The Market-Based Asset Price Probability

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Abstract

This paper introduces the *market-based* asset price probability during time averaging interval *Δ*. We substitute the present problem of guessing the "correct" form of the asset price probability by description of the price probability as function of the market trade value and volume statistical moments during *Δ*. We define *n-th* price statistical moments as ratio of *n-th* statistical moments of the trade value to *n-th* statistical moments of the trade volume. That definition states no correlations between time-series of *n-th* power of the trade volume and price during *Δ,* but doesn't result statistical independence between the trade volume and price. The set of price *n-th* statistical moments defines Taylor series of the price characteristic function. Approximations of the price characteristic function that reproduce only first *m* price statistical moments, generate approximations of the *market-based* price probability. That approach unifies probability description of *market-based* asset price, price indices, returns, inflation and their volatilities. *Market-based* price probability approach impacts the asset pricing models and uncovers hidden troubles and usage bounds of the widespread risk hedging tool – Value-at-Risk, lets you determine the price autocorrelations and revises the classical option pricing from one to two dimensional problem. Market-based approach doesn't simplify the price probability puzzle but establishes direct economic ties between asset pricing, market randomness and economic theory. Description of the *market-based* price and returns volatility, Skewness and Kurtosis requires development of economic theories those model relations between second, third and forth order macroeconomic variables. Development of these theories will take a lot of efforts and years.

Keywords : asset price, price probability, returns, inflation, market trades

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1. Introduction

Aspirations to have price predictions for the next day, month or year are as old as market trades. Merchants and pawnbrokers, investors and traders, bankers and stockbrokers, academics and households for years, decades and centuries seek for reliable, secure and precise price forecasts. However, little by little it became clear that exact price guesses as well as tomorrow fortune forecasts are too fickle and variable. Thus, exact predictions of a single price for the next time term were replaced by variety predictions of possible set of price values. Ambiguity of the future was projected in uncertainty of price prognosis. Centuries of asset pricing studies (Dimson and Mussavian, 1999) track price probability up to Bernoulli's studies 1738, but probably, Bachelier (1900) was one of the most influential paper that outlines probabilistic character of the price behavior and forecasting. “The probabilistic description of financial prices, pioneered by Bachelier”(Mandelbrot, et.al., 1997). “in fact the first author to put forward the idea to use a random walk to describe the evolution of prices was Bachelier.” (Shiryaev, 1999). During last century the endless number of studies discussed asset pricing models and described different hypothesis, laws and properties of random asset price (Kendall and Hill, 1953; Muth, 1961; Sharpe, 1964; Fama, 1965; Stigler and Kindahl, 1970; Black and Scholes, 1973; Merton, 1973; Tauchen and Pitts, 1983; Mackey, 1989; Friedman, 1990; Cochrane and Hansen, 1992; Campbell, 2000; Heaton and Lucas, 2000; Cochrane, 2001; Poon and Granger, 2003; Andersen et.al., 2005a; 2005b; 2006; Cochrane, 2005; Wolfers and Zitzewitz, 2006; DeFusco et.al., 2017; Weyl, 2019; Cochrane, 2022). Rigorous mathematical treatment of stochastic description and probabilistic modelling of asset price is given in (Shiryaev, 1999; Shreve, 2004). It is obvious, that we referred only a negligible part of endless studies on asset pricing.

Asset price dynamics is under action of numerous factors of different nature that result irregular or random price change during almost any time interval. A century ago Fetter (1912) mentioned 117 price definitions, and that for sure increase varieties of asset price considerations, treatments and forecasting. That generates enormous range of price studies that outline price variations and dependence on market (Fama, 1965; Tauchen and Pitts, 1983; Odean, 1998; Poon and Granger, 2003; Andersen et.al., 2005b; DeFusco et.al., 2017..), macroeconomic (Cochrane and Hansen, 1992; Heaton and Lucas, 2000; Diebold and Yilmaz, 2008), business cycles (Mills, 1946; Campbell, 1998), expectations (Muth, 1961; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014), trading volumes (Karpoff, 1987; Campbell et.al., 1993; Gallant et.al., 1992; Brock and LeBaron, 1995;

Llorente et.al., 2001) and many other factors that for sure impact price trends and fluctuations. The line of factors and references can be continued forever (Goldsmith and Lipsey, 1963; Andersen et.al., 2001; Hördahl and Packer, 2007; Fama and French, 2015).

However, we have no idea to review the asset pricing studies, but chose one "simple" problem of the asset price universe. That problem concerns the notion and definition of the asset price probability density function (PDF). It seems to be one of the most common and well-studied issues of financial economics. Almost all standard probability distributions (Forbes, et.al., 1992; Walck, 2011) were tested to check how they can model, describe and predict price PDF and random price properties. A lot was done but the asset price PDF puzzle seems to be still inconceivable. Actually, it is twice interesting to have a fresh look at the conventional matter. Indeed, since Bachelier (1900) the joint efforts of economists and statisticians were directed to uncover the "correct" model of the random price change and its PDF. May be credibility and domination of Bachelier and his famous followers put studies of the price probability not that side? We don't critique any notable studies but remind that the asset price is not a single, main and independent issue of economics and financial markets. Asset pricing is woven deeply into relations, laws and properties of the economy and finance. We consider the asset price PDF problem as a puzzle of the economic and financial relations, as a result of market evolution and not as a standing separately question. As we mentioned above, there are a lot of different definitions of price. Fetter (1912) presents 117 price definitions and hence there are probably a lot of different treatments and approaches to price probability modelling. We don't review that variety of price definitions but consider single, simple and conventional market-based price notion that is trivially determined by each market trade. Indeed, each particular market trade at time t can be determined by the trade value $C(t)$, trade volume $U(t)$ and trade price $p(t)$ those match simple relations:

$$C(t) = p(t)U(t) \tag{1.1}$$

It is well known that the time-series records of the values, volumes and prices of the performed market trades are very irregular and usually assumed as random. Trivial equation (1.1) establishes important requirement on the probabilities of time-series that match (1.1). Indeed, PDF of the time-series of market trades performed at moments t_i of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ those match (1.1) cannot be determined independently. Given probabilities of the trade value and volume (1.1) determine the asset price probability.

That approach to definition of the market-based asset price probability doesn't simplify the price probability puzzle, but establishes direct links between the stochasticity of the market trade value and volume on one hand and PDF of the market price on the other hand. Actually, we replace the initial problem: what is the "correct" price probability by a different one. We consider how probabilities of the market trade value and volume determine the asset price PDF, but don't study the properties of the market trade value and volume probabilities.

