Effective Rank and Dimensionality Reduction: From Complex Disaggregation Back to a Simple World

Tsoulfidis, Lefteris

University of Macedonia

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Lefteris Tsoulfidis
Professor, Department of Economics
University of Macedonia
Thessaloniki, Greece
Email: Lnt@uom.edu.gr

Abstract
In recent years there is a revival of political economy and discussions are about the near linearities of price rate of profit trajectories. In this article, we argue that the economy's input-output data are of low effective dimensionality, meaning that there is overfitting of both data and dimensions and that the fundamental behavior of the economy can be tracked down with the use of a low dimensional system.

Key words: Near-linearity, price trajectories, eigendecomposition, effective rank, Shannon entropy

JEL Classifications: B24, B51, C67, D46, D57, E11, E32

1. Introduction

In recent years, the research has repeatedly shown that the shape of the price rate of profit trajectories and the wage rate of profit curves are near linear. Curved trajectories do exist, but they are relatively few, and even fewer are the trajectories with a single extremum and we do not exclude the possibility of two extrema in the relevant region. The explanations offered for these linearities were based on the characteristic distribution of the eigenvalues of the system matrices. More specifically, in the usual dimensions of input-output matrices, the dominant eigenvalue is significantly higher (by 40% to 60%) than the second, followed by the third and a few more, their exact number depending on the size of the matrices. The remainder eigenvalues form a long tail and paint an exponentially falling distribution.

Three hypotheses have been put forward to explain this distribution of eigenvalues and the associated with this linearities:

1. The randomly distributed input-output coefficients (Bródy 1997; Schefold 2022).
2. The closeness of vertically integrated compositions of capital (VICC) between sectors (Shaikh 2016, Shaikh 2022, Ferrer-Hernández and Torres-González 2022 and the literature cited there).
3. The low effective-rank or effective dimensionality of the utilized matrices shapes the exponential fall in their eigenvalues, which in turn determines the near-linear features of PRP and WRP curves (Mariolis and Tsoulfidis 2018, Tsoulfidis 2021 and 2022).

The purpose of this study is to examine the extent to which these three hypotheses are consistent with the available evidence and proceed with the less researched third hypothesis by operationalizing a new metric of effective rank based on Shannon entropy.

The remainder of the article is structured as follows: Section 2 examines the realism of these competing explanations and introduces the concept of effective rank (and dimensionality) to identify the number of eigen- or singular-values that condition the behavior of the entire economic system. Section 3 illustrates the theoretical discussion by utilizing actual input-output data of the US economy of 15 sectors for (the most recent) year 2020, so the reader may have a better grasp of the usefulness and reliability of the approach. Section four discusses the results of the analysis and brings additional evidence in support of the view that the effective rank of the empirical input-output matrices is by far lower than their nominal rank. Section five concludes with the idea that there is overfitting of data and that fewer data and dimensions compressed in two or three sectors would be adequate to convey the essential behavioral features of the system.

2. Effective rank and dimensionality

Our research has shown that the first of the above hypotheses does not corroborate with the available evidence. The reason is that although a random or rather a near random matrix gives rise to an exponentially falling distribution of eigenvalues. However, it does follow that every skew distribution of eigen- or singular-values comes from a random matrix. Our empirical analysis in Tsoulfidis (2021) and (2022) has shown that the random matrix hypothesis does not pass the statistical tests. First, because the actual output vector, $\mathbf{x}$ of the input-output coefficient matrix is quite different from the standard or right-hand-side (r.h.s.) output vector, $\mathbf{s}$ derived either directly from matrix $\mathbf{A}$ or by its multiplication by the Leontief inverse, $\mathbf{H} = \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$. The idea is that if the two vectors are no different to each other, it follows the price – rate of profit trajectories and the wage – rate of profit curves will be linear. Second the employment coefficients vector, $\mathbf{l}$ also differs significantly from the left hand side (l.h.s.) unique positive eigenvector, $\boldsymbol{\pi}$ of the matrix $\mathbf{A}$ or of $\mathbf{H}$. The idea is that if differences between these two vectors are minimal, it follows that the economy is described by the case of an equal composition of capital between sectors. From the above it follows that for the randomness hypothesis to hold the following difference vectors, namely, $\mathbf{d}_1 = \mathbf{x} - \mathbf{s}$ and $\mathbf{d}_2 = \mathbf{l} - \boldsymbol{\pi}$, must display both zero correlation and zero covariance.

