



Munich Personal RePEc Archive

Information Revelation in Sequential Ascending Auctions

Said, Maher

Yale University

7 October 2008

Online at <https://mpra.ub.uni-muenchen.de/11315/>
MPRA Paper No. 11315, posted 31 Oct 2008 06:25 UTC

INFORMATION REVELATION IN SEQUENTIAL ASCENDING AUCTIONS

MAHER SAID[†]
OCTOBER 7, 2008

ABSTRACT. We examine a model in which buyers with single-unit demand are faced with an infinite sequence of auctions. In each period, a new buyer probabilistically arrives to the market, and is endowed with a constant private value. We demonstrate by way of a simple example the inefficiency of the second-price sealed-bid auction in this setting, and therefore focus instead on the ascending auction.

We then show that the mechanism in which the objects are sold via ascending auctions has an efficient, fully revealing, and Markov perfect Bayesian equilibrium which is *ex post* optimal for all buyers in each period, given their expectations about the future. In equilibrium, all buyers completely reveal their private information in every period. However, equilibrium bidding behavior is memoryless. Bids depend only upon the information revealed in the current auction, and not on any information revealed in previous periods. This lack of memory is crucial, as it allows buyers to behave symmetrically, despite the informational asymmetry arising from the arrival of uninformed buyers. This provides the appropriate incentives for these new buyers to also reveal their information.

KEYWORDS: Sequential auctions, Ascending auctions, Random arrivals, Information revelation, Dynamic Vickrey-Clarke-Groves mechanism, Marginal contribution.

JEL CLASSIFICATION: C73, D44, D83.

[†]DEPARTMENT OF ECONOMICS, YALE UNIVERSITY, 28 HILLHOUSE AVENUE, NEW HAVEN, CT 06511
E-mail address: maher.said@yale.edu.

I would like to thank my advisor, Dirk Bergemann, for his constructive advice and comments, as well as Alessandro Bonatti, Rahul Deb, Johannes Hörner, Nenad Kos, Larry Samuelson, Itai Sher, and Rakesh Vohra for their many helpful discussions and suggestions. This work was generously supported by a National Science Foundation Graduate Research Fellowship and a Cowles Foundation Anderson Fellowship.

1. INTRODUCTION

Many markets, including those for art, wine, government-issued debt, and timber and other natural resources sell multiple objects via sequential auctions, while eBay has built a market capitalization of over \$40 billion by providing a marketplace for these auctions. In this paper, we examine a model of such markets in which a single object is sold at a time and new buyers arrive on the market at random times. Each bidder has an independently drawn private value for purchasing an object. In contrast to much of the literature that makes use of sealed-bid auctions, we focus on the ascending auction. Although the various auction formats are in many respects equivalent in a static private-values setting, this equivalence does not hold in a dynamic environment, primarily due to the information revelation inherent in the ascending auction format. The difference between the two formats is further exacerbated in the sequential auction setting when we allow for dynamically changing populations of buyers. In particular, the entry of a new buyer introduces an additional informational asymmetry. We show, however, that this asymmetry may be easily resolved by employing ascending auctions. In equilibrium, each buyer's bids and payoffs depend only on the buyer's rank amongst their current competitors and the (revealed) values of those opponents with lower values. Furthermore, these strategies have the remarkable property of being memoryless—in each auction conducted, bids are independent of the information revealed in previous periods, despite the fact that *all* private information is revealed during every auction.

We feel that this model serves as a useful abstraction of online auction sites such as eBay or uBid, especially when considering the extensive market on these sites for individual units of brand-new homogenous goods. Typically, a variety of auctions for identical items are open simultaneously, but may be ordered by their closing time. Thus, abstracting away from intra-auction dynamics, a sequential auction model yields a good approximation.¹ With this in mind, many authors (see Sailer (2006) or Zeithammer (2006), for instance) make use of the second-price sealed-bid auction, citing evidence from Roth and Ockenfels (2002) and Bajari and Hortaçsu (2003) about the prevalence of “sniping” (last-second bidding) in online auctions in defense of their modeling choice. However, as shown by Cai, Wurman, and Chao (2007), pure-strategy symmetric equilibria do *not* exist in sequential sealed-bid auctions when buyer values are fixed across time and bids are made publicly observable after each auction. As most online auctions bear a close resemblance to English auctions in regards to intra-auction dynamics as well as the visibility of submitted bids (both during an auction and after an auction has closed), we believe that the ascending auction is better-suited than the sealed-bid second-price auction for modeling online auction markets.²

What is more, we feel that the sequential ascending auction is important for another, independent, reason. Bergemann and Välimäki (2008) demonstrate the suitability of sequential ascending auctions as a simple way to provide for the truthful implementation of the socially efficient allocation in a task scheduling problem. In particular, they provide an example in which sequential

¹Nekipelov (2007) examines a different aspect of these markets, choosing to ignore the sequential nature of these auctions in order to study within-auction dynamics.

²Either choice is, of course, a compromise, abstracting away important features of the real-world environment for the sake of tractability.

ascending auctions are equivalent to their dynamic generalization of the classic Vickrey-Clarke-Groves mechanism.³ Cavallo, Parkes, and Singh (2007) generalize this mechanism to settings in which agents may be “inaccessible” for periods of time. The present work complements these papers, as we show that the sequential ascending auction serves as an (easily implemented and understood) indirect mechanism that is equivalent to their direct mechanisms in a complex environment, and is therefore an incentive compatible mechanism for inducing socially efficient choices.

The present work is closely related to several papers in the sequential auctions literature. Milgrom and Weber (2000) examine the properties of a variety of auction formats for the (simultaneous or sequential) sale of multiple objects with a fixed set of buyers and objects. In regards to the ascending auction with private values, they show that, in equilibrium, buyers bid exactly their values. However, they allow for neither discounting nor the entry of new buyers, features that play a central role in our model. The vast majority of the literature following that work has chosen to focus on sealed-bid auctions; for example, the previously referenced Sailer (2006) and Zeithammer (2006) conduct empirical studies of eBay auctions making use of sequential second-price sealed-bid auctions and assumptions of an effectively static environment. Kittsteiner, Nikutta, and Winter (2004) examine the role of discounting in sequential sealed-bid auctions, and prove a revenue equivalence result for auctions in which the only information revealed is the valuations of bidders who have already left the market. Meanwhile, Jeitschko (1998) considers a model of first-price sealed-bid auctions in which winner’s bids are revealed, allowing the remaining buyers to update their beliefs about their opponents’ valuations. On the other hand, Cai, Wurman, and Chao (2007) demonstrate the nonexistence of pure-strategy symmetric equilibria in sealed-bid sequential auction models in which *all* bids are revealed. The only paper that we are aware of that examines sequential ascending auctions is that of Caillaud and Mezzetti (2004), who examine reserve prices in a model with a sequence of only two auctions.

Certain papers within the literature on bargaining with incomplete information are also related to our model. Inderst (2008) considers a bargaining model in which a seller is randomly visited by heterogeneous buyers. If the seller is currently engaged in bargaining with one agent when another arrives, she may choose to switch from bargaining with one buyer to bargaining with the other. However, this switch is permanent, implying that the arrival of a new buyer either “restarts” the game or is completely irrelevant. Fuchs and Skrzypacz (forthcoming) take a different approach: they consider an incomplete information bargaining problem between a buyer and a seller, and allow for the possibility of the arrival of various “events.” These events end the game and yield an exogenously determined expected payoff to each agent. The suggested interpretation is that these events may be viewed as triggers for some sort of multi-lateral mechanism involving new entrants (a second-price auction, for example) for which the expected payoffs are a reduced-form representation. Thus, while both works are primarily concerned with characterizing the endogenous option value that results from the potential arrival of additional participants to the

³Their example differs substantially from the present work, as it does not take into account the complications of random buyer and seller arrivals, and hence is essentially a static problem.

market, they do this in a framework of bilateral bargaining which fails to capture the dynamic nature of direct competition between several current and future potential market participants.

In addition, we would be remiss in not noting the relationship between our model and that of Peters and Severinov (2006). Also primarily motivated by auction markets such as eBay, their work considers a setting with multiple buyers and sellers interacting simultaneously. They find a perfect Bayesian equilibrium that supports efficient trade at Vickrey prices; moreover, if the numbers of buyers and sellers are sufficiently large, then trade is also ex post efficient. While their model has the advantage of considering the effects of competing auctions on the strategic behavior of buyers and sellers, it does not take into account what we believe are two key features of the markets in question (and the two key features of our model): auctions are conducted asynchronously, and new agents arrive on the market at random times.

To this end, Nekipelov (2007) models entry *during* a single ascending auction in order to study entry deterrence and within-auction price dynamics. On the other hand, Said (2008) examines the role of random entry in a model of sequential second-price auctions in which objects are stochastically equivalent; values are independently and identically distributed across both buyers and objects. In the present work, however, buyers' values are constant—each agent's independent and private value for the good does not vary over time. Since buyers' values are *not* drawn anew in each period, the information revealed in equilibrium becomes an important factor, affecting the option value arising from participation in future auctions. In particular, not only are expected payoffs affected by the number of agents present on the market, but the realizations of their values are integral to price determination as well.

