractable compared to other economic geography models. This allows us to obtain some analytical proofs that would otherwise be unattainable, namely the expressions for the indirect utility in the central city and any periphery (or potential city) $i \neq k$, and proofs on the existence and uniqueness of bifurcation points from the full agglomeration. The MT model, although not analytically solvable, is remarkably simple for the case of full agglomeration (a single city) in terms of expressions for the indirect utilities. The MT model differs significantly in some assumptions, as we shall see further.

[11] Akamatsu et al. (2021) give more information on this typology, its meaning, and consequences.

3.1. Common ground

We first lay down some general assumptions and establish a common ground of the economic geography models. We shall omit most derivations and write down just the main assumptions and results.

There are two factors of production (skilled and unskilled workers) and two sectors (manufacturing, M, and traditional, A). Both types of workers consume final goods of two types: manufacturing sector goods and a traditional sector good. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile between cities. The number of skilled workers in city $i \in N$ is denoted by λ_i under the constraint $\sum_{i \in N} \lambda_i = 1$. The unskilled workers are immobile and distributed equally with $L_i = L/K$ ($^{\forall} i \in N$).

The traditional sector uses immobile labor to produce a perfectly tradable good under perfect competition and constant returns to scale (each agent produces one unit of the agricultural good). Choosing the traditional good as numeraire, we set its price and the wage of immobile agents at unity in all regions. In the M-sector, many varieties of imperfectly tradable manufactured goods are produced under monopolistic competition and increasing returns to scale. Each firm produces a single variety using one unit of mobile labour (fixed cost) and, in addition, one unit of immobile labour per unit of output (variable cost). There is free entry in the manufacturing sector, thus firm profits are driven to zero (the nominal wage of mobile labour, w_i, totally absorbs operating profits).

The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from city i to city $j \neq i$, only a fraction $1/\tau_{ij} < 1$ actually arrives ($\tau_{ii} = 1$). It is assumed that $\tau_{ij} = \exp(\tau m(i, j)\tilde{L})$ is a function of a transport cost parameter $\tau > 0$, where $m(i, j)$ is an integer expressing the road distance between cities i and j and $\tilde{L}$ is the distance unit. We have $m(i, j) = |i - j|$ for the long narrow economy. As our bifurcation parameter, we introduce the trade freeness (a converse measure of transport costs):

$$\phi = \exp[-\tau(\sigma - 1)\tilde{L}] \in (0, 1),$$

where σ (> 1) is a parameter that expresses the constant elasticity of substitution between manufactured varieties. Then we represent by $\phi_{ij} = \phi^{|i-j|}$ the friction between cities i and j that decays in proportion to the transportation distance.

Agents in region $i \in N$ maximize utility $u(C_i, A_i)$, where C_i is the consumption level of a composite good of manufactures and A_i is that of the traditional good, subject to the budget constraint $P_i C_i + A_i = y_i$, where P_i is the regional price index of the manufacturing goods composite, and y_i is the agents' income. Product market and labor market equilibrium yield a unique short-run equilibrium wage, a price and the consumption levels as a function of the spatial distribution of mobile agents $\boldsymbol{\lambda} = (\lambda_i | i \in N)$.

3.2. The FE model

The utility from consumption in the FE model is given by:

$$u_i^{FE} = \mu \ln C_i + (1 - \mu) \ln A_i,$$

where $\mu \in (0, 1)$ is the share of income spent on manufactures and: $C_i = \left[\int_{s \in S} c_i(s)^{\frac{\sigma-1}{\sigma}} ds\right]^{\frac{\sigma}{\sigma-1}}$ is the CES composite of manufactures, $c_i(s)$ is the consumption in region i of a variety s of manufactures, and $\sigma > 1$ is the elasticity of substitution between varieties. We hereafter assume the no-black-hole condition $\mu < \sigma - 1$ (Forslid and Ottaviano, 2003) since its violation is quite exceptional and empirically unrealistic.[12]

[12] Anderson and Wincoop (2004), for instance, find that σ is likely to range between 5 and 10.

The indirect utility in region i is given by (see Akamatsu et al., 2021):

$$v_i(\boldsymbol{\lambda}) = \frac{\mu}{\sigma - 1} \ln \Delta_i(\boldsymbol{\lambda}) + \ln w_i(\boldsymbol{\lambda}) + \zeta,$$

where $\Delta_i(\boldsymbol{\lambda}) = \sum_{j \in N} \phi_{ij} \lambda_j$ and ζ is a constant term. The nominal wage in region i is given by (see Gaspar et al., 2019):

$$w_i(\boldsymbol{\lambda}) = \frac{\mu}{\sigma} \sum_{j \in N} \frac{\phi_{ij} \left(1 + w_j \lambda_j\right)}{\sum_{m \in N} \phi_{mj} \lambda_m} \tag{4}$$

where we have made use of the simplification $L = K$ so that $l_i = L_i/L = 1, \forall i \in N$.[13]

For the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^{\mathrm{FA}}$ at the central place k, the indirect utilities at $i = k,\ k \pm \delta\ (1 \leq \delta \leq k)$ are expressed explicitly as (see Appendix B.1)

$$v_k = \ln \frac{\hat{\mu}}{1 - \hat{\mu}} (2k + 1), \qquad \hat{\mu} = \frac{\mu}{\sigma} \in (0, 1); \tag{5}$$

$$v_{k \pm \delta} = \ln \frac{\hat{\mu}}{1 - \hat{\mu}} + \frac{\delta \mu}{\sigma - 1} \ln \phi + \ln \left\{ (\hat{\mu} k + k + 1) \phi^{\delta} + (1 - \hat{\mu}) \left[(k - \delta) \phi^{-\delta} + \sum_{p=1}^{\delta} \phi^{\delta - 2p} \right] \right\}. \tag{6}$$

By bilateral symmetry of the full agglomeration, we have $v_{k-\delta} = v_{k+\delta}$.

3.3. The PF model

The PF model differs from the FE model only in the agents' upper-tier utility function, which is now given by a quasi-linear logarithmic form: $u_i^{PF} = \alpha \ln C_i + A_i$, where $\alpha > 0$ is a preference parameter towards manufactured goods that does <u>not</u> have the same meaning as μ in the FE model.

The indirect utility is given by (see Gaspar et al., 2018; 2021):

$$v_i(\boldsymbol{\lambda}) = \frac{\alpha}{\sigma - 1} \ln \Delta_i(\boldsymbol{\lambda}) + \ln w_i(\boldsymbol{\lambda}) + \eta,$$

[13] The choice of the total population L of low skilled workers is not influential on the results as the payoff is linear in L (see Gaspar et al. (2019, pp. 9) for details). Thus, setting L = K entails no loss of generality.

where η is a constant. The nominal wage is given by:

$$w_i(\boldsymbol{\lambda}) = \frac{\alpha}{\sigma} \sum_{j \in N} \frac{\phi_{ij}\left(1 + \lambda_j\right)}{\sum_{m \in N} \phi_{mj} \lambda_m}. \tag{7}$$

It was shown that varying the number $L_i = l \geq 0$ of unskilled workers has qualitative impacts in the PF model (see Gaspar et al., 2018). However, we keep $l = 1$ for simplicity, because any qualitative changes in l can be obtained by varying, e.g., σ instead.

The indirect utilities for $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^{\mathrm{FA}}$ are given by (see Appendix B.2)

$$\begin{aligned} v_k &= \frac{\alpha}{\sigma}(2k + 2) - \alpha + \bar{A}, \\ v_{k \pm \delta} &= \frac{\alpha}{\sigma}\left[\phi^{\delta}(k + 2) + \phi^{-\delta}(k - \delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1}\right] + \frac{\alpha\delta}{\sigma - 1}\ln\phi - \alpha + \bar{A}, \quad 1 \leq \delta \leq k. \end{aligned} \tag{8}$$

Here $\bar{A}$ is the initial endowment of the numeraire good.

3.4. The PFSU model

The PFSU model (Pflüger and Südekum, 2008) builds on Pflüger (2004), with the only difference being that it introduces a housing sector (denoted by H), which produces a local dispersion force. The upper-tier utility of an agent is given by:

$$u_i^{PFSU} = \alpha \ln C_i + \gamma \ln H_i + A_i,$$

where $\alpha > 0$, H_i stands for the consumption of housing goods, and $\gamma > 0$ is a parameter representing preferences towards consumption of housing. The indirect utility of a mobile agent is given by:

$$v_i = \frac{\alpha}{\sigma - 1} \ln \Delta_i(\boldsymbol{\lambda}) + \ln w_i(\boldsymbol{\lambda}) - \ln \frac{\lambda_i + 1}{h_i} + \xi,$$

where ξ is a constant term and $h_i = H_i/H$ denotes the share of housing stock in region i. For simplicity, we assume that $h_i = 1$, $^{\forall} i \in N$. Again, we let $l_i = 1$ everywhere for simplicity and for comparison purposes with the other models.

The nominal wage in region i is just the same as in the PF model in (7). The indirect utilities for $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^{\mathrm{FA}}$ are given by (see Appendix B.3)

$$\begin{aligned} v_k &= \frac{\alpha}{\sigma}(2k+2) - \gamma \ln 2, \\ v_{k\pm\delta} &= \frac{\alpha}{\sigma}\left[\phi^{\delta}(k+2) + \phi^{-\delta}(k-\delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1}\right] + \frac{\alpha\delta}{\sigma - 1}\ln\phi, \quad 1 \le \delta \le k. \end{aligned} \tag{9}$$

3.5. The MT model

Murata and Thisse (2005) studied the interplay between commuting costs and inter-regional transport costs by employing a simplified yet reasonable specification. We closely follow Takayama et al. (2020), who extended the MT model to several regions.

The internal structure of each region is assumed to be one-dimensional and featureless except that there is a given CBD; the city expands symmetrically around the origin. There are only skilled and mobile workers, who choose their own residential region $i \in N$ and location x in that region, where the CBD is located at $x = 0$. Land endowment equals unity everywhere in a region and agents are assumed to inelastically consume one unit of land. The opportunity cost of land is normalized to zero in every region. Then, the city spreads over the interval $\mathcal{X}_i \equiv [-\lambda_i/2, \lambda_i/2]$, where $-\lambda_i/2$ and $\lambda_i/2$ denote the city boundaries.

Commuting costs take an iceberg form. Specifically, the effective labour supply of a worker located at x is given by $s(x) = 1 - 4\theta|x|$, for $x \in \mathcal{X}_i$, where $\theta \in [0, 1/2)$ is the commuting rate that ensures that $s(x) \ge 0$ for all $x \in \mathcal{X}_i$ and for all $i \in N$. Then, total effective labor supply at the CBD of region i is given by:

$$L_i = \int_{x_i \in \mathcal{X}_i} s(x)dx = \lambda_i(1 - \theta\lambda_i).$$

Letting $r_i(x)$ be the land rent at x, a residential equilibrium implies that the wage net of commuting costs and land rent must be equal across locations: $s(x)w_i - r_i(x) = s(\lambda_i/2)w_i$. We thus have: $r_i(x) = 2\theta(\lambda_i - 2|x|)w_i$, which implies an aggregate land rent at region i of:

$$R_i = \int_{x_i \in \mathcal{X}_i} r_i(x)dx = \theta w_i \lambda_i^2.$$

Finally, since land is locally owned, the income of a worker in region i at any location x is given by $y_i = (1 - \theta\lambda_i)w_i$. Preferences are given by the following upper-tier utility function $u_i^{MT} = C_i$. The remaining difference between the MT model and the footloose entrepreneur models is that each firm now requires a fixed input of one worker and a variable input of one worker per output of the manufactured good that is produced.

Following Takayama et al. (2020), we reach the short-run equilibrium whereby firms earn zero profits, which yields the following wage equation for city i:

$$L_i w_i = L_i w_i^{1-\sigma} \sum_{j \in N} \frac{\phi_{ij} L_j w_j}{\sum_{m \in N} L_m w_m^{1-\sigma} \phi_{jm}}.$$

We normalize $\sum_{j \in N} \lambda_j w_i = w_k$.