In Sec.2 we describe how probabilities of the trade value and volume determine the asset price probability. Further we briefly consider consequences for the market-based probability on returns and inflation, consider asset pricing models and Value-at-Risk (VaR) as most conventional risk management tool, and argue some other issues of financial economics. Sec. 5 – Conclusion. We assume that our readers are familiar with standard issues of probability theory as statistical moments, characteristic functions and etc. This paper for sure, is not for novices and we propose that readers already know or able find on their own definitions and explanations of the notions, terms and models that are nor given in the text.

2. Market-based price probability

To start with, let us consider the time-series of the market trades with selected asset performed at moments t_i, $i=1,\dots$. Economic analysis of time-series has a long history and references (Davis, 1941; Anderson, 1971; Cochrane, 2005; Diebold, 2019) indicate author's preferences only. We take that the market trades are performed at moments t_i and are presented by records of time-series of the trade value $C(t_i)$, trade volume $U(t_i)$ and price $p(t_i)$. The moments of time t_i of the performed market trades introduce the initial time division of the time axis of the problem under consideration. We study random properties of market trades using records of performed trade's time-series only. Thus, any considerations of possible impact of agent's price expectations, price forecasts, economic or financial factors and any possible influence on the market price can be omitted. Actually, all possible factors those impact market pricing are already collected into the records of the performed market trades that are described by time-series of the trade value $C(t_i)$ and volume $U(t_i)$. Hence, one can state that the randomness of the market price time-series is completely determined by stochastic properties of time-series of the market trade value and volume (1.1).

For simplicity we take that all trades are performed at times t_i multiply of small interval ε:

$$t_i = \varepsilon \cdot i \;;\; i = 0, 1, 2, \dots \tag{2.1}$$

Time-series (2.1) establish time axis division multiplied of ε. Let us take that we study the market trade time series during time horizon T and assume that initial time division $\varepsilon \ll T$. High frequency trading can deliver market trade records with ε as fraction of second. Such precise time division generates high irregular time-series values and of little help for modelling price at long time horizon T. Description of market price at horizon T that equals weeks, months or years requires aggregation of the initial market time-series during some reasonable time interval Δ, that takes intermediate value

$$\varepsilon \ll \Delta \ll T \tag{2.2}$$

Market time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ aggregated or averaged during the interval Δ result time-series with time axis division multiple of Δ. For simplicity let take the interval Δ multiple of ε for some n as:

$$\Delta = 2n \cdot \varepsilon \quad ; \quad N = 2n + 1 \quad ; \quad \varepsilon \ll \Delta \ll T \tag{2.3}$$

Aggregation of time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ during the interval Δ generates corresponding time-series at moments t_k and results time axis division multiple of Δ

$$t_k = \Delta \cdot k \quad ; \quad \Delta_k = \left[t_k - \frac{\Delta}{2}; t_k + \frac{\Delta}{2}\right] \quad ; \quad k = 0, 1, 2, \dots \tag{2.4}$$

Let us take that each member of time-series of the trade value $C(t_k)$ at moment t_k (2.4) is determined by aggregation or averaging of time-series $C(t_i)$ during time interval Δ_k (2.4). Total trade value $C(t_k)$ and total trade volume $U(t_k)$ during the interval Δ_k are determined as

$$C(t_k) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U(t_k) = \sum_{i=1}^{N} U(t_i) \tag{2.5}$$

$$t_k - \frac{\Delta}{2} \le t_i \le t_k + \frac{\Delta}{2} \tag{2.6}$$

Due to our assumption (2.3) there are $N=2n+1$ members of time-series $C(t_i)$ or $U(t_i)$ in each time interval Δ_k. We consider time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ as random variables during the intervals Δ_k (2.4) and determine the mean market trade value $C(t_k;1)$ and the mean trade volume $U(t_k;1)$ at time t_k averaged during Δ_k as

$$C(t_k;1) = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \quad ; \quad U(t_k;1) = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{2.7}$$

We underline that mean values of market trade (2.7) are determined for the averaging interval Δ_k (2.2-2.4). Different choice of the averaging interval Δ (2.2) results in different values of the average trade value and volume (2.7).

For the given averaging interval Δ_k (2.2-2.4) we consider time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.6) as random variables. Probability measures of the trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.6) are determined in a conventional way (Shiryaev, 1999; Shreve, 2004). Probability of trade value $P(C)$ is proportional to frequency of trades at value C. If during the time interval Δ_k (2.6) there are m_c trades at value C and m_u trades at volume U then, due to (2.3) probability $P(C)$ and $P(U)$:

$$P(C) \sim \frac{m_c}{N} \quad ; \quad P(U) \sim \frac{m_u}{N} \tag{2.8}$$

Further we note conventional approach to probability definition similar to (2.8) as the *frequency-based* in contrary to the *market-based* definition of the price probability below. Statistical moments of the trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.6) are determined as usual:

$$C(t_k;n) = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(t_k;n) = \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \;\; ; \; n = 1, \dots \tag{2.9}$$

For $n=1,2,\dots$ statistical moments (2.9) completely determine properties of the trade value $C(t_i)$ and volume $U(t_i)$ treated as random variables for t_i inside Δ_k (2.6).

Now we can explain how statistical moments (2.9) of the market trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.6) determine random properties of the market price $p(t_i)$ that match (1.1) for t_i inside Δ_k (2.6). It is well known that random variable can equally be described by its PDF or by set of statistical moments (Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005). First price statistical moment – mean price $p(t_k;1)$ at moment t_k during the interval Δ_k (2.6) for price $p(t_i)$ time-series that match equation (1.1) is determined as

$$C(t_k;1) = p(t_k;1)U(t_k;1) \tag{2.10}$$

It is obvious that this definition of the mean price $p(t_k;1)$ (2.10) is completely coincides with definition of volume weighted average price (VWAP) (2.11) that was introduced at least 30 years ago and is widely used now (Berkowitz et.al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; CME Group, 2020; Duffie and Dworczak, 2021)

$$p(t_k;1) = \frac{\sum_{i=1}^{N} p(t_i)U(t_i)}{\sum_{i=1}^{N} U(t_i)} \tag{2.11}$$