The paper is organized as follows. We present our model in Section 2, and then provide a simple example in Section 3 that demonstrates some of the advantages of the ascending auction format over the second-price sealed-bid auction in a dynamic setting with buyer entry. Section 4 solves for the equilibrium in our model with buyer entry and demonstrates some of its desirable properties. In Section 5, we discuss the relationship between our model and the dynamic Vickrey-Clarke-Groves mechanism. Finally, Section 6 concludes.

2. MODEL

We consider a market in which time is discrete; periods are indexed by $t \in \mathbb{N}$. There is a finite number n_t of risk-neutral buyers on the market in any given period t . Each buyer $i \in \{1, \dots, n_t\}$ has a valuation v_i for a single unit of the good in question, where v_i is drawn from the distribution F with continuous density f and support $\mathbf{V} := [0, \bar{v}]$. We assume that valuations are private information, and are independently and identically distributed across buyers. Moreover, additional buyers may arrive on the market in each period. We will assume that at most one new buyer arrives in any given period, and that this arrival occurs with some exogenously given probability $q \in [0, 1]$. Finally, we assume that all buyers discount the future exponentially with discount factor $\delta \in (0, 1)$.

In each period, there is exactly one object available for sale via an ascending (English) auction. The auction begins with the price at zero and all bidders participating in the auction. Each bidder may choose any price at which to drop out of the auction. This exit decision is irreversible (in

the current period), and is observable by all agents currently present in the market. Finally, the auction ends whenever exactly 1 active bidder remains, and the price paid by this winning bidder is the price at which the last exit occurred. Note that we assume that the number of active bidders is commonly known throughout the auction.⁴ With this in mind, each bidder’s decision problem *within* a given period is not to choose a single bid, but rather a sequence of functions, each of which is a exit price contingent on the (observed) exit prices of the bidders who have already dropped out of the current auction.

Throughout, we will denote by \hat{y} the *ordered* vector of realized values for those buyers currently present on the market, where

$$\hat{y}_1 > \hat{y}_2 > \cdots > \hat{y}_n.$$

Furthermore, for any $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$, we will denote by $V_{k,n}(\hat{y})$ the expected payoff at the beginning of a period (before any entry has occurred) of the buyer with the k -th highest of n values. For example, if there are three bidders present, with $v_2 > v_3 > v_1$, then $\hat{y} = \{v_2, v_3, v_1\}$, bidder 1’s payoff is $V_{3,3}(\hat{y})$, bidder 2’s payoff is $V_{1,3}(\hat{y})$, and bidder 3’s payoff is $V_{2,3}(\hat{y})$.

3. MOTIVATING EXAMPLE

Suppose that there are two buyers on the market with values $v_1, v_2 \in [0, 1]$, where, without loss of generality, we assume that $v_1 > v_2$. In addition, a third potential buyer with value $v_3 \sim F$, where F is the uniform distribution on $[0, 1]$ may enter the market with some exogenously given probability $q \in [0, 1]$. For simplicity, we assume that new entry may occur only in the initial period. Each buyer wishes to purchase exactly one unit of an object which is being sold via a sequence of auctions. All buyers discount time with a common discount factor $\delta \in (0, 1)$. Furthermore, we make the assumption that v_1 and v_2 are commonly known amongst all buyers, which may be viewed as the result of information being revealed via bidding behavior in some (unmodeled) previous periods. On the other hand, the new entrant’s value (assuming she arrives on the market in the first period) is her own private information. We will consider two variants of this example; first, we will assume that objects are sold via second-price auctions in which the buyers’ bids are revealed after each round, and then we will assume that objects are sold via ascending auctions.

3.1. *Second-Price Auction*

We begin with the second-price auction. Note that in any round in which there is only one bidder present, that bidder receives the object at a price of zero, regardless of her bid. Therefore, if there are two bidders present in the second period, each bidder i has an option value of δv_i from losing. Thus, regardless of the information that each bidder has about the other, it is weakly dominant for each bidder to submit a bid of their true value less their option value—the optimal bid for each bidder i is $(1 - \delta)v_i$. Thus, denoting the payoff of a bidder in the second round when

⁴This assumption is a consequence of the standard Milgrom and Weber (1982) “button auction” approach to modeling English auctions.

there are two bidders present as $U(v_i, v_j)$, we have

$$(1) \quad U(v_i, v_j) = \begin{cases} v_i - (1 - \delta)v_j, & \text{if } v_i > v_j; \\ \delta v_i, & \text{if } v_i \leq v_j. \end{cases}$$

Note that, using this expression, we may write the payoff of a lone bidder with value v_i as $U(v_i, 0)$.

Now consider the third bidder (when present). Under the assumption that bidder 1 bids a greater amount than bidder 2 (that is, that $b_1 > b_2$), the third bidder faces a choice between winning the auction and receiving a payoff of $v_3 - b_1$ or losing the auction and facing bidder 2 in the next period, yielding a payoff of $\delta U(v_3, v_2)$. Thus, bidder 3 prefers to win if, and only if, $v_3 - b_1 \geq \delta U(v_3, v_2)$, or, equivalently, $b_1 \leq v_3 - \delta U(v_3, v_2)$. She can then win the auction if, and only if, it is optimal for her to do so by bidding

$$(2) \quad b_3(v_3) = v_3 - \delta U(v_3, v_2) = \begin{cases} (1 - \delta)v_3 + \delta(1 - \delta)v_2, & \text{if } v_3 > v_2; \\ (1 - \delta^2)v_3, & \text{if } v_3 \leq v_2. \end{cases}$$

Note that b_3 is strictly increasing in v_3 , and hence fully identifies bidder 3's valuation in the next period when bids are revealed. For convenience, we will denote by u_1 and u_2 the values of bidder 3 that submit bids equal to those of bidders 1 and 2, respectively; that is,

$$u_1 := b_3^{-1}(b_1) \text{ and } u_2 := b_3^{-1}(b_2).$$

Now consider the case of bidder 2's bid in the first period of the game. If she submits a winning bid in the first period, she receives a payoff of $v_2 - b^*$, where b^* is the highest competing bid that she faces. On the other hand, if she loses the first-round auction, she receives a payoff of $\delta \mathbb{E}[U(v_2, v^*)]$, where

$$v^* = \begin{cases} 0, & \text{with probability } 1 - q; \\ v_3, & \text{with probability } qF(u_1); \\ v_1, & \text{with probability } q(1 - F(u_1)). \end{cases}$$

Thus, bidder 2 prefers to win if, and only if, $v_2 - b^* \geq \delta \mathbb{E}[U(v_2, v^*)]$. She may then guarantee that she wins only when it is desirable to do so by bidding

$$(3) \quad \begin{aligned} b_2 &= v_2 - \delta \mathbb{E}[U(v_2, v^*)] \\ &= v_2 - \delta \left[\begin{aligned} &(1 - q)v_2 + q(1 - F(u_1))\delta v_2 \\ &+ q \int_0^{v_2} (v_2 - (1 - \delta)v_3) dF(v_3) + q \int_{v_2}^{u_1} \delta v_2 dF(v_3) \end{aligned} \right] \\ &= (1 - \delta)(1 + \delta q)v_2 - (1 - \delta)\delta q \frac{v_2^2}{2}. \end{aligned}$$

Finally, let us consider buyer 1's bidding behavior in the first period of the game. Note first that $u_2 < v_2 < v_1$, implying that if bidder 1 loses today, she will definitely win the auction in the next period. To see this, note that if bidder 3 enters and wins the first round, bidder 1 faces $v_2 < v_1$ in the next period. On the other hand, if bidder 2 is the high bidder in the first round, then bidder 1 is either alone or faces $v_3 < u_2 < v_1$ in the second round. Thus, when the high opponent bid is b^* ,

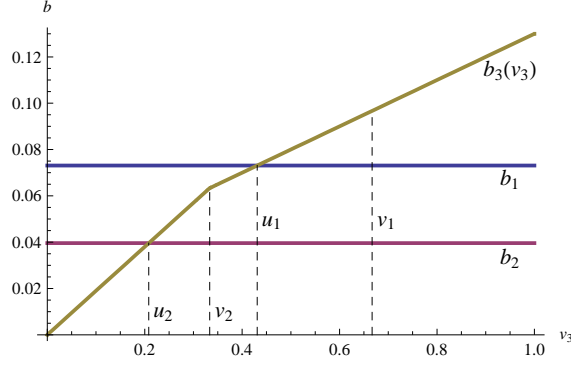


FIGURE 1. Initial bids when $v_1 = \frac{2}{3}$, $v_2 = \frac{1}{3}$, $v_3 \sim U[0, 1]$, $\delta = \frac{9}{10}$, and $q = \frac{1}{4}$.

winning yields bidder 1 a payoff of $v_1 - b^*$, while losing yields a payoff of $\delta U(v_1, v^*)$, where

$$v^* = \begin{cases} 0, & \text{with probability } 1 - q; \\ v_3, & \text{with probability } qF(u_2); \\ v_2, & \text{with probability } q(1 - F(u_2)). \end{cases}$$

Thus, similar to the cases of bidders 2 and 3, bidder 1 may guarantee that she wins only when it is desirable for her to do so by bidding

$$b_1 = v_1 - \delta \mathbb{E}[U(v_1, v^*)] = v_1 - \delta \left[(1 - q)v_1 + \delta q(1 - F(u_2))(v_1 - (1 - \delta)v_2) + q \int_0^{u_2} (v_1 - (1 - \delta)v') dF(v') \right].$$