Our findings in testing the USA input output tables of the years 2007 and 2014 of dimensions 54 industries (Timmer *et al*. 2015) and of the years 2012 of 70 industries (www.bea.gov) suggest that the correlation between $\mathbf{d}_1$ and $\mathbf{d}_2$ is statistically significant, and therefore the randomness hypothesis is not consistent with the data despite of the fact that the covariance of the two vectors was found zero.[1] The zero covariance does not help, because of its dependence on the normalization condition. However, the same is not true with the correlation coefficient, and if the correlation coefficient is positive and statistically significant, the randomness hypothesis does not hold on purely statistical reasoning. Besides, there are other more intuitive and systematic reasons related to the nature of technological change and the associated input-output coefficients, whose value is declining over time (Carter 1970 and Tsoulfidis and Tsaliki 2019). The persistence of the ranking of industries according to backward, forward and their total linkages is another reason that renders the randomness hypothesis not coming to terms with the empirical evidence (Tsoulfidis and Athanasiadis 2022).

The exponentially decreasing distribution of eigenvalues is also consistent with the remaining two hypotheses from which the closeness of VICCs to the economy-wide average is quite appealing to researchers. The idea is that if the VICCs are too close to each other, except for just a few, it follows the maximal eigenvalue (along with a few

[1] All four vectors are normalized in the unit simplex, that is, the sum of their elements is equal to one.

others) will be crucial for the behavior of the entire economy lending support to the conceptualization of one-commodity world economies. The remainder of eigenvalues will be flocking together at negligibly small values, whose effect will not be felt in the economy. The trouble with this hypothesis is that the estimation of VICCs depends on equilibrium prices for which we need the VICCs. In short, there is cyclicality, which can be hardly overcome unless the estimations are carried out in terms of labor values or market prices or simply by stipulating that all three kinds of prices end up in quite close estimates. However, the question becomes, how can one decide between too different or too similar VICCs? There is no such metric, and the notion of the VICC, although intuitively in the right direction, nonetheless requires further qualifications. Thus, it becomes imperative to invoke (if not contrive) a metric that is independent of prices.

Consequently, we are left with the third in line hypothesis which we need to introduce first and then discuss its explanatory content. Roy and Vetterli (2007) are from the first that proposed a metric for the estimation of the effective rank of a matrix.[2] In order to find the required number of terms to be included in the representation, they employ the Shannon (1948) entropy index or the spectral entropy defined as

$$S = -\sum_i^n \sigma_i \log \sigma_i \qquad (1)$$

where $\sigma_i{}'$s stand for the normalized singular values of the matrix, whose effective rank we want to estimate, with i=1, 2, …, n. Thus,

$$\sigma_i = s_i / \sum_i^n s_i \text{ where } s_i = s_1 \geq s_2 \geq \cdots \geq s_n \geq 0 \text{ are the singular values.}$$

By stipulating that $0\log(0) = 0$, the effective rank ($erank$) of our matrix can be written

$$erank(\mathbf{H}R) = e^S \qquad (2)$$

It follows that the more similar the singular values, the higher the entropy, whose maximum is attained when $\sigma_i = 1/n$ for all $i = 1, 2, \dots, n$. In the hypothetical case, that all the $\sigma_i{}'$s are of the same value, the entropy will be $-\log n^{-1}$. The exponential of this term gives an effective rank equal to one whereas the maximal nominal rank might be n, that is, the number of linearly independent rows or columns. In the case of a random matrix its effective rank will be 1 and the nominal n.

[2] Their metric is inspired by the work of Campbell (1960).