However, derivation of VWAP (2.10; 2.11) as consequences of (1.1) hides interesting "minor" issue. Let note math expectation as $E[\dots]$. Then math expectation of (1.1) gives mean price $p(t_k;1)$ (2.10) or equally VWAP (2.11) if and only if one take correlations between random time-series $U(t_i)$ and price $p(t_i)$ averaged during the time interval Δ_k (2.4) to be equal zero:

$$E[C(t_i)] = E[p(t_i)U(t_i)] = E[p(t_i)]E[U(t_i)] \;\; \leftrightarrow E[\delta p(t_i)\,\delta U(t_i)] = 0 \tag{2.12}$$

$$p(t_i) = E[p(t_i)] + \delta p(t_i) \;\; ; \;\; U(t_i) = E[U(t_i)] + \delta U(t_i) \tag{2.13}$$

$$E[\delta p(t_i)] = \; E[\delta U(t_i)] = 0 \tag{2.14}$$

No correlations (2.12) between VWAP and trade volume is a result of definition of the price averaging procedure. Actually, numerous studies describe "observed" correlations between price and trading value (Tauchen and Pitts, 1983; Karpoff, 1987; Gallant et.al., 1992; Campbell et.al., 1993; Llorente et.al., 2001; DeFusco et.al., 2017). That underlines the different approaches to definition of the asset price PDF and price averaging procedure. Neglect of the trivial equation (1.1) that prohibit independent definitions of the trade value, volume and price probabilities results "observation" of correlations between trade volume and price. Usage of VWAP states no correlations between trade volume and price. However, in App.A we derive expressions that describe correlations between the price $p(t_i)$ and the square of trade volume $U^2(t_i)$ and correlations between the squares of price $p^2(t_i)$ the trade volume $U(t_i)$. The choice of price averaging procedure between the *frequency-based* independent approach and the *market-based* approach determines the viewpoint for consideration the random price properties.

Actually, the mean price $p(t_k;1)$ (2.10) or VWAP (2.11) don't define PDF of the random price during the averaging interval Δ_k (2.4). To fully define random price properties one should determine price PDF or price characteristic function or introduce all price *n-th* statistical moments. All three methods give equal description of random variable (Shephard, 1991; Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005). To define price *n-th* statistical moments for all $n=1,2,\ldots$ we use (1.1) and state no-correlation assumption similar to (2.12). Let us take *n-th* power of equation (1.1):

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 1,2,3,\ldots \tag{2.15}$$

Let assume no-correlation (2.16-2.18) between time-series of *n-th* power of the trade volume $U^n(t_i)$ and *n-th* power of the price $p^n(t_i)$:

$$E[\delta p^n(t_i)\delta U^n(t_i)] = 0 \tag{2.16}$$

$$p^n(t_i) = E[p^n(t_i)] + \delta p^n(t_i) \;\; ; \;\; U^n(t_i) = E[U^n(t_i)] + \delta U^n(t_i) \tag{2.17}$$

$$E[\delta p^n(t_i)] = \; E[\delta U^n(t_i)] = 0 \tag{2.18}$$

Then for (2.9) math expectation of (2.15) and (2.16-2.18) for all $n=1,2,\ldots$ determine price *n-th* statistical moments $p(t_k;n)$ during the averaging interval Δ_k (2.4) as:

$$E[C^n(t_i)] = E[p^n(t_i)]\, E[U^n(t_i)] \quad \leftrightarrow \quad C(t_k;n) = p(t_k;n)U(t_k;n) \tag{2.19}$$

$$E[C^n(t_i)] = C(t_k;n)\; ; \;\; E[U^n(t_i)] = \; U(t_k;n) \;\; ; \;\; E[p^n(t_i)] = \; p(t_k;n) \tag{2.20}$$

As we show in App.A, no correlations assumption (2.16) don't cause statistical independence between the trade volume and price random variables. Time-series of *n-th* power of trade volume $U^n(t_i)$ can correlate with time-series of *m-th* power of price $p^m(t_i)$ for $n \neq m$ and we derive correlations between the square of price p^2 and the trade volume U (A.4) and between the square of the trade volume U^2 and the price p (A.2).

It is well known, that the set of *n-th* statistical moments for all $n=1,2,\ldots$ of random variable determines its characteristic function as Taylor series (Shephard, 1991; Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005). Asset price characteristic function $F(t_k;x)$ as Taylor series at moment t_k for price treated as random variable during the interval Δ_k (2.4) takes form:

$$F(t_k;x) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} p(t_k;n)\, x^n \tag{2.21}$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \frac{d^n}{(i)^n dx^n} F(t_k;x)|_{x=0} \tag{2.22}$$

The most important result of our derivation of the *market-based* asset price characteristic function is follows: price characteristic function $F(t_k;x)$ (2.21) depends on set of market trade value $C(t_k;n)$ and volume $U(t_k;n)$ *n-th* statistical moments for all $n=1,2,..$ and hence the price PDF also depends on market trade statistical moments (2.9). Any predictions of the *market-based* asset price PDF at horizon T require forecasts of the market trade value and volume PDF at same horizon T or forecasts of all market trade *n-th* statistical moments.

Direct calculation of the asset price PDF $\eta(t_k;p)$ (we omit 2π factors in Fourier transform for brevity):

$$\eta(t_k;p) = \int dx\, F(t_k;x) \exp(-ixp) \tag{2.23}$$

as Fourier transform of price characteristic function $F(t_k;x)$ as Taylor series (2.21) is not possible. However, one can take m-approximations of the price characteristic function $F_m(t_k;x)$ (2.24; 2.25) that allows Fourier transform and first m price statistical moments coincides with (2.22). We refer (Olkhov, 2021d) for details:

$$F_m(t_k;x) = \exp\left\{\sum_{j=1}^{m} \frac{i^j}{j!}\, a_j\, x^j\right\} \quad ; \quad m = 1,2,.. \tag{2.24}$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \frac{d^n}{(i)^n dx^n} F_m(t_k;x)|_{x=0} \quad ; \quad n = 1,2,..m \tag{2.25}$$

Such m-approximation of characteristic function $F_m(t_k;x)$ reproduces first m price statistical moments (2.22) and generates m-approximation of the PDF $\eta_m(t_k;p)$ at moment t_k for price treated as random variable during the interval Δ_k (2.4):

$$\eta_m(t_k;p) = \int dx\, F_m(t_k;x) \exp(-ixp) \tag{2.26}$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \int dp\ \ p^n \eta_m(t_k;p) \quad ; \quad n \le m \tag{2.27}$$

For $n=2$ approximation of the asset price probability measure $\eta_2(t_k;p)$ takes form of simple Gaussian distribution with the *market-based* asset price volatility $\sigma_p^2(t_k)$:

$$\eta_2(t_k;p) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_p(t_k)} \exp\left\{-\frac{\left(p - p(t_k;1)\right)^2}{2\sigma_p^2(t_k)}\right\} \tag{2.28}$$

$$\sigma_p^2(t_k) = p(t_k;2) - p^2(t_k;1) = \frac{C(t_k;2)}{U(t_k;2)} - \frac{C^2(t_k;1)}{U^2(t_k;1)} \tag{2.29}$$

We underline that simple Gaussian approximation of the asset price probability measure (2.28) depends on second statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$. Prediction of Gaussian price probability $\eta_2(t_k;p)$ (2.28) at horizon T requires forecasts of the second statistical moments of the market trade value and volume (2.9) at the same horizon T.