Recall that $u_2 = b_3^{-1}(b_2) < v_2$, implying that $u_2 = b_2 / (1 - \delta^2)$. Combining this with the assumption that $F(x) = x$ implies that

$$u_2 = \frac{b_2}{1 - \delta^2} = \frac{1 + \delta q}{1 + \delta} v_2 - \frac{\delta q}{1 + \delta} \frac{v_2^2}{2}.$$

Thus, we may conclude that

$$(4) \quad b_1 = (1 - \delta)v_1 + (1 - \delta)\delta q v_2 - \frac{\delta(1 - \delta)(1 + \delta q)(1 + \delta(2 - q))q v_2^2}{2(1 + \delta)^2} + \frac{\delta^3(1 - \delta)(1 - q)q^2 v_2^3}{2(1 + \delta)^2} + \frac{\delta^3(1 - \delta)q^3 v_2^4}{8(1 + \delta)^2}.$$

For clarity, [Figure 1](#) plots the bids of all three buyers for fixed parameter values.⁵ The key features to note are that $u_1 < v_1$ and $u_2 < v_2$; use of the second-price auction in this context may lead to inefficient outcomes, as “low” values of bidder 3 may outbid bidders 1 and 2 despite their having higher values. This result is driven by two main features of our setting: first, agents discount the future and hence the order in which objects are allocated matters; and second, there is a fundamental asymmetry in information—bidder 3’s value is private information, while the values of bidders 1 and 2 are commonly known. Thus, in addition to the nonexistence of symmetric equilibria in sequential second-price sealed-bid auctions as demonstrated by [Cai, Wurman, and](#)

⁵The qualitative features of the equilibrium do not depend on these parameter values. Moreover, the result remains true even if the presence of the new entrant is made common knowledge or contingent bidding is used. This may be easily seen by examining the bids of buyers 1 and 2 in the cases where $q = 0$ and $q = 1$.

Chao (2007), allowing for the entry of new buyers may induce inefficient outcomes, even in the asymmetric equilibria of the sequential auction game.

3.2. Ascending Auction

We now demonstrate that the ascending auction does not share the inefficiency of the second-price auction in this setting. Note that when there are only two bidders present, the losing bidder is guaranteed a payoff of δv_i in the next period. Therefore, bidders are willing to remain active in an auction until the price reaches $(1 - \delta)v_i$. Thus, the expected payoff of a bidder when she has only one opponent present on the market is given by $U(v_i, v_j)$ from Equation 1.

When there are three bidders present, matters are slightly different. In particular, the very nature of an ascending auction immediately reveals to all bidders the number of participants. Thus, bidder 3 is unable to keep private her presence on the market. This implies that, in a symmetric equilibrium, the first bidder to drop out of the auction knows that they have the lowest value among three bidders, and will not be allocated an object until the third period, yielding an expected payoff of $\delta^2 v_i$. Thus, each of the three bidders remains active until the price reaches

$$(1 - \delta^2)v_i.$$

Denoting by \hat{y}_3 the lowest of the three values, the two remaining bidders now know that they are guaranteed a payoff of $U(v_i, \hat{y}_3)$ in the following period, and are hence willing to remain active until they are indifferent between winning at the current price and winning the object in the following period; that is, until the price reaches

$$(1 - \delta)v_i + \delta(1 - \delta)\hat{y}_3.$$

Notice that these cutoff prices are strictly increasing in each bidder's value, and hence are both efficient and fully revealing.⁶ Thus, we have established that the ascending auction does not suffer from the same shortcomings as the second-price auction in this relatively simple setting. We will therefore focus exclusively on the ascending auction from this point forward. Moreover, we will relax the assumption that new entry may occur only once—a new buyer may arrive on the market in *any* period.

4. EQUILIBRIUM ANALYSIS

4.1. Preliminaries and Equilibrium Strategies

One of the most critical features of the equilibrium that we construct in this model is that buyer's bids and payoffs do not depend upon the valuations of higher-ranked bidders (neither in expectation nor realization), even if that information is publicly available. Recall that \hat{y} is the *ordered* vector of realized buyer valuations, where

$$\hat{y}_1 > \dots > \hat{y}_n,$$

⁶In addition, it is straightforward to verify that these strategies do, in fact, constitute an equilibrium. Conditional on participation, no bidder wishes to deviate from these strategies. Furthermore, no bidder wishes to postpone their participation to a future period.

and that we denote by $V_{k,n}(\hat{y})$ the expected payoff at the beginning of a period (before any entry has occurred) of the buyer with the k -th highest of n values. To show the property described above, we will show that (abusing notation slightly) we may write

$$V_{k,n}(\hat{y}_1, \dots, \hat{y}_n) = V_{k,n}(\hat{y}_k, \dots, \hat{y}_n).$$

A formal statement of this result may be found in the subsequent section; in the meantime, we will describe the equilibrium taking this property as a given.

Suppose that an auction is in progress with n bidders with (ordered) values \hat{y} . When all bidders are still active, a bidder with valuation v_i who drops out of the bidding learns (and reveals) that, in equilibrium, she has the lowest value; that is, that $\hat{y}_n = v_i$. Therefore, her expected payoff in the next period is $V_{n-1,n-1}(v_i)$, as at the *beginning* of the next period, before a new entrant may arrive, there will be $n - 1$ bidders present (the current period's n bidders less the winning buyer) and she will have the lowest value. Therefore, each bidder i should remain in the auction until the current price p is such that

$$v_i - p = \delta V_{n-1,n-1}(v_i).$$

At this price, bidder i is indifferent between purchasing the object today and waiting until the next period when i will be the lowest-valued buyer. Thus, when no one has dropped out, bidder i will remain in the auction until the price reaches

$$(5) \quad \beta_{n,n}(v_i) := v_i - \delta V_{n-1,n-1}(v_i).$$

Once someone drops out of the auction, the remaining $n - 1$ bidders learn the realization of \hat{y}_n and that they are not the lowest-valued competitor.⁷ Therefore, the next bidder (with value v_j) to drop out reveals herself to be the second-lowest of the n bidders; therefore, her expected payoff in the next period is $V_{n-2,n-1}(v_j, \hat{y}_n)$, as she will be the second-lowest of the $n - 1$ buyers remaining in the following period. Thus, each bidder j who has not already dropped out should remain in the auction until the current price p is such that she is indifferent between purchasing the object in the present period and waiting until the next period—that is, when

$$v_j - p = \delta V_{n-2,n-1}(v_j, \hat{y}_n).$$

Thus, when only one bidder has dropped out, bidder j remains in the auction until the price reaches

$$(6) \quad \beta_{n-1,n}(v_j, \hat{y}_n) := v_j - \delta V_{n-2,n-1}(v_j, \hat{y}_n).$$

Proceeding inductively, we define for each $k = 2, \dots, n$ the bidding function

$$(7) \quad \beta_{k,n}(v_i, \hat{y}_{k+1}, \dots, \hat{y}_n) := v_i - \delta V_{k-1,n-1}(v_i, \hat{y}_{k+1}, \dots, \hat{y}_n).$$

These bidding functions define the drop-out points for a bidder with value v_i when there are k buyers still active in the auction. Notice that this implies that the final price in this auction will be

$$\beta_{2,n}(\hat{y}_2, \dots, \hat{y}_n) = \hat{y}_2 - \delta V_{1,n-1}(\hat{y}_2, \dots, \hat{y}_n).$$

⁷This of course requires $\beta_{n,n}$ to be invertible, something that we will verify in short order.

Keep in mind, however, that we must verify that these bid functions are invertible (so that values are revealed), and also that these bidding strategies indeed form an equilibrium. This requires a characterization of the expected payoff functions $V_{k,n}$.

4.2. Payoff Functions

As a preview of our results, consider first the case of a lone buyer present on the market at the beginning of a period with valuation v_1 , and that a second buyer may arrive with probability q . Once the price clock starts rising, it is immediately revealed whether there are one or two bidders present. Thus, there is no asymmetric information regarding the number of active bidders.

Note that if the second bidder does not arrive, the lone bidder receives the object for free. In the case of two bidders present, however, each bidder $i = 1, 2$ will stay in the auction until the price rises to $\beta_{2,2}(v_i) = v_i - \delta V_{1,1}(v_i)$. Thus,

$$V_{1,1}(v_1) = (1 - q)v_1 + q \left[\int_0^{v_1} (v_1 - \beta_{2,2}(v')) dF(v') + \int_{v_1}^{\bar{v}} \delta V_{1,1}(v_1) dF(v') \right].$$

The first term in this expression is bidder 1's payoff if she is alone on the market. The second term is her expected payoff if a second bidder arrives, and is the sum of her expected winnings if the second bidder has a lower value than her and her expected continuation payoff if she loses the auction. Differentiation of this expression with respect to v_1 and substituting for $\beta_{2,2}(v_1)$ yields

$$V'_{1,1}(v_1) = \frac{1 - q(1 - F(v_1))}{1 - \delta q(1 - F(v_1))}.$$

Note that we can rewrite this expression as

$$V'_{1,1}(v_1) = \sum_{t=0}^{\infty} (\delta q(1 - F(v_1)))^t [1 - q(1 - F(v_1))],$$

which is the summation of the expected per-period gain from a marginal increase in v_1 , discounted by the probability of that gain being realized in any given period.