However, the following statement by Roy and Vetterli (2007): "In the sequel, all logarithms are to the base e and we adopt the convention that $0\log0 = 0$", unfortunately, made the present author utilize natural logarithms and the derived results were not of any help at all. But as they say "every cloud has a silver lining", which in this case led the research to indirect estimates of the effective rank through an eigendecomposition of the matrix $\mathbf{H}R = \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-\mathbf{1}}R$, where R the reciprocal of the maximal eigenvalue of $\mathbf{H}$. The matrix $\mathbf{H}R$, which can be rewritten into the following eigen or spectral decomposition form (Meyer 2001, pp. 243-4, Mariolis and Tsoulfidis 2018)

$$\mathbf{H}R = (\mathbf{y}_1\mathbf{x}'_1)^{-1}\mathbf{x}'_1\mathbf{y}_1 + \lambda_2(\mathbf{y}_2\mathbf{x}'_2)^{-1}\mathbf{x}'_2\mathbf{y}_2 + \cdots + \lambda_n(\mathbf{y}_n\mathbf{x}'_n)^{-1}\mathbf{x}'_n\mathbf{y}_n \quad (3)$$

where, $\lambda_i,\ i = 1, 2, \ldots, n$ stand for the normalized eigenvalues of the matrix $\mathbf{H}$ with the dominant $\lambda_1 = 1$, and $\mathbf{y}$ and $\mathbf{x}$ are the left-hand side (l.h.s.) and right-hand side (r.h.s.) eigenvectors, respectively. The prime over the vector $\mathbf{x}$ indicates its transpose. The first or the maximal eigenvalue is denoted by $\lambda_1 = 1$ whereas the second eigenvalue by λ_2 and the remainder or subdominant eigenvalues by λ_n. Since each of the formed matrices is the result of multiplication by the two vectors, it follows that their respective rank will be equal to one. In adding more terms, we merely increase the rank of the resulting matrices according to the number of their terms. It would be of great interest to see if the two measures; namely, the eigendecomposition and effective matrix rank, give rise to the same answer. Furthermore, whether or not their combination leads to more definitive, from a practical point of view, conclusions about the effective rank of the matrix. This is the reason, in the section below, we proceed with an illustration based on an actual input-output table based on fifteen sectors.

3. A Simple Illustrative Realistic Example

We utilize the more recent input output table of the US economy of the year 2020 starting with the 15x15 sectoral structure of total requirements, or what is the same as the Leontief inverse, $[\mathbf{I} - \mathbf{A}]^{-1}$.[3] The matrix of input-output coefficients $\mathbf{A}$ is obtained

[3] The following are the fifteen sectors: 1. Agriculture etc., 2. Mining, 3. Utilities, 4. Construction, 5. Manufacturing, 6. Wholesale trade, 7. Retail trade, 8. Transportation and warehousing, 9. Information, 10. Finance, insurance, real estate, 11. Professional and business services, 12. Educational services,

by inverting the Leontief inverse and subtracting it from the identity matrix, $\mathbf{I}$. Thus we arrive at the matrix $\mathbf{H} = \mathbf{A}[\mathbf{I} - \mathbf{A}]^{-1}$, that is, the matrix of vertically integrated input-output coefficients. Furthermore, we estimate the vector of employment coefficients, $\mathbf{l}$ by dividing the sectoral wages by the respective output both available in the commodity by industry table of the same source. We adjust these findings by dividing by the economy-wide average wage as this is given by the social security administration (https://www.ssa.gov). The workers consumption goods vector, $\mathbf{b}$ is obtained by multiplying the so-obtained average money wage times the share of consumption goods in the total of each sector. With the help of these vectors and matrices (see Tsoulfidis 2021, 2022, and the literature cited there) we estimate the actual trajectories through the following formula

$$\mathbf{p} = (1 - \rho)\mathbf{v}[\mathbf{I} - \mathbf{H}R\rho]^{-1} \tag{4}$$

Where $\rho \equiv r/R$ is the relative rate of profit, that is, the ratio of the rate of profit, r corresponding to the reciprocal maximal eigenvalue of the matrix $\mathbf{A}[\mathbf{I} - \mathbf{A} - \mathbf{bl}]^{-1}$ and R, the maximal rate of profit corresponding to the reciprocal eigenvalue of matrix $\mathbf{H}$. Finally, $\mathbf{v}$ stands for the vector of labor values $\boldsymbol{\lambda} = \mathbf{l}[\mathbf{I} - \mathbf{A}]^{-1}$ which obtain their monetary expression (direct prices), when they are normalized according to $\mathbf{v} = \boldsymbol{\lambda}(\mathbf{ex})(\boldsymbol{\lambda}\mathbf{x})^{-1}$, where $\mathbf{e}$ is the row summation vector.