It is important to underline that the market-based asset price volatility $\sigma_p^2(t_k)$ (2.29) depends on second statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$ (2.9) (Olkhov, 2020). Description of the market trade 2-d statistical moments arises a problem of description second statistical moments of all macroeconomic variables those depend on market trading. That requires development of the second-order economic theory that describes relations between 2-d statistical moments of market trades and second-order macroeconomic variables (Olkhov, 2020; 2021a; 2021e). Usage of the *market-based* approach to price probability opens the way for introduction of price autocorrelations (Olkhov, 2022a; 2022b).

We should mention that our consideration of price statistical moments $p(t_k;n)$ through statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ only complement well-known properties of the random variable determined as difference of two random variables Indeed, taking logarithm of (1.1) one easy obtains that logarithm of price $ln(p)$ equals logarithm of the trade value $ln(C)$ minus logarithm of the trade volume $ln(U)$. That case is described in many probability introductory notes (Papoulis and Pillai, 2002; p.181) and we consider it briefly in App. B.

However, definition of price statistical moments $p(t_k;n)$ (2.19), price characteristic function (2.21; 2.22; 2.24; 2.25) and approximations of the price PDF (2.26; 2.27) describe price random properties through statistical moments of the random market trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.9). That approach seems to be more useful for econometric and theoretical studies of the market price probability, than assessment of log-probabilities (App.B), as it is based on the direct assessments of the trade statistical moments (2.9) during interval Δ_k (2.4). Records of market trade time series allow calculate trade statistical moments (2.9) and derive price statistical moments (2.19). However, observed market trade time series on their own don't identify the exact form of the trade value PDF, trade volume PDF and their joint PDF that are required to derive log price PDF (App.B). At that point usage of trade statistical moments is more preferable.

3. Market-based returns and inflation

Description of the market-based probability of returns follows the same frame as the above description of the market-based asset price PDF. Actually, returns $r(t_1,t_2)$ are determined as

$$r(t_1, t_2) = \frac{p(t_2)-p(t_1)}{p(t_1)} = \frac{p(t_2)}{p(t_1)} - 1 \tag{3.1}$$

Let's take price index $a(t_1,t_2)$ (3.2) as:

$$p(t_2) = p(t_1)a(t_1,t_2) \tag{3.2}$$

$$r(t_1,t_2) = a(t_1,t_2) - 1 \tag{3.3}$$

In Sec.2 we already derived the *market-based* asset price probability statistical moments $p(t_k;n)$ (2.19), price characteristic function $F(t_k;x)$ (2.21; 2.22) and approximations of the *market-based* price probability measure $\eta_m(t_k;p)$ (2.26-2.29). Let's use the same approach to describe the *market-based* probability of the price index $a(t_1,t_2)$ (3.2). We shall consider two cases. First, we assume that the price index $a(t_1,t_2)$ (3.2) is determined with respect to the fixed price $p(t_1)$ and consider statistical properties of the price index by time t_2 during the averaging interval Δ_k (2.4). In the second case we consider random properties of the price index $a(t_1,t_2)$ (3.2) by both time moments t_1 and t_2 averaged during different averaging intervals Δ_k and Δ_{k+m} (2.4).

1-st case - Returns. Take into account (2.9; 2.19; 3.2) and for the *n-th* statistical moments of the price index $a(t_1,t_k;n)$ averaged during the interval Δ_k (2.4) obtain:

$$a(t_1,t_k;n) = E[a^n(t_1,t_2)] \tag{3.4}$$

$$a(t_1,t_k;n)p^n(t_1) = p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} \tag{3.5}$$

E[...] – math expectation during the interval Δ_k (2.4). From (3.4; 3.5) obtain expressions for the *n-th* statistical moments of returns $r(t_1,t_k;n)$:

$$r(t_1,t_k;n) = E[r^n(t_1,t_2)] \tag{3.6}$$

$$r(t_1,t_k;n) = E[(a(t_1,t_2) - 1)^n] \tag{3.7}$$

Due to (3.4; 3.5; 3.7) the *n-th* returns statistical moment $r(t_1,t_k;n)$ is a simple sum of *m-th* statistical moments of the price index $a(t_1,t_k;m)$, $m \le n$:

$$r(t_1,t_k;n) = \sum_{m=0}^{n} (-1)^{(n-m)} \frac{n!}{m!(n-m)!} a(t_1,t_k;m) \;\; ; \;\; a(t_1,t_k;0) = 1 \tag{3.8}$$

Due to (3.5) returns *n-th* statistical moments $r(t_1,t_k;n)$ can be presented through the market trade value and volume statistical moments (3.9)

$$r(t_1,t_k;n) = \sum_{m=0}^{n} (-1)^{(n-m)} \frac{n!}{m!(n-m)!} p^{-m}(t_1) \frac{C(t_k;m)}{U(t_k;m)} \;\; ; \;\; \frac{C(t_k;0)}{U(t_k;0)} = 1 \tag{3.9}$$

or equally through the *n-th* price statistical moment (see (3.5)).

In particular, one can easy derive relations (3.11) between the *market-based* price volatility $\sigma_p^2(t_k)$ (2.29), price $p(t_1)$ at moment t_1 and *market-based* volatility of returns $\sigma_r^2(t_1,t_k)$ (3.10)

$$\sigma_r^2(t_1,t_k) = r(t_1,t_k;2) - r^2(t_1,t_k;1) = E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^2\right] \tag{3.10}$$

$$\sigma_p^2(t_k) = p^2(t_1)\, \sigma_r^2(t_1,t_k) \tag{3.11}$$

Using (3.4-3.9) one can easy derive that return's Skewness $Sk_r(t_1,t_k)$ (3.13) that describe asymmetry of the market-based return's distribution equals (3.14) the *market-based* asset price Skewness $Sk_p(t_k)$ (3.12):

$$Sk_p(t_k)\sigma_p{}^3(t_k) = E\left[\left(p(t_2) - p(t_k;1)\right)^3\right] \tag{3.12}$$

$$Sk_r(t_1,t_k)\sigma_r{}^3(t_1,t_k) = E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^3\right] \tag{3.13}$$

$$Sk_p(t_k) = Sk_r(t_1,t_k) \tag{3.14}$$

One can easy obtain that the *market-based* price Kurtosis $Ku_p(t_k)$ (3.15) equals (3.17) returns Kurtosis $Ku_r(t_1,t_k)$ (3.16):

$$Ku_p(t_k)\sigma_p{}^4(t_k) = E\left[\left(p(t_2) - p(t_k;1)\right)^4\right] \tag{3.15}$$

