A Precondition-Free Demonstration That the Wold-Juréen (1953) Utility Function (Always) Generates a Giffen Good

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A Precondition-Free Demonstration That the Wold-Juréen (1953) Utility Function (Always) Generates a Giffen Good

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Abstract: Both Weber (1997) and Sproule (2020) offered a precondition for the existence of Giffenity when the decision maker's utility function is the Wold-Juréen (1953) utility function. The purpose of this paper is to show that all preconditions (including those due to Weber (1997) and Sproule (2020)) are superfluous, in the present demonstration that the inferior good in the Wold-Juréen (1953) utility function is also a Giffen good. Our success relies upon our use of both of the Marshallian demand functions, rather than just one, as is the case in both Weber (1997) and Sproule (2020). In brief, the present paper offers a precondition-free demonstration that the Wold-Juréen (1953) utility function will (always) generate a Giffen good.

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1. Introduction

In a paper entitled “The Case of a Giffen Good: Comment” which appears in This Journal, Christian Weber (1997) explored the properties of two utility functions, each of which has the potential to exhibit Giffen bebavior. These two utility functions are due to Spiegel (1994) and due to Wold and Juréen (1953).

Regarding the latter of the two [hereafter referred to as the Wold-Juréen (1953) utility function], Weber (1997, page 36) reported that “analysis of the utility function of Wold and Jureen (1953) shows that, whatever else may have caused the general absence from microeconomics text books of utility functions leading to Giffen behavior (the exception being Katzner 1988[2]), such utility functions need not be ‘hard to handle’ (Spiegel 1994; 137) and can even make quite tractable classroom examples.” The present paper shares and wholly endorses Weber’s viewpoint.

In subsequent pages, Weber (1997, page 38) went on to show that the first good (Good 1, which is an inferior good) exhibits Giffen behavior[3] if the decision-maker's (hereafter DM) income exceeds the price of the second good (the price of Good 2). This, Weber argues, is tenable if the DM’s income is “relatively large” and if the price of Good 2 is “relatively small”. Sometime later, Sproule (2020) offered an alternative to Weber’s (1997) somewhat specious precondition for Giffenity: Sproule’s (2020) alternative precondition is that the price of Good 1 be greater than or equal to the price of Good 2.

The purpose of this paper is to simplify Weber’s (1997) and Sproule’s (2020) analyses by rendering superfluous any and all preconditions for the Giffenity of Good 1

[2] In the case of the 2006 edition, see Katzner (2006, Exercise 2.16, page 79).
[3] Simply put, “Giffen behavior” is tantamount to an upward sloping demand curve [Jensen and Miller (2008, page 1553)]. Stated differently, “Giffen behavior” is said to occur when any market price and the related quantity demanded move in the same direction.

(including Weber's and Sproule's preconditions). Our new precondition-free result is facilitated by our use of the demand functions for both Goods 1 and 2, rather than the use of only one demand function (viz., the demand function for Good 1) as is the case in the papers by Weber (1997) and Sproule (2020).

This paper is organized as follows. In Section 2, we offer a review of the core literature, which is comprised of the papers by Weber (1997) and Sproule (2020). In Section 3, we present our principal finding, and that is this: If the DM's utility function is the Wold-Juréen (1953) utility function, then Good 1 is always a Giffen good, and this is because Good 2 is a complementary good to Good 1 Our final remarks are offered in Section 4.

2. Previous Research

Consider a DM, whose utility function is the Wold-Juréen (1953) utility function. After Wold and Juréen (1953), Weber (1997), and Sproule (2020), this utility function is defined as $U = (x_1 - 1)(x_2 - 2)^{-2}$ where U denotes the DM's utility level, where x_1 denotes the quantity of Good 1, where x_2 denotes the quantity of Good 2, and where $x_1 > 1$ and where $0 < x_2 < 2$