For reasons of clarity of presentation and economy in space, in Figure 1 below, we select to display eight out of fifteen price trajectories for illustrative purposes. And in each of the panel of eight graphs, we also display the three approximations, the linear, quadratic and cubic, according to relation (3). In the panels of graphs of Figure 1, we display just a few sectors (with the most curved trajectories except for the last one) for illustrative purposes. On the horizontal axis of each of the graphs, we display the relative rate of profit, ρ and on the vertical axis the ratio of estimated price, $\mathbf{p}$ over the values, $\mathbf{v}$, or p_i/v_i. The straight lines refer to the linear approximations, the dashed black lines stand for the square approximation, the red dotted lines represent the cubic approximations, finally, the blue continues with the round markers lines are the actual

health care, and social assistance, 13. Arts, entertainment, recreation, accommodation, and food services, 14. Other services, 15. Government.

estimated prices whose paths we want to approximate. The crossing of line of price-value equality indicates a change in the characterization of capital intensity (Sraffa 1960).

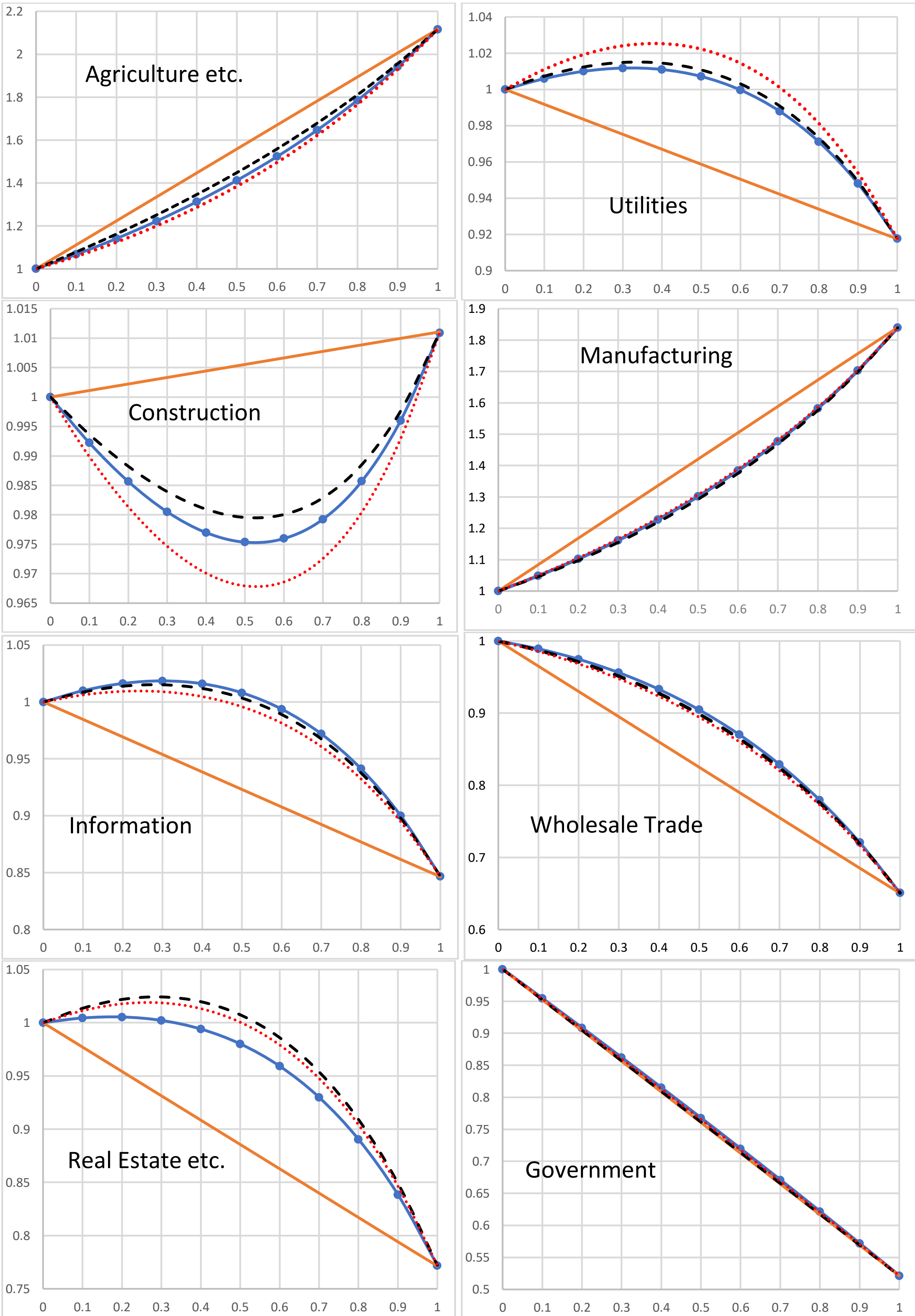

Figure 1. Linear, Quadratic and Cubic Approximations

The linear approximation in and of itself is satisfactory even in case one would tolerate a relatively small deviation. The quadratic approximation, in general, is an improvement over the linear, even for this small size of input-output description, but one cannot say the same with the cubic, which we find, in most cases, excessive and therefore redundant.

One would be wondering of whether the same answer we would derive through the exponential of Shannon index of entropy. We apply the singular value decomposition (SVD) method in the matrix **H**. The idea is that the eigenvalues might be negative and also maybe complex; hence, the use of absolute eigenvalues might be a solution but we think the singular values are the most appropriate for the task at hand. They differ from the eigenvalues of the same matrix **H**, in that they are the positive square roots of the eigenvalues of the matrix $\mathbf{H'H}$, which are no different from those of the matrix $\mathbf{HH'}$. Our estimates are shown in Table 1 below:

Table 1. Singular values, Shannon's entropy and Effective rank,

Ranking of Singular Values	Singular Values (1)	Normalized Singular Values (2)	Common Logarithms of (2) (3)	The Product of (2)x(3) (4)