$$Ku_r(t_1,t_k)\sigma_r{}^4(t_1,t_k) = E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^4\right] \tag{3.16}$$

$$Ku_p(t_k) = Ku_r(t_1,t_k) \tag{3.17}$$

2-d case - Inflation. Now assume that (3.2) is averaged by time t_1 during time interval Δ_k and by t_2 during the time interval Δ_{k+m} (2.4). Similar to (2.15; 2.16) we assume no correlations between price $p(t_1)$ and price index $a(t_1,t_2)$ while averaging by t_1 during time interval Δ_k. That is the way to define mean price index $a(t_k,t_{k+m};1)$ similar to (2.10). Then obtain

$$p(t_{k+m};1) = p(t_k;1)a(t_k,t_{k+m};1)$$

For *n-th* power of (3.2) similar considerations result:

$$p(t_{k+m};n) = p(t_k;n)a(t_k,t_{k+m};n) \tag{3.18}$$

$$p(t_k;n) = E_{\Delta_k}[p^n(t_i)] \quad ; \quad p(t_{k+m};n) = E_{\Delta_{k+m}}\left[p^n(t_j)\right]$$

$$a(t_k,t_{k+m};n) = E_{\Delta_k}E_{\Delta_{k+m}}\left[a^n(t_i,t_j)\right]$$

We denote as $E_{\Delta k}[...]$ – math expectation during the time interval Δ_k. This case can be more useful for description of inflation. Actually, if averaging interval Δ equals 1 week, month, quarter, year – one can obtain price index (3.18) or inflation measured week-to-week, month-to-month or year-to-year. The choice of particular asset, particular market sector, industry sector and etc. gives corresponding inflation.

Relations (3.18) give direct dependence of the price index $a(t_k,t_{k+m}:n)$ on market trade statistical moments

$$a(t_k,t_{k+m};n) = \frac{p(t_{k+m};n)}{p(t_k;n)} = \frac{C(t_{k+m};n)}{C(t_k;n)}\frac{U(t_k;n)}{U(t_{k+m};n)} \tag{3.19}$$

Let's introduce the trade value index $c(t_k,t_{k+m};n)$ and the volume index $u(t_k,t_{k+m};n)$ as

$$c(t_k,t_{k+m};n) = \frac{C(t_{k+m};n)}{C(t_k;n)} \quad ; \quad u(t_k,t_{k+m};n) = \frac{U(t_{k+m};n)}{U(t_k;n)} \tag{3.20}$$

Then (3.19) takes form

$$c(t_k, t_{k+m}; n) = a(t_k, t_{k+m}; n)u(t_k, t_{k+m}; n) \tag{3.21}$$

Inflation *n-th* statistical moment $In(t_k,t_{k+m};n)$ takes form similar to (3.7 - 3.10):

$$In(t_k, t_{k+m}; n) = E_{\Delta_k} E_{\Delta_{k+m}}[(a(t_1, t_2) - 1)^n] \tag{3.22}$$

$$I_n(t_1, t_k; n) = \sum_{j=0}^{n} (-1)^{(n-j)} \frac{n!}{j!(n-j)!} a(t_k, t_{k+m}; j) \ ; \ a(t_k, t_{k+m}; 0) = 1 \tag{3.23}$$

or, taking into account (3.21):

$$I_n(t_1, t_k; n) = \sum_{j=0}^{n} (-1)^{(n-j)} \frac{n!}{j!(n-j)!} \frac{c(t_k, t_{k+m}; n)}{u(t_k, t_{k+m}; n)} \tag{3.24}$$

Mean inflation $In(t_k,t_{k+m};1)$ taken at time term Δ_{k+m} with respect to time term Δ_k takes form:

$$In(t_k, t_{k+m}; 1) = a(t_k, t_{k+m}; 1) - 1 \ = \frac{c(t_k, t_{k+m}; 1)}{u(t_k, t_{k+m}; 1)} - 1 \tag{3.25}$$

and its volatility $\sigma^2_{In}(t_k,t_{k+m})$

$$\sigma^2_{In}(t_k, t_{k+m}) = a(t_k, t_{k+m}; 2) - a^2(t_k, t_{k+m}; 1) \tag{3.26}$$

$$\sigma^2_{In}(t_k, t_{k+m}) = \frac{c(t_k, t_{k+m}; 2)}{u(t_k, t_{k+m}; 2)} - \frac{c^2(t_k, t_{k+m}; 1)}{u^2(t_k, t_{k+m}; 1)} \tag{3.27}$$

The trade value index $c(t_k,t_{k+m};n)$ and trade volume index $u(t_k,t_{k+m};n)$ (3.20) describe growth of the market trade value and volume during the interval Δ_{k+m} with respect to the interval Δ_k. It is obvious, that market trade is the only direct indicator of the economic growth and development. Growth of trade value and volume reflect economic growth. Thus relations (3.20-3.24) link description of asset returns and inflations with studies of economic growth during selected time interval Δ_k and Δ_{k+m} (2.4).

We leave further investigation of above relations between economic growth and market trade indexes (3.20) for future.

4. Consequences for asset pricing and value-at-risk

Asset-pricing

All asset-pricing models deal with prices averaged by some probability (Cochrane, 2001; Campbell, 2002). Predictions of the price probability at certain time horizon T play the core role for the assessments of price forecasts at horizon T. Introduction of the *market-based* asset price probability determined by statistical properties of the market trade value and volume (2.19 - 2.22) during the averaging time interval Δ_k (2.4) makes predictions of the price PDF as one of the core problem of macroeconomics and finance. Indeed, any prediction of the price PDF at time horizon T for the averaging interval Δ_k (2.4) requires forecasting of the market trade value and volume probabilities at the same horizon T and during the same interval Δ_k (2.4). In simple words, to predict price PDF one should forecast the market trade values and volumes probabilities during the interval Δ_k (2.4) at horizon T. That causes forecasting of supply and demand, production function and investment, economic development and growth and etc., and other factors those impact market trade on one hand and depend on results of market trade on the other hand at horizon T. Introduction of the *market-based* price probability ties up the problems of prediction of financial markets price with problems of forecasting of market trade, economic development and growth. It is obvious, that exact prediction of economic future is impossible. However, development of the approximations may give assessments of future market trade probability and price probability. One should take into account basic relations (2.19-2.22) those determine price statistical moments through statistical moments of the market trade value and volume. Approximations (2.24-2.27) that take into account first 2,3,4 statistical moments should check how approximate price probability forecasts match predictions of the market trade value and volatility statistical moments during selected time interval Δ_k (2.4). We underline that duration of the interval Δ_k is very important for assessments of statistical moments and for sustainability of predictions at the given time horizon T.

