Evolutionary Stability of Behavioural Rules

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Abstract

I develop a notion of evolutionary stability of behavioural rules when individuals simultaneously interact in a family of strategic games. An individual's strategy choice is determined by his behavioural rule which may take into account the manner in which the games have been played in the past. The payoffs obtained by individuals following a particular behavioural rule determine that rule's fitness. A population is stable if, whenever some individuals from an incumbent behavioural rule mutate and follow a mutant behavioural rule, the fitness of each incumbent behavioural rule exceeds that of the mutant behavioural rule. The behavioural rules approach thus conceptualises stability when individuals simultaneously interact in a variety of strategic environments. I first show the lack of stability whenever individuals exhibit heterogeneity in their behavioural rules. Furthermore, when all individuals follow the same behavioural rule, I find that the behavioural rules approach to stability is a refinement of the evolutionary stability of strategies approach in that the necessary condition for stability of behavioural rules is stronger than the corresponding condition for evolutionary stability of strategies. Finally, I present a sufficient condition for stability that is reasonably close to the necessary condition alluded to above.

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1 Introduction

A basic tenet of evolution is the selection of the fitter over the less fit. In the context of decision-making in game-theoretic strategic situations, this principle has traditionally been expressed in terms of evolutionary stable strategies (abbreviated as ESS). A game is a representation of a strategic situation that is defined by three elements: *(i)* the set of players, *(ii)* each player's strategy set, which specifies the strategies that may chosen by the player, and *(iii)* a payoff function that describes the payoff received by each player corresponding to each combination of strategies chosen by the players. A particular strategy (or a particular mix of strategies) is said to be evolutionarily stable if it is able to withstand any mutant strategy in the sense of being fitter (i.e. obtaining a higher payoff) than the mutant strategy. The interpretation is that if an ESS is adopted by a population of players, then it is not possible for any other mutant strategy to invade the population.

In contrast to the ESS framework where each individual is associated with a particular strategy, I forward a notion of evolutionary stability where each individual is associated with a behavioural rule, and stability is based on the fitness of behavioural rules. These individuals simultaneously interact in a family of strategic games, and an individual's behavioural rule determines his strategy choice in each game, possibly taking into consideration the manner in which the games have unfolded in the past. Examples of behavioural rules include playing a best-response to some empirical distribution of strategies played in the past, imitation of most successful/popular strategies, or choosing strategies from the strategy set according to some probability distribution. I define a particular behavioural rule (or, combination of behavioural rules) to be evolutionarily stable if its (their) fitness exceeds that of any mutant behavioural rule – the interpretation is that a stable population cannot be invaded by any other mutant behavioural rule. Thus, the behavioural rules approach conceptualises stability when individuals simultaneously interact in a variety of strategic environments.

I show that if the individuals display heterogeneity in their behavioural rule, then the population is unstable. Next, when all individuals follow the same behavioural rule, I find that stability obtains only if the behavioural rule leads all individuals to choose exactly one strategy in each game, and, in addition, the condition on this strategy is stronger than the corresponding condition for it to be an evolutionary stable strategy of the game in question. Thus, stability of behavioural rules leads to a refinement of evolutionary stable strategies. Finally, I present a sufficient condition for stability that is 'reasonably' close to the necessary condition alluded to above.

The primary contribution of this paper is to develop a framework for evolutionary stability in the context of two novel features: firstly, the focus on behavioural rules, and secondly, the interaction environment wherein individuals may play multiple games. Evidently, the behavioural rules approach in this paper generalises the ESS approach pioneered by Smith and Price (1973) – particularly useful expositions on the properties and applicability of the ESS framework include Weibull (1995), Samuelson (1997), and Sandholm (2010). Existing papers that examine the interaction between individuals who use different behavioural rules include Kaniovski, Kryazhimskii and Young (2000), Juang (2002), Josephson (2009) and Khan (2021b); however, the aim of these papers is not to study the stability properties of the behavioural rules but rather, to examine the outcomes that obtain when individuals display heterogeneity in their decision-making process.

On the other hand, the feature of individuals interacting in a number of games is generally understudied relative to how natural and commonplace this appears to be in practical settings. There is, however, an existing body of work which examines situations where individuals, due to reasons of complexity or incomplete information, play a particular game by extrapolating their experience in similar games. For instance, LiCalzi (1995) studies convergence of fictitious play when individuals possibly play different games on the basis of prior experience with similar games, Steiner and Stewart (2008) focus on learning when beliefs are formed through extrapolation, and Mengel (2012) compares learning across games to learning in a single game.

The most closely related paper is Khan (2021a) which uses the evolutionary stability of behavioural rules approach in a bargaining game; in contrast, in this paper, not only do I substantially generalise this framework to any game but also conceptualise stability when individuals simultaneously interact in a number of different games.

The plan of the paper is as follows. I introduce the framework in Section 2, define evolutionary stability of behavioural rules in Section 3, present the results in Section 4, and conclude in Section 5.

2 The Framework

There is a population of unit mass, and individuals in this population interact in a family of strategic environments. The strategic situations under consideration are symmetric two player games with a finite strategy space. Consider such a game G with the identical finite strategy set for each player being denoted by $S_G = \{s^1, \ldots, s^{P(G)}\}$. The payoff function $\pi_G : S_G \times S_G \to \mathbb{R}$ maps from the set of pure strategy combinations that can be used by two players in the game G to the real line. The payoff received by playing a pure strategy $s^i \in S_G$ against a pure strategy $s^j \in S_G$ in the game G is denoted by $\pi_G(s^i, s^j)$.

A strategy $s^j \in S_G$ is a *pure strategy best-response* to the strategy $s^i \in S_G$ if, for all $s^k \in S_G$, the inequality $\pi_G(s^j, s^i) \geq \pi_G(s^k, s^i)$ holds. The *set of pure strategy best-responses* to $s^i \in S_G$ in the game G is denoted by $BR_G(s^i)$, where $BR_G(s^i) = \{s^j \in S_G : \forall s^k \in S_G$, it holds that $\pi_G(s^j, s^i) \geq \pi_G(s^k, s^i)\}$. The finiteness of the strategy set S_G implies that, for each $s^i \in S_G$, the set $BR_G(s^i)$ is always non-empty.

A strategy combination (s^i, s^j) is a *pure strategy Nash equilibrium* of the game G if $s^i \in BR_G(s^j)$ and $s^j \in BR_G(s^i)$. That is, in a Nash equilibrium, none of the two players experience an improvement in the payoff from a unilateral deviation from the chosen strategy. The strategy combination (s^i, s^i) is a *symmetric pure strategy Nash equilibrium* if $s^i \in BR_G(s^i)$; in this case, s^i is said to support a symmetric pure strategy Nash equilibrium of the game G.

A strategy $s^i \in S_G$ is an *evolutionary stable strategy* (ESS) of the game G if *(i)* $s_i \in BR_G(s^i)$, and *(ii)* for any $s^j \in BR_G(s^i) \setminus \{s^i\}$, the inequality $\pi_G(s^i, s^j) > \pi_G(s^j, s^j)$ holds. So, an evolutionary stable strategy always supports a symmetric pure Nash equilibrium.