Note that, since F is continuous, $V'_{1,1}$ is also a continuous function. Furthermore, we have $V_{1,1}(0) = 0$, implying by way of the Fundamental Theorem of Calculus that

$$(8) \quad V_{1,1}(v_i) = V_{1,1}(0) + \int_0^{v_i} V'_{1,1}(v') dv' = \int_0^{v_i} \frac{1 - q(1 - F(v'))}{1 - \delta q(1 - F(v'))} dv'.$$

Note that $0 < V'_{1,1}(v') \leq 1$ for all $v' \in \mathbf{V}$. Hence, $V_{1,1}$ is strictly increasing, as is $\beta_{2,2}$.

Proceeding inductively, we arrive at the following

PROPOSITION 1 (Existence and uniqueness of $V_{k,n}$).

Fix any $n > 1$, and suppose that the expected payoff to a buyer when a period starts with $n - 1$ bidders present depends only on the rank of that bidder and the values of those with values lower than her; that is, given (known) values $\hat{y} \in \mathbf{V}^{n-1}$, the k -th highest of the $n - 1$ bidders receives expected payoff $V_{k,n-1}(\hat{y}_k, \dots, \hat{y}_{n-1})$. Then the expected payoff of the k -th highest of n bidders, for all $k = 1, \dots, n$, is given by $V_{k,n}(\hat{y}_k, \dots, \hat{y}_n)$. Furthermore, given $\{V_{k,n-1}\}_{k=1}^{n-1}$, the functions $\{V_{k,n}\}_{k=1}^n$ are uniquely determined.

Proof. The proof may be found in [Appendix A](#). □

Thus, the strategies in [Equation 7](#) lead to well-defined and unique value functions for the buyers. In addition, following these strategies implies that these expected payoffs do not depend upon the bids or prices paid in any previous periods, but instead depend only upon the values of those currently present buyers ranked below a bidder.

We are also able to use the indifference inherent in the definition of our conjectured equilibrium strategy in order to illustrate the link between the various payoff functions. In particular, we have the following

PROPOSITION 2 (Relationship between $V_{k,n}$ and $V_{1,n}$).

Fix any $n \in \mathbb{N}$. Then for all $k = 1, \dots, n$, the expected payoff to the k -th ranked of n buyer is equal to that of the highest-ranked buyer when she is tied with $k - 1$ of her opponents; that is,

$$V_{k,n}(\hat{y}_k, \dots, \hat{y}_n) = V_{1,n}(\hat{y}_k, \dots, \hat{y}_k, \hat{y}_{k+1}, \dots, \hat{y}_n).$$

Proof. The proof may be found in [Appendix A](#). □

As mentioned above, this result makes heavy use of the indifference conditions built into the bidding strategies described in [Equation 7](#), and in particular the indifference of the buyer with the second-highest value. This bidder drops out at a price at which she is indifferent between winning immediately or waiting one period. Unsurprisingly, when the top two buyers have the same value, they must receive the same payoff, regardless of the tie-breaking rule used to determine which one of the two should receive the object when they drop out simultaneously. The intuition behind the relationship between lower-ranked buyers' payoff functions is analogous. Moreover, [Proposition 2](#) implies that knowledge of the functions $\{V_{1,n}\}_{n=1}^{\infty}$ is sufficient to determine the remaining value functions. Thus, define the function $\lambda : \mathbf{V} \rightarrow [0, 1]$ by

$$(9) \quad \lambda(v) := \delta \frac{1 - q(1 - F(v))}{1 - \delta q(1 - F(v))}.$$

We have the following

PROPOSITION 3 (Characterization of $V_{k,n}$).

For all $n \in \mathbb{N}$ and all $k = 1, \dots, n$,

$$(10) \quad V_{k,n}(\hat{y}) = \delta^{-1} \sum_{j=k}^n \int_{\hat{y}_{j+1}}^{\hat{y}_j} \lambda^j(v') dv',$$

where we take $\hat{y}_{n+1} := 0$.

Proof. Note that we may write $V_{1,n}(\hat{y})$ as

$$(11) \quad \begin{aligned} V_{1,n}(\hat{y}) = & (1 - q) [\hat{y}_1 - \beta_{2,n}(\hat{y}_2, \dots, \hat{y}_n)] \\ & + q \left[\sum_{j=0}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} (\hat{y}_1 - \beta_{2,n+1}(\hat{y}_{-1}(v'))) dF(v') \right. \\ & \left. + \int_{\hat{y}_1}^{\bar{v}} \delta V_{1,n}(\hat{y}) dF(v') \right], \end{aligned}$$

where $\hat{y}(v')$ is the ordered vector that arises from adding v' to \hat{y} . We will denote by $V_{1,n}^{(j)}$ the partial derivative of $V_{1,n}$ with respect to its j -th argument. Differentiation with respect to \hat{y}_1 then implies that

$$\begin{aligned} V_{1,n}^{(1)}(\hat{y}) &= 1 - q(1 - F(\hat{y}_1)) + \delta q(1 - F(\hat{y}_1))V_{1,n}^{(1)}(\hat{y}) \\ &= \delta^{-1}\lambda(\hat{y}_1). \end{aligned}$$

Notice that this result is independent of n . Furthermore, note that this implies that $V_{1,n}^{(j)}(\hat{y})$ does not depend on \hat{y}_1 for any $j \neq 1$; equivalently, for all $n \in \mathbb{N}$,

$$V_{1,n}^{(1,j)}(\hat{y}) = 0 \text{ for all } \hat{y} \in \mathbf{V}^n \text{ and } j \neq 1.$$

Differentiating [Equation 11](#) with respect to \hat{y}_2 now leads to

$$\begin{aligned} V_{1,n}^{(2)}(\hat{y}) &= -(1 - q(1 - F(\hat{y}_2)) + \delta(1 - q)V_{1,n-1}^{(1)}(\hat{y}_2, \dots, \hat{y}_n) \\ &\quad + \delta q \sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}^{(1)}(\hat{y}_{-1}(v')) dF(v') \\ &\quad + \delta q \int_{\hat{y}_2}^{\hat{y}_1} V_{1,n}^{(2)}(v', \hat{y}_2, \dots, \hat{y}_n) dF(v') + \delta q(1 - F(\hat{y}_1))V_{1,n}^{(2)}(\hat{y}) \end{aligned}$$

Note first that

$$\int_{\hat{y}_2}^{\hat{y}_1} V_{1,n}^{(2)}(v', \hat{y}_2, \dots, \hat{y}_n) dF(v') = (F(\hat{y}_1) - F(\hat{y}_2)) V_{1,n}^{(2)}(\hat{y})$$

since $V_{1,n}^{(1,2)} = 0$. Moreover,

$$\begin{aligned} \sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}^{(1)}(\hat{y}_{-1}(v')) dF(v') &= \sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta^{-1}\lambda(\hat{y}_2) dF(v') \\ &= \int_0^{\hat{y}_2} \delta^{-1}\lambda(\hat{y}_2) dF(v') = \delta^{-1}\lambda(\hat{y}_2)F(\hat{y}_2). \end{aligned}$$

Thus, we have

$$\begin{aligned} V_{1,n}^{(2)}(\hat{y}) &= \frac{-(1 - q(1 - F(\hat{y}_2))) + (1 - q(1 - F(\hat{y}_2)))\lambda(\hat{y}_2)}{1 - \delta q(1 - F(\hat{y}_2))} \\ &= \frac{1 - q(1 - F(\hat{y}_2))}{1 - \delta q(1 - F(\hat{y}_2))} (\lambda(\hat{y}_2) - 1) = \delta^{-1} (\lambda^2(\hat{y}_2) - \lambda(\hat{y}_2)). \end{aligned}$$

Note that, similar to the case of $V_{1,n}^{(1)}$, $V_{1,n}^{(2)}$ depends only on the second argument of $V_{1,n}$. Thus, for all $n \in \mathbb{N}$,

$$V_{1,n}^{(2,j)}(\hat{y}) = 0 \text{ for all } \hat{y} \in \mathbf{V}^n \text{ and } j \neq 2.$$

Proceeding inductively, fix any $k \in \{3, \dots, n\}$ for arbitrary $n \in \mathbb{N}$, and suppose that

$$V_{1,n}^{(j)}(\hat{y}) = \delta^{-1} (\lambda^j(\hat{y}_j) - \lambda^{j-1}(\hat{y}_j))$$

for all $j = 2, \dots, k-1$. Differentiating [Equation 11](#) with respect to \hat{y}_k yields

$$\begin{aligned} V_{1,n}^{(k)}(\hat{y}) &= \delta(1-q)V_{1,n-1}^{(k-1)}(\hat{y}_2, \dots, \hat{y}_n) + \delta q \sum_{j=0}^{n-k} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}^{(k-1)}(\hat{y}_{-1}(v')) dF(v') \\ &\quad + \delta q \sum_{j=n-k+1}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}^{(k)}(\hat{y}_{-1}(v')) dF(v') + \delta q(1-F(\hat{y}_1))V_{1,n}^{(k)}(\hat{y}). \end{aligned}$$

Since $V_{1,n}^{(k-1)}$ does not depend on any of its arguments but the $(k-1)$ -th,

$$\sum_{j=0}^{n-k} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}^{(k-1)}(\hat{y}_{-1}(v')) dF(v') = F(\hat{y}_k)V_{1,n-1}^{(k-1)}(\hat{y}_{-1}).$$

In addition, $V_{1,n}^{(j,k)} = 0$ for all $j < k$ implies that

$$\sum_{j=n-k+1}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}^{(k)}(\hat{y}_{-1}(v')) dF(v') = (F(\hat{y}_1) - F(\hat{y}_k))V_{1,n}^{(k)}(\hat{y}).$$

Thus, we have

$$\begin{aligned} V_{1,n}^{(k)}(\hat{y}) &= \delta(1-q)V_{1,n-1}^{(k-1)}(\hat{y}_{-1}) + \delta q F(\hat{y}_k)V_{1,n-1}^{(k-1)}(\hat{y}_{-1}) + \delta q(1-F(\hat{y}_k))V_{1,n}^{(k)}(\hat{y}) \\ &= \delta \frac{1-q(1-F(\hat{y}_k))}{1-\delta q(1-F(\hat{y}_k))} V_{1,n-1}^{(k-1)}(\hat{y}_{-1}) = \delta^{-1} \left(\lambda^k(\hat{y}_k) - \lambda^{k-1}(\hat{y}_k) \right). \end{aligned}$$

By induction, the above expression holds for all $k = 2, \dots, n$, where $n \in \mathbb{N}$ is arbitrary.