In the state of full agglomeration to the central city k, the indirect utility in a potential city i is given by $v_i = \zeta \Delta_i^{\frac{1}{\sigma-1}} y_i$, where $y_i = (1 - \theta\lambda_i)w_i$, $\Delta_i = \sum_{j \in N} L_j w_j^{1-\sigma} \phi_{ij}$ and $\zeta > 0$ is a constant. The indirect utilities for $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0^{\mathrm{FA}}$ are given by (see Appendix B.4)

$$v_k = \zeta(1-\theta)^{\frac{\sigma}{\sigma-1}},$$

$$v_{k\pm\delta} = \zeta(1-\theta)^{\frac{1}{\sigma-1}} \phi^{\frac{\delta}{\sigma}\frac{2\sigma-1}{\sigma-1}} \quad 1 \le \delta \le k. \tag{10}$$

4. How and where satellite cities form for economic geography models

Being equipped with the sustain bifurcation mechanism of the full agglomeration at the center (Section 2.3), we are ready to elucidate "how and where (new) satellite cities form" for the four economic geography models (Section 3). This full agglomeration is shown to be superior in stability to full agglomerations elsewhere (Section 4.1). The sustainability and stability of the full agglomeration are investigated analytically (Section 4.2). Bifurcation from the full agglomeration is studied numerically (Section 4.3).

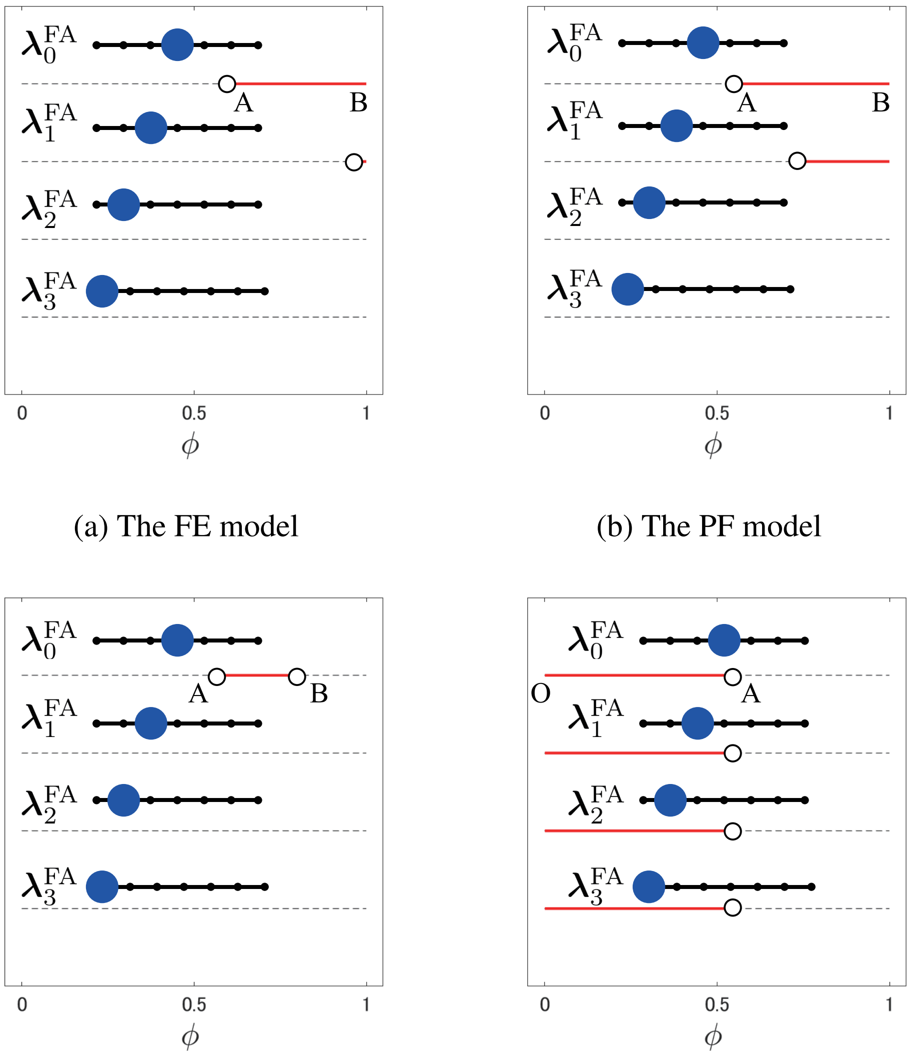
4.1. Stability of full agglomerations at various locations: Numerical studies

We investigate the stability of the full agglomerations at various locations to demonstrate the superior stability of the full agglomeration at the center. Such superiority is presumed in the theoretical study in Section 2.3.

By Proposition 1, the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\rm FA}$ for any δ steps away from the center is a stationary point of the replicator dynamics for any value of ϕ (and any value of any other parameter). Among the stationary points of the full agglomeration, we are interested in stable ones which can be identified as those satisfying Proposition 2, i.e., investigating if the utility $v_{k-\delta}$ is strictly larger than the utility v_i elsewhere.

We numerically investigated the range of $\phi \in (0,1)$ in which the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\rm FA}$ is stable for the four typical economic geography models presented in Section 3. We used $K = 7$ cities and typical values of economic parameters.[14] This stable range

[14]Referring to various empirical results (e.g., Bergstrand et al., 2013), we set $\sigma = 6.0$ for all the four models. For the FE model, we set $\mu = 0.4$ which is often used as a benchmark case and satisfies the no-black-hole condition ($\mu < \sigma - 1$). We set $\alpha = 0.8$ for the PF model and $(\alpha, \gamma) = (0.8, 0.1)$ for the PFSU model. For the MT model, we set $\theta = 0.2$ in accordance with Murata and Tisse (2005).



(a) The FE model

(b) The PF model

(c) The PFSU model

(d) The MT model

Figure 6: The range of ϕ of stable full agglomerations $\boldsymbol{\lambda} = \lambda_\delta^{\mathrm{FA}}$ ($(\sigma, \mu) = (6.0, 0.4)$ for the FE model, $(\sigma, \alpha) = (6.0, 0.8)$ for the PF model, $(\sigma, \alpha, \gamma) = (6.0, 0.8, 0.1)$ for the PFSU model, and $(\sigma, \theta) = (6.0, 0.2)$ for the MT model; red solid line: stable; broken line: unstable; $\circ$: global sustain point)

is shown by the red solid line in Fig. 6. For the FE, PF and PFSU models, the full agglomeration $\lambda = \lambda_0^{\mathrm{FA}}$ at the center has the longest range of stable state $\phi \in (\phi^{\mathrm{s}}, 1)$ and is the one which becomes stable (sustainable) first among the full agglomerations, when the trade freeness increases from a low value. In contrast, full agglomerations $\lambda_\delta^{\mathrm{FA}}$ $(\delta \geq 1)$ in the cities away from the center are much inferior in stability. Thus the central city has a better trade environment and workers living there are endowed with a larger indirect utility.[15] Hence we hereafter focus on the full agglomeration at the center.

4.2. *Sustainability and stability of full agglomeration at the center: Theoretical study*

As stated in Section 2.3, the global sustain point for the full agglomeration at the center plays an important role in determining the location of the satellite cities. For the four models, the existence and uniqueness of a local sustain point (possibly a global sustain point) are described as below.

Proposition 6. *(i) The FE model and the PF model has a local sustain point* $\phi_i^{\mathrm{s}} \in (0, 1)$ *for every* $i \in \{0, \dots, k\}$*. For* $\delta = |i - k| \in \{1, \dots, 6\}$*, this point is unique for any* $k \geq \delta$*.*

(ii) The PFSU model has zero, one or two local sustain points for $\delta \in \{1, \dots, 6\}$ *and for any* $k \geq \delta$*.*

(iii) The MT model has a unique local sustain point $\phi_i^{\mathrm{s}} \in (0, 1)$ *for every* $i \in \{0, \dots, k\}$*.*

Proof. See Appendix B for the proof. □

Regarding the FE and PF models, although Proposition 6(i) establishes uniqueness only for $\delta \in \{1, \dots, 6\}$, its extension to any k and any $\delta \in N_\delta$ is very likely to hold (see

[15] This superiority of the central city is in line with the limit behavior of the Krugman model for a long narrow economy with three places (Ago et al., 2006).

Conjectures 1 and 2 in Appendix B).

Based on Proposition 6, the global sustain point and the stability of the full agglomeration can be described by the following proposition.

Proposition 7. *(i) For the FE model and the PF model, there exists a global sustain point* $\phi^{\mathrm{s}} = max_i \phi_i^{\mathrm{s}} \in (0, 1)$ *and the full agglomeration is stable for* $\phi \in (\phi^{\mathrm{s}}, 1)$. *For* $\delta \in \{1, \ldots, 6\}$*, the full agglomeration is unstable for* $\phi \in (0, \phi^{\mathrm{s}})$.

(ii-1) For the PFSU model, for large k and/or large γ*, there is no local sustain point and the full agglomeration is unstable for any* $\phi \in (0, 1)$.

(ii-2) For the PFSU model, if there are only two local sustain points ϕ_i^{s1} *and* ϕ_i^{s2} *for any* $i \in N$*, we can define two global sustain points:* $\phi^{\mathrm{s1}} = \max_i \phi_i^{\mathrm{s1}}$ *and* $\phi^{\mathrm{s2}} = \min_i \phi_i^{\mathrm{s2}}$.

Then if $\phi^{\mathrm{s1}} < \phi^{\mathrm{s2}}$, ϕ^{s1} *and* ϕ^{s2} *are global sustain points and the full agglomeration is stable for* $\phi \in (\phi^{\mathrm{s1}}, \phi^{\mathrm{s2}})$ *and is unstable for* $\phi \in \left\{(0, \phi^{\mathrm{s1}}) \cup (\phi^{\mathrm{s2}}, 1)\right\}$.

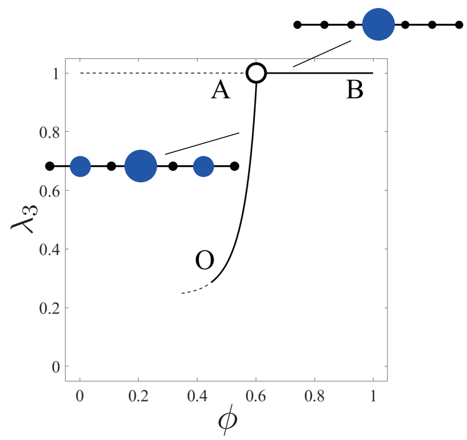
(iii) The MT model has a unique global sustain point $\phi^{\mathrm{s}} \in (0, 1)$ *and the full agglomeration is stable for* $\phi \in (0, \phi^{\mathrm{s}})$ *and unstable for* $\phi \in (\phi^{\mathrm{s}}, 1)$. *At this global sustain point, satellite cities emerge one step away from the center (*$\delta = 1$*).*

Proof. See Appendix B for the proof. □

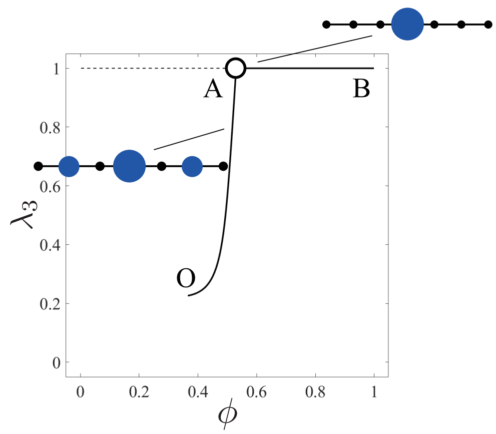
4.3. Bifurcation from full agglomeration: Numerical study

By numerical bifurcation analysis of the four economic geography models, we demonstrate the usefulness of (1) the bifurcation theory in Section 2.3 in elucidating "how (new) satellite cities form around a large city" and (2) the two step procedure to determine "where (new) satellite cities form" presented at the end of Section 2.