A *mixed strategy* in the game G is a probability vector $p = (p^1, \ldots, p^{P(G)})$ such that, for any $i \in \{1, \ldots, P(G)\}$, $p^i \geq 0$ denotes the probability of playing strategy $s^i \in S_G$, and $\Sigma_{i=1}^{P(G)} p^i = 1$. The set of all mixed strategies of the game G is represented by Δ_G. The payoff obtained by a pure strategy $s^i \in S_G$ against a mixed strategy $p \in \Delta_G$ is given by $\pi_G(s^i, p) = \Sigma_{i=1}^{P(G)} p^j \pi_G(s^i, s^j)$.

I now discuss one implication of evolutionary stable strategies. Consider the set of mixed strategies $\Delta_{G|i,j} \subset \Delta_G$ where all the pure strategies apart from s^i and s^j are played with zero probability. That is, $\Delta_{G|i,j} = \{p \in \Delta_G : p^i + p^j = 1\}$. Also, consider the sets of mixed strategies $\Delta_{G|i<x,j} \subset \Delta_{G|i,j}$, and $\Delta_{G|i>x,j} \subset \Delta_{G|i,j}$, where the probability of playing the pure strategy s^i is at most x, and at least x, respectively. So, $\Delta_{G|i<x,j} = \{p \in \Delta_{G|i,j} : p^i < x\}$, and $\Delta_{G|i>x,j} = \{p \in \Delta_{G|i,j} : p^i > x\}$. Then, one obtains the lemma below – I refer to the appendix for a proof of the lemma.

Lemma 1. *Consider any two distinct evolutionarily stable strategies s^i and s^j in S_G. Then there exists $\bar{p}^i \in (0, 1)$ such that, for every $p \in \Delta_{G|i<\bar{p}^i,j}$ and every $p' \in \Delta_{G|i>\bar{p}^i,j}$, the inequalities $\pi_G(s^i, p) < \pi_G(s^j, p)$ and $\pi_G(s^j, p') < \pi_G(s^i, p')$ hold.*

Thus, the payoff obtained by the evolutionarily stable strategy s^i (similarly, s^j) is higher than that obtained by the evolutionarily stable strategy s^j (similarly, s^i) against each mixed strategy in $\Delta_{G|i,j}$ where s^i is played with probability greater (less) than a threshold $\bar{p}^i \in (0,1)$. I illustrate this in the example below.

Example 1. In the game G below, the strategy set $S_G = \{s^1, s^2, s^3\}$.

	s^1	s^2	s^3
s^1	2, 2	1, 0	1, 0
s^2	0, 1	3, 3	1, 0
s^3	0, 1	0, 1	1, 1

Figure 1

In this game, it is easily verified that both s^1 and s^2 are evolutionarily stable strategies. In the context of Lemma 1 above, let $p = (p^1, p^2, p^3)$ denote any mixed strategy such that $p^1 + p^2 = 1$. Then, $\pi_G(s^1, p) = 2p^1 + p^2 = 1 + p^1$, and $\pi_G(s^2, p) = 3p^2 = 3 - 3p^1$. Then, $\pi_G(s^1, p) > \pi_G(s^2, p)$ if and only if $p^1 > 0.5$, and $\pi_G(s^1, p) < \pi_G(s^2, p)$ if and only if $p^1 < 0.5$. That is, for every $p \in \Delta_{G|i<0.5,j}$ and every $p' \in \Delta_{G|i>0.5,j}$, the inequalities $\pi_G(s^1, p) < \pi_G(s^2, p)$ and $\pi_G(s^2, p') < \pi_G(s^1, p')$ hold. ■

I will now describe the structure of interaction in each time period, and then illustrate the same with an example. I consider a finite set of games $\mathcal{G}$ whose power set is denoted by $\mathcal{P}(\mathcal{G})$. A subset $\mathcal{P}_t(\mathcal{G})$ of $\mathcal{P}(\mathcal{G})$ gives the collection of games that are relevant for time period t, and I remain agnostic over the manner in which the set $\mathcal{P}_t(\mathcal{G})$ is determined. In time period t, the individuals in the population play a set of games $\mathcal{G}_t \in \mathcal{P}_t(\mathcal{G})$. For any time period t, the only restriction on $\mathcal{P}_t(\mathcal{G})$ is that each game in $\mathcal{G}$ belongs to at least one element in $\mathcal{P}_t(\mathcal{G})$.

Example 2. Consider a finite set of games $\mathcal{G} = \{G_1, G_2, G_3\}$. The power set of $\mathcal{G}$ is $\mathcal{P}(\mathcal{G}) = \{\{G_1\}, \{G_2\}, \{G_3\}, \{G_1, G_2\}, \{G_1, G_3\}, \{G_2, G_3\}, \{G_1, G_2, G_3\}\}$. I now illustrate two selection rules that determine, in any time period t, the set $\mathcal{P}_t(\mathcal{G})$:

(a) Suppose that the structure of interactions in any period t is such that individuals must interact in exactly two games. Then, $\mathcal{P}_t(\mathcal{G}) = \{\{G_1, G_2\}, \{G_1, G_3\}, \{G_2, G_3\}\}$, and the set of games played in period t, namely $\mathcal{G}_t$ is an element of $\mathcal{P}_t(\mathcal{G})$.

(b) Suppose that in each time period t where t is even, individuals interact in exactly two games, and, in addition, G_2 must always be played; when t is odd, then individuals play exactly one game. Then, in any even time period t, $\mathcal{P}_t(\mathcal{G}) = \{\{G_1, G_2\}, \{G_2, G_3\}\}$, while for any odd time period t, $\mathcal{P}_t(\mathcal{G}) = \{\{G_1\}, \{G_2\}, \{G_3\}\}$, and $\mathcal{G}_t$ is an element in $\mathcal{P}_t(\mathcal{G})$.

In both cases (a) and (b), the only restriction imposed on $\mathcal{P}_t(\mathcal{G})$ is satisfied – each game in $\mathcal{G}$ belongs to at least one element in $\mathcal{P}_t(\mathcal{G})$. On the other hand, if for some time period t,

$\mathcal{P}_t(\mathcal{G}) = \{\{G_1\}, \{G_3\}, \{G_1, G_3\}\}$, then the restriction is not satisfied. ■

The population of unit mass is represented by individuals being uniformly distributed over the unit interval $[0, 1]$. Each individual is identified by his respective location on the unit interval. For a game $G \in \mathcal{G}_t$, the relative frequency of the individuals in the population playing a pure strategy $s^i \in S_G$ in time period t is denoted by $f^i_{G|t}$. In time period t, the vector of relative frequencies with which each strategy is played in the population, and in a subset A of individuals, is denoted by $f_{G|t}$ and $f_{G|A,t}$, respectively. I refer to $f_{G|t}$, and $f_{G|A,t}$, as the period t population strategy profile in the game G, and the period t strategy profile of the individuals in the set A in the game G, respectively. In order to simplify notation, I omit the time subscript whenever no confusion arises from doing so.