Since λ^k is continuous for all $k \in \mathbb{N}$, we may use the Fundamental Theorem of Calculus and the boundary condition $V_{1,n}(0, \dots, 0) = 0$ to show that

$$V_{1,n}(\hat{y}) = V_{1,n}(0, \dots, 0) + \sum_{j=1}^n \int_0^{\hat{y}_j} V_{1,n}^{(j)}(v') dv' = \delta^{-1} \sum_{j=1}^n \left[\int_0^{\hat{y}_j} \lambda^j(v') dv' - \int_0^{\hat{y}_{j+1}} \lambda^j(v') dv' \right],$$

where we take $\hat{y}_{n+1} := 0$. We may then make use of [Proposition 2](#) and some arithmetic manipulation in order to achieve the desired result. \square

To better understand this result, let us consider two ‘‘corner’’ cases. In particular, notice that if $q = 0$ (that is, if no new buyers ever arrive on the market), then

$$V_{k,n}(\hat{y}) = \sum_{j=k}^n \delta^{j-1}(\hat{y}_j - \hat{y}_{j+1})$$

for all $k = 1, \dots, n$ and any $n \in \mathbb{N}$. Thus, the expected payoff to a buyer in this case is the discounted difference between consecutively ranked valuations. Note that this is also exactly the externality imposed by the k -th highest buyer on all those ranked below her when there is no entry, as she postpones each one’s receipt of an object by exactly one period. On the other hand, if $\delta = 1$ (implying that buyers are ‘‘infinitely patient’’), then for any $q < 1$, we have

$$V_{k,n}(\hat{y}) = \hat{y}_k$$

for all $k = 1, \dots, n$ and any $n \in \mathbb{N}$. In this case, buyers care only about their eventual receipt of an object, but not about the timing of that event. Therefore, their bids are all equal to zero, and any random assignment of objects leaves the buyers equally well off.

4.3. Equilibrium

With the characterization derived in [Proposition 3](#), we may now reformulate the bidding strategies from [Equation 7](#) as

$$(12) \quad \beta_{k,n}(v_i, \hat{y}_{k+1}, \dots, \hat{y}_n) = v_i - \int_{\hat{y}_{k+1}}^{v_i} \lambda^{k-1}(v') dv' - \sum_{j=k+1}^n \int_{\hat{y}_{j+1}}^{\hat{y}_j} \lambda^{j-1}(v') dv'.$$

This expression allows us to demonstrate the properties of bids in the following

PROPOSITION 4 (Information revelation and sequential consistency of $\beta_{l,k}$).

The buyers' bids $\beta_{k,n}$, where $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$, are strictly increasing in each buyers' own valuation. Furthermore, when the buyers use these bidding functions, the exit of a lower-ranked bidder does not induce the immediate exit of any higher-ranked bidders.

Proof. The proof may be found in [Appendix A](#). □

Note that this result verifies our previous assumption that buyers' values are revealed after each round—since the bidding functions are strictly increasing in each buyers' own private valuation, the price at which they drop out of the auction is an invertible function, thereby allowing the inference of their value by their competitors. Furthermore, since the bidding functions are “sequentially consistent,” a higher-ranked bidder remains in the auction instead of immediately exiting after a lower-ranked bidder drops out, thereby allowing the other buyers to (eventually) deduce their value.

Finally, it remains to be shown that the bidding strategies described are, in fact, an equilibrium of this model. We demonstrate this in the following

PROPOSITION 5 (Equilibrium verification).

Suppose that in each period, buyers bid according to the cutoff strategies given in [Equation 12](#). This strategy profile forms a perfect Bayesian equilibrium of the sequential auction game.

Proof. Consider any period with $n \in \mathbb{N}$ buyers on the market, and fix an arbitrary bidder i . Suppose that all bidders other than i are using the conjectured strategy. We must show that bidder i has no incentive to make a one-shot deviation from the collection of bidding functions $\{\beta_{k,n}\}_{k=2}^n$. Specifically, we must show that i prefers not to exit an auction either earlier or later than specified.

Note first that if $v_i < \hat{y}_1$, dropping out of the auction early has no effect on future actions due to the memorylessness of the bidding strategies—in future periods, current actions and the beliefs they induce are ignored, and the process of information revelation is repeated anew. Hence, a one-shot deviation to an early exit will not affect the bidding behavior, outcomes, or payoffs in future periods. On the other hand, suppose that bidder i has the highest realized valuation among bidders present on the market. Following the conjectured equilibrium leads to a payoff of $\hat{y}_1 - \beta_{2,n}(\hat{y}_{-1})$, while deviating to an early exit leads to the second-ranked bidder winning and a

payoff to i of $\delta V_{1,n-1}(\hat{y}_{-2})$. Letting $\hat{w} := (\hat{y}_3, \dots, \hat{y}_n)$, we then have

$$\begin{aligned} \hat{y}_1 - \beta_{2,n}(\hat{y}_2, \hat{w}) - \delta V_{1,n-1}(\hat{y}_1, \hat{w}) &= \hat{y}_1 - \hat{y}_2 + \delta (V_{1,n-1}(v_2, \hat{w}) - V_{1,n-1}(v_1, \hat{w})) \\ &= \hat{y}_1 - \hat{y}_2 + \left(\int_0^{\hat{y}_2} \lambda(v') dv' - \int_0^{\hat{y}_1} \lambda(v') dv' \right) \\ &= \int_{\hat{y}_2}^{\hat{y}_1} (1 - \lambda(v')) dv' > 0 \end{aligned}$$

since $\hat{y}_1 > \hat{y}_2$ and $0 < \lambda(v') < 1$ for all $v' \in \mathbf{V}$. Thus, deviating and exiting the auction early leads to a strict decrease in utility if the realized values are such that bidder i has the highest value, and does not affect payoffs otherwise.

Of course, bidder i also has the option of remaining active *beyond* the cutoffs specified by $\{\beta_{k,n}\}_{k=2}^n$. If the realized values are such that $v_i = \hat{y}_1$, delaying exit will have no effect, as the other bidders will have already dropped out of the auction earlier than i . If, on the other hand, $v_i = \hat{y}_k$ for some $k > 1$, then delaying exit may have an effect on i 's payoffs. To be precise, if i exits before the eventual winner, her payoff will remain unchanged as behavior in future periods does not depend upon information already revealed. Thus, in order to influence her payoff, i must win the auction, remaining active until all other bidders have dropped out. Winning the auction yields a payoff of $\hat{y}_k - \beta_{2,n}(\hat{y}_{-k})$, while following the strategy in [Equation 12](#) leads to an expected payoff of $\delta V_{k-1,n-1}(\hat{y})$. Letting $\hat{w} := (\hat{y}_{k+1}, \dots, \hat{y}_n)$, we have

$$\begin{aligned} \hat{y}_k - \beta_{2,n}(\hat{y}_{-k}) - \delta V_{k-1,n-1}(\hat{y}_k, \hat{w}) &= \hat{y}_k - \hat{y}_1 + \delta (V_{1,n-1}(\hat{y}_1, \dots, \hat{y}_{k-1}, \hat{w}) - V_{k-1,n-1}(\hat{y}_k, \hat{w})) \\ &= \hat{y}_k - \hat{y}_1 + \sum_{j=1}^{k-1} \int_{\hat{y}_{j+1}}^{\hat{y}_j} \lambda^j(v') dv' \\ &< \hat{y}_k - \hat{y}_1 + \sum_{j=1}^{k-1} (\hat{y}_j - \hat{y}_{j+1}) = 0, \end{aligned}$$

where the inequality follows from the fact that $0 < \lambda(v') < 1$ for all $v' \in \mathbf{V}$. Hence, deviating and exiting the auction later than prescribed has no effect if i has the highest value, but may lead to a decrease in utility if the realized values are such that $v_i < \hat{y}_1$.