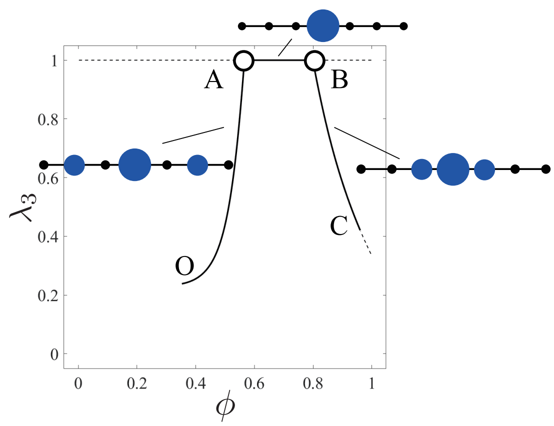
We conducted the comparative static analysis regarding $K = 7$ cities for the four models for the same parameter values as those used in Fig. 6. Figure 7 plots the paths of



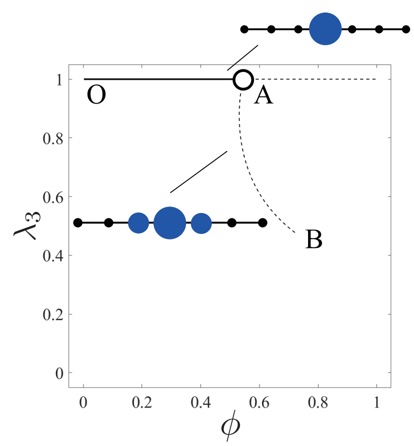
(a) The FE model



(b) The PF model



(c) The PFSU model



(d) The MT model

Figure 7: Bifurcating paths of equilibria emanating from the state of the full agglomeration at the center (parameter values are $(\sigma, \mu) = (6.0, 0.4)$ for the FE model, $(\sigma, \alpha) = (6.0, 0.8)$ for the PF model, $(\sigma, \alpha, \gamma) = (6.0, 0.8, 0.1)$ for the PFSU model, and $(\sigma, \theta) = (6.0, 0.2)$ for the MT model; solid line: stable; broken line: unstable; ○: global sustain point)

full agglomeration λ_0^{FA} at the center and bifurcating paths that branch at the global sustain points shown by $\circ$. The vertical axis is the population λ_3 at the central region ($k = 3$) and the horizontal axis is the trade freeness $\phi \in (0, 1)$. Stable equilibria are shown by solid curves, while unstable ones by broken ones.

First, we investigate the stability of the full agglomeration λ_0^{FA}. A solid horizontal line at $\lambda_3 = 1$ denotes stable full agglomeration, while the broken line at $\lambda_3 = 1$ stands for unstable one. There is a global sustain point (shown by $\circ$) at an end of this solid horizontal line: the point A for all the four models and the point B for the PFSU model. The stability and the existence of global sustain point(s) agree with Proposition 7:

- For the FE and PF models, the full agglomeration has a unique global sustain point ϕ^{s} at A, and is stable for $\phi \in (\phi^{\mathrm{s}}, 1)$ (during AB) and unstable for $\phi \in (0, \phi^{\mathrm{s}})$ (during OA) in agreement with Proposition 7(i).

- For the PFSU model, the full agglomeration has two global sustain points $\phi^{\mathrm{s}1}$ and $\phi^{\mathrm{s}2}$ (respectively, at A and B), stable for $\phi \in (\phi^{\mathrm{s}1}, \phi^{\mathrm{s}2})$ (during AB), and unstable for $\phi \in \left\{(0, \phi^{\mathrm{s}1}) \cup (\phi^{\mathrm{s}2}, 1)\right\}$ (during OA and BC) in agreement with Proposition 7(ii-2).

- For the MT model, the full agglomeration has a unique global sustain point ϕ^{s} at A, and is stable for $\phi \in (0, \phi^{\mathrm{s}})$ (during OA) and unstable for $\phi \in (\phi^{\mathrm{s}}, 1)$ (during AB) in agreement with Proposition 7(iii).

Next, we investigate stable bifurcating paths for the global sustain points. From each global sustain point, we found two kinds of bifurcating paths: one with one satellite city and another with two satellite cities in agreement with Proposition 3. The paths with one satellite city are all unstable, while those with two satellite cities are stable except for the global sustain point A for the MT model. Therefore, only the curves with two satellite cities (OA and BC), which are superior in stability, are included in Fig. 7.

For the FE, PF, and PFSU models, each global sustain point has one stable bifurcating path just after bifurcation in agreement with Proposition 4. As ϕ increases from a small value, the stable bifurcating path OA transits to a stable full agglomeration λ_0^{FA} (the path AB). In agreement with Proposition 5(i), there is a continuation of stable paths. The global sustain point A has $\delta_{\mathrm{sat}} = 2$ and the bifurcated path OA displays a core–satellite pattern with a large central city surrounded by two satellite cities located two steps away from the center. We see an emergence of a sustainable full agglomeration at the center by steadily absorbing and finally nullifying the (mobile) population of satellite cities.

For the PFSU model, as ϕ further increases, we can see a further transition from the stable full agglomeration (path AB) to stable bifurcating equilibria (path BC), expressing a diffused state of agglomeration with two satellite cities at $\delta = 1$. Thus the PFSU model, unlike the FE and PF models, accommodates both the coalescence and the emergence of satellite cities as the freeness of trade increases. This might tell a more compelling story regarding the historical evolution of the spatial economy due to the spectacular decrease in transport costs. During the first stage, we observe the formation of very large cities (monocentric patterns), but eventually further drops in transport costs give rise to the emergence of satellite cities, thus forming chains of cities that develop into megalopolises or megaregions as a result of the increase in economic integration.

For the MT model, as ϕ increases, the stable full agglomeration becomes unstable at the global sustain point A. Bifurcating path of equlibria branches in the reversed direction of $\phi < \phi^{\mathrm{s}}$ and is unstable just after bifurcation in agreement with Proposition 5(ii).

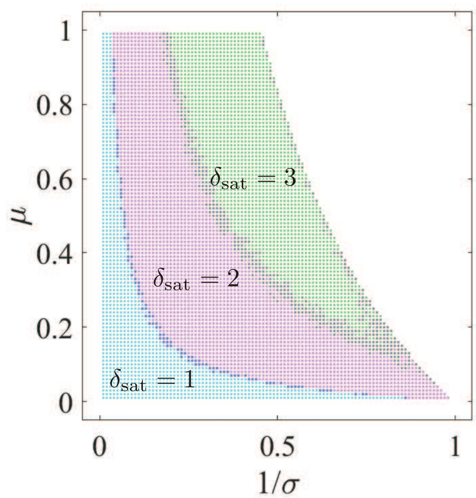
5. Location of satellite cities for economic geography models

As we have seen for economic geography models in Section 4.3, there are stable bifurcating paths with a pair of satellite cities located δ_{sat} steps away from the full agglomeration λ_0^{FA} at the center. In this section, (1) the location δ_{sat} of satellite cities is shown to be dependent on the models and their parameters (Section 5.1) and (2) the location δ_{sat} for a large number of places is investigated (section 5.2).

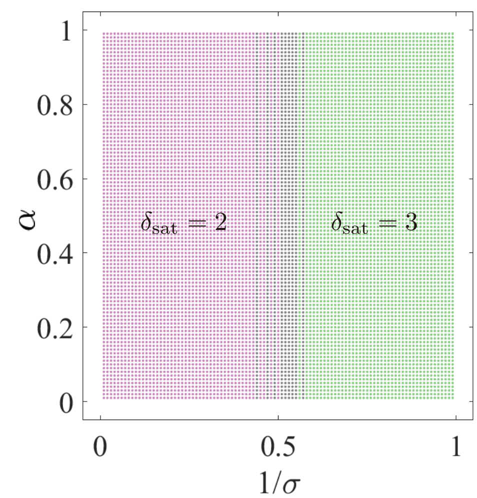
5.1. Model and parameter dependency of the location of satellite cities

We numerically obtained the value of δ_{sat} of the FE model for $K = 7$ cities by varying the values of the parameters σ and μ. Figure 8(a) depicts the contours of δ_{sat} in the range $(0, 1) \times (0, 1)$ of the space of $(1/\sigma, \mu)$. As $1/\sigma$ and/or μ increases, δ_{sat} increases one by one from the smallest value of $\delta_{\text{sat}} = 1$. That is, as agglomeration forces increase due to stronger scale economies or a larger size of the manufacturing sector (respectively, a decrease in σ and/or an increase in μ), the satellite cities tend to form away from the primary city at the center, thereby forming an *agglomeration shadow* (Arthur, 1990; Ikeda et al., Fig. 5, 2017). By contrast, as agglomeration forces decrease, the satellite cities tend to locate closer to the primary city, thereby forming a discrete version of a hump-shaped megalopolis around this city for $\delta_{\text{sat}} = 1$. Thus, we have observed the dependence of agglomeration patterns on the values of economic parameters, which possibly are a source of the diversity of the population distribution of a chain of cities observed worldwide and help the explanation of the historical emergence of megalopolises or megaregions.

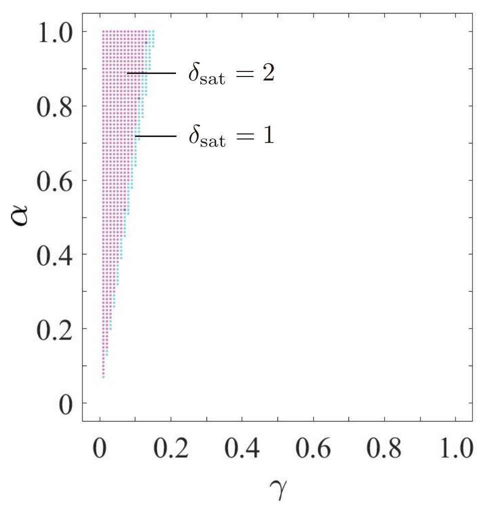
This is in accordance with the analytical results on the stability of the single agglomeration in the pioneering work by Fujita and Krugman (1995) who considered a similar economic geography model with a one-dimensional unbounded continuous location



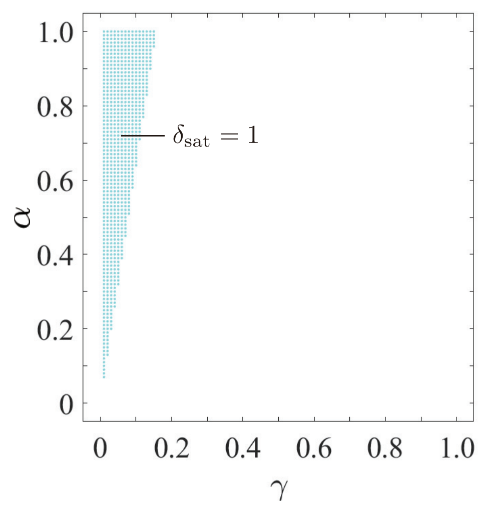
(a) The FE model



(b) The PF model



The lower global sustain point



The higher global sustain point

(c) The PFSU model ($\sigma = 6.0$)

Figure 8: Contour maps of δ_{sat}, which indexes the location of satellite cities for $K = 7$ cities emerging from the global sustain point, drawn on parameter spaces (sky blue area: $\delta_{\text{sat}} = 1$; pink: $\delta_{\text{sat}} = 2$; green: $\delta_{\text{sat}} = 3$; dark blue: $\delta_{\text{sat}} = 1$ and $\delta_{\text{sat}} = 2$ are coincidental; black: $\delta_{\text{sat}} = 2$ and $\delta_{\text{sat}} = 3$ are coincidental)

space. They, for instance, show that a satellite city may emerge away from the center, but not *how far* from the center. They suggest that, with a so-called potential function, we could investigate locations of satellite cities. They, however, do not explicitly explain the parameter dependence of such locations. By contrast, we take a step further by investigating the stability of agglomeration patterns that form the large central city and satellite cities as shown in Fig. 8(a) and explicitly showing the parameter dependence of the location of satellite cities around a large central city.