The pure strategy used by player $i \in [0, 1]$ in time period t in a game $G \in \mathcal{G}_t$ is represented by $s_{G|i,t}$. I clarify that in contrast to $s_{G|i,t}$, I use s^i to denote a particular strategy in the strategy set without any reference to which players choose that strategy. The payoff received by player i from playing this game with another player j is $\pi_G(s_{G|i,t}, s_{G|j,t})$. The total payoff of player i in game G in time period t on choosing $s_{G|i,t}$ when the population strategy profile is $f_{G|t}$ is given by $\pi_{G|i,t} = \Sigma_{j=1}^{P(G)} f^j_{G|t}\, \pi_G(s_{G|i,t}, s^j)$. Since the strategy set S_G is finite, the set of feasible payoffs is finite, and I represent the vector of relative frequencies of payoffs received by the players in the population in time period t by $\pi_{G|t}$. The entire history of strategy profiles and payoff profiles of each and every period till period t is denoted by $f_{G|1\to t}$ and $\pi_{G|1\to t}$.

In order to focus on the process of strategy choice, I assume $\mathcal{G}_1 = \mathcal{G}$, i.e. in the very first period, each game in $\mathcal{G}$ is played by the individuals in the population, and I describe the manner in which they choose their respective strategies in period $t+1$ (for any integer $t > 1$). Player i is associated with a *behavioural rule* R_i that, for each $G \in \mathcal{G}$, maps from the history of strategy and payoff profiles into a subset of the finite set of pure strategies S_G. This latter subset, denoted by $R_i(f_{G|1\to t}, \pi_{G|1\to t}) \subset S_G$, is the period $t+1$ *response set* of player i in game G. The strategy chosen by player i in period $t+1$ in the game G belongs to $R_i(f_{G|1\to t}, \pi_{G|1\to t})$, and each strategy in this set can be chosen. Hence, each strategy in $R_i(f_{G|1\to t}, \pi_{G|1\to t})$ is called a *feasible period* $t+1$ *strategy* for player i in the game G. A strategy profile (i.e. the vector of relative frequency of strategies in S_G) that can be generated by taking one such feasible period $t+1$ strategy for each individual in the population is defined as a *feasible period* $t+1$ *population strategy profile* of the game G. In the game G, a typical feasible period $t+1$ population strategy profile, and the set of all feasible period $t+1$ population strategy profiles, is denoted by $f_G^{t\to t+1}$ and $\Delta_G^{t\to t+1}$.

The behavioural rule of a player does not change across different games in a time period. The finite set of behavioural rules in use in the population at the end of time period t is denoted by $\mathcal{R}_t$. R^I (with an upper case subscript) denotes a typical behavioural rule in the population without any reference to the players using that rule while R_i (with a lower case subscript) refers to the behavioural rule of the player located at point i in $[0,1]$.

The population is *uniform at time period* $t+1$ if, for all $i, j \in [0,1], R_i(f_{G|1\to t}, \pi_{G|1\to t})$ $= R_j(f_{G|1\to t}, \pi_{G|1\to t})$ holds for all $G \in \mathcal{G}$. That is, the set of feasible period $t+1$ strategies is the same for all individuals in each game that may be played in period $t+1$. The population is said to be *diverse at time period* t otherwise. Hence, a sufficient (necessary) condition for a uniform (diverse) population at is that all individuals (not all individuals) in the population use the same rule. Some examples of behavioural rules are as follows:

(i) Best-response: An individual plays a best response to some strategy profile – this may be the strategy profile till date, or the strategy profile of the previous period, or the strategy profile of some selected time periods.

(ii) Imitation: An individual plays the strategy of the individual who received the highest payoff in some past period, or the highest average payoff in some selected time periods.

(iii) Stochastic play: An individual chooses a strategy from the strategy set according to some probability distribution.

For a behavioural rule R^I, SoR_t^I and $\mid SoR_t^I \mid$ are the set of players in the population, and the relative frequency of the players in the population, who play as per the behavioural rule R^I at the end of time period t. In the context of a game $G \in \mathcal{G}_t$, the period t strategy profile of the individuals in the set SoR_t^I, and their payoff profile, is given by $f_{G|t}^I$, and $\pi_{G|t}^I$, respectively. Hence $f_{G|t}^I$ is equivalent to $f_{G|SoR_t^I,t}$. A strategy profile (i.e. the vector of relative frequency of strategies in S_G) that can be generated by taking one feasible period $t+1$ strategy for each individual in SoR_t^I is referred to as a feasible period $t+1$ strategy profile for R^I in the game G. A typical feasible period $t+1$ strategy profile for R^I in the game G is denoted by $f_G^{I,t\to t+1}$. The set of all feasible period $t+1$ strategy profiles for the behavioural rule R^I is denoted by $\Delta_G^{I,t\to t+1}$. The example below illustrates the notion of feasible period $t+1$ strategy profiles.

Example 3. Consider the game below. The strategy set comprises of three strategies s^1, s^2, and s^3, and the payoff function is depicted via the payoff matrix below.

	s^1	s^2	s^3
s^1	$1,1$	$5,0$	$4,0$
s^2	$0,5$	$2,2$	$4,0$
s^3	$0,4$	$0,4$	$3,3$

Figure 2

Suppose that all individuals play strategy s^3 in the very first time period so that the population level strategy profile is $(0, 0, 1)$. Also suppose that the population has the following composition in terms of behavioural rules: each individual in the interval $[0, 0.5)$ plays a best-response to previous period's population level strategy profile, while each individual in the interval $[0.5, 1]$ imitates the strategy that has yielded the highest payoff in the previous time period. Then, in time period 2, the response set of each individual in the interval $[0, 0.5)$ is $\{s^1, s^2\}$, and the response set of each individual in the interval $[0.5, 1]$ is $\{s^3\}$. As a result, the feasible period 2 strategy profiles for the sub-population $[0, 0.5)$ is given by $(x, 1-x, 0)$ where x takes any value in $[0, 1]$, while the feasible period 2 strategy profiles for the sub-population $[0.5, 1]$ is given by $[0, 0, 1]$. Finally, the feasible period 2 strategy profile for the entire population $[0, 1]$ is $[y, 0.5 - y, 0.5]$ where y takes any value between $[0, 0.5)$. ■

The *game specific fitness* of a behavioural rule $R^I \in \mathcal{R}_t$ in time period t in a game $G \in \mathcal{G}_t$ is determined by a real valued *fitness function* $F^I_{G|t}$ that maps from the period t payoff profile $\pi_{G|t}$ to the set of real numbers. So, $F^I_{G|t}(\pi_{G|t})$ is the game specific fitness of the rule R^I in time period t in the game $G \in \mathcal{G}_t$. For ease of notation, I drop the argument of the game specific fitness and refer to it by $F^I_{G|t}$. For the rule $R^I \in \mathcal{R}_t$, the vector of game specific fitness in all games in $\mathcal{G}_t$ is denoted by $(F^I_{G|t})_{G \in \mathcal{G}_t}$. The collection of game specific fitness of all behavioural rules in $\mathcal{R}_t$ in all games in $\mathcal{G}_t$ is denoted by $(F^I_{G|t})^{R^I \in \mathcal{R}_t}_{G \in \mathcal{G}_t}$.

Finally, the *fitness* of a behavioural rule $R^I \in \mathcal{R}_t$ in time period t is given by a real valued *aggregate fitness function* $F^I_t((F^I_{G|t})^{R^I \in \mathcal{R}_t}_{G \in \mathcal{G}_t})$. This is the fitness of R^I across all games in this time period – I drop the argument to simplify notation and simply write F^I_t to refer to this fitness level. The criterion for evolutionary stability of behavioural rules that I define in the next section is based on this fitness level.