Thus, we may conclude that bidder i has no incentive to make a one-shot deviation regardless of the realized values; thus, bidding according to [Equation 12](#) is optimal along the equilibrium path. In order to determine optimality *off* the equilibrium path, we need to consider the behavior of bidders after a deviation. Since these post-deviation histories are never reached, we are free to choose arbitrary off-equilibrium beliefs—Bayes' rule has no bite in this situation. In particular, we will suppose that after a deviation, buyers ignore the history of the game and believe that the deviator is *currently* truthfully revealing her value in accordance with the bidding functions $\{\beta_{k,n}\}_{k=2}^n$. The arguments used above therefore imply that, with these beliefs, continuing to bid according to this strategy remains optimal for all agents, including any that may have deviated in the current or previous periods. Thus, bidding according to [Equation 12](#) is optimal along the entire game tree, and hence is a perfect Bayesian equilibrium of the sequential auction game. \square

5. DYNAMIC VICKREY-CLARKE-GROVES MECHANISM

Bergemann and Välimäki (2008) develop the dynamic pivot mechanism (also referred to as the dynamic marginal contribution or dynamic Vickrey-Clarke-Groves mechanism), a direct mechanism that implements the socially efficient allocation in a dynamic private value environment in which agents receive private information over time. In the mechanism that they propose, agents receive in each period their marginal contribution to the social welfare in a dynamic generalization of the standard Vickrey-Clarke-Groves Mechanism. In this mechanism, the truth-telling strategy is periodic ex post individually rational and incentive compatible.⁸ Moreover, the authors show that the sequential ascending auction yields an identical implementation in the case of a scheduling problem with a fixed finite set of independent tasks. Cavallo, Parkes, and Singh (2007) take the model one step further, demonstrating that dynamic VCG truthfully implements the socially efficient allocation in more general dynamic settings. In this section, we show that the equilibrium in the sequential ascending auction discussed above is equivalent to the truth-telling equilibrium of the dynamic VCG mechanism.

Consider the model examined above in which exactly one object is available for sale in every period. In this setting, the socially efficient policy is to allocate each object to the buyer with the highest valuation present on the market.⁹

Let us define W_0 to be the expected value to the social planner at the beginning of a period in which no buyers are present on the market. Then, letting $\mathbb{E}[v]$ denote the expected value of the distribution F , we may write

$$W_0 = q \int_0^{\bar{v}} (v' + \delta W_0) dF(v') + (1 - q)\delta W_0 = \frac{q}{1 - \delta} \mathbb{E}[v].$$

Denote by $W_n(\hat{y})$ the expected value to the social planner at the beginning of a period when there are n buyers with values $\hat{y}_1 > \dots > \hat{y}_n$, before the realization of the new buyer arrival process. We may recursively solve for this function; in particular, we have the following

PROPOSITION 6 (Planner's payoff function).

The social planner's expected value at the beginning of a period in which there are n buyers present on the market with values $\hat{y}_1 > \dots > \hat{y}_n$ is given by

$$(13) \quad W_n(\hat{y}) = W_0 + \delta^{-1} \sum_{j=1}^n \int_0^{\hat{y}_j} \lambda^j(v') dv'.$$

Proof. The proof is similar to that of [Proposition 3](#), and may be found in [Appendix A](#). □

⁸In dynamic settings with the arrival of new information, the concepts of *periodic* ex post individual rationality and incentive compatibility are the natural counterparts of ex post individual rationality and incentive compatibility, as they account for the possibility of the arrival of additional information in the future.

⁹Intuitively, since a new object arrives in every period and future entrants' values are independent of the current state, there is no benefit to not allocating the object in any particular period. Moreover, allocating an object to a lower-valued buyer is inefficient due to the fact that the common discount factor δ is smaller than one; therefore, postponing a higher-valued buyer for the benefit of a lower-valued one is costly. A more formal exposition of this argument may be found in [Appendix B](#).

Essentially, this result provides an analogue to the social planner's payoff in the case of a fixed number of buyers without any entry. With n buyers whose values are given by $\hat{y}_1 > \dots > \hat{y}_n$, the efficient allocation yields a value to the planner given by

$$\sum_{j=1}^n \delta^{j-1} v_j.$$

In our setting, however, potential entrants can and do rearrange the ordering of agents in the efficient allocation, postponing the time at which buyers with lower valuations receive an object. Thus, their contribution to social welfare must take this effect into account. So, consider the buyer with the highest valuation \hat{y}_1 . If we increase his valuation by an infinitesimal amount, the planner gains an equal amount with probability $1 - q(1 - F(\hat{y}_1))$, the probability that no higher-valued new entrant arrives. On the other hand, with the complementary probability $q(1 - F(\hat{y}_1))$, assignment of the object to our buyer (and the realization of the planner's gain) is postponed. Thus, the benefit from the increase in \hat{y}_1 is

$$\begin{aligned} \sum_{t=0}^{\infty} (\delta q(1 - F(\hat{y}_1)))^t (1 - q(1 - F(\hat{y}_1))) &= \frac{1 - q(1 - F(\hat{y}_1))}{1 - \delta q(1 - F(\hat{y}_1))} \\ &= \delta^{-1} \lambda(\hat{y}_1). \end{aligned}$$

Integrating this ratio therefore captures the total contribution of the high-value buyer over a "null" buyer. Analogous reasoning follows for the remaining (lower-ranked) buyers.

We may then use this result to provide an interpretation for the buyer payoff functions characterized in [Proposition 3](#) by relating the expression for buyer payoffs in [Equation 10](#) to the planner's payoff function. In particular, we have the following

PROPOSITION 7 (Relationship between V and W).

For any $n \in \mathbb{N}$ and any $k \in \{1, \dots, n\}$, the expected payoff of the k -th ranked buyer in the sequential auction game is equal to her marginal contribution to the social welfare; that is,

$$(14) \quad V_{k,n}(\hat{y}) = W_n(\hat{y}) - W_{n-1}(\hat{y}_{-k}).$$

Proof. Note that we may write

$$W_{n-1}(\hat{y}_{-k}) = W_0 + \delta^{-1} \sum_{j=1}^{k-1} \int_0^{\hat{y}_j} \lambda^j(v') dv' + \delta^{-1} \sum_{j=k+1}^n \int_0^{\hat{y}_j} \lambda^{j-1}(v') dv'.$$

Straightforward arithmetic then yields

$$\begin{aligned} W_n(\hat{y}) - W_{n-1}(\hat{y}_{-k}) &= \left[W_0 + \delta^{-1} \sum_{j=1}^n \delta^{j-1} \int_0^{\hat{y}_j} \lambda^j(v') dv' \right] \\ &\quad - \left[W_0 + \delta^{-1} \sum_{j=1}^{k-1} \int_0^{\hat{y}_j} \lambda^j(v') dv' + \delta^{-1} \sum_{j=k+1}^n \int_0^{\hat{y}_j} \lambda^{j-1}(v') dv' \right] \\ &= \delta^{-1} \left[\sum_{j=k}^n \int_0^{\hat{y}_j} \lambda^j(v') dv' - \sum_{j=k+1}^n \int_0^{\hat{y}_j} \lambda^{j-1}(v') dv' \right] \end{aligned}$$

Combining the two summations, we arrive, as desired, at the definition of $V_{k,n}(\hat{y})$. \square

Thus, the marginal contribution of a buyer, and hence their expected payoff in the equilibrium of the sequential ascending auction game, is exactly the buyer's marginal contribution to the social welfare, which is determined by the difference in the scheduling of object assignments to those bidders who have lower values. Moreover, this demonstrates the equivalence between the dynamic pivot mechanism and the sequential ascending auction in this setting. Not only are continuation payoffs identical in the two settings, but the timing of payments and object allocations are also the same. Thus, the sequential ascending auction serves, in this setting, as a straightforward and intuitive indirect mechanism that is equivalent to the dynamic Vickrey-Clarke-Groves mechanism.

6. DISCUSSION

This paper solves for an equilibrium in a model of sequential auctions. In particular, we show that in sequential ascending auctions, objects can be allocated efficiently in a manner that employs the truthful revelation of private information. Moreover, the bidding strategy employed by buyers in this equilibrium has the striking property of being robust to the random entry of new buyers whose valuations are private information—in each period, *all* private information is revealed anew, and hence there is no incentive for new entrants to attempt to manipulate the outcome of future periods by altering the information that they (truthfully) reveal upon their entry.

Furthermore, we show that the sequential ascending auction format may be used as an indirect mechanism for the efficient allocation of goods in a dynamic setting. In particular, the equilibrium we characterize preserves the efficiency, individual rationality, and incentive compatibility of the dynamic Vickrey-Clarke-Groves mechanism developed and characterized by Bergemann and Välimäki (2008) and Cavallo, Parkes, and Singh (2007): the allocations, transfers, and payoffs of the two mechanisms are identical.

There are several interesting avenues for future research in this area. For example, it would be desirable to have a fully developed model of seller behavior and competition in “overlapping” auctions, perhaps applying some of the insights of Peters and Severinov (2006) in a setting with multiple simultaneously running auctions. Such a setting also allows for the introduction and study of endogenous arrival and entry deterrence in a manner similar to Nekipelov (2007) but while accounting for the endogenously determined option value of participating in future auctions.

Another important question regards the usefulness of sequential ascending auctions as an indirect mechanism that implements socially efficient policies when agents are not constrained to have single-unit demand. Bergemann and Välimäki (2008) provide an example that demonstrates the failure of the sequential ascending auction in implementing the efficient policy in one such setting; it would be useful to understand how this example may be generalized so as to better understand when indirect implementation via an auction mechanism is possible. These questions are, however, left for future work.