We further look at the PF model to show the model dependence of δ_{sat}. Figure 8(b) depicts the contours of δ_{sat} in the range $(0, 1) \times (0, 1)$ in $(1/\sigma, \alpha)$-space. The parameter α does not influence δ_{sat}. As $1/\sigma$ increases, δ_{sat} increases from 2 to 3 but $\delta_{\text{sat}} = 1$ cannot be realized, unlike the FE model. This demonstrates the model dependence of δ_{sat}.

Model dependence can be seen also for the PFSU model. Figure 8(c) plots the contours of δ_{sat} in the range $(0, 1) \times (0, 1)$ in (α, γ)-space for $\sigma = 6.0$. There are two possible global sustain points, called the lower global sustain point and the higher one (cf., the points A and B in Fig. 7(c)). While $\delta_{\text{sat}} = 1$ and 2 are attainable at the lower global sustain point, only $\delta_{\text{sat}} = 1$ at the upper one. For the lower global sustain point, an increase in the preferences for manufactured goods (higher α) or a decrease in the preference for housing (lower γ) also brings about an increase in δ_{sat} thus corroborating the results that higher agglomeration forces push potential satellite cities away from the central region.

For the MT model, $\delta_{\text{sat}} = 1$ holds for any parameter values by Proposition 7(iii). This model, accordingly, does not have model dependence of δ_{sat}, unlike the other three models. The model, for this reason, is not suitable for the investigation of "where satellite cities form."

5.2. Location of satellite cities for a large number of cities

We would like to investigate where (new) satellite cities form for a very large number of potential cities K. As an index for the optimal location of the satellite cities, we introduce a normalized length from the center, being defined as δ_{sat}/k.

We computed the values of δ_{sat}/k for $k = 3, 50, 100$, and 200 ($K = 2k+1$ cities) for the four models using the same parameter values as those employed in Fig. 7 in Section 4.3. Table 1 lists the values of δ_{sat}/k,[16] where $k = 3$ corresponds to the analysis in Fig. 7.

For the FE model, as k increases to a large value, such as $k = 50$, the optimal location is convergent quickly to $\delta_{\text{sat}}/k \approx 0.58$, as can be seen from the plot of δ_{sat}/k against k in Fig. 9. For the PF model, the convergence is much slower and the location of the satellite cities tends to be much closer to the central city than the FE model. For the MT model, $\delta_{\text{sat}} = 1$ holds for any k and, accordingly, δ_{sat}/k converges to 0, i.e., the center of the economy.

These results may be interpreted as a discretized line segment counterpart of the infinite line economy and are consistent with preexisting analytical results on this economy by Fujita and Krugman (1995) who show that a satellite city may emerge away from the center, but not *how far* from the center. In contrast to their study, our result explicitly determines the location of satellite cities around a central city. It is empirically relevant as it fits several real world examples of a chain of cities as to be seen in Section 6.

[16] The results of the PFSU model are not included in this table, because sustain points do not exist for large k by Proposition 7(ii-1). No sustain point exists for large values of $k = 50$, 100, and 200, while two sustain points exist for $k = 3$ (Fig. 7(c)).

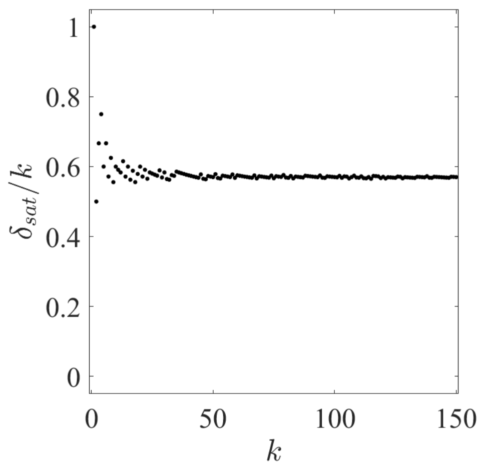


Figure 9: Convergence of normalized length δ_{sat}/k from the center for a large k computed for the FE model ($\mu = 0.4$ and $\sigma = 6.0$)

Table 1: The convergence of δ_{sat}/k as k increases

	$k = 3$	50	100	200
The FE model	$2/3 = 0.67$	0.58	0.57	0.57
The PF model	$2/3 = 0.67$	0.33	0.22	0.17
The MT model	$1/3 = 0.33$	0.02	0.01	0.005

6. Use of economic geography analysis in the study of real data

We have seen several theoretical patterns of satellite cities around a large city at the center and ensured their existence for the typical economic geography models. In this section, comparing these patterns with worldwide city size distributions, we demonstrate the usefulness of the proposed procedures in the analysis of real population data.

6.1. Mainland in Japan

Taking Japan in Fig. 1(b) as reference with $K = 5$ regions, we conducted the comparative static analysis for the FE model (Forslod and Ottaviano, 2003). Figure 10 shows the paths of equilibria, in which the vertical axis denotes the population $\lambda_2 \in [0, 1]$ at the central city and the horizontal axis denotes the trade freeness $\phi \in (0, 1)$. As the trade is liberated (ϕ increases from 0), there appears a series of equilibria A, B, . . ., I; the associated spatial population distributions are depicted at the right of this figure. The bifurcation occurs at the global sustain point H engendering two satellite places at $\delta = 1$ leading to a stable state A–G.

As ϕ increases from a low value, there emerge three stable stages: (1) the twin cities at $\delta = 1$ (Tokyo and Osaka) for an intermediate value of ϕ slightly below 0.5 along the path DE, (2) the central city surrounded by the twin cities in an intermediate stage along the path EG, containing E′ and E′′, and (3) the full agglomeration at the center (Nagoya) for ϕ over 0.5 along the path HI. Then one could infer that the configuration of Japan resembles that of the pattern E′′ with Nagoya at the center being highly populated but scarcely if compared to the gigantic satellites of Osaka and Tokyo at $\delta = 1$, and with the border of the regions of Hiroshima and Sendai at $\delta = 2$ a little smaller than Nagoya.

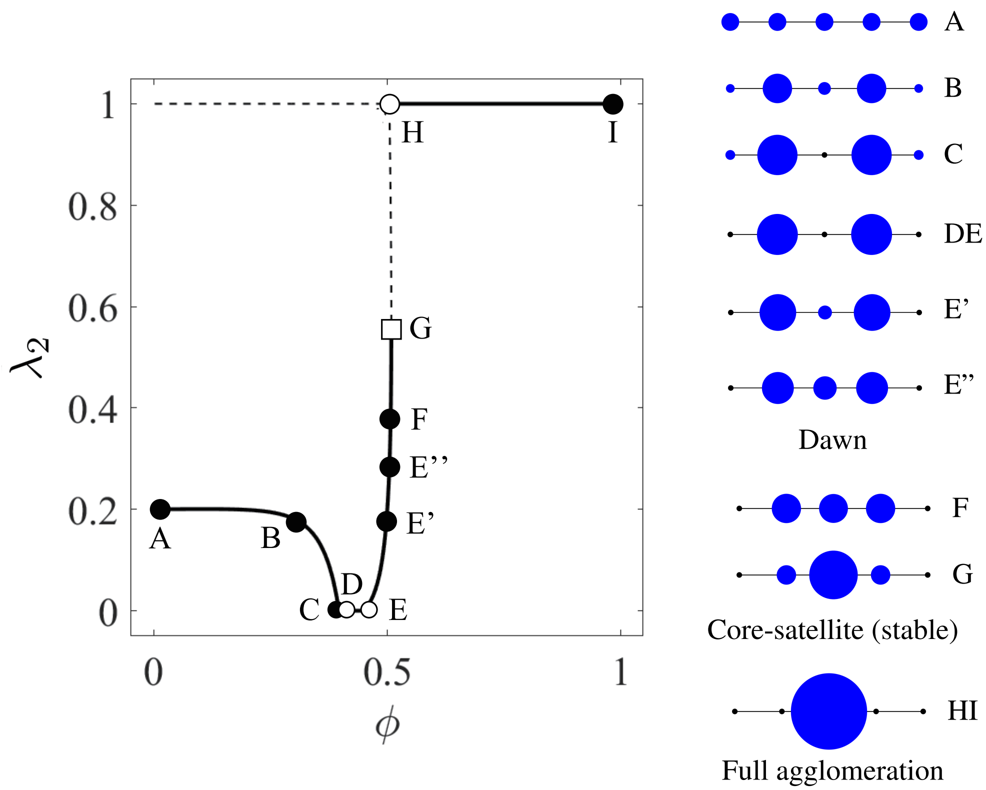


Figure 10: Paths of equilibria for the FE model with $(\sigma, \mu) = (6.0, 0.4)$ for $K = 5$ cities (solid line: stable; broken line: unstable; ○: global sustain point; □: local maximum point of ϕ; •: reference point)

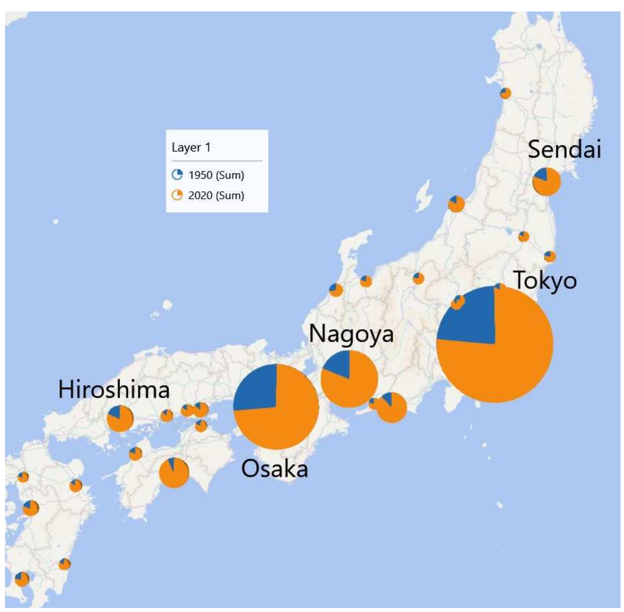


Figure 11: Population of cities in Japan. Blue arc is the population in 1950 and orange arc is the population in 2020 (source: the UN, Department of Economic and Social Affairs, Population Division (2018)

When we compare the five cities' population in the 1950's (shown by blue arcs in Fig. 11) with the population in 2020 (shown by orange arcs), we observe that Nagoya has grown significantly more compared to Tokyo and Osaka, thus suggesting that Japan may be in the stage of $E'E''$, expressing the growth of the central city (Nagoya). Such growth is in line with the recent numerical experiment on Shinkansen extension by Hayakawa et al. (2021), which reports that the investment on Shinkansen network would lead to the population change of large metropolitan areas of Tokyo, Osaka, and Nagoya by −0.3%, 0.6%, and 9.8%, respectively. Thus, the largest growth of population in response to the investment is expected to occur at Nagoya located at the geographical center of the metropolitan areas of Tokyo and Osaka.

6.2. The Atlantic Axis

Regarding the Atlantic Axis in Fig. 12, when restricted to 7 cities, from Porto southwards to La Coruña in the North, we can see that these two cities at the borders of the corridor acting as large cities three steps ($\delta_{\text{sat}} = 3$) away from the bigger central city (Vigo). Indeed, both in Portugal and Spain there is a large size of the business and industrial tissue concentrated in the northwestern coastal provinces, thus corroborating our predictions that higher agglomeration forces push satellite cities away from the center, as we have seen for the FE model (see the parameter zone for $\delta_{\text{sat}} = 3$ in Fig. 8(a)).

A few notes of caution are warranted at this point. First, the choice of parameter values for our simulations in Figs. 7 and 10 is plausible for illustration purposes but may be arbitrary from an empirical perspective. This means that obtaining specific estimates for the megaregions discussed throughout this paper would likely improve our understanding and predictive capacity regarding the location of satellite cities around a central city.