I underline that both the game specific fitness function and the aggregate fitness function defined above may differ across time periods, across games, and across behavioural rules as well. In context of the game specific fitness functions, the only assumption I make is as follows. I denote the relative frequency distribution of the payoffs obtained in game $G \in \mathcal{G}_t$ by individuals following the behavioural rule R^I in time period t by $\underline{F}^I_{G|t}(\cdot)$. If, in time period t, the relative frequency distribution of payoffs of one particular behavioural rule first order stochastically dominates (strictly first order stochastically dominates) that of another behavioural rule, then the time period t game specific fitness of the former rule is at least as much as (higher than) the latter. That is, if $\underline{F}^I_{G|t}(x) \leq \underline{F}^J_{G|t}(x)$ holds for each real number x, then $F^I_{G|t} \geq F^J_{G|t}$, and if, in addition, $\underline{F}^I_{G|t}(x) < \underline{F}^J_{G|t}(x)$ holds for some real number x, then $F^I_{G|t} > F^J_{G|t}$.

I also make a similar assumption in case of the aggregate fitness function. The relative frequency distribution of the game specific fitness of the behavioural rule R^I in time period t is denoted by $\underline{F}^I_t(\cdot)$. Then, if $\underline{F}^I_t(x) \leq \underline{F}^J_t(x)$ holds for each real number x, then $F^I_t \geq F^J_t$, and if, in addition, $\underline{F}^I_t(x) < \underline{F}^J_t(x)$ holds for some real number x, then $F^I_t > F^J_t$.

I highlight that no other assumption is made on either the game specific fitness function or the aggregate fitness function. I will now introduce the notion of evolutionary stability of behavioural rules.

3 Evolutionary Stability of Behavioural Rules

The stability criterion compares the fitness of each incumbent behavioural rule in the population to the fitness of a mutant behavioural rule in the face of an "effective mutation" in an incumbent behavioural rule. In order to introduce the concept of an effective mutation, consider a game $G \in \mathcal{G}_{t+1}$, and a feasible period $t+1$ strategy profile $f_G^{t\to t+1}$ of the game G. By definition, $f_G^{t\to t+1}$ is a strategy profile that can be realised if each individual in the population chooses a strategy from his response set that is determined by his behavioural rule. I will now define an effective mutation in this feasible strategy profile $f_G^{t\to t+1}$.

Consider $\mathcal{R}_t$, namely the set of behavioural rules at the end of time period t. Suppose that a strict subset of individuals who follow one particular behavioural rule, say $R^I \in \mathcal{R}_t$, mutate at the very beginning of period $t+1$, and adopt a different (mutant) behavioural rule, say $R^{I'}$. The mutating behavioural rule is called the *source behavioural rule*, the mass of mutating individuals is denoted by ε, where $\varepsilon <\mid SoR^I_t \mid$, and the set of mutating individuals is denoted by M_ε. The strategy profile post-mutation is denoted by $f_{G|t+1}$, and the strategy profile played by the set of the mutating individuals M_ε due to the mutation is $f_{G|M_\varepsilon,t+1}$. A mutation is said to be *effective* at the profile $f_G^{t\to t+1}$ if and only if $f_{G|M_\varepsilon,t+1}$ is different the strategy profile that would be played by the individuals in M_ε in the feasible strategy profile $f_G^{t\to t+1}$ in the absence of a mutation. Recall that the strategy profile of the individuals in the subset $M_\varepsilon \subsetneq SoR^I_t$ at the profile $f_G^{I,t\to t+1}$ is denoted by $f_{G|M_\varepsilon}^{I,t\to t+1}$.

Then, the mutation by individuals in M_ε in their behavioural rule R^I is *effective at the feasible strategy profile* $f_G^{I,t\to t+1}$ if the following conditions hold:

(i) $f_{G|M_\varepsilon,t+1} \neq f_{G|M_\varepsilon}^{I,t\to t+1}$

(ii) $f_{G|SoR_t^I\setminus M_\varepsilon,t+1} = f_{G|SoR_t^I\setminus M_\varepsilon}^{I,t\to t+1}$

(iii) for all other incumbent behavioural rules $R^J \in \mathcal{R}_t \setminus \{R^I\}$, $f_{G|t+1}^J = f_G^{J,t\to t+1}$.

When these conditions hold, then I refer to $f_{G|t+1}$ as an effectively mutated strategy profile.

Condition (i) above states that in the case of an effective mutation in the strategy profile $f_G^{t\to t+1}$, the strategy profile played by the mutating individuals is not the same as what is played by them at $f_G^{I,t\to t+1}$ if they did not mutate. Condition (ii) states that the individuals who continue to follow the source behavioural rule choose the strategy profile that they would play at $f_G^{I,t\to t+1}$. Condition (iii) states the same for individuals following all other behavioural rules (if any). Thus, in order for the mutation to be effective at particular feasible strategy profile, it must be that the post-mutation strategy profile differs from it.

For any $\varepsilon > 0$, there is an *effective ε mutation in the population* in period $t+1$ if the there exists a game $G \in \mathcal{G}_{t+1}$, a source behavioural rule $R^I \in \mathcal{R}_t$ with some subset of mutating individuals $M_\varepsilon \subsetneq SoR_t^I$ of mass ε, and a feasible period $t+1$ strategy profile $f_G^{t\to t+1} \in \Delta_G^{t\to t+1}$ such that the mutation effective at $f_G^{t\to t+1}$.

The notion of stability of behavioural rules that I present next is based on a comparison of fitness of behavioural rules when there is an effective mutation in the population.

A population with set of incumbent behavioural rules $\mathcal{R}_t$ satisfies *evolutionary stability of behavioural rules* at period $t+1$ if, for every collection of games $\mathcal{G}_{t+1} \subset \mathcal{P}_{t+1}(\mathcal{G})$ and all $\varepsilon > 0$ small enough, any effective ε mutation in the population by any mutant behavioural rule $R^{I'}$ results in $F_{t+1}^{I'} < F_{t+1}^J$ holding for all incumbent behavioural rules $R^J \in \mathcal{R}_t$.

Thus, stability requires that for all effective mutations in the population, each incumbent behavioural rule should have a higher fitness than any mutant behavioural rule, irrespective of the source behavioural rule, and the collection of games that may be played in period $t+1$. The results relating to stability are presented next.

4 Results

I begin by stating in Proposition 1 that a necessary condition for the stability of any population in time period $t+1$ is that each incumbent behavioural rule should have a higher game specific fitness than any mutant behavioural rule in each and every game $G \in \mathcal{G}$ – this is in spite of the behavioural rule based stability notion comparing the aggregate fitness (across of all games) of the incumbent behavioural rules with that of a mutant behavioural rule.