In our review of the literature (that is, our review of Weber (1997) and Sproule (2020)), we begin with the following Lagrange function:

$$\text{maximize}_{x_1, x_2, \lambda}\ L = U(x_1, x_2) + \lambda\left(p_1 x_1 + p_2 x_2 - \bar{m}\right) \tag{1}$$

where

$U(x_1, x_2) = (x_1 - 1)(x_2 - 2)^{-2}$ such that $x_1 > 1$ and $0 < x_2 < 2$,
p_1 denotes the price of Good 1,
p_2 denotes the price of Good 2, and
$\bar{m}$ denotes the DM's income such that

$$\bar{m} = p_1 x_1 + p_2 x_2 \ . \tag{2}$$

The solution to Equation (1) requires that the first-order conditions (hereafter FOCs) and the second-order condition (hereafter SOC) for an interior maximum be met, If they are met, then there exists a Marshallian demand function for both Goods 1 and 2. Since Weber (1997, page 39) has already shown that the SOC holds,[4] then we accept that the Marshallian demand functions for both Goods 1 and 2 exist and that they take the form:

$$x_1^* \;=\; x_1^*(p_1,p_2,m) \;=\; 2 \;+\; \frac{2p_2 \;-\; m}{p_1} \tag{3}$$

$$x_2^* \;=\; x_2^*(p_1,p_2,m) \;=\; 2\left(\frac{m \;-\; p_1}{p_2} \;-\; 1\right) \tag{4}$$

We turn next to the precondition and prediction of Weber (1997), followed by the precondition and prediction of Sproule (2020), for the Giffenity of Good 1 when the DM's utility function is the Wold-Juréen (1953) utility function. Both of these rely on Equation (3), and they ignore Equation (4), The coherency of Weber's (1997) precondition and prediction for Giffenity, and Sproule's (2020) precondition and prediction, for Giffenity, can be seen within the context of Propositions 1 and 2, which follow next.

Proposition 1 [Weber (1997)]: If the DM has the Wold-Juréen (1953) utility function, then Good 1 is a Giffen good if and only if $m - 2p_2 \;>\; 0$; that is, if m is "relatively large" and if p_2 is "relatively small".

Proof: (a) $\text{Sign}\left(\dfrac{\partial x_1^*}{\partial p_1}\right) \;=\; \text{sign}(m - 2p_2)$ [Weber (1997, page 40)].

(b) If m is "relatively large" and if p_2 is "relatively small"., then $\dfrac{\partial x_1^*}{\partial p_1} \;>\; 0$ and Good 1 is a Giffen good [Weber (1997, page 40)]. •

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[4] $$|\bar{B}| = \begin{vmatrix} 0 & U_{x_1} & U_{x_2} \\ U_{x_1} & U_{x_1x_1} & U_{x_1x_2} \\ U_{x_2} & U_{x_1x_2} & U_{x_2x_2} \end{vmatrix} = \begin{vmatrix} 0 & (x_2-2)^{-2} & -2(x_1-1)(x_2-2)^{-3} \\ (x_2-2)^{-2} & 0 & -2(x_2-2)^{-3} \\ -2(x_1-1)(x_2-2)^{-3} & -2(x_2-2)^{-3} & 6(x_1-1)(x_2-2)^{-4} \end{vmatrix}$$
$$= \; 2(x_1-1)(x_2-2)^{-8} \;>\; 0 \text{ since } x_1 \;>\; 1 \text{ [Weber (1997, page 39)].}$$

As outlined in Proposition 1, the essence of Weber's (1997) precondition for Giffenity is that the DM's income be "relatively large" and the price of Good 2 be "relatively small". As we shall see in Proposition 2, Sproule's (2020) precondition for Giffenity is that the price of Good 1 be greater than or equal to the price of Good 2.

To delve into the details of Sproule's (2020) Proposition 2, we first require the following three lemmas:

Lemma 1 [Sproule (2020)]: If the DM's utility function is the Wold-Juréen (1953) utility function, then $m < p_1 + 2p_2$.

Lemma 2 [Sproule (2020)]: If the DM's utility function is the Wold-Juréen (1953) utility function, then $p_1 + p_2 < m$

Lemma 3 [Sproule (2020)]: If the DM's utility function is the Wold-Juréen (1953) utility function, then (by Lemmas 1 and 2) $p_1 - p_2 < m - 2p_2 < p_1$.

Proposition 2 [Sproule (2020)]: If the DM's utility function is the Wold-Juréen (1953) utility function, and if $p_1 \geq p_2$, then Good 1 is a Giffen good.