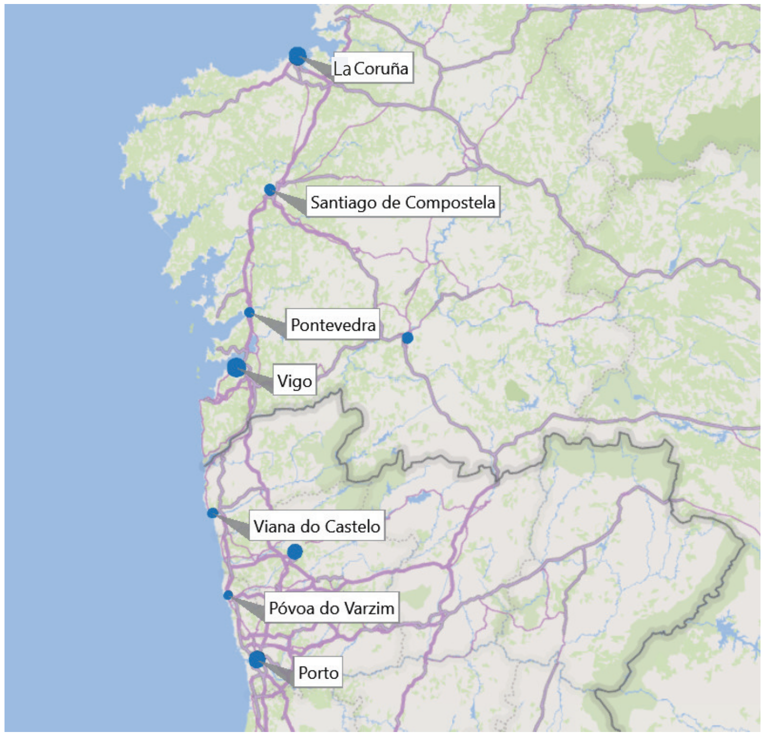


Figure 12: Atlantic Axis mega-region (population from 2017, Eurostat)

This, however, is left for future research.

Second, a great part of the distribution and size of cities may well be related with activities regarding services instead of manufacturing goods, something which the models used here disregard completely.[17]

Third, for most of the empirically depicted cases in this work, particularly in Europe, the mega-regions seem to always have significant populations at the border regions. This could be due to an underestimation of agglomeration forces by our choice of parameter values. If agglomeration forces are instead very strong in these megaregions, then more population at the borders lies in accordance with our predictions.

[17]We thank Ricardo Gonalves for pointing this out to us.

7. Conclusion

We have elucidated the bifurcation mechanism of the formation of twin satellite cities around a large core city in a long narrow economy. This mechanism is independent on models and is readily applicable to many spatial economic models under replicator dynamics. A pertinent combination of this model-independent general bifurcation mechanism with model-dependent properties, such as stability/sustainability and parameter dependency, is vital in the successful elucidation of the mechanism of the formation of satellite cities.

Analyzing the FE, PF and PFSU models, we have shown that the higher the agglomeration forces (less preference for housing, a higher industry size, or stronger scale economies), the farther satellite cities emerge from the central city. Conversely, if agglomeration forces are weak, there emerges a hump-shaped megalopolis with satellite cities located side-by-side with the primary central city.

It is pertinent to relate our use of the long narrow economy with Thisse et al. (2021), who study urbanization patterns in a linear city with three discrete locations and heterogeneous agents as increasing returns to scale in production increase. Granted, the settings differ fundamentally in that their model falls under the class of urban economics whereas the models here belong to the field of geographical economics.[18] However, what we wish to highlight is what makes both works relatable: the line segment with discrete locations.

Thisse et al. (2021) find that, once increasing returns become sufficiently strong,

[18]Note that this does not imply in any way that we consider geographical economics and urban economics competing theories. Rather, they can and should be very much complementary (see Gaspar, 2018; 2021).

stronger returns to scale may induce the dispersion of economic activities toward the peripheral (border regions). Their results hinge heavily on the heterogeneity of agents regarding residential location preferences. Thus, to increase comparability between the models we could introduce heterogeneity in our setting as in Castro et al. (2022).

It would thus be interesting to check if this dispersion process for high increasing returns to scale also implies higher residence in outer regions with more locations in their setting. If the answer is positive, the results are similar to ours, not in the conventional sense of dispersion (as in uniformity of the spatial distribution), but in the sense of increasing distance regarding the centre for locations that become more concentrated/populated. This could potentially add to a weak conjecture of geographical scale invariance which would suggest a fractal relationship between spatial configurations at low scales and very large scales.

We have thus observed diverse agglomeration patterns dependent on the values of trade freeness and on microeconomic parameters. Such dependence possibly is a source of the diversity of the population distribution of a chain of cities observed worldwide. The population distribution in a chain of cities in the mainland of Japan was successfully explained using the result of the analysis for five cities, while that in the Atlantic Axis by that for seven cities. Thus, the analysis by the economic geography model gives an insight into the investigation of real data. This motivated us to conduct the study of the bifurcation mechanism from the full agglomeration presented in this paper.

A remark is on the standpoint of this paper. While it is customary to start from the uniform state, we highlight agglomeration patterns emanating from the completely agglomerated state. Nowadays it would be far more important to investigate the competition between a large central city and satellite cities than to investigate the self-organization of cities in a flat land envisaged in Central Place Theory. Future work will extend this theory to different spatial topologies, such as a "star economy", a "racetrack economy" or regions in two-dimensional space.

It is a topic for future research to study the progress of satellite cities' formation for other spatial economic models based on the bifurcation mechanism proposed in this paper, thereby selecting a model that is on purpose. Among these, we have early (new) economic geography models such as the original Core-Periphery model by Krugman (1991) and similar ones: the modified version with land instead of immobile workers by Puga (1999), the models with dispersive congestion effects by Helpman (1998) and Tabuchi (1998), or the quasi-linear upper tier utility footloose entrepreneur model (e.g. Ottaviano et al., 2002).

More recently, the structural model-based approach used to evaluate the causal effects of regional agglomerations summarized by Redding and Rossi-Hansberg (2017) has lead to a sprawl of the so-called quantitative spatial economics models, such as Redding and Sturm (2008), Allen and Arkolakis (2014) and Behrens and Murata (2021), to name a few. The model by Allen and Arkolakis (2014), for instance, is a very interesting candidate as it relies on exogenous differences but also incorporates agglomeration externalities in a way to preserve sufficient analytical tractability to provide conditions under which a unique spatial distribution equilibrium exists and additionally allowing for some comparative statics. On the other hand, there is a caveat in these structural model-based approaches in that under the uniqueness of equilibrium, by construction, they cannot explain the endogenous formation of multiple agglomerations (Akamatsu et al., 2021).

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Appendix A. Theoretical details of bifurcation analysis

Appendix A.1. Proof of Proposition 1

For $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}_{\delta}$, we have $\lambda_i = 0$ $(i \neq k-\delta)$ and $v_{k-\delta} - \bar{v} = 0$ since $\bar{v} = \sum_{i \in N} \lambda_i v_i = 1 \times v_{k-\delta}$; accordingly, the governing equation (3) with (2) is satisfied for any $i \in N$.

Appendix A.2. Proof of Proposition 2

At the state of the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}_{\delta}$ at the place $i = k-\delta$, the eigenvalues of the Jacobian matrix of the static governing equation $\boldsymbol{F}(\boldsymbol{\lambda}, \phi)$ in (3) are given by $-v_{k-\delta}(< 0)$ and $v_i - \bar{v} = v_i - v_{k-\delta}$ $(i \neq k-\delta)$. Hence, if $v_{k-\delta} > v_i$ for any $i \neq k-\delta$, all eigenvalues are negative and this state is stable.

Appendix A.3. Proof of Proposition 3

We consider the sustain point $(\boldsymbol{\lambda}^{\mathrm{FA}}_{0}, \phi^{\mathrm{c}}_{\delta})$, where $v_{k-\delta} - v_k = v_{k+\delta} - v_k = 0$ is satisfied. In the neighborhood of this point, the governing equation $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0}$ in (3) can be reduced to a two-dimensional bifurcation equation $\tilde{F}_i = 0$ $(i = k-\delta,\ k+\delta)$ in two independent variables $(x, y) = (\lambda_{k-\delta}, \lambda_{k+\delta})$ with an incremental parameter $\psi = \phi - \phi^{\mathrm{c}}_{\delta}$ (cf., Ikeda and Murota, 2019) as shown in Lemma 1 below.

Lemma 1. *The bifurcation equation at the critical point* $(\boldsymbol{\lambda}^{\mathrm{FA}}_{0}, \phi^{\mathrm{c}}_{\delta})$ *is expressed as*

$$\begin{aligned} \tilde{F}_{k-\delta}(x, y, \psi) &= x\,(a\psi + bx + cy + \text{higher order terms}) = 0, \\ \tilde{F}_{k+\delta}(x, y, \psi) &= y\,(a\psi + by + cx + \text{higher order terms}) = 0 \end{aligned} \tag{A.1}$$

with the symmetry condition $\tilde{F}_{k+\delta}(x, y, \psi) = \tilde{F}_{k-\delta}(y, x, \psi)$ *and expansion coefficients:*

$$(a, b, c) = \left.\left(\frac{\partial g}{\partial \phi}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\right|_{(x,y,\psi)=(0,0,\phi^{\mathrm{c}}_{\delta})}, \qquad g(x, y, \psi) = v_{k-\delta}(\tilde{\boldsymbol{\lambda}}) - v_k(\tilde{\boldsymbol{\lambda}});$$

$$\tilde{\boldsymbol{\lambda}} = (\boldsymbol{0}_{k-\delta-1}, x, \boldsymbol{0}_{\delta}, 1-x-y, \boldsymbol{0}_{\delta}, y, \boldsymbol{0}_{k-\delta-1}, \phi^{\mathrm{c}}_{\delta} + \psi), \qquad \boldsymbol{0}_p = (\underbrace{0, \ldots, 0}_{p \text{ times}}).$$

Proof. In the neighborhood of the critical point $(\boldsymbol{\lambda}_0^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$, $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0}$ in (3) reduces to three equations $F_j = 0$ with three variables v_j ($j = k, k \pm \delta$), while the other variables are equal to 0. Then $F_{k-\delta} + F_k + F_{k+\delta} = 0$ gives the conservation law: $\lambda_{k-\delta} + \lambda_k + \lambda_{k+\delta} = 0$. The variable λ_k can be eliminated from $F_{k-\delta}$ and $F_{k+\delta}$ to arrive at (A.1). The symmetry condition arises from the bilateral symmetry of the long narrow economy. □

The bifurcation equation (A.1) with the symmetry condition has solutions $(x, y) = (\lambda_{k-\delta}, \lambda_{k+\delta}) = (0, 0)$, $(w, 0)$, $(0, w)$, and (w, w) $(w > 0)$; $(x, y) = (0, 0)$ corresponds to the pre-bifurcation solution $(\boldsymbol{\lambda}_0^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$ and others to bifurcating solutions. Since the solutions $(w, 0)$ and $(0, w)$ are identical, we hereafter consider only the former solution.

