Asset Demand: A Simple Dual Approach

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Abstract

This paper provides a simple framework for obtaining asset demand using indirect utility functions. Assuming expected utility maximization, we show that assets are held according to their mean returns' proportional marginal utility. We also show that an asset's equilibrium equity premium is given by the ratio of the indirect utility function's mean and standard deviation elasticities. Furthermore, we show that we can extend these results to a non-expected utility framework.

KEYWORDS: Moments, Indirect Utility Function, Asset Demand, Duality.

JEL Classification: G11, D14, D80, D81

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1 Introduction

Starting with the seminal paper by H.M. Markowitz,[1] there is now a vast literature on portfolio choice theory.[2] The most widely used model is based on the mean-variance approach. A more general model, first introduced by Arrow (1965), assumes expected utility maximization. Although the latter is a more general model, the former is simpler and has become the basis for modern portfolio choice theory.

This paper proposes a simple dual approach for obtaining asset demand functions. It shows that an individual's demand for assets can be derived from the indirect utility function (IUF) in moments space and is given by the proportional marginal utility of means' returns. We also show that an asset's proportional equity premium is given by the ratio of the IUF's elasticities with respect to the mean and standard deviation. Moreover, we show that, in both cases, all moments of the distribution matter. Finally, our derived relationships are based on estimable "right-hand-side" variables and are consistent with expected and non-expected utility maximization models.

2 Portfolio Choice and the Indirect Utility Function

Consider an individual who invests in $n+1$ assets whose total returns are given by the vector $R = (R_0, R_1 ... R_n)$. We assume that R_0 is a risk-free total return, but $R_1 ... R_n$, are bounded continuous random variables with a cumulative distribution function, G, a finite support $\mathcal{Z} \equiv (R : R \in [\underline{R}, \overline{R}])$ and finite (at least) first and second-order moments, given by the vector $\mu = E(R)$ and the covariance matrix Σ.

Let the continuous and bounded vector $x = (x_0, x_1 ... x_n)$ denote the amounts invested in the $n+1$ assets. Assuming that the individual has an amount of $A = 1$ to invest, the feasible set, $\mathcal{X}$, is given by:

$$\mathcal{X} \equiv \{x : \sum_{i=0}^{n} x_i = 1\}. \tag{1}$$

The individual's terminal wealth, w, is given by,

$$w = \sum_{i=0}^{n} R_i x_i = R'x \tag{2}$$

[1] See Markowitz (1952).

[2] See, for example, classic early papers by Merton (1969) and Samuelson (1970).

where R' is the row vector of total returns. Since $w = R'x$ is a linear function of R (and the vector x is bounded), terminal wealth is also a continuous and finite random variable whose distribution can be derived from G.

It is useful to define the vector of return as:

$$R_i \equiv \mu_i + \sigma_i e_i,\ i = 0...n \tag{3}$$

where $\mu_0 = R_0,\ \sigma_0 = 0,\ e = (e_0, e_1...e_n)$ and $(e_1...e_n)$ is a vector of continuous and bounded random variables whose cumulative distribution is G_e, with $E(e_i) = 0,\ i = 1...n$. Using this in equation (2), we can write w as:

$$w = \sum_{i=0}^{n} (\mu_i + \sigma_i e_i) x_i \tag{4}$$

Given that the distribution G (and hence G_e) is uniquely characterized by its moments,[3] and assuming expected utility maximization, the solution to the individual's problem can be characterized by the indirect utility function, V, defined as:

$$\max_x \{E\{u(\sum_{i=0}^{n}(\mu_i + \sigma_i e_i)x_i\} :\ \ x \in \mathcal{X}\} \equiv V(\mu, \Sigma, M_{-2}) \tag{5}$$

where M_{-2} denotes the moments of the distribution other than μ and Σ.

Appelbaum (2006) shows that the convexity of V is a necessary (but not a sufficient) condition for expected utility maximization.[4] On the other hand, V may or may not be convex in moments without expected utility maximization.[5]

In addition to the convexity of V, it is easy to show that V is increasing in the first moments but not necessarily in the second moments. To show this, we use the envelope theorem and obtain from (5) that for all utility functions with $\frac{\partial u(w)}{\partial w} > 0$, we have:

$$\frac{\partial V}{\partial \mu_i} = E\{\frac{\partial u(w)}{\partial w}\} x_i > 0,\ i = 0...n \tag{6}$$

From equation (5) we also have:

$$\frac{\partial V}{\partial \sigma_i} = E\{\frac{\partial u(w)}{\partial w} e_i\} x_i = Cov[\frac{\partial u(w)}{\partial w}, e_i] x_i \ \ i = 1...n \tag{7}$$

Unfortunately, even for a risk-averse individual, the sign of $E\{\frac{\partial u(w)}{\partial w} e_i\}$ is ambiguous unless there is only one risky asset (in which case $Cov[\frac{\partial u(w)}{\partial w}, e_i] < 0$ for the single risky asset if the individual

[3] This is the so-called "moments problem." For proof, see Wilks (1964), theorem 5.5.1. p. 126.

[4] The proof is based on Machina (1984), who shows that expected utility maximization.implies that the IUF (defined in equation (5)) is convex in the distribution.

[5] See non-expected utility examples in Appelbaum (2006).

is risk-averse). With many risky assets, the sign of $Cov[\frac{\partial u(w)}{\partial w}, e_i]$ is unknown because it depends on the correlation between e_i and all other $e'_j s,\ j \neq i$. Therefore, even for a risk-averse individual, an increase in the second moment may not decrease V.