Proposition 1. *A necessary and sufficient condition for any population to satisfy evolutionary stability of behavioural rules is that in each game $G \in \mathcal{G}$, each incumbent behavioural rule should have a higher game specific fitness than any mutant behavioural rule.*

The sufficiency part of the theorem is obvious and follows simply from the strict first order stochastic dominance property of the relative frequency distribution of the game specific fitness. In the proof for the necessity part of the proposition (presented in the appendix), I develop the argument that there exists an effective mutation in the population such that: *(i)* all individuals who follow the same behavioural rule choose the same strategy in a game in $\mathcal{G}_{t+1}$, and this holds for all games in $\mathcal{G}_{t+1}$, and *(ii)* in all but one specific game $G \in \mathcal{G}_{t+1}$, the mutant individuals play the same strategy as the individuals who continue to follow the source behavioural rule. It then follows that all individuals who follow the same behavioural rule obtain the same payoffs; furthermore, in all but one game, the mutant individuals obtain the same payoff as the individuals who continue to follow the source behavioural rule. This implies that in all but the one specific game G, the mutant behavioural rule has the same game specific fitness as the source behavioural rule. So, in order for the population to be stable, the game specific fitness of the source behavioural rule must be higher than that of the mutant behavioural rule in the game G. The proposition then follows from the fact that any incumbent behavioural rule may be the source behavioural rule, and that any game in $\mathcal{G}$ can be the specific game in which the strategy chosen by the mutant individuals differ from that chosen by those who continue to follow the source behavioural rule game.

Next, in Proposition 2 below, I state that there does not exist any diverse stable population. In the proof of this proposition (presented in the appendix), I show that in any game $G \in \mathcal{G}$, there exists a feasible strategy profile and an effective ε mutation in the population at that feasible strategy profile, such that the game specific fitness of the source behavioural rule does not exceed that of the mutant behavioural rule whenever the population is diverse. This, along with Proposition 1, gives the result of Proposition 2.

Proposition 2. *There does not exist any diverse population of behavioural rules that satisfies evolutionary stability of behavioural rules.*

The instability of diverse populations leads me to analyse the stability of a uniform population. Proposition 3, and Proposition 4, below present a necessary condition, and a sufficient condition, respectively, for stability of a uniform population. A strengthening of the necessary condition is sufficient for stability, and after presenting the propositions, I discuss the reason for this gap between the necessary and sufficient condition.

Proposition 3. *Suppose that a population satisfies evolutionary stability of behavioural rules at time period* $t+1$*. Then, in each game* $G \in \mathcal{G}$*, all individuals have the same unique feasible period* $t+1$ *strategy* $s^i \in S_G$*, and, in addition,* s^i *is an ESS such that for any two distinct strategies* $s^j, s^k \in BR_G(s^i) \setminus \{s^i\}$*,* $\pi_G(s^i, s^k) \geq \pi_G(s^j, s^k)$ *or* $\pi_G(s^i, s^j) \geq \pi_G(s^k, s^j)$*.*

Corollary 1. *A population satisfies evolutionary stability of behavioural rules only if each game has an evolutionarily stable strategy, and hence a symmetric pure strategy Nash equilibrium.*

Proposition 4. *A population satisfies evolutionary stability of behavioural rules at time period* $t+1$ *if, in each game* $G \in \mathcal{G}$*, all individuals in the population have the same evolutionary stable strategy* $s^i \in S_G$ *as the unique feasible period* $t+1$ *strategy, and, in addition,* s^i *is such that, for any* $s^j \in BR_G(s^i) \backslash \{s^i\}$ *and* $s^k \in S_G$*, the inequality* $\pi_G(s^i, s^k) \geq \pi_G(s^j, s^k)$ *holds.*

Proposition 3 reveals that even the necessary condition for stability is stronger than the criterion for ESS. This is due to the fact that ESS requires an incumbent strategy to be fitter than any mutant strategy but not against multiple mutant strategies. On the other hand, in the evolutionary stability of behavioural rules approach, the response set of the mutant behavioural rule may be non-singleton as a result of which all the mutant individuals need not play the same strategy. Hence, each incumbent behavioural rule must now be fitter than any mutant behavioural rule, even when the mutation expresses itself in multiple strategies. As a result, the stability criterion now is more stringent than in the ESS approach.

A comparison of Proposition 3 and Proposition 4 shows that while both propositions require the same evolutionary stable strategy to be the only feasible strategy in the response set of all individuals, the additional condition is stronger in case of sufficiency. Necessity implies that an inequality involving two other evolutionary strategies must hold while sufficiency requires that a similar inequality must hold for two other strategies of which only one must be an ESS. The reason for this difference, as I discuss below, can be traced back to the considerable degree of freedom of the fitness functions.

In order to convey the intuition behind the necessary condition, I consider a game $G \in \mathcal{G}$ where the unique feasible strategy for all individuals in the population is an evolutionarily stable strategy s^i, and there exist two different strategies $s^j, s^k \in BR_G(s^i) \setminus \{s^i\}$. Now, suppose that a mass ε of the incumbent individuals mutate, and both s^j and s^k are feasible period $t+1$ strategies for the mutant behavioural rule so that a mass ν and $\varepsilon - \nu$ of the mutant individuals choose s^j and s^k, respectively. So, the payoff received from playing s^i, s^j, and s^k in this effectively mutated strategy profile equals $(1-\varepsilon)\pi(s^i,s^i)+\nu\pi(s^i,s^j)+(\varepsilon-\nu)\pi(s^i,s^k)$, $(1-\varepsilon)\pi(s^j,s^i)+\nu\pi(s^j,s^j)+(\varepsilon-\nu)\pi(s^j,s^k)$, and $(1-\varepsilon)\pi(s^k,s^i)+\nu\pi(s^k,s^j)+(\varepsilon-\nu)\pi(s^k,s^k)$, respectively. Since $s^j, s^k \in BR(s^i)\setminus\{s^i\}$, one obtains $(1-\varepsilon)\pi(s^i,s^i) = (1-\varepsilon)\pi(s^j,s^i) = (1-\varepsilon)\pi(s^j,s^i)$. Then, if neither $\pi(s^i,s^j) \geq \pi(s^k,s^j)$ nor $\pi(s^i,s^k) \geq \pi(s^j,s^k)$ do not hold, i.e. $\pi(s^i,s^j) < \pi(s^k,s^j)$ and $\pi(s^i,s^k) < \pi(s^j,s^k)$ hold instead, then payoff obtained from playing s^i does not exceed the payoff obtained from playing either of s^j or s^k whenever $\frac{\nu}{\varepsilon} <\min\{\frac{\pi(s^j,s^k)-\pi(s^i,s^k)}{\pi(s^j,s^k)-\pi(s^i,s^k)+\pi(s^i,s^j)-\pi(s^j,s^j)}, \frac{\pi(s^k,s^k)-\pi(s^i,s^k)}{\pi(s^k,s^k)-\pi(s^i,s^k)+\pi(s^i,s^j)-\pi(s^k,s^j)}\}$. In this event, each mutant individual's payoff is at least as high as that of any individual who follows the source behavioural rule. So, the relative frequency distribution of payoffs of the mutant behavioural rule first order dominates that of the incumbent behavioural rule, and the mutant behavioural rule's game specific fitness is at least as much as that of the incumbent behavioural rule, thereby implying (due to Proposition 1) instability of the population. This explains the necessary condition in Proposition 3.