1	1.266776	0.476572	-0.32187	-0.15339
2	0.500926	0.188453	-0.72480	-0.13659
3	0.252810	0.095109	-1.02178	-0.09718
4	0.171247	0.064425	-1.19095	-0.07673
5	0.138969	0.052281	-1.28165	-0.06701
6	0.096752	0.036399	-1.43891	-0.05237
7	0.073059	0.027486	-1.56089	-0.04290
8	0.045746	0.017210	-1.76422	-0.03036
9	0.035570	0.013382	-1.87348	-0.02507
10	0.024936	0.009381	-2.02774	-0.01902
11	0.017067	0.006421	-2.19242	-0.01408
12	0.015397	0.005792	-2.23714	-0.01296
13	0.008659	0.003258	-2.48711	-0.00810
14	0.006393	0.002405	-2.61887	-0.00630
15	0.003790	0.001426	-2.84591	-0.00406
Sum:	2.658	1.000	Shannon (S)	-0.746
			$\text{erank} = e^{-S}$	2.109

From Shannon's index of entropy, S whose exponential is equal to 2.109, the effective rank of the system matrix is equal to 2. A result absolutely consistent with the approximations through the eigendecomposition, where we found only marginal improvements adding the quadratic term whereas the cubic term did not improve the approximation in our 15x15 input-output structure, an indication that we should not go beyond the quadratic term in dimensions of this size input-output structure. The same effective rank equal to two, we got by using the absolute eigenvalues instead of the singular ones.

4. Results and their evaluation

We have also experimented with input-output data of various dimensions for the same country and years. The differences, we found were in the decimals, which however do not play any role because at the end the rank must be a one-digit number. More specifically, our estimates for the US economy of the benchmark years 2007 and 2012 showed that the 15x15 dimensions gave that a quadratic approximation would be adequate. When we increased the dimensions to 70x70 and, for the same years (www.bea.gov), the effective rank doubled; however, the eigen- or spectral-decomposition indicated that, for all practical purposes, a cubic term is more than satisfactory while the fourth or fifth terms do not necessarily improve the approximation (Tsoulfidis 2022). We have also tested the 405x405 dimensions input-output data, which gave an effective rank equal to eight. We did not, at present, try eigen approximations for these super high input-output tables. We have also estimated the old 65x65 tables of the BEA, which, also gave effective ranks or dimensions equal to four. Not surprisingly, since the distribution of eigen- or singular-values pretty much remains the same over the years.