Lemma 2. *The critical point $(\boldsymbol{\lambda}_0^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$ is a bifurcation point with two kinds of branches:*

$$(\boldsymbol{\lambda}, \phi) = (\boldsymbol{\lambda}_0^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}}) + (\Delta\boldsymbol{\lambda}_p, \psi_p), \quad p = 1, 2;$$

$$\Delta\boldsymbol{\lambda}_1 = w(\boldsymbol{e}_\delta^1, -2, \boldsymbol{e}_\delta^2), \quad \psi_1 \approx -(b + c)w/a; \quad \boldsymbol{e}_\delta^1 = (\boldsymbol{0}_{k-\delta-1}, 1, \boldsymbol{0}_\delta), \quad 0 < w \ll 1, \tag{A.2}$$

$$\Delta\boldsymbol{\lambda}_2 = w(\boldsymbol{e}_\delta^1, -1, \boldsymbol{0}_k), \quad \psi_2 \approx -bw/a; \quad \boldsymbol{e}_\delta^2 = (\boldsymbol{0}_\delta, 1, \boldsymbol{0}_{k-\delta-1}). \tag{A.3}$$

Proof. We see that $(x, y) = (\lambda_{k-\delta}, \lambda_{k+\delta}) = (w, w)$ corresponds to $\Delta\boldsymbol{\lambda}_1 = w(\boldsymbol{e}_\delta^1, -2, \boldsymbol{e}_\delta^2)$ and satisfies (A.1) in Lemma 1 for $\psi = \psi_1 \approx -(b + c)w/a$. Also, $(x, y) = (w, 0)$ corresponds to $\Delta\boldsymbol{\lambda}_2 = w(\boldsymbol{e}_\delta^1, -1, \boldsymbol{0}_k)$ and satisfies (A.1) for $\psi = \psi_2 \approx -bw/a$. □

Appendix A.4. Proof of Proposition 4

Let a local sustain point ϕ_δ^{c} not be a global sustain point. Then there exists δ' $(\delta' \neq \delta)$ such that $v_{k-\delta'} - v_k > 0$ at this point. By continuity of $v_{k-\delta'}$ and v_k as functions in ϕ, $v_{k-\delta'} - v_k > 0$ is satisfied in a neighborhood of $(\boldsymbol{\lambda}_0^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$. Therefore, the bifurcation solution is unstable just after bifurcation.

Appendix A.5. Proof of Proposition 5

The Jacobian matrix for the bifurcation equation (A.1) reads

$$\hat{J} \approx \begin{pmatrix} a\psi + 2bx + cy & cx \\ cy & a\psi + 2by + cx \end{pmatrix}.$$

The use of $(x, y) = w(1, 1)$ and $\psi = \psi_1 \approx -(b + c)w/a$ (cf., (A.2)) in $\hat{J}$ leads to $\hat{J}_1$ below and the use of $(x, y) = w(1, 0)$ and $\psi = \psi_2 \approx -bw/a$ (cf, (A.3)) leads to $\hat{J}_2$ as

$$\hat{J}_1 \approx w \begin{pmatrix} b & c \\ c & b \end{pmatrix}, \qquad \hat{J}_2 \approx w \begin{pmatrix} b & c \\ 0 & c - b \end{pmatrix}.$$

Lemma 3. *The bifurcating solution* $(\Delta\boldsymbol{\lambda}_1, \psi_1)$ *has the eigenvalues:* $e_1 \approx (b+c)w$ *and* $e_2 \approx (b - c)w$. *On the other hand,* $(\Delta\boldsymbol{\lambda}_2, \psi_2)$ *has the eigenvalues:* $e_1 \approx bw$ *and* $e_2 \approx (c - b)w$.

Lemma 4. *Under the assumption that the state of the full agglomeration is stable for* $\psi < 0$ *or* $\psi > 0$*, there are three cases: (i) If* $-b > |c|$*, only the first bifurcating path* $(\Delta\boldsymbol{\lambda}_1, \psi_1)$ *is stable. (ii) If* $c < b < 0$*, only the second bifurcating path* $(\Delta\boldsymbol{\lambda}_2, \psi_2)$ *is stable. (iii) Otherwise, both paths are unstable. (iv) A stable bifurcating path branches in the direction of* $\psi < 0$ *(respectively,* $\psi > 0$*) if the full agglomeration state is stable for* $\psi > 0$ *(respectively,* $\psi < 0$*).*

Proof. For the fully agglomerated state $(x, y) = (0, 0)$, we have $\hat{J} = a\psi I$ with the eigenvalue $a\psi$ (twice repeated). If the state is sustainable for $\psi > 0$ (respectively, $\psi < 0$), we have $a < 0$ (respectively, $a > 0$). (i) The first bifurcating solution $(\Delta\boldsymbol{\lambda}_1, \psi_1)$ with $e_1 \approx (b + c)w$ and $e_2 \approx (b - c)w$ (cf., Lemma 3) is stable if $-b > |c|$. Since $b + c < 0$, $a < 0$, and $w > 0$, $\psi = \psi_1 \approx -(b + c)w/a$ in (A.2) gives $\psi = \psi_1 < 0$ (respectively, $\psi_1 > 0$). (ii) The second bifurcating solution $(\Delta\boldsymbol{\lambda}_2, \psi_2)$ with $e_1 \approx bw$ and $e_2 \approx (c - b)w$ $(w > 0)$ is stable if $c < b < 0$. Since $b < 0$, $a < 0$ and $w > 0$, $\psi = \psi_2 \approx -bw/a$ in (A.3) gives $\psi = \psi_2 < 0$ (respectively, $\psi_2 > 0$). The two bifurcating solutions cannot be stable simultaneously since $-b > |c|$ and $c < b < 0$ are contradictory. (iii) and (iv) are apparent form (i) and (ii). □

Appendix B. Indirect utilities and sustain points at full agglomeration

We consider the full agglomeration at the center ($i = k$). By bilateral symmetry about the center ($v_i = v_{K-i}$), it suffices to consider places $i = \{0, \dots, k-1\}$ at the left. We often use $\delta = k - i$, which expresses steps away from the center, instead of i.

For the FE, PF and PFSU models, the wage in city $i \in N$ is given by a general form:

$$w_i(\boldsymbol{\lambda}) = \frac{\xi}{\sigma} \sum_{j \in N} \frac{\phi_{ij} b_j}{\sum_{m \in N} \phi_{mj} \lambda_m}, \tag{B.1}$$

where ξ and b_j are model dependent. We have $\xi = \mu$ and $b_i = (1 + w_i \lambda_i)$ for the FE model and $\xi = \alpha$ and $b_i = (1 + \lambda_i)$ for the PF and PFSU models.

For the full agglomeration in region k ($\lambda_k = 1$), for which $\lambda_j = 0$ and $b_j = 1$ ($j \neq k$) hold, the wage in region $i \neq k$ is given by

$$w_i = \frac{\xi}{\sigma} \left(\phi^{\delta} b_k + \sum_{j \neq k}^{2k} \frac{\phi_{ij}}{\phi_{kj}} \right). \tag{B.2}$$

The second term of the nominal wage is systematically given, using $\phi_{ij} = \phi^{|i-j|}$, by:

$$\begin{aligned}
\sum_{j \neq k}^{2k} \frac{\phi_{ij}}{\phi_{kj}} &= \sum_{j=0}^{i-1} \frac{\phi_{ij}}{\phi_{kj}} + \sum_{j=i}^{k-1} \frac{\phi_{ij}}{\phi_{kj}} + \sum_{j=k+1}^{2k} \frac{\phi_{ij}}{\phi_{kj}} = \sum_{j=0}^{i-1} \frac{\phi^{i-j}}{\phi^{k-j}} + \sum_{j=i}^{k-1} \frac{\phi^{j-i}}{\phi^{k-j}} + \sum_{j=k+1}^{2k} \frac{\phi^{j-i}}{\phi^{j-k}} \\
&= \sum_{j=0}^{i-1} \phi^{i-k} + \sum_{j=i}^{k-1} \phi^{2j-i-k} + \sum_{j=k+1}^{2k} \phi^{k-i} = i\phi^{i-k} + \frac{\phi^{-i-k}\left(\phi^{2k} - \phi^{2i}\right)}{\phi^2 - 1} + k\phi^{k-i} \\
&= \phi^{-\delta}(k - \delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1} + k\phi^{\delta}.
\end{aligned}$$

Thus, the wage in region $i \neq k$ is finally given by:

$$w_i = \frac{\xi}{\sigma} \left[\phi^{\delta}(k + b_k) + \phi^{-\delta}(k - \delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1} \right]. \tag{B.3}$$

Appendix B.1. The FE model

Derivation of v_i (Proof of (5) and (6)): We have $b_i = (1 + w_i\lambda_i)$. By setting $i = k$ in (B.1), the wage in region k is obtained as

$$w_k = \frac{\hat{\mu}}{1-\hat{\mu}}(2k+1), \quad \hat{\mu} = \frac{\mu}{\sigma}.$$

The nominal wage in any potential city $i \neq k$ is given by:

$$w_i = \hat{\mu}\left\{\phi^{\delta}\left[k+1+\frac{\hat{\mu}}{1-\hat{\mu}}(2k+1)\right]+\left[\phi^{-\delta}(k-\delta)+\frac{\phi^{\delta}-\phi^{-\delta}}{\phi^2-1}\right]\right\}.$$

The indirect utilities for the FE model in the central region at $i = k$ and in peripheries at $i = k \pm \delta$ ($1 \leq \delta \leq k$) with distance $\delta = |i - k|$ from the center are given, respectively, by:

$$\begin{aligned} v_k &= \ln\frac{\hat{\mu}}{1-\hat{\mu}} + \ln(2k+1), \\ v_{k\pm\delta} &= \ln\frac{\hat{\mu}}{1-\hat{\mu}} + \frac{\delta\mu\ln\phi}{\sigma-1} + \ln\left\{\phi^{\delta}(\hat{\mu}k+k+1)+(1-\hat{\mu})\left[\phi^{-\delta}(k-\delta)+\frac{\phi^{\delta}-\phi^{-\delta}}{\phi^2-1}\right]\right\}. \end{aligned} \tag{B.4}$$

These equations give (5) and (6).

Existence of local sustain point (Proof of Proposition 6(i)): The sustain point must satisfy $\mathcal{S}^{FE} \equiv v_{k\pm\delta} - v_k = 0$ for some δ (≥ 1) with

$$\mathcal{S}^{FE} = \frac{\delta\mu\ln\phi}{\sigma-1} + \ln\left\{\phi^{\delta}(\hat{\mu}k+k+1)+(1-\hat{\mu})\left[\phi^{-\delta}(k-\delta)+\frac{\phi^{\delta}-\phi^{-\delta}}{\phi^2-1}\right]\right\} - \ln(2k+1). \tag{B.5}$$

Notice that

$$\lim_{\phi\to+0}\mathcal{S}^{FE} = +\infty, \quad \lim_{\phi\to1}\mathcal{S}^{FE} = 0. \tag{B.6}$$

Differentiating $\mathcal{S}^{FE}$ with respect to ϕ, we get:

$$\frac{\partial\mathcal{S}^{FE}}{\partial\phi} = \frac{A_1 - A_2}{d\phi\,(\phi^2-1)\,A_3},$$

where:

$$A_1 = \delta\phi^{2\delta}c[k(\mu+\sigma)+\mu] + \delta\phi^{2\delta+4}c\left[k(\mu+\sigma)+\sigma\right] + \phi^2 a\left[\delta b(2k-2\delta+1)+2d\right] > 0,$$

$$A_2 = \delta ab\left[(k-\delta)(\phi^4+1)+1\right] + 2ad\phi^{2\delta+2} + \delta c(2k+1)\phi^{2\delta+2}(\mu+\sigma) > 0,$$

$$A_3 = -a\left[\delta\left(\phi^2-1\right) - k\phi^2 + k + 1\right] + \phi^{2\delta}\left[\phi^2(k(\mu+\sigma)+\sigma) - k(\mu+\sigma) - \mu\right] < 0,$$

where $A_3 < 0$ can be shown by performing the variable change $\phi^\delta = x \in (0,\phi)$. Hence, the zeros of the derivative depend on the zeros of $A_1 - A_2$. First, notice that:

$$\lim_{\phi\to 1}\frac{\partial \mathcal{S}^{FE}}{\partial\phi} = \delta\left[\frac{\delta(\sigma-\mu)}{2k\sigma+\sigma} + \mu\left(\frac{1}{\sigma} + \frac{1}{\sigma-1}\right)\right] > 0.$$

This means, together with $\lim_{\phi\to 1}\mathcal{S}^{FE} = 0$ in (B.6), that

$$\mathcal{S}^{FE} < 0 \quad \text{for } \phi = 1-\epsilon, \tag{B.7}$$

while $\mathcal{S}^{FE} > 0$ for $\phi = \epsilon$, with $\epsilon > 0$ arbitrarily small. Thus, by the Intermediate Value Theorem, there exists at least one local sustain point $\phi^c_\delta \in (0,1)$ such that $\mathcal{S}^{FE}(\phi^c_\delta) = 0$.