On the other hand, suppose that $s^k \notin BR_G(s^i)$, and all else remains the same as in the preceding paragraph. Now, $s^k \notin BR_G(s^i)$, along with s^i being an ESS, implies $\pi(s^k,s^i) < \pi(s^i,s^i)$. Also, suppose that the sufficient condition $\pi(s^i,s^k) \geq \pi(s^j,s^k)$ does not hold, i.e. $\pi(s^i,s^k) < \pi(s^j,s^k)$ holds instead. Since $s^k \notin BR_G(s^i)$, necessity does not impose any condition on the payoff of any strategy against strategy s^k. At the same time, $s^k \notin BR_G(s^i)$ implies that, for any ε small enough, the payoff from playing s^i in the effectively mutated strategy profile described in the paragraph above exceeds that of playing s^k. However, irrespective of how small ε is, $s^j \in BR_G(s^i)$ along with $\pi(s^i,s^k) < \pi(s^j,s^k)$ implies that the payoff from playing s^j is higher than that of s^i whenever $\frac{\nu}{\varepsilon} < \frac{\pi(s^j,s^k)-\pi(s^i,s^k)}{\pi(s^j,s^k)-\pi(s^i,s^k)+\pi(s^i,s^j)-\pi(s^j,s^j)}$. In this case, the relative frequency distribution of one behavioural rule does not first order stochastically dominate that of the other; hence, there is no restriction on the form that the fitness functions may take, and so, suppose that game specific fitness function of any behavioural rule equals the maximum payoff obtained by an individual following that rule. Then, the game specific fitness of the mutant behavioural rule exceeds that of the incumbent behavioural rule; by Proposition 1, the population is not stable. Hence, the particular condition in Proposition 4 is sufficient to impose stability.

Finally, in the example below, I show that evolutionary stability of behavioural rules approach can represent a refinement of the evolutionary stable strategies approach, thus implying that the converse of Proposition 3 (and Corollary 1) is not true in general.

Example 4. Suppose that the individuals in the population interact only in the game below in each and every time period. It is easily verified that both s^1 and s^4 are evolutionarily stable strategies of this game G.

	s^1	s^2	s^3	s^4
s^1	$3,3$	$2,3$	$3,3$	$0,1$
s^2	$3,2$	$1,1$	$5,5$	$0,1$
s^3	$3,3$	$5,5$	$1,1$	$0,1$
s^4	$1,0$	$1,0$	$1,0$	$4,4$

Figure 3

I will argue that the population satisfies evolutionary stability of behavioural rules if and only if s^4 is the only strategy in each individual's response set, thus implying that the evolutionary stable strategy s^1 can never be in the response set of any individual.

Now, by Proposition 3, a population is stable at time period $t+1$ only if the population is uniform, and the only feasible period $t+1$ strategy profiles is either $(p^1, p^2, p^3, p^4) = (1, 0, 0, 0)$ or $(p^1, p^2, p^3, p^4) = (0, 0, 0, 1)$.

So, first suppose that $(p^1, p^2, p^3, p^4) = (1, 0, 0, 0)$ is the only feasible strategy profile, and a mass ε of the individuals mutate to another behavioural rule of which a mass $\frac{\varepsilon}{2}$ of the mutant individuals choose s^2 while the remaining mass $\frac{\varepsilon}{2}$ of the mutant individuals choose s^3. Then, in period $t+1$, the payoff obtained by each incumbent individual, each mutant individual who chooses s^2, and each mutant individual who plays s^3 is $3(1-\varepsilon)+2\frac{\varepsilon}{2}+3\frac{\varepsilon}{2} = 3-\frac{\varepsilon}{2}$, $3(1-\varepsilon)+\frac{\varepsilon}{2}+5\,\frac{\varepsilon}{2} = 3$, and $3(1-\varepsilon)+5\frac{\varepsilon}{2}+\frac{\varepsilon}{2} = 3$, respectively. Hence, for any positive ε, no matter how small, the payoff received by each mutant individual exceeds the payoff received by each incumbent individual. As a result, the relative frequency distribution of period $t+1$ payoffs of the mutant behavioural rule strictly first order stochastically dominates that of the incumbent behavioural rule. Consequently, the mutant behavioural rule is fitter than the incumbent behavioural rule in period $t+1$, and so, the population is not stable in period $t+1$.

Secondly, suppose that $(p^1, p^2, p^3, p^4) = (0, 0, 0, 1)$ is the only feasible strategy profile. Now, for all $s^k \in S_G \setminus \{s^4\}$, the inequality $\pi(s^4, s^4) > \pi(s^k, s^4)$ holds. This satisfies the sufficient condition in Proposition 4, and so the population is stable if and only the unique feasible strategy profile in every period is $(p^1, p^2, p^3, p^4) = (0, 0, 0, 1)$ ■

5 Conclusion

I develop a novel notion of evolutionary stability when individuals in a population interact simultaneously in a family of strategic environments. The manner in which an individual chooses his action in a strategic situation is captured by a behavioural rule. Therefore, the population can be thought of as being comprised of a set of behavioural rules, and this set of behavioural rules is stable if it is able to withstand the invasion by any other mutant behavioural rule in the sense of being fitter than the mutant rule. This represents a substantial generalisation over the traditional concept of evolutionary stability of strategies by, firstly, describing each individual in terms of his decision-making process and secondly, by conceptualising stability when individuals interact in a number of strategic games simultaneously.

I show that any population which comprises of more than one incumbent behavioural rule is unstable. Next, I present fairly close necessary and sufficient conditions for stability of a population comprising of a single behaviorial rule, and show that evolutionary stability of the behavioural rule approach is a refinement of the evolutionary stability of strategies approach in that it leads individuals to necessarily choose in each game an action that must satisfy a condition that is stricter than the requirement for an evolutionary stable strategy.

Appendix

Proof of Lemma 1. Consider any game G, and any two distinct evolutionarily stable strategies s^i and s^j of G. Since s^i is an ESS of G, one of the following two conditions hold: either $\pi_G(s^i, s^i) > \pi_G(s^j, s^i)$, or $\pi_G(s^i, s^i) = \pi_G(s^j, s^i)$ and $\pi_G(s^i, s^j) > \pi_G(s^j, s^j)$. Similarly, since s^j is an ESS of G, either $\pi_G(s^i, s^j) < \pi_G(s^j, s^j)$, or $\pi_G(s^i, s^j) = \pi_G(s^j, s^j)$ and $\pi_G(s^i, s^i) < \pi_G(s^j, s^i)$. Now, because both s^i and s^j are ESS, combining the conditions above, it must be that $\pi_G(s^i, s^i) > \pi_G(s^j, s^i)$ and $\pi_G(s^i, s^j) < \pi_G(s^j, s^j)$.