The results in the case of matrices of lower dimensions 54x54 of the USA, 2007 and 2014 (Timmer et al. 2015) were quite similar. In both matrices, we found that the quadratic approximation of the price trajectories is more than satisfactory (Tsoulfidis 2021). In contrast, the cubic and the quartic terms did not add much information, even in those trajectories characterized by the highest curvature. These particular trajectories are those of the minimal difference between prices and labor values, indicating the closeness of their VICCs to the economy-wide average or the standard ratio. The results for the other countries, to the extent tested, were no different from those of the US economy. The distribution of eigen- and singular-values displayed a repeated pattern described by the exact same parameters of an exponential equation whose fit in the distribution of the eigenvalues of all years and countries tested has been excellent. These results lead to the idea that there are certain regularities embedded deeply in the available input-output data and they are manifested through the skew distribution of eigen- or singular values, which in turn determine the effective rank and dimensions of the system matrices. From a mathematical point of view, the idea of the effective rank and dimensions and their estimation through the above formula is quite reasonable. After all, the top few singular values are distinct and quite different from the bulk of singular values, and these top singular values are those that compress a lot more information than the rest of the singular values combined.

Finally, the matrix of fixed capital stock, derived through the capital flows tables indicated much lower dimensions, and the quadratic term would be more than enough. After all, the second eigen- or spectral-value in these matrices is markedly lower than the maximal. Besides, in capital stock matrices, as expected, there are too many rows with zero elements. The idea is that neither the consumer goods industries nor services produce any capital goods, so their rows are filled either with zeros or with relatively small numbers. It is important to point out that the multiplication of the capital stock matrix by the Leontief inverse gives rise to a new matrix whose form resembles that of the capital stock matrix. In counting, the number of zeros in our 65x65 capital stock matrix, we found 39 rows which when added to the zeros scattered to the rest of cells, they amounted to 61 percent of total figures of the capital stock matrix, without counting the near-zero negligibly small elements (Tsoulfidis and Tsaliki 2019, Tsoulfidis 2021).

From our discussion so far follows that both the spectral decomposition and the effective rank operate complementary to each other and help us approximate economic reality with solid analytical tools capable of extracting its essential features. The hitherto analysis has shown that Samuelson’s (1962) one-commodity world description of the economy was an oversimplification, but so was Ricardo’s corn model, Marx’s schemes of simple reproduction based on the assumption of equal organic composition of capital between departments, and the currently in use growth models. Our findings of near-linear price trajectories by no means suggest that the neoclassical theory is adequate in dealing with real-world features. On the contrary, we argue that the problems are in the assumption of given endowments, near-perfect substitutability, and the subjective nature of preferences that permeate the whole neoclassical analysis, regardless of whether it refers to the pure exchange economy or production, which is theorized as indirect exchange. There is a by far better alternative couched on the labor theory of value that was abandoned for purely ideological reasons. Our analysis so far has shown that the first two eigen- or singular-values are adequate to construct models that mimic the operation of the entire economy. In this respect, the principal components analytical method may be used and it has been used profitably in this direction (Tsoulfidis and Athanasiadis 2022).

5. Concluding remarks

In short, the applied factorization method revealed that the structure of the economies is simpler than is commonly thought, and a lot of information is compressed in the maximal eigenvalue of the system matrices while the remaining eigen- or rather singular-values add relatively little additional information. Thus, by limiting ourselves to the first few terms of the eigendecomposition, we obtain a satisfactory approximation of the price trajectories consequent upon changes in income distribution. In so doing, we end up with the view that the actual economies are not similar to a one-commodity world. The latter would require equal capital intensities between industries, which is another way to say that the system's matrices would have nominal and effective rank equal to one. This does not mean that our multi-commodity world requires all commodities and dimensions to uncover its structural features. In a nutshell, we are dealing with overfitting data and over-dimensional representations of the actual economies. Our analysis has shown that the deep laws of motion of the system can be laid bare by de-noising our data and meaningfully compressing the dimensions of the system to just a few.

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