APPENDIX A. PROOFS

In order to prove [Proposition 1](#), we will make repeated use of the following technical result. Denote by $\mathbf{C}(\mathbf{V}^n)$ the set of continuous real-valued functions on \mathbf{V}^n . In addition, for any $k < n$, let $\mathbf{C}_k(\mathbf{V}^n) \subseteq \mathbf{C}(\mathbf{V}^n)$ denote the subset of such functions that do *not* depend on their first k arguments. We endow $\mathbf{C}(\mathbf{V}^n)$ with the sup-metric d_∞ , where

$$d_\infty(f, g) := \sup \{|f(x) - g(x)| : x \in \mathbf{V}^n\} \text{ for all } f, g \in \mathbf{C}(\mathbf{V}^n).$$

This implies that $\mathbf{C}(\mathbf{V}^n)$ is a complete metric space. We have the following

LEMMA 1 ($\mathbf{C}_k(\mathbf{V}^n)$ is closed).

For any $k \leq n$, $\mathbf{C}_k(\mathbf{V}^n)$ is a closed subset of $\mathbf{C}(\mathbf{V}^n)$.

Proof. Fix any convergent sequence $\{f_m\}_{m=1}^\infty$ in $\mathbf{C}(\mathbf{V}^n)$ such that $f_m \in \mathbf{C}_k(\mathbf{V}^n)$ for all $m \in \mathbb{N}$, and let $f^* \in \mathbf{C}(\mathbf{V}^n)$ denote the (uniform) limit of this sequence. Suppose, however, that $f^* \notin \mathbf{C}_k(\mathbf{V}^n)$. Then there exist $x, y \in \mathbf{V}^n$ such that $x_i = y_i$ for $i = k+1, k+2, \dots, n$, but $f^*(x) \neq f^*(y)$. Let

$$\epsilon := |f^*(x) - f^*(y)| > 0.$$

Since f_m converges to f^* , there exists $M_x \in \mathbb{N}$ such that $|f_m(x) - f^*(x)| < \frac{\epsilon}{2}$ for all $m > M_x$. Similarly, there exists $M_y \in \mathbb{N}$ such that $|f_m(y) - f^*(y)| < \frac{\epsilon}{2}$ for all $m > M_y$. Therefore, for any $m > \max\{M_x, M_y\}$,

$$\begin{aligned} \epsilon &= |f^*(x) - f^*(y)| \leq |f^*(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f^*(y)| \\ &< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

a contradiction. The first inequality above follows from the triangle inequality, and the second is due to the fact that $f_m \in \mathbf{C}_k(\mathbf{V}^n)$ implies $f_m(x) = f_m(y)$. Therefore, we must have $f^*(x) = f^*(y)$; that is, there are no $x, y \in \mathbf{V}^n$ such that x and y agree on their last $n - k$ arguments but $f^*(x) \neq f^*(y)$. Thus, we may conclude that $f^* \in \mathbf{C}_k(\mathbf{V}^n)$. \square

Proof of Proposition 1. Let $\hat{y} \in \mathbf{V}^n$ denote the ordered vector of values of those bidders present at the beginning of the period, and suppose that they are commonly known. Furthermore, suppose that all buyers use the bidding strategies described in [Equation 7](#). If there are no entrants, then the highest-valued buyer (without loss of generality, bidder 1) wins the object, and pays the price

$$\beta_{2,n}(\hat{y}_2, \dots, \hat{y}_n) = \hat{y}_2 - \delta V_{1,n-1}(\hat{y}_2, \dots, \hat{y}_n).$$

On the other hand, if a new entrant enters with value v' , bidder 1 may no longer win the object. Furthermore, even if she does win, the price she pays will depend upon the realization of v' . In particular, we may write the expected payoff of bidder 1 as

$$\begin{aligned} (A.1) \quad V_{1,n}(\hat{y}) &= (1 - q) [\hat{y}_1 - \beta_{2,n}(\hat{y}_2, \dots, \hat{y}_n)] \\ &+ q \left[\sum_{j=0}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} (\hat{y}_1 - \beta_{2,n+1}(\hat{y}_{-1}(v'))) dF(v') + \int_{\hat{y}_1}^{\bar{v}} \delta V_{1,n}(\hat{y}) dF(v') \right], \end{aligned}$$

where $\hat{y}(v')$ is the ordered vector of values including the new entrant, and we define $\hat{y}_{n+1} := 0$. The first term (multiplied by $1 - q$) is bidder 1's payoff when no entrant arrives, while the second term is the (probability-weighted) sum of the payoffs for each possible realized ranking of the new entrant.

Substituting the definition of $\beta_{2,n}$ and $\beta_{2,n+1}$ from [Equation 7](#) and simplifying, we see that $V_{1,n}$ is given by the fixed point of the operator $T_{1,n} : \mathbf{C}(\mathbf{V}^n) \rightarrow \mathbf{C}(\mathbf{V}^n)$ defined by

$$(A.2) \quad \begin{aligned} [T_{1,n}(W)](\hat{y}) &:= (1 - q) [\hat{y}_1 - \hat{y}_2 + \delta V_{1,n-1}(\hat{y}_2, \dots, \hat{y}_n)] \\ &+ q \left[\int_0^{\hat{y}_1} \hat{y}_1 dF(v') - \int_0^{\hat{y}_2} \hat{y}_2 dF(v') - \int_{\hat{y}_2}^{\hat{y}_1} v' dF(v') \right. \\ &\left. + \delta \sum_{j=0}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} W(\hat{y}_{-1}(v')) dF(v') + \int_{\hat{y}_1}^{\bar{v}} \delta W(\hat{y}) dF(v') \right]. \end{aligned}$$

Fix any $W, W' \in \mathbf{C}(\mathbf{V}^n)$ such that $W \geq W'$. Then

$$\begin{aligned} [T_{1,n}(W) - T_{1,n}(W')](\hat{y}) &= \delta q \left[\sum_{j=0}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} [W - W'](\hat{y}_{-1}(v')) dF(v') \right. \\ &\left. + (1 - F(\hat{y}_1)) [W - W'](\hat{y}) \right] \geq 0. \end{aligned}$$

In addition, for any $W \in \mathbf{C}(\mathbf{V}^n)$ and any $\alpha \in \mathbb{R}_{++}$,

$$[T_{1,n}(W + \alpha)](\hat{y}) = [T_{1,n}(W)](\hat{y}) + \delta q \alpha.$$

Thus, $T_{1,n}$ satisfies the monotonicity and discounting conditions of Blackwell's Contraction Lemma, and hence we may apply the Banach Fixed Point Theorem to show that $V_{1,n}$ is the unique fixed point of $T_{1,n}$.¹⁰

Now consider $V_{2,n}$. Suppose (again without loss of generality) that bidder 1 has the second-highest of the n values; that is, that $v_1 = \hat{y}_2$. If there are no new entrants, then bidder 1 loses the auction, but has the highest value in the next period. On the other hand, if a new entrant arrives, bidder 1 will still lose the auction. However, in the next period, her ranking depends on the realization of the new entrant's value. Thus, we may write her payoff as the fixed point of the operator $T_{2,n} : \mathbf{C}(\mathbf{V}^n) \rightarrow \mathbf{C}(\mathbf{V}^n)$ defined by

$$(A.3) \quad \begin{aligned} [T_{2,n}(W)](\hat{y}) &:= \delta(1 - q) V_{1,n-1}(\hat{y}_2, \dots, \hat{y}_n) + q \left[\sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta V_{1,n}(\hat{y}_{-1}(v')) dF(v') \right. \\ &\left. + \int_{\hat{y}_2}^{\hat{y}_1} \delta W(\hat{y}_{-1}(v')) dF(v') + \int_{\hat{y}_1}^{\bar{v}} \delta W(\hat{y}) dF(v') \right]. \end{aligned}$$

Applying exactly the same technique and steps as with $T_{1,n}$, we see that $T_{2,n}$ is a contraction mapping on $\mathbf{C}(\mathbf{V}^n)$. Notice that $T_{2,n}$ in fact maps elements of $\mathbf{C}_1(\mathbf{V}^n)$ into $\mathbf{C}_1(\mathbf{V}^n)$ itself; thus, applying [Lemma 1](#), the unique fixed point of $T_{2,n}$ does not depend upon its first argument. We may therefore, with a slight abuse of notation, write $V_{2,n}(\hat{y}) = V_{2,n}(\hat{y}_2, \dots, \hat{y}_n)$.

¹⁰See Section C.6 of [Ok \(2007\)](#) for precise statements of the results we are applying.

Now consider any arbitrary k such that $1 < k \leq n$, and suppose that $V_{k-1,n} \in \mathbf{C}_{k-2}(\mathbf{V}^n)$. Then $V_{k,n}$ is given by a fixed point of the operator $T_{k,n} : \mathbf{C}(\mathbf{V}^n) \rightarrow \mathbf{C}(\mathbf{V}^n)$, where $T_{k,n}$ is defined by

$$(A.4) \quad [T_{k,n}(W)](\hat{y}) := \delta(1-q)V_{k-1,n-1}(\hat{y}_k, \dots, \hat{y}_n) + q \left[\sum_{j=0}^{n-k} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta V_{k-1,n}(\hat{y}_{-1}(v')) dF(v') \right. \\ \left. + \sum_{j=n-k+1}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta W(\hat{y}_{-1}(v')) dF(v') + \int_{\hat{y}_1}^{\bar{v}} \delta W(\hat{y}) dF(v') \right].$$

We may again apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem to show that $V_{k,n}$ is the unique fixed point of $T_{k,n}$. Furthermore, it is straightforward to show that $T_{k,n}$ maps elements of $\mathbf{C}_{k-1}(\mathbf{V}^n)$ into $\mathbf{C}_{k-1}(\mathbf{V}^n)$. Therefore, again using [Lemma 1](#), we may write $V_{k,n}(\hat{y}) = V_{k,n}(\hat{y}_k, \dots, \hat{y}_n)$.