Proof: If the DM's utility function is the Wold-Juréen (1953) utility function, and if $p_1 \geq p_2$, then $m - 2p_2 > 0$ [Lemma 3] and $\dfrac{\partial x_1^*}{\partial p_1} = \dfrac{m - 2p_2}{(p_1)^2} > 0$ •

In the next section, we will demonstrate that there exists a precondition-free alternative to Propositions 1 and 2.

3. Without Precondition, The Wold-Juréen (1953) Utility Function (Always) Generates a Giffen Good

In this section, we demonstrate our precondition-free alternative to Propositions 1 and 2. By a "precondition-free alternative" we mean an alternative which avoids the need to use Weber's (1997) precondition [viz., the DM's income is "relatively large" and the price of Good 2 is "relatively small"], Sproule's (2020) precondition [viz., $p_1 \geq p_2$], or any other such precondition. Stated differently, in this section, we show that, without precondition, the Wold-Juréen (1953) utility function (always) generates a Giffen good.

We are able to offer this new result only because (in our analysis) we use both of the Marshallian demand functions [viz., both Equations (3) and (4)], rather than only one of the two Marshallian demand functions as is done in Section 2 [viz., Equation (3)].

Proposition 3: If the DM's utility function is the Wold-Juréen (1953) utility function, then: (a) Good 2 is a complement of Good 1 and therefore (b) Good 1 is a Giffen good.

Proof: (a) In view of Equation (4), it follows that $\frac{\partial x_2^*}{\partial p_1} < 0$. (b) When Equation (2) is evaluated at the equilibrium level [that is, when $\bar{m} = p_1 x_1^*\left(p_1, p_2, m\right) + p_2 x_2^*\left(p_1, p_2, m\right)$], it follows that $\text{sign}\left(\frac{\partial x_1^*}{\partial p_1}\right) = -\text{sign}\left(\frac{\partial x_2^*}{\partial p_1}\right)$ and that $\frac{\partial x_2^*}{\partial p_1} < 0$ implies $\frac{\partial x_1^*}{\partial p_1} > 0$ •

4. Conclusion

The present paper demonstrates that any and all preconditions for Giffenity are superfluous when the DM's utility function is the Wold-Juréen (1953) utility function. Such preconditions include those due to Weber (1997) and Sproule (2020). In sum, this paper offers an new approach in the study of Giffen behavior, and this new approach offers a precondition-free demonstration that the Wold-Juréen (1953) utility function (always) generates a Giffen good**.**

The present paper is organized as follows. In Section 2, we offered a review of the core literature, which is comprised of Weber (1997) and Sproule (2020). In Section 3, we presented our principal finding, and that is this: If the DM's utility function is the Wold-Juréen (1953) utility function], then Good 2 is a complementary good to Good 1 and therefore Good 1 is (always) a Giffen good. In sum, Proposition 3 in Section 3 contains our precondition-free demonstration that the Wold-Juréen (1953) utility function (always) generates a Giffen good.

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AN EDITORIAL NOTE:

THE FOLLOWING MATERIAL IS TEMPORARY, AND EVENTUALLY IT IS TO BE DELETED

A PROOF OF EQUATIONS (3) AND (4)

Proposition [A Proof of Equations (3) and (4)]: If $U = U(x_1, x_2) = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, and if the SOC holds, which it does [see Weber (1997, page 39)], then the solution set for the consumer problem is:

$$x_1^* = 2 + \frac{2p_2 - m}{p_1} \tag{3}$$

$$x_2^* = 2\left(\frac{m - p_1}{p_2} - 1\right) \tag{4}$$

Proof: The consumer's problem is:

$$\max_{x_1, x_2, \lambda} L = (x_1 - 1)(x_2 - 2)^{-2} + \lambda(p_1 x_1 + p_2 x_2 - m)$$

where L denotes the Lagrange function and λ denotes the Lagrange multiplier. The first-order conditions (FOC) associated with an interior maximum are:

$$\frac{\partial L}{\partial x_1} = 0 = (x_2 - 2)^{-2} + \lambda p_1 \tag{A1}$$

$$\frac{\partial L}{\partial x_2} = 0 = (x_1 - 1)(-2)(x_2 - 2)^{-3} + \lambda p_2 \qquad \text{(A2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 = p_1 x_1 + p_2 x_2 - m \qquad \text{(A3)}$$