Value-at-risk

Dependence of predictions of the asset price probability approximations on forecasting the finite number of statistical moments of the market trade value and volume impact accuracy and reliability of Value-at-Risk (VaR) - the most widespread risk tool for hedging risk of market price change. Economic ground of VaR was developed more than 30 years ago (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Choudhry, 2013). "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon" (Longerstaey and Spencer, 1996). Nevertheless huge progress in VaR performance was achieved since then, the core features of VaR remain the same. To assess the VaR at horizon T one should calculate integral of the left tail of the returns PDF predicted at horizon T to estimate "potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon". Such assessment limits the possible capital loss due to market price variations for selected time horizon T for given probability. Usage of any VaR version requires predictions of returns PDF at horizon T. As we show above, returns probability and returns statistical moments are completely determined by asset price probability and price statistical moments (3.1-3.14) and by market trade statistical moments (2.9; 3.9). Prediction of returns probability is the problem of the asset price probability forecasting. As we discussed in Sec.2, the *market-based* asset price probability and price statistical moments are determined by random properties of the market trade value and volume. In simple words: VaR assessment on base of predicted returns PDF at horizon T depends on forecasting the market trade value and volume probabilities at the same horizon T. The problem of VaR assessment almost equals the problem of market trade PDF prediction.

Accuracy and reliability of VaR assessment depends on accuracy of predictions of the market trade value and volume probabilities. The more statistical moments of market trades are predicted, the higher accuracy of VaR can be obtained. However, imaginable exact forecast of the market trade PDF at horizon T would provide for that lucky man a unique opportunity to manage and beat the market alone. That is much more profitable then any VaR calculations. One who will succeed in exact prediction of the market PDF will forget about VaR assessments and will enjoy beating the market alone! However, there is still remains a small, negligible and trivial problem – how one can *exactly* predict the market trade PDF?

These obstacles arises the problem of assessment of accuracy of any price probability prediction in compare with imaginable price probability determined by market trade probability forecasts. That problem may help establish economic ground and introduce possible limits on reliability of possible usage of VaR based on the *market-based* asset price probability. We leave this problem for future.

5. Conclusion

The asset price probability plays the core role in macroeconomics and finance. Introduction of the *market-based* asset price PDF through statistical moments of the market trade value and volume ties up the asset pricing theories and financial market studies with description of market trade evolution, economic development and growth. We don't solve the price probability puzzle but substitute the problem of the "correct" description of the price PDF by the problem of description of PDF of the market trades. Investigation of the market trade value and volume PDF and prediction of their statistical moments remain the problems for future.

The *market-based* approach to price probability establishes the unified description of the price statistical moments, price indices and returns statistical moments and ties up the predictions of price, price indices, returns probabilities with the problem of predictions of the market trade value and volume probabilities and statistical moments. Market-based asset price probability implies that the classical option pricing models should be considered not at one, but at the two-dimensional spaces (Olkhov, 2021c). We study bounds of reliability for usage of Value-at-Risk determined by the accuracy of forecasting of the market trade probability (Olkhov, 2021b). We use *market-based* asset price PDF to describe dependence of the asset price autocorrelations on the market trade value and volume correlations (Olkhov, 2022a; 2022b). As we show, price volatility depends on the second statistical moments of the market trade value and volume (2.29) and *n-th* price statistical moments are determined by *n-th* statistical moments of market trades (2.9). Description of the asset price volatility as well as description of price Skewness and Kurtosis require description of the *2-d*, *3-d* and *4-th* market trade statistical moments. As we show (Olkhov, 2021a; 2021e) *n-th* market trade statistical moments depend on and impact at similar statistical moments of macroeconomic variables. Thus, description of the market-based price volatility implies development of the second-order economic theory. Description of price Skewness requires development of the 3-d order economic theory and so on. Studies of theoretical economics (Olkhov, 2022c) have an endless and interesting list of problems for many years ahead.

However, agents and academics are free in their expectations and preferences in choosing conventional *frequency-based* approach to asset price probability. Numerous studies in game theory, expectations, market studies and etc., propose various methods and factors that may impact the price change, define evolution of the asset price, returns, volatility and etc. All such studies investigate complex problems those govern price evolution, market dynamics and economic development. However, all possible factors, expectation, relations and laws, those can impact and define price choice and price change, are imprinted in and are recorded by time-series of the performed market trades and are available as market time-series for our analysis (2.1-2.29).

We complement the current studies in asset pricing with a new look and demonstrate hidden complexity, unity and beauty of the *market-based* asset price probability theory. Further development of that theory will take a lot of efforts and years.

Appendix A

For simplicity we omit here time t_i of random variables and denote trade value $C(t_i)$ as C, trade volume as U and price as p. Taking into account (2.19) we denote here statistical moments of trade value $C(n)$, volume $U(n)$ and price $p(n)$ as:

$$C(n) = E[C^n] = E[p^n U^n] = E[p^n]E[U^n] = p(n)U(n)$$

Let us take into account (2.15; 2.19) and consider

$$E[pU^2] = E[CU] \tag{A.1}$$

We denote correlation $corr\{CU\}$ between trade value C and trade volume U:

$$corr\{CU\} = E[\delta C\ \delta U] \ ; \quad corr\{pU^2\} = E[\delta p\ \delta U^2]$$

Then from (A.1) obtain:

$$E[pU^2] = p(1)U^2(1) + corr\{pU^2\} = E[CU] = E[C]E[U] + corr\{CU\}$$

$$corr\{pU^2\} = p(1)[U^2(1) - U(2)] + corr\{CU\}$$

$$corr\{pU^2\} = corr\{CU\} - p(1)\sigma^2(U) \tag{A.2}$$

Similar consideration for relations between square of price p^2 and trade volume U give:

$$E[p^2U] = E[C^2U^{-1}] \tag{A4}$$

$$E[p^2U] = p(2)U(1) + corr[p^2U] = [C^2U^{-1}] = p(2)U(2)U(-1) + corr\{C^2U^{-1}\}$$

$$U(-1) = E[U^{-1}]$$

$$corr[p^2U] = corr\{C^2U^{-1}\} + p(2)[U(2)U(-1) - U(1)] \tag{A.5}$$

The value and sign of correlations depend on the duration of the averaging time interval Δ and that important dependence should be investigated. Let us underline that correlations (A.5) between trade volume U and square of price p^2 determine relations between trade volume and volatility studies (Ito and Lin, 1993; Brock and LeBaron, 1995; Ciner and Sackley, 2007; Bogousslavsky and Collin-Dufresne, 2019), but one should take into account that correlations (A.5) are determined by the market-based price probability (2.19; 2.21; 2.22).