Uniqueness of local sustain point (Proof of Proposition 6(i)): In the study of the uniqueness, we employ the no-black-hole condition $\sigma - 1 > \mu$, and introduce several positive constants

$$a = \sigma-\mu > 0, \quad b = \sigma-\mu-1 > 0, \quad c = \sigma+\mu-1 > 0, \quad d = \sigma-1 > 0.$$

Let us define $F(\phi) \equiv A_1 - A_2$. Notice that:

$$F(0) = -\delta(k-\delta+1)ab < 0, \quad F(1) = 0, \quad F'(0) = 0, \quad F'(1) = 0,$$

$$F''(0) = 2a\left[\delta(-2\delta+2k+1)b+2d\right] > 0,$$

$$F''(1) = 8\delta\left\{\delta d\sigma + \mu\left[\delta+\sigma(-\delta+4k+2)-2k-1\right]\right\} > 0. \tag{B.8}$$

Then, given the six results in (B.8), the Intermediate Value Theorem establishes that $F(\phi)$ has at least one zero for $\phi \in (0, 1)$ for any k and any $\delta \in N_\delta$.

Next, we will demonstrate the uniqueness of the critical point for $\delta \in \{1, \ldots, 6\}$, by showing that $F(\phi)$ has exactly one zero for $\phi \in (0, 1)$ for each δ. This establishes that $\mathcal{S}^{FE}$ has at most one turning point (a minimum), which implies that the critical point is unique and corresponds to a local sustain point. The proof consists on using Descartes' rule of signs after reordering the monomials of $F(\phi)$ and determine the maximum number of positive real roots of $F(\phi)$. For this purpose, we compute $F_j(\phi) = F(\phi)|_{\delta=j}$ $(j = 1, \ldots, 6)$:

$$F_1(\phi) = \left(\phi^2 - 1\right)^2 \left\{\phi^2 c\,[k(\mu + \sigma) + \sigma] - kab\right\},$$

$$F_2(\phi) = 2\left(\phi^2 - 1\right)^2 \left\{\phi^4 c\,[k(\mu + \sigma) + \sigma] + \phi^2 \mu a - (k - 1)ab\right\},$$

$$F_3(\phi) = \left(\phi^2 - 1\right)^2 \left\{3\phi^6 c\,[k(\mu + \sigma) + \sigma] + \phi^4 a(3\mu + d) + \phi^2 a\langle 2\mu - b\rangle - 3(k - 2)ab\right\},$$

$$F_4(\phi) = 2\left(\phi^2 - 1\right)^2 \left\{2\phi^8 c\,[k(\mu + \sigma) + \sigma] + \phi^6 a(2\mu + d) + 2\phi^4 \mu a + \phi^2 a\langle \mu - b\rangle - 2(k - 3)ab\right\},$$

$$\begin{aligned} F_5(\phi) = \left(\phi^2 - 1\right)^2 \Big\{&5\phi^{10} c\,[k(\mu + \sigma) + \sigma] + \phi^8 a(5\mu + 3d) + \phi^6 a(5\mu + d) \\ &+ \phi^4 a\langle 4\mu - b\rangle + \phi^2 a\langle 2\mu - 3b\rangle - 5(k - 4)ab\Big\}, \end{aligned}$$

$$\begin{aligned} F_6(\phi) = 2\left(\phi^2 - 1\right)^2 \Big\{&3\phi^{12} c\,[k(\mu + \sigma) + \sigma] + \phi^{10} a(3\mu + 2d) + \phi^8 a(3\mu + d) + 3\phi^6 \mu a \\ &+ \phi^4 a\langle 2\mu - b\rangle + \phi^2 a\langle \mu - 2b\rangle - 3(k - 5)ab\Big\}, \end{aligned}$$

where $\langle\cdot\rangle$ denotes a term that can take both signs dependent on the values of the parameters μ and $b = \sigma - \mu - 1$. We focus on the polynomial of ϕ^2 in the curly bracket $\{\cdot\}$ and, in turn, to note that the sign change of the coefficients of the polynomial occurs only once for each F_j $(j \in \{1, \ldots, 6\})$. Thus, $F_j = F(\phi)|_{\delta=j}$ has one zero for $\delta \in \{1, \ldots, 6\}$ by Descartes' rule of signs. This concludes the proof.

Note that, for F_1, the coefficient for ϕ^2 is positive (+) and the coefficient for the constant term is negative (−); accordingly, the sign changes once from positive to negative. For F_3, the signs for the coefficients are +, +, ±, −; accordingly, the sign change occurs once ($\langle\cdot\rangle$ denotes a term that becomes positive or negative dependent on the values of the parameters therein). For F_5, although the two terms $\langle 4\mu - b\rangle$ and $\langle 2\mu - 3b\rangle$ can take both signs, $\langle 2\mu - 3b\rangle$ becomes negative first as b increases since $\langle 4\mu - b\rangle > \langle 2\mu - 3b\rangle$. Hence, the sign change occurs once for any values of μ and b. Other cases of F_2, F_4, and F_6 can be treated similarly.

Although we cannot prove the uniqueness for a general δ, an examples with a large values of $\delta = 10$ can be provided in the following conjecture. For exceedingly high values of δ (like 100 or 1000), it is possible, but extremely cumbersome to demonstrate.

Conjecture 1. *The uniqueness of a local sustain point holds for any k and for any $\delta \in N_\delta$ in the FE model.*

Proof. Substituting arbitrarily large values for δ by increasing order, for any $k \geq \delta$, it seems to be always possible to factor out the term $\left(\phi^2 - 1\right)^2$ that multiplies a polynomial whose coefficients only change sign once, after rearranging all the monomials by decreasing order. For the purpose of illustration, we set $\delta = 10$:

$$\begin{aligned} F_{10}(\phi) = 2\left(\phi^2 - 1\right)^2 \Big\{ & 5\phi^{20} c\left[k(\mu + \sigma) + \sigma\right] + \phi^{18} a(5\mu + 4d) + \phi^{16} a(5\mu + 3d) \\ & + \phi^{14} a(5\mu + 2d) + \phi^{12} a(5\mu + d) + 5\phi^{10}\mu a + \phi^{8} a\langle 4\mu - b\rangle \\ & + \phi^{6} a\langle 3\mu - 2b\rangle + \phi^{4} a\langle 2\mu - 3b\rangle + \phi^{2} a\langle \mu - 4b\rangle - 5(k-9)ab \Big\}. \end{aligned}$$

Although there are several terms that can take both signs, since $\langle 4\mu - b\rangle > \langle 3\mu - 2b\rangle > \langle 2\mu - 3b\rangle > \langle \mu - 4b\rangle$, the sign change occurs once for any values of μ and b. Thus, $F_{10}(\phi)$ has one zero for $\phi \in (0, 1)$. □

Stability of full agglomeration (Proof of Proposition 7(i)): From $\mathcal{S}^{FE} < 0$ for $\phi = 1 - \epsilon$ in (B.7), with $\epsilon > 0$ low enough, we see that the full agglomeration is stable in the neighborhood of $\phi = 1$ and becomes unstable at a local sustain point ϕ_i^{s} for some place i. Such local sustain point serves as the global sustain point and is defined as $\phi^{\text{s}} = \max_i \phi_i^{\text{s}}$. The full agglomeration is stable for $\phi \in (\phi^{\text{s}}, 1)$. If each local sustain point is unique, the full agglomeration is unstable for $\phi \in (0, \phi^{\text{s}})$.

Appendix B.2. The PF model

Derivation of v_i (Proof of (8)): We have $b_i = 1 + \lambda_i$. The wage in the core region at $i = k$ is now given by: $w_k = \dfrac{\alpha}{\sigma}(2k + 2)$. The indirect utility in the core is given by:

$$v_k = \frac{\alpha}{\sigma}(2k + 2) - \alpha + \bar{A}. \tag{B.9}$$

With the use of $b_k = 2$ in (B.3), the nominal wage in region $i \neq k$ is obtained as

$$w_i = \frac{\alpha}{\sigma}\left[\phi^{\delta}(k + 2) + \phi^{-\delta}(k - \delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1}\right].$$

The indirect utility of peripheries at $i = k \pm \delta$ $(1 \leq \delta \leq k)$ is given by

$$v_i = v_{k\pm\delta} = \frac{\alpha}{\sigma}\left[\phi^{\delta}(k + 2) + \phi^{-\delta}(k - \delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1}\right] + \frac{\alpha\delta}{\sigma - 1}\ln\phi - \alpha + \bar{A}. \tag{B.10}$$

Existence of local sustain point (Proof of Proposition 6(i)): Recall that $\sigma - 1 > 0$. The sustain point must satisfy $\mathcal{S}^{PF} \equiv v_{k\pm\delta} - v_k = 0$ for some δ (≥ 1). From (B.9) and (B.10), we have

$$\mathcal{S}^{PF} = \frac{\alpha}{\sigma}\left[\phi^{\delta}(k + 2) + \phi^{-\delta}(k - \delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1}\right] + \frac{\alpha\delta}{\sigma - 1}\ln\phi - \frac{\alpha}{\sigma}(2k + 2). \tag{B.11}$$

Notice that

$$\lim_{\phi\to+0}\mathcal{S}^{PF} = +\infty, \qquad \lim_{\phi\to 1}\mathcal{S}^{PF} = 0. \tag{B.12}$$

Differentiating $\mathcal{S}^{PF}$ with respect to ϕ, we get

$$\frac{\partial \mathcal{S}^{PF}}{\partial \phi} = \frac{\alpha\, \phi^{-\delta}\,(B_1 - B_2)}{(\sigma-1)\sigma\,(\phi^2-1)^2\,\phi},$$

where:

$$B_1 = \delta\sigma\left(\phi^4+1\right)\phi^{\delta} - (\sigma-1)\phi^2\left[\delta(2\delta-2k-1)-2\right] + \delta(\sigma-1)\phi^{2\delta}\left[(k+2)\phi^4+k+1\right] > 0,$$

$$B_2 = 2\delta\sigma\phi^{\delta+2} + (\sigma-1)\phi^{2\delta+2}\left[\delta(2k+3)+2\right] + \delta(\sigma-1)\left[-\delta+\phi^4(k-\delta)+k+1\right] > 0.$$

The zeros of the derivative depend on the zeros of $B_1 - B_2$. First, notice that:

$$\lim_{\phi\to 1}\frac{\partial \mathcal{S}^{PF}}{\partial \phi} = \frac{\delta\alpha\left[\delta(\sigma-1)+2\sigma-1\right]}{(\sigma-1)\sigma} > 0.$$

This, together with $\lim_{\phi\to 1}\mathcal{S}^{PF} = 0$ in (B.12), means that

$$\mathcal{S}^{PF} < 0 \quad \text{for } \phi = 1-\epsilon, \tag{B.13}$$

with $\epsilon > 0$ arbitrarily small, while $\mathcal{S}^{PF} > 0$ for $\phi = \epsilon$. Thus, by the Intermediate Value Theorem, there exists at least one sustain point $\phi \in (0,1)$ such that $\mathcal{S}^{PF}(\phi) = 0$.

Uniqueness of local sustain point (Proof of Proposition 6(i)): We demonstrate the uniqueness of the critical point. Let us define $G(\phi) \equiv B_1 - B_2$. Then we have

$$G(0) = -\delta(\sigma-1)(-\delta+k+1) < 0, \quad G(1) = 0, \quad G'(0) = 0, \quad G'(1) = 0,$$

$$G''(0) = 2(\sigma-1)\left(-2\delta^2+\delta+2\delta k+2\right) > 0, \quad G''(1) = 8\delta(\delta(\sigma-1)+2\sigma-1) > 0.$$

Then, given these six results, the Intermediate Value Theorem establishes that $G(\phi)$ has at least one zero for $\phi \in (0,1)$, for any k and any $\delta \in N_\delta$.