Now, consider any mixed strategy $p \in \Delta_{G|i,j}$. Then, one obtains $\pi_G(s^i, p) = p^i \pi_G(s^i, s^i) + (1 - p^i)\pi_G(s^i, s^j)$ and $\pi_G(s^j, p) = p^i \pi_G(s^j, s^i) + (1 - p^i)\pi_G(s^j, s^j)$. So, $\pi_G(s^i, p) > \pi_G(s^j, p)$ if and only if $p^i \pi_G(s^i, s^i) + (1 - p^i)\pi_G(s^i, s^j) > p^i \pi_G(s^j, s^i) + (1 - p^i)\pi_G(s^j, s^j)$. Then, there exists a real number $\bar{p}^i \equiv \frac{\pi_G(s^j,s^j) - \pi_G(s^i,s^j)}{\pi_G(s^i,s^i) - \pi_G(s^j,s^i) + \pi_G(s^j,s^j) - \pi_G(s^i,s^j)}$ such that $\pi_G(s^i, p) > \pi_G(s^j, p)$ if and only if $p^i > \bar{p}^i$, and $\pi_G(s^i, p) < \pi_G(s^j, p)$ if and only if $p^i < \bar{p}^i$. Further, $\pi_G(s^j, s^j) > \pi_G(s^i, s^j)$ and $\pi_G(s^i, s^i) > \pi_G(s^j, s^i)$ imply $\bar{p}^i \in (0, 1)$. Thus, $\pi_G(s^i, p') > \pi_G(s^j, p')$ and

$\pi_G(s^i, p) < \pi_G(s^j, p)$ hold for all $p' \in \Delta_{G|i>\bar{p}^i,j}$ and all $p \in \Delta_{G|i<\bar{p}^i,j}$. ■

Proof of Proposition 1. I will only prove the necessity part of the proposition because the sufficiency part is obvious. Suppose that the population is stable in time period $t+1$. Consider, for any ε small enough, a mutation in an incumbent behavioural rule $R^I \in \mathcal{R}_t$ with the set of mutating individuals of mass ε being denoted by $M_\varepsilon \subsetneq SoR^I_t$, and the mutant behavioural rule being denoted by $R^{I'}$. Thus, $SoR^I_{t+1} = SoR^I_t \setminus M_\varepsilon$. Also consider the following effective ε mutation in the population:

(a) All individuals following the same incumbent behavioural rule choose the same strategy in each game that is played in period $t+1$, and this holds for all behavioural rules. That is, for any incumbent behavioural rule $R^J \in \mathcal{R}_t$, and for all $i, j \in SoR^J_{t+1}$, one has $s_{G|i,t+1} = s_{G|j,t+1}$, and this holds for all games in $\mathcal{G}_{t+1}$.

(b) Only in one particular game in $\mathcal{G}_{t+1}$ do the mutant individuals choose a strategy that is different from that chosen by those who continue to follow the source behavioural rule; in all other games, the mutant individuals choose the same strategy as the latter individuals. That is, for all $i, j \in M_\varepsilon$ and all $k \in SoR^I_{t+1}$, there exists only one particular game $G \in \mathcal{G}_{t+1}$ such that $s_{G|i,t+1} = s_{G|j,t+1}$ but $s_{G|i,t+1} \neq s_{G|k,t+1}$; for all other games $G' \in \mathcal{G}_{t+1} \setminus \{G\}$, it holds that $s_{G'|i,t+1} = s_{G'|j,t+1} = s_{G'|k,t+1}$.

Then, all individuals who follow the same behavioural rule obtain the same payoff in each game in this period. Furthermore, since the payoff obtained in each game in $\mathcal{G}_{t+1} \setminus \{G\}$ by each individual in SoR^I_{t+1} is equal to that obtained by each individual in M_ε, the game specific fitness of R^I equals the game specific fitness of $R^{I'}$ in all these games. Now, a necessary condition for stability is that, in the game G, the game specific fitness of R^I should exceed that of $R^{I'}$; for, if not, then the game specific fitness of R^I and $R^{I'}$ would be the same in each and every game, leading to both rules having the same aggregate fitness, thereby implying that the population is not stable.

Finally, the proposition follows from the fact that this must be true for each incumbent behavioural rule $R^I \in \mathcal{R}_t$, each collection of games $\mathcal{G}_{t+1} \in \mathcal{P}_{t+1}(\mathcal{G})$, and each $G \in \mathcal{G}_{t+1}$. ■

Proof of Proposition 2. If the population in period $t+1$ is diverse, then there exists a game G, and two different incumbent behavioural rules $R^I, R^J \in \mathcal{R}_t$, such that the period $t+1$ response set of these two behavioural rules in the game G is not identical. So, without loss of generality, suppose that the strategy $s^i \in S_G$ belongs to the period $t+1$ response set of the behavioural rule R^I, and that the strategy $s^j \in S_G$ belongs to the period $t+1$ response set of the behavioural rule R^J but not to the response set of R^I. Then, in the absence of a mutation, there exists a feasible period $t+1$ strategy profile where individuals who follow the behavioural rule R^I and R^J choose s^i and s^j, respectively. I do not specify the period $t+1$ strategy choice of individuals who follow other behavioural rules (if any).

Now, consider ε small enough, and suppose that in period $t+1$, a mass ε of the individuals who follow the behavioural rule R^I mutate. Also suppose that the mutant behavioural rule is such that s^j is in its period $t+1$ response set, and each mutant individual chooses s^j. It follows that this represents an effective ε mutated strategy profile. Then, in the game G, each mutant individual obtains a payoff that is identical to the payoff obtained by each individual who follows the incumbent behavioural rule R^J. So, in period $t+1$, the relative frequency distribution of the payoffs of these two behavioural rules in the game G is identical, due to which their game specific fitness in the game G is also the same. Then, Proposition 1 implies that the population is not stable. Thus, a stable diverse population does not exist. ■

Proof of Proposition 3. In Step 1, I will show that if the uniform population is stable at time period $t+1$, then each strategy in the response set must be an ESS that satisfies the inequalities stated in the proposition. Next, in Step 2, I will demonstrate that all individuals must have the same unique feasible period $t+1$ strategy.

Step 1. Consider a game $G \in \mathcal{G}$. Since the population is uniform, all individuals have the same response set. So, suppose that there exists a strategy s^i in the response set that is not an ESS of the game. Then, consider the specific feasible period $t+1$ strategy profile where all individuals choose s^i. Now, for any ε small enough, there exists $s^j \in S_G$ such that if ε mass of the individuals mutate and choose s^j, then an effectively mutated strategy profile is obtained, and the mutant individuals obtain a higher payoff than the incumbent individuals in this effectively mutated strategy profile. So, the relative frequency distribution of payoffs of the mutant behavioural rule strictly first order stochastically dominates that of the incumbent behavioural rule. Hence, in the game G, the mutant behavioural has a higher game specific fitness than the incumbent behavioural rule, and, by Proposition 1, the population is unstable. Consequently, a necessary condition for stability is that each strategy in the individuals' response set must be an ESS of the game.

Next, consider the specific feasible period $t+1$ strategy profile where all individuals choose s^i, where, following from the above, s^i must be an ESS of the game. Also, consider the effectively mutated strategy profile $f_{G|t+1}$ where a mass ε of the individuals mutate, and mass ν of the mutant individuals choose $s^j \in BR_G(s^i) \setminus \{s^i\}$ while the mass $\varepsilon - \nu$ of the mutant individuals choose $s^k \in BR_G(s^i) \setminus \{s^i, s^j\}$. Then, in the strategy profile $f_{G|t+1}$, the payoff obtained by each incumbent individual is $\pi_G(s^i, f_{G|t+1}) = (1-\varepsilon)\pi_G(s^i, s^i) + \nu\pi_G(s^i, s^j) + (\varepsilon - \nu)\pi_G(s^i, s^k)$, the payoff obtained by a mutant individual choosing s^j is $\pi_G(s^j, f_{G|t+1}) = (1-\varepsilon)\pi_G(s^j, s^i) + \nu\pi_G(s^j, s^j) + (\varepsilon-\nu)\pi_G(s^j, s^k)$, and the payoff obtained by a mutant individual choosing s^k is equal to $\pi_G(s^k, f_{G|t+1}) = (1-\varepsilon)\pi_G(s^k, s^i) + \nu\pi_G(s^k, s^j) + (\varepsilon - \nu)\pi_G(s^k, s^k)$.