An implication of (A1) and (A2) is:

$$-\lambda = \frac{(x_2 - 2)^{-2}}{p_1} = \frac{(x_1 - 1)(-2)(x_2 - 2)^{-3}}{p_2}$$

$$\Rightarrow \frac{1}{p_1 (x_2 - 2)^2} = \frac{-2(x_1 - 1)}{p_2 (x_2 - 2)^3}$$

$$\Rightarrow \frac{1}{p_1} = \frac{2(1 - x_1)}{p_2 (x_2 - 2)}$$

$$\Rightarrow p_2 (x_2 - 2) = 2 p_1 (1 - x_1)$$

$$\Rightarrow p_2 x_2 = 2 p_1 (1 - x_1) + 2 p_2$$

$$\Rightarrow x_2 = 2 \frac{p_1}{p_2}(1 - x_1) + 2$$

$$\Rightarrow x_2 = 2 \frac{p_1}{p_2} - 2 \frac{p_1}{p_2} x_1 + 2$$

$$\Rightarrow x_2 = 2\left(\frac{p_1}{p_2} + 1\right) - 2 \frac{p_1}{p_2} x_1$$

$$\Rightarrow x_2 = 2\left(\frac{p_1 + p_2}{p_2}\right) - 2 \frac{p_1}{p_2} x_1 \qquad \text{(A4)}$$

(a) The Demand Function for Good 1 [viz. Equation (3)]: An implication of (A3) and (A4) is:

$$p_1 x_1 + 2 p_2 \left(\frac{p_1 + p_2}{p_2} - \frac{p_1}{p_2} x_1\right) = m$$

$$\Rightarrow p_1 x_1 + 2(p_1 + p_2 - p_1 x_1) = m$$

$$\Rightarrow p_1 x_1 - 2 p_1 x_1 + 2(p_1 + p_2) = m$$

$$\Rightarrow\ -\,p_1 x_1 \ =\ m \ -\ 2\left(p_1 + p_2\right)$$

$$\Rightarrow\ p_1 x_1 \ =\ 2p_1 \ +\ 2p_2 \ -\ m$$

$$\Rightarrow\ x_1^* \ =\ 2 \ +\ \frac{2p_2 \ -\ m}{p_1} \qquad \text{(A5)}$$

(b) **The Demand Function for Good 2 [viz. Equation (4)]:** An implication of (A4) is:

$$x_2 \ -\ 2\left(\frac{p_1 + p_2}{p_2}\right) \ =\ -\,2\frac{p_1}{p_2}x_1$$

$$\Rightarrow\ x_1 \ =\ \frac{p_2}{2p_1}\left(2\left(\frac{p_1 + p_2}{p_2}\right) \ -\ x_2\right)$$

$$\Rightarrow\ x_1 \ =\ \left(\frac{p_1 + p_2}{p_1}\right) \ -\ \frac{p_2}{2p_1}x_2 \qquad \text{(A6)}$$

An implication of (A3) and (A6) is:

$$p_1\left(\left(\frac{p_1 + p_2}{p_1}\right) \ -\ \frac{p_2}{2p_1}x_2\right) \ +\ p_2 x_2 \ =\ m$$

$$\Rightarrow\ p_1\left(\frac{p_1 + p_2}{p_1}\right) \ -\ p_1\frac{p_2}{2p_1}x_2 \ +\ p_2 x_2 \ =\ m$$

$$\Rightarrow\ p_1 + \ p_2 \ -\ \frac{p_2}{2}x_2 \ +\ p_2 x_2 \ =\ m$$

$$\Rightarrow\ \frac{p_2}{2}x_2 \ =\ m \ -\ p_1 \ -\ p_2$$

$$\Rightarrow\ x_2^* \ =\ \frac{2}{p_2}\left(m \ -\ p_1 \ -\ p_2\right) \ =\ 2\left(\frac{m \ -\ p_1}{p_2} \ -\ 1\right) \qquad \text{(A7)}$$

In summary, Equations (A5) and (A7) are what we seek [viz., a proof of Equations (3) and (4)]. •