Appendix B

Taking *ln* of equation (1.1) one obtains sums of random variables:

$$\pi(t) = c(t) - u(t) \tag{B.1}$$

$$\pi = \ln p \quad ; \quad c = \ln C \quad u = \ln U \tag{B.2}$$

That well-know problem is described in many introductory notes on probability (Papoulis and Pillai, 2002; p.181). Let function $g(c,u)$ determines the joint probability density function (PDF) of variables c and u (B.2). Probability distribution $Q(\pi)=P\{c-u< \pi\}$ is determined by conditions

$$Q(\pi) = P\{c < \pi + u\} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\pi+u} dc\, g(c,u)$$

Then PDF $q(\pi)$ is determined as:

$$q(\pi) = \frac{d}{d\pi} Q(\pi) = \int_{-\infty}^{\infty} dx\, g(\pi + x, x) = \int_{-\infty}^{\infty} dx\, g(x, x - \pi) \tag{B.3}$$

To assess volume weighed average price (VWAP) (2.10; 2.11) from (1.1) and (B.1; B.2) obtain

$$E[C] = E[exp(c)] = E[exp(\pi + u)] = E[exp(\pi)]E[exp(u)] = E[p]\, E[U] \tag{B.4}$$

Respectively, relations (2.15-2.20) result:

$$E[C^n] = E[exp(nc)] = E[exp\{n(\pi + u)\}] = E[exp(n\pi)]E[exp(nu)] = E[p^n]\, E[U^n]$$

However, PDF $q(\pi)$ (B.3) is not too useful for calculation of price *n-th* statistical moments $E[p^n(t)]$. Moreover, time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ give the opportunity to assess *n-th* statistical moments of the trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.9) but don't derive sufficient information to guess the exact form of PDF and joint PDF $g(c,u)$ to be able to calculate (B.3). On the contrary, usage of assessments of first 1,2,3.. statistical moments allow select PDF approximations those match first statistical moments (2.9).

However, one should remind that above consideration of market-based asset price probability through assessment of the market trade statistical moments (2.9; 2.19-2.27) derive successive approximations only. Relations (B.1-B.3) present a different way to derive approximations. No "exact" solution of the PDF of the asset price exists, simply because no exact PDF of the market trade value and volume are known. Different averaging intervals *Δ,* different markets, assets, phases of business cycles and etc., result variations of market trade PDF and corresponding variations of the asset price PDF.

References

Anderson, T.W. (1971). The Statistical Analysis of Time-Series. Wiley&.Sons, Inc., 1-757

Andersen, T., Bollerslev, T., Diebold, F.X, Ebens, H. (2001). The Distribution of Realized Stock Return Volatility, Journal of Financial Economics, 61, 43-76

Andersen, T.G, Bollerslev, T., Christoffersen, P.F. and F.X. Diebold, (2005a). Volatility Forecasting, CFS WP 2005/08, 1-116

Andersen, T.G., Bollerslev, T., Christoffersen, P.F. and F.X. Diebold, (2005b). Practical Volatility And Correlation Modelling For Financial Market Risk Management, NBER, WP11069, Cambridge, MA, 1-41

Andersen, T.G., Bollerslev, T., Christoffersen, P.F., and Diebold, F.X. (2006). Volatility and Correlation Forecasting, in G. Elliot, C.W.J. Granger, and Allan Timmermann (eds.), Handbook of Economic Forecasting. Amsterdam: North-Holland, 778-878

Bachelier, L., (1900). Théorie de la speculation, Annales scientifiques de l'É.N.S. 3e série, **17**, 21-86

Berkowitz, S.A., Dennis, E., Logue, D.E., Noser, E.A. Jr. (1988). The Total Cost of Transactions on the NYSE, *The Journal of Finance*, 43, (1), 97-112

Black, F. and M. Scholes, (1973). The Pricing of Options and Corporate Liabilities, J. Political Economy, 81 (3), 637-654

Bogousslavsky, V. and P. Collin-Dufresne, (2019). Liquidity, Volume, and Volatility, Swiss Finance Institute, RPS 19-69, 1-59

Brock, W.A. and B.D. LeBaron, (1995). A Dynamic structural model for stock return volatility and trading volume. NBER, WP 4988, 1-46

Buryak, A. and I. Guo, (2014). Effective And Simple VWAP Options Pricing Model, Intern. J. Theor. Applied Finance, 17, (6), 1450036, https://doi.org/10.1142/S0219024914500356

Busseti, E. and S. Boyd, (2015). Volume Weighted Average Price Optimal Execution, 1-34, arXiv:1509.08503v1

Campbell, J.Y. and R.J. Shiller, (1988). Stock Prices, Earnings And Expected Dividends. NBER, WP 2511, 1-42

Campbell, J.Y., Grossman, S.J. and J.Wang, (1993). Trading Volume and Serial Correlation in Stock Return. Quatr. Jour. Economics, 108 (4), 905-939

Campbell, J.Y. (1998). Asset Prices, Consumption, and the Business Cycle. NBER, WP6485

Campbell, J.Y. (2000). Asset Pricing at the Millennium. Jour. of Finance, 55(4), 1515-1567

Campbell, J.Y. (2002). Consumption-Based Asset Pricing. Harvard Univ., Cambridge, Discussion Paper # 1974, 1-116

Choudhry, M., (2013). An Introduction to Value-at-Risk, 5th Edition. Wiley, 1-224

Ciner, C. and W. H. Sackley, (2007). Transactions, volume and volatility: evidence from an emerging market, Applied Financial Economics Letters, **3**, 161–164

CME Group, (2020). https://www.cmegroup.com/search.html?q=VWAP

Cochrane, J.H. and L.P. Hansen, (1992). Asset Pricing Explorations for Macroeconomics. Ed., Blanchard, O.J., Fischer, S. NBER Macroeconomics Annual 1992, v. 7, 115 – 182

Cochrane, J.H. (2001). Asset Pricing. Princeton Univ. Press, Princeton, US

Cochrane, J.H. (2005). Time Series for Macroeconomics and Finance. Graduate School of Business, Univ. Chicago, 1-136

Cochrane, J.H. (2022). Portfolios For Long-Term Investors, Rev. Finance, 26(1), 1–42

CreditMetrics™, (1997). Technical Document. J.P. Morgan & Co, NY. 1-212

Davis, H.T., (1941). The Analysis of Economic Time-Series. Cowles Commission for Research in Economics. The Principia Press, Inc., 1-627

DeFusco, A.A., Nathanson, C.G. and E. Zwick, (2017). Speculative Dynamics of Prices and Volume, Cambridge, MA, NBER WP 23449, 1-74