Now, if none of the conditions of the proposition hold, then $\pi_G(s^i, s^k) < \pi_G(s^j, s^k)$, and $\pi_G(s^i, s^j) < \pi_G(s^k, s^j)$ hold. One also obtains from s^i being an ESS and from $s^j, s^k \in BR_G(s^i)$ that $\pi_G(s^i, s^i) = \pi_G(s^j, s^i) = \pi_G(s^k, s^i)$, and $\pi_G(s^i, s^j) > \pi_G(s^j, s^j)$ and $\pi_G(s^i, s^k) > \pi_G(s^k, s^k)$. So, the relative magnitudes of the payoffs $\pi_G(s^i, f_{G|t+1})$, $\pi_G(s^j, f_{G|t+1})$ and $\pi_G(s^k, f_{G|t+1})$ is determined by comparing $\nu\pi_G(s^i, s^j) + (\varepsilon - \nu)\pi_G(s^i, s^k)$, $\nu\pi_G(s^j, s^j) + (\varepsilon - \nu)\pi_G(s^j, s^k)$, and $\nu\pi_G(s^k, s^j) + (\varepsilon - \nu)\pi_G(s^k, s^k)$. It is then easily verified that if $\frac{\nu}{\varepsilon} \leq \min\{\frac{\pi_G(s^j,s^k)-\pi_G(s^i,s^k)}{\pi_G(s^i,s^j)-\pi_G(s^j,s^j)+\pi_G(s^j,s^k)-\pi_G(s^i,s^k)}, \frac{\pi_G(s^k,s^k)-\pi_G(s^i,s^k)}{\pi_G(s^i,s^j)-\pi_G(s^i,s^k)+\pi_G(s^k,s^k)-\pi_G(s^k,s^j)}\}$, then the inequality $\pi_G(s^i, f_{G|t+1}) \leq \min\{\pi_G(s^i, f_{G|t+1}), \pi_G(s^i, f_{G|t+1})\}$ holds. In this case, there does not exist any incumbent individual who obtains a higher payoff than any mutant individual. So, the relative frequency distribution of payoffs of the mutant individuals strictly first order stochastically dominates that of the incumbent behavioural rule, due to which the former has a higher game specific fitness. Then, by Proposition 1, the population is unstable. Thus, a necessary condition for stability is for the inequality in the proposition to hold.

Step 2. I will show that if there is more than one feasible period $t+1$ strategy profile in any game, then there exists an effective ε mutation whereby the game specific fitness of a mutant behavioural rule exceeds that of the incumbent behavioural rule. This, by Proposition 1, implies that the population in not stable. Consequently, each game must have exactly one feasible period $t+1$ strategy profile.

Consider any game $G \in \mathcal{G}$, and suppose that there exists more than one feasible period $t+1$ strategy profile of this game. By Step 1, each of these strategies – say s^i and s^j – must be an ESS of the game. Then, for all $q \in [0, 1]$, there is a feasible period $t+1$ population strategy profile in the game G given by $f_G^{t\to t+1}$ such that a mass q of the individuals choose s^i while the other individuals choose s^j.

Now, by Lemma 1, there exists $\bar{p}^i \in (0, 1)$ such that for every $p \in \Delta_{G|i<\bar{p}^i,j}$ and every $p' \in \Delta_{G|i>\bar{p}^i,j}$, the inequalities $\pi_G(s^i, p) < \pi_G(s^j, p)$ and $\pi_G(s^j, p') < \pi_G(s^i, p')$ are satisfied. So, let $q < \bar{p}^i$, and consider the effective ε mutation whereby the mutation occurs in the set of individuals who, in the absence of a mutation, would have played s^i, and these mutant individuals play s^j instead. Then, the strategy profile in period $t+1$ is given by $f_{G|t+1}$, where $f^i_{G|t+1} = q - \varepsilon$, and $f^j_{G|t+1} = 1 - q + \varepsilon$, with $q - \varepsilon < \bar{p}^i$. Clearly, this represents an effective ε mutated strategy profile in which all the mutant individuals obtain the payoff $\pi(s^j, f_{G|t+1})$; on the other hand, some of the incumbent individuals receive $\pi(s^i, f_{G|t+1})$ while the other incumbent individuals receive $\pi(s^j, f_{G|t+1})$. Since $f^i_{G|t+1} < \bar{p}^i$, the inequality $\pi_G(s^j, f_{G|t+1}) > \pi_G(s^i, f_{G|t+1})$ holds. As a result, the relative frequency distribution of payoffs of the mutant behavioural rule strictly first order stochastically dominates that of the incumbent behavioural rule, and the former has a higher game specific fitness than the latter. By Proposition 1, the population in not stable. Thus, a necessary condition for stability is that each game must have exactly one feasible period $t + 1$ strategy profile. ■

Proof of Proposition 4. Consider any game $G \in \mathcal{G}$. Suppose that the conditions of the propositions hold, and in time period $t+1$, a mass $\varepsilon > 0$ of the individuals mutate to another behavioural rule. Each incumbent individual chooses $s^i \in S_G$ in period $t + 1$. Hence, in any ε effectively mutated strategy profile, a positive mass of the mutant individuals must choose a strategy other than s^i. So, let $\nu \in [0, \varepsilon)$ denote the mass of the mutant sub-population who choose s^i, and let $f_{G|t+1}$ denote any effectively ε mutated period $t + 1$ strategy profile of the game G.

Then, the payoff received by each incumbent individual, and a mutant individual choosing s^j (where $s^j \neq s^i$), in the strategy profile $f_{G|t+1}$ equals $(1 - \varepsilon + \nu)\pi_G(s^i, s^i) + \Sigma_{k \neq i} f^k_{G|t+1}\pi_G(s^i, s^k)$ and $(1 - \varepsilon + \nu)\pi_G(s^j, s^i) + \Sigma_{k \neq i} f^k_{G|t+1}\pi_G(s^j, s^k)$, where $\Sigma_{k \neq i} f^k_{G|t+1} = \varepsilon - \nu$. Whenever ε is small enough, the inequality $(1 - \varepsilon + \nu)(\pi_G(s^i, s^i) - \pi_G(s^j, s^i)) + \Sigma_{k \neq i} f^k_G(\pi_G(s^i, s^k) - \pi_G(s^j, s^k)) > 0$ holds under the conditions of the proposition. So, in every effectively ε mutated strategy profile, the payoff received by playing each strategy s^j (where $s^j \neq s^i$) is lower than the payoff received by playing the strategy s^i. Hence, in the game G, the relative frequency distribution of payoffs of the incumbent behavioural rule strictly first order dominates that of the mutant behavioural rule, and the former has a higher game specific fitness than the latter. Since this holds for each $G \in \mathcal{G}$ and for any mutant behavioural rule, the aggregate fitness of the incumbent behavioural rule is also higher, and population is stable. ■

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