Let us now consider the optimal portfolio. Summing over all assets in equation (6), we have:

$$\sum_{i=0}^{n} \frac{\partial V}{\partial \mu_i} = E\{\frac{\partial u(w)}{\partial w}\} \sum_{i=0}^{n} x_i = E\{\frac{\partial u(w)}{\partial w}\},\ i = 0...n \tag{8}$$

Then, substituting $\sum_{i=0}^{n} \frac{\partial V}{\partial \mu_i}$ back into equation (6), we get the demand for asset $i = 0...n$, as:

$$x_i = \frac{\partial V(\mu, \Sigma, M_{-2})}{\partial \mu_i} / \sum_{i=0}^{n} \frac{\partial V(\mu, \Sigma, M_{-2})}{\partial \mu_i},\ i = 0...n \tag{9}$$

Thus, we have:

Proposition 1: Each asset's demand is given by the proportional marginal utility of its mean return.

Naturally, since the IUF is a function of all moments, so are asset demands. The "proportionality rule," in equation (9), is simple, intuitive, based on estimable "right-hand-side" variables, easy to use in empirical applications, consistent with expected utility maximization, and can also be extended to non-expected utility models.[6] To estimate asset demand, first, we need to choose a functional form for V that is increasing in means and convex in all moments. Second, asset holdings data is required. If moments data is not available, we can use GARCH, or nonparametric[7] models to estimate the relevant moments.

Finally, in a non-expected utility framework, the IUF will still be a function of all moments; however, it may or may not be convex. Nevertheless, Proposition 1 will still hold.

2.1 Equilibrium Asset Prices

The equilibrium relationship between expected asset retuns and the moments can be easily obtaind from the IUF. First, we use the budget constraint to re-write w as:

$$w = R_0 + \sum_{i=1}^{n} (\mu_i - R_0) x_i + \sum_{i=1}^{n} \sigma_i e_i x_i \tag{10}$$

[6] A non-expected utility model, in the context of firms' decisions, is given in Appelbaum (2006).
[7] See Appelbaum and Ullah (1997).

Using equation (10), the solution to problem (5) is described by the IUF $V(\mu, \Sigma, M_{-2})$, where:

$$\max_{x_1,...x_n} \{E\{u(R_0 + \sum_{i=1}^{n}(\mu_i - R_0)x_i + \sum_{i=1}^{n}\sigma_i e_i x_i\} : \; x \in \mathcal{X}\} \equiv V(\mu, \Sigma, M_{-2}) \tag{11}$$

For all assets with an interior solution, the first-order conditions for this problem can be written as:

$$[\mu_i - R_0]E\{\frac{\partial u(w)}{\partial w}\} = -\sigma_i E\{\frac{\partial u(w)}{\partial w}e_i\}, \; i = 0...n \tag{12}$$

But, substituting equations (6) and (7) into equation (12), it can be re-written as:

$$\mu_i = R_0 - \sigma_i \frac{\partial V}{\partial \sigma_i}/\frac{\partial V}{\partial \mu_i} \equiv F(V(\mu, \Sigma, M_{-2}), \tag{13}$$

where $-\sigma_i \frac{\partial V}{\partial \sigma_i}/\frac{\partial V}{\partial \mu_i}$ is the equity premium and $-\frac{\partial V}{\partial \sigma_i}/\frac{\partial V}{\partial \mu_i}, \; i = 1...n$, is the slope of the individual's IUF indifference curve in (μ_i, σ_i) space. Equation (13) gives us the equilibrium relationship between asset expected returns and the moments of the distribution.[8] Alternatively, the (equilibrium) proportional equity premium (PEP), $(\mu_i - R_0)/\mu_i$, can be written as:

$$\frac{\mu_i - R_0}{\mu_i} = -\frac{\partial \ln V}{\partial \ln \sigma_i}/\frac{\partial \ln V}{\partial \ln \mu_i} \equiv \frac{\theta_{\sigma_i}}{\theta \mu_i}, \tag{14}$$

where $\theta_{k_i} = \left|\frac{\partial \ln V}{\partial \ln k_i}\right|$, $k = \sigma, \; \mu$, is the absolute value of $V's$ elasticity with respect to k_i. Thus, we have:

Proposition2: The proportional equity premium is given by the ratio of IUF 's elasticities with respect to the mean and standard deviation.

Equation (13) does not consider asset markets' general equilibrium conditions, as in the CAPM.[9] However, unlike the CAPM equation, equity premia depend on all the distribution moments in this model. Thus, skewness, kurtosis and even the fifth moment (a measure of tails' asymmetry) may affect the PEP.

As with asset demand, here, too, once we choose a functional form for V and given estimates of (relevant) moments, we could estimate equation (14) or (13), together with the system if asset demand equations to study the effects of the moments on both asset demand and PEP. Finally, Proposition 2 still holds for non-expected utility maximization models.

[8] Note that, using equation (13), we obtain the Sharpe Ratio, SR_i, as $SR_i \equiv \frac{\mu_i - r_0}{\sigma_i} = -\frac{\partial V}{\partial \sigma_i}/\frac{\partial V}{\partial \mu_i}, \quad i = 1...n$. This is the slope of the individual's IUF indifference curve in (μ_i, σ_i) space. A similar result is obtained in Appelbaum and Basu (2010) in the context of the macro equity premium puzzle.

[9] For example, see Sharpe (1964).

3 Conclusion

This paper uses the IUF in moments space to derive asset demand functions and the equilibrium relationship between expected asset returns and the moments of the distribution. The rules we derive are simple, intuitive and easy to use in empirical applications. Furthermore, we show that both asset demand functions and equilibrium asset expected returns depend on all the distribution moments, are easy to apply empirically, consistent with expected utility maximization, and can be extended to non-expected utility models.

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