Diebold, F.X. and K. Yilmaz, (2008). Macroeconomic Volatility And Stock Market Volatility, Worldwide, NBER WP 14269, 1-35

Diebold, F.X. (2019). Time Series Econometrics, Univ. Pennsylvania, 1-215

Dimson. E. and M.Mussavian, (1999). Three centuries of asset pricing. J.Banking&Finance, 23(12) 1745-1769

Duffie, D., Dworczak, P. (2021). Robust Benchmark Design. Journal of Financial Economics, 142(2), 775–802

Fama, E.F. (1965). The Behavior of Stock-Market Prices. J. Business, 38 (1), 34-105

Fama, E.F. and K.R. French, (2015). A five-factor asset pricing model. J. Financial Economics, 116, 1-22

Fetter, F.A., (1912). The Definition of Price. American Economic Rev., 2 (4), 783-813

Forbes, C., Evans, M., Hastings, N., Peacock, B. (2011). Statistical Distributions. Wiley

Friedman, D.D. (1990). Price Theory: An Intermediate Text. South-Western Pub. Co., US

Gallant, A.R., Rossi, P.E. and G. Tauchen, (1992). Stock Prices and Volume, The Review of Financial Studies, 5(2), 199-242

Goldsmith, R.W. and R.E. Lipsey, (1963). Asset Prices and the General Price Level, NBER, 166 – 189, in Studies in the National Balance Sheet of the United States, Ed. Goldsmith,

R.W. and R. E. Lipsey
Greenwood, R. and A. Shleifer, (2014). Expectations of Returns and Expected Returns. The Review of Financial Studies, 27 (3), 714–746
Heaton, J. and D. Lucas, (2000). Stock Prices and Fundamentals. Ed. Ben S. Bernanke, B.S and J. J. Rotemberg, NBER Macroeconomics Annual 1999, v. 14., 213 – 264
Hördahl, P. and F. Packer, (2007). Understanding asset prices: an overview. Bank for International Settlements, WP 34, 1-38
Ito, T. and W.L. Lin, (1993). Price Volatility And Volume Spillovers Between The Tokyo And New York Stock Markets, NBER WP 4592, 1-35
Karpoff, J.M. (1987). The Relation Between Price Changes and Trading Volume: A Survey. The Journal of Financial and Quantitative Analysis, 22 (1), 109-126
Kendall, M.G and A.B. Hill, (1953). The Analysis of Economic Time-Series-Part I: Prices. Jour. Royal Statistical Soc., Series A, 116 (1), 11-34
Klyatskin, V.I. (2005). Stochastic Equations through the Eye of the Physicist, Elsevier B.V.
Llorente, G., Michaely R., Saar, G. and J. Wang. (2001). Dynamic Volume-Return Relation of Individual Stocks. NBER, WP 8312, Cambridge, MA., 1-55
Longerstaey, J., and M. Spencer, (1996). RiskMetrics -Technical Document. J.P.Morgan & Reuters, N.Y., Fourth Edition, 1-296
Mackey, M.C. (1989). Commodity Price Fluctuations: Price Dependent Delays and Nonlinearities as Explanatory Factors. J. Economic Theory, 48, 497-509
Malkiel, B. and J.G. Cragg, (1980). Expectations and the valuations of shares, NBER WP 471, 1-70
Mandelbrot. B, Fisher, A. and L. Calvet, (1997). A Multifractal Model of Asset Returns, Yale University, Cowles Foundation Discussion WP1164, 1-39
Merton, R.C. (1973). An Intertemporal Capital Asset Pricing Model, Econometrica, 41, (5), 867-887
Mills, F.C. (1946). Price-Quantity Interactions in Business Cycles. NBER, Prins.Univ., NY
Muth, J.F. (1961). Rational Expectations and the Theory of Price Movements, Econometrica, 29, (3) 315-335
Olkhov, V. (2020). Volatility Depend on Market Trades and Macro Theory. MPRA, WP102434, 1-18
Olkhov, V. (2021a). Price, Volatility and the Second-Order Economic Theory. ACRN Jour. Finance & Risk Perspectives, Sp. Issue, 18th FRAP Conference, 10, 139-165
Olkhov, V. (2021b). To VaR, or Not to VaR, That is the Question. SSRN, WP 3770615

Olkhov, V. (2021c). Classical Option Pricing and Some Steps Further, SSRN, WP 3587369, 1-16

Olkhov, V., (2021d), Three Remarks On Asset Pricing, SSRN WP 3852261, 1-24.

Olkhov, V. (2021e). Theoretical Economics and the Second-Order Economic Theory. What is it? SSRN, WP 3975585, 1-14

Olkhov, V. (2022a). Price and Payoff Autocorrelations in the Consumption-Based Asset Pricing Model, SSRN, WP 4050652, 1- 18

Olkhov, V. (2022b). Introduction of the Market-Based Price Autocorrelation, SSRN, WP 4035874, 1-13

Olkhov, V. (2022c). Economic Policy - The Forth Dimension of the Economic Theory, SSRN, WP 4079069, 1-20

Odean,T. (1998). Volume, Volatility, Price, And Profit When All Traders Are Above Average, The Journal of Finance, LIII, (6), 1887-1934

Papoulis, A. and S.U. Pillai, (2002). Probability Random Variables and Stochastic Processes, 4th Ed., McGraw-Hill, NY.,1-860

Poon, S-H. and C.W.J. Granger, (2003). Forecasting Volatility in Financial Markets: A Review, J. of Economic Literature, 41, 478–539

Sharpe, W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. The Journal of Finance, 19 (3), 425-442

Shephard, N.G. (1991). From Characteristic Function to Distribution Function: A Simple Framework for the Theory. *Econometric Theory*, 7 (4), 519-529

Shiryaev, A.N. (1999). Essentials Of Stochastic Finance: Facts, Models, Theory. World Sc. Pub., Singapore. 1-852

Shreve, S. E. (2004). Stochastic calculus for finance, Springer finance series, NY, USA

Stigler, G.J. and J.K. Kindahl, (1970). The Dispersion of Price Movements, NBER, 88 - 94 in Ed. Stigler, G.J, Kindahl, J.K. The Behavior of Industrial Prices

Tauchen, G.E. and M. Pitts, (1983). The Price Variability-Volume Relationship On Speculative Markets, Econometrica, 51, (2), 485-505

Walck, C. (2011). Hand-book on statistical distributions. Univ.Stockholm, SUF–PFY/96–01

Weyl, E.G. (2019). Price Theory, AEA J. of Economic Literature, 57(2), 329–384

Wolfers, J. and E. Zitzewitz, (2006). Interpreting Prediction Market Prices As Probabilities, NBER WP12200, 1-22