Next, we will demonstrate the uniqueness of the critical point for $\delta \in \{1,\ldots,6\}$, by showing that $G(\phi)$ has exactly one zero for $\phi \in (0,1)$. This establishes that $\mathcal{S}^{PF}$ has at most one turning point (a minimum), which implies that the critical point is unique and corresponds to a local sustain point. The proof consists on using Descartes' rule of signs after reordering the monomials of $G(\phi)$ and determine the maximum number of positive real roots of $G(\phi)$. We compute $G_j(\phi) = G(\phi)|_{\delta=j}$ $(j = 1, \ldots, 6)$:

$$
\begin{aligned}
G_1(\phi) =& \left(\phi^2 - 1\right)^2 (\sigma - 1)\left[(k+2)\phi^2 + \frac{\sigma}{\sigma - 1}\phi - k\right],\\
G_2(\phi) =& 2\left(\phi^2 - 1\right)^2 (\sigma - 1)\left[(k+2)\phi^4 + \frac{\sigma}{\sigma - 1}\phi^2 - (k-1)\right],\\
G_3(\phi) =& \left(\phi^2 - 1\right)^2 (\sigma - 1)\left[3(k+2)\phi^6 + \phi^4 + \frac{3\sigma}{\sigma - 1}\phi^3 - \phi^2 - 3(k-2)\right],\\
G_4(\phi) =& 2\left(\phi^2 - 1\right)^2 (\sigma - 1)\left[2(k+2)\phi^8 + \phi^6 + \frac{2\sigma}{\sigma - 1}\phi^4 - \phi^2 - 2(k-3)\right],\\
G_5(\phi) =& \left(\phi^2 - 1\right)^2 (\sigma - 1)\left[5(k+2)\phi^{10} + 3\phi^8 + \phi^6 + \frac{5\sigma}{\sigma - 1}\phi^5 - \phi^4 - 3\phi^2 - 5(k-4)\right],\\
G_6(\phi) =& 2\left(\phi^2 - 1\right)^2 (\sigma - 1)\left[3(k+2)\phi^{12} + 2\phi^{10} + \phi^8 + \frac{3\sigma}{\sigma - 1}\phi^6 - \phi^4 - 2\phi^2 - 3(k-5)\right].
\end{aligned}
$$

We see that the coefficients of the polynomial for each $G_i(\phi)$ changes sign once. Thus, $G(\phi)$ has one zero for $\phi \in (0, 1)$ by Descartes' rule of signs.

Conjecture 2. *The uniqueness of a local sustain point holds for any k and any $\delta \in N_\delta$ in the PF model.*

Proof. Substituting arbitrarily large values for δ by increasing order, for any $k \geq \delta$, it seems to be always possible to factor out the term $\left(\phi^2 - 1\right)^2$ that multiplies a polynomial whose coefficients only change sign once, after rearranging all the monomials by decreasing order. For the purpose of illustration, we set $\delta = 20$:

$$
\begin{aligned}
G_{20}(\phi) =& 2\left(\phi^2 - 1\right)^2 (\sigma - 1)\left[10(k+2)\phi^{40} + 9\phi^{38} + 8\phi^{36} + 7\phi^{34} + 6\phi^{32} + 5\phi^{30} + 4\phi^{28}\right.\\
&+ 3\phi^{26} + 2\phi^{24} + \phi^{22} + \frac{10\sigma}{\sigma - 1}\phi^{20} - \phi^{18} - 2\phi^{16} - 3\phi^{14}\\
&\left.- 4\phi^{12} - 5\phi^{10} - 6\phi^8 - 7\phi^6 - 8\phi^4 - 9\phi^2 - 10(k-19)\right].
\end{aligned}
$$

One can see that the polynomial in the brackets changes sign only once. Thus, $G(\phi)$ has one zero for $\phi \in (0, 1)$. This concludes the proof. □

Stability of full agglomeration (Proof of Proposition 7(i)): The proof is similar to that for the FE model.

Appendix B.3. The PFSU model

Derivation of v_i (Proof of (9)): As shown by Akamatsu et al. (2021), the nominal wage in the PFSU model is just the same as in the PF model. The indirect utility becomes

$$v_i(\boldsymbol{\lambda}) = w_i(\boldsymbol{\lambda}) + \frac{\alpha}{\sigma - 1} \ln \Delta_i(\boldsymbol{\lambda}) - \gamma \ln(\lambda_i + 1),$$

where $\Delta_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} \phi_{ij}\lambda_j$ and $w_i(\boldsymbol{\lambda})$ is given by (1). Further, we have: $b_i = l + \lambda_i$. Since we keep $l = 1$ for simplicity, we have $b_i = 1 + \lambda_i$. The indirect utilities are given by $(1 \leq \delta \leq k)$

$$v_k = \frac{\alpha}{\sigma}(2k+2) - \gamma \ln 2; \quad v_i = v_{k\pm\delta} = \frac{\alpha}{\sigma}\left[\phi^{\delta}(k+2) + \phi^{-\delta}(k-\delta) + \frac{\phi^{\delta} - \phi^{-\delta}}{\phi^2 - 1}\right] + \frac{\alpha\delta}{\sigma - 1}\ln\phi.$$

Existence of local sustain point (Proof of Proposition 6(ii)): The sustain point must satisfy $\mathcal{S}^{PFSU} \equiv v_{k\pm\delta} - v_k = 0$ for some $\delta\ (\geq 1)$ with

$$\mathcal{S}^{PFSU} \equiv v_i - v_k = \mathcal{S}^{PF} + \gamma \ln 2 \tag{B.14}$$

with $\mathcal{S}^{PF}$ in (B.11) and the second term is a constant that does not depend on ϕ. We consider the case of $\delta \in \{1, \ldots, 6\}$ where $\mathcal{S}^{PF}$ has a unique zero at $\phi = \phi^{\mathrm{s}}$ with $\mathcal{S}^{PF} < 0$ for $\phi \in (\phi^{\mathrm{s}}, 1)$ and $\mathcal{S}^{PF} > 0$ for $\phi \in (0, \phi^{\mathrm{s}})$ (cf., Appendix B.2). An increase in the value of the parameter γ shifts $\mathcal{S}^{PF}$ upwards up to a point where $\mathcal{S}^{PFSU}$ may have two, one, or no zeros.

Non-existence of sustain point (Proof of Proposition 7(ii-1)) First, we consider the case of $\gamma \to +\infty$. From (B.14), it is apparent that $\mathcal{S}^{PFSU} > 0$ for any i and ϕ. Hence there is no sustain point and the full agglomeration is always unstable.

Next, we consider the case of $k \to +\infty$ and show that a sustain point does not exist and the full agglomeration is always unstable. To prove these, it is sufficient to show $v_{k-1} - v_k > 0$ for $\phi \in (0,1)$. For $i = k-1$ ($\delta = 1$), (B.14) with (B.11) gives

$$\mathcal{S}^{PFSU}\Big|_{\delta=1} = \frac{\alpha}{\sigma}\Big[k(\phi + \phi^{-1} - 2) + 2\phi - 2\Big] + \frac{\alpha}{\sigma - 1}\ln\phi + \gamma\ln 2. \tag{B.15}$$

Since we have $\phi + \phi^{-1} - 2 = \phi^{-1}(\phi - 1)^2 > 0$ for $\phi \in (0,1)$, we can show:

$$\lim_{k\to+\infty}(v_{k-1} - v_k) = +\infty \quad \forall\phi \in (0,1). \tag{B.16}$$

Stability of full agglomeration (Proof of 7(ii-2)): If there are two zeros $\mathcal{S}^{PFSU}$ in $\phi \in (0,1)$ for each i, we have two local sustain points, $\phi_i^{s1} \in (0,1)$ and $\phi_i^{s2} \in (0,1)$ with $\phi_i^{s2} > \phi_i^{s1}$. Then the full agglomeration is unstable for $\phi \in \Big\{(0,\phi_1^s) \cup (\phi_2^s, 1)\Big\}$. If we introduce

$$\phi^{s1} = \max_i \phi_i^{s1} \in (0,1), \quad \phi^{s2} = \min_i \phi_i^{s2} \in (0,1),$$

we note that the full agglomeration is unstable for $\phi \in \{(0,\phi_{s1}) \cup (\phi_{s2}, 1)\}$. If $\phi_{s1} > \phi_{s2}$, there is no stable zone of ϕ. If $\phi_{s1} < \phi_{s2}$, there is a stable zone of $\phi \in (\phi_{s1}, \phi_{s2})$.

Appendix B.4. The MT model

Derivation of v_i (Proof of (10)**):** In the MT model, we have the wage equation:

$$w_i L_i = \sum_{j=0}^{n} \frac{\phi_{ij} L_i w_i^{1-\sigma}}{\sum_{m=0}^{n} L_m w_m^{1-\sigma}\phi_{jm}} L_j w_j,$$

where $\rightarrow L_i = \lambda_i(1-\theta\lambda_i)$. Let us normalize $\sum_j \lambda_j w_i = w_k$, where w_k is the wage in the central region at full agglomeration. The wage at a potential city i for any L_i is given by:

$$w_i^\sigma = \sum_{j=0}^{n} \frac{\phi_{ij} L_j w_j}{\sum_{m=0}^{n} L_m w_m^{1-\sigma} \phi_{jm}} \iff w_i = \left(\sum_{j=0}^{n} \frac{\phi_{ij} L_j w_j}{\sum_{m=0}^{n} L_m w_m^{1-\sigma} \phi_{jm}} \right)^{1/\sigma}.$$

At the full agglomeration, for which the locations $i \neq k$ have zero population, the wage becomes $w_i = \phi^{\frac{\delta}{\sigma}} w_k$. The indirect utility is given by $v_i = \zeta \Delta_i^{\frac{1}{\sigma-1}} y_i$, where $y_i = (1-\theta\lambda_i)w_i$, $\Delta_i = \sum_{j=0}^{n} L_j w_j^{1-\sigma} \phi_{ij}$ and $\zeta > 0$ is a constant. Using $w_i = \phi^{\frac{\delta}{\sigma}} w_k$, we get the indirect utility:

$$v_k = \zeta(1-\theta)^{\frac{\sigma}{\sigma-1}}; \quad v_i = v_{k\pm\delta} = \zeta(1-\theta)^{\frac{1}{\sigma-1}} \phi^{\frac{\delta}{\sigma}\frac{2\sigma-1}{\sigma-1}}, \quad 1 \leq \delta \leq k.$$

Unique existence of sustain point (Proof of Propositions 6(iii) and 7(iii)): We have

$$\mathcal{S}^{MT} \equiv v_{k\pm\delta} - v_k = (1-\theta)^{\frac{1}{\sigma-1}} \phi^{\frac{\delta}{\sigma}\frac{2\sigma-1}{\sigma-1}} - (1-\theta)^{\frac{\sigma}{\sigma-1}}. \tag{B.17}$$

We have $\mathcal{S}^{MT} < 0$ for $\phi = 0$ and $\mathcal{S}^{MT} > 0$ for $\phi = 1$. The solution to $\mathcal{S}^{MT} = 0$ with (B.17) is

$$\phi = \phi_\delta^c \equiv (1-\theta)^{\frac{(\sigma-1)\sigma}{\delta(2\sigma-1)}}. \tag{B.18}$$

The critical point is uniquely determined or each δ so that the local sustain point is also uniquely determined. Further, notice that

$$\frac{\partial \phi_\delta^c}{\partial \delta} = -\frac{(\sigma-1)\sigma \ln(1-\theta)(1-\theta)^{-\frac{(\sigma-1)\sigma}{\delta-2\delta\sigma}}}{\delta^2(2\sigma-1)} > 0,$$

which means that ϕ_δ^c is strictly increasing in δ. Since full agglomeration is stable below the sustain point, we have that $\delta_{sat} = 1$ and

$$\phi = \phi_s \equiv (1-\theta)^{\frac{(\sigma-1)\sigma}{2\sigma-1}}. \tag{B.19}$$

The sustain point in terms of commuting costs is given by:

$$\theta_s = 1 - \phi^{\frac{1}{\sigma}\frac{2\sigma-1}{\sigma-1}}.$$