Thus, by induction, the bidding strategies in [Equation 7](#) lead to unique value functions $V_{k,n}$ such that, for all n and all $k = 1, \dots, n$, $V_{k,n} \in \mathbf{C}_{k-1}(\mathbf{V}^n)$. \square

Proof of Proposition 2. Recall from [Equation A.2](#) in the proof of [Proposition 1](#) that $V_{1,n}$ is defined as the unique fixed point of $T_{1,n}$. Letting $\hat{w} := \hat{y}_{-1} = (\hat{y}_2, \hat{y}_3, \dots, \hat{y}_n)$, we have

$$V_{1,n}(\hat{y}_2, \hat{w}) = [T_{1,n}(V_{1,n})](\hat{y}_2, \hat{w}) = \delta(1-q)V_{1,n-1}(\hat{w}) \\ + \delta q \left[\sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} V_{1,n}(\hat{w}(v')) dF(v') + \int_{\hat{y}_2}^{\bar{v}} \delta V_{1,n}(\hat{w}) dF(v') \right].$$

However, this is identical to the definition of $T_{2,n}$ given in [Equation A.3](#), implying that, for all $n \in \mathbb{N}$,

$$V_{2,n}(\hat{w}) = V_{1,n}(\hat{y}_2, \hat{w}).$$

Fix $k > 1$, and suppose that $V_{k,n}(\hat{y}_k, \hat{y}_{k+1}, \dots, \hat{y}_n) = V_{k-1,n}(\hat{y}_k, \hat{y}_k, \hat{y}_{k+1}, \dots, \hat{y}_n)$ for all $n \geq k$. Redefine $\hat{w} := (\hat{y}_{k+1}, \hat{y}_{k+2}, \dots, \hat{y}_n)$, and consider $V_{k,n}(\hat{y}_{k+1}, \hat{w})$. Recalling from [Equation A.4](#) the definition of $T_{k,n}$, we have

$$V_{k,n}(\hat{y}_{k+1}, \hat{w}) = [T_{k,n}(V_{k,n})](\hat{y}_{k+1}, \hat{w}) = \delta(1-q)V_{k-1,n-1}(\hat{w}) \\ + q \left[\sum_{j=0}^{n-k} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta V_{k-1,n}(\hat{w}(v')) dF(v') \right. \\ \left. + \sum_{j=n-k+1}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta V_{k,n}(\hat{w}(v')) dF(v') + \int_{\hat{y}_1}^{\bar{v}} \delta V_{k,n}(\hat{w}) dF(v') \right].$$

Taking into account the fact that the fixed point of this operator lies in $\mathbf{C}_k(\mathbf{V}^n)$ allows us to rewrite the above as

$$V_{k,n}(\hat{y}_{k+1}, \hat{w}) = \delta(1-q)V_{k,n-1}(\hat{w}) + q \left[\sum_{j=0}^{n-k-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} \delta V_{k,n}(\hat{w}(v')) dF(v') + \int_{\hat{y}_{k+1}}^{\bar{v}} \delta V_{k,n}(\hat{w}) dF(v') \right].$$

Notice that the above is a reformulation of the expression for $T_{k+1,n}$. Since $V_{k,n}(\hat{w})$ is a fixed point of the operator, the uniqueness result from [Proposition 1](#) implies that

$$V_{k+1,n}(\hat{w}) = V_{k,n}(\hat{y}_{k+1}, \hat{w})$$

for all $n \geq k + 1$. Thus, by induction on k , we have established that

$$V_{k,n}(\hat{y}_k, \dots, \hat{y}_n) = V_{1,n}(\hat{y}_k, \dots, \hat{y}_k, \hat{y}_{k+1}, \dots, \hat{y}_n)$$

for arbitrary $n \in \mathbb{N}$ and for all $k = 1, \dots, n$. \square

Proof of Proposition 4. To prove the first part of this result, it suffices to simply differentiate the bid function $\beta_{k,n}$ with respect to the bidder's own value v_i . In particular, we have, for all $n \in \mathbb{N}$ and $k = 1, \dots, n$,

$$\frac{\partial}{\partial v_i} \beta_{k,n}(v_i, \hat{y}_{k+1}, \dots, \hat{y}_n) = 1 - \lambda^{l-1}(v_i).$$

However, $0 < \lambda(v) < 1$ for all $v \in \mathbf{V}$, and so $\frac{\partial}{\partial v_i} \beta_{k,n}(v_i, \hat{y}_{k+1}, \dots, \hat{y}_n) > 0$.

As for the second claim, let $\hat{w} := (\hat{y}_{k+1}, \dots, \hat{y}_n)$ and note that

$$\begin{aligned} \beta_{k,n}(v_i, \hat{w}) - \beta_{k+1,n}(\hat{w}) &= v_i - \delta V_{k-1,n-1}(v_i, \hat{w}) - \hat{y}_{k+1} + \delta V_{k,n-1}(\hat{w}) \\ &= v_i - \hat{y}_{k+1} - \delta (V_{1,n-1}(v_i, \dots, v_i, \hat{w}) - V_{1,n-1}(\hat{y}_{k+1}, \dots, \hat{y}_{k+1}, \hat{w})) \\ &= v_i - \hat{y}_{k+1} - \int_{\hat{y}_{k+1}}^{v_i} \lambda^k(v') dv' \\ &= \int_{\hat{y}_{k+1}}^{v_i} (1 - \lambda^k(v')) dv'. \end{aligned}$$

Since $0 < \lambda(v') < 1$ for all $v' \in \mathbf{V}$, this expression is positive if, and only if, $v_i > \hat{y}_{k+1}$. Thus, the exit of a lower-ranked bidder does not induce the immediate exit of a higher-ranked bidder who is using the bidding strategy given in [Equation 12](#). \square

Proof of Proposition 6. We begin by showing that W_1 has the desired form and then proceed inductively. Note that W_1 is a fixed point of the operator $\hat{T}_1 : \mathbf{C}(\mathbf{V}) \rightarrow \mathbf{C}(\mathbf{V})$ defined by

$$(A.5) \quad [\hat{T}(g)](x) := (1 - q)(x + \delta W_0) + q \left[\int_0^x (x + \delta g(y)) dF(y) + \int_x^{\bar{v}} (y + \delta g(x)) dF(y) \right].$$

This operator is clearly a self-map from $\mathbf{C}(\mathbf{V})$ into itself. Furthermore, it is straightforward to see that \hat{T}_1 is a contraction. Fix any $g, g' \in \mathbf{C}(\mathbf{V})$ such that $g' > g$. Then

$$[\hat{T}_1(g' - g)](x) = \delta q \left[\int_0^x (g'(y) - g(y)) dF(y) + (1 - F(x))(g'(x) - g(x)) \right] > 0.$$

Furthermore, for any $g \in \mathbf{C}(\mathbf{V})$ and any $\alpha \in \mathbb{R}_{++}$,

$$[\hat{T}_1(g + \alpha)](x) = [\hat{T}_1(g)](x) + \delta q \alpha.$$

Since $\delta q < 1$, we may apply Blackwell's Contraction Lemma and the Banach Fixed Point Theorem, implying that \hat{T}_1 has a unique fixed point W_1 such that

$$W_1(\hat{y}_1) = (1 - q)(\hat{y}_1 + \delta W_0) + q \left[\int_0^{\hat{y}_1} (\hat{y}_1 + \delta W_1(v')) dF(v') + \int_{\hat{y}_1}^{\bar{v}} (v' + \delta W_1(\hat{y}_1)) dF(v') \right].$$

Differentiating this expression with respect to \hat{y}_1 yields

$$W_1'(\hat{y}_1) = (1 - q) + qF(\hat{y}_1) + \delta q(1 - F(\hat{y}))W_1'(\hat{y}) = \delta^{-1}\lambda(\hat{y}_1).$$

Finally, note that $W_1(0) = W_0$, since a buyer with value zero adds nothing to the social welfare. Since the continuity of F implies the continuity of W_1' , we may apply the Fundamental Theorem of Calculus, yielding

$$(A.6) \quad W_1(\hat{y}_1) = W_0 + \delta^{-1} \int_0^{\hat{y}_1} \lambda(v') dv'.$$

Now consider $W_n(\hat{y})$ for arbitrary $n > 1$, and suppose that W_{n-1} takes the desired form.¹¹ W_n is defined to be a fixed point of the operator $\hat{T}_n : \mathbf{C}(\mathbf{V}^n) \rightarrow \mathbf{C}(\mathbf{V}^n)$ given by

$$(A.7) \quad \begin{aligned} [\hat{T}_n(g)](x) &:= (1 - q)(x_1 + \delta W_{n-1}(x_{-1})) \\ &+ q \left[\sum_{j=0}^{n-1} \int_{x_{n-j+1}}^{x_{n-j}} (x_1 + \delta g(x_2, \dots, x_{n-j}, y, x_{n-j+1}, \dots, x_n)) dF(y) \right. \\ &\quad \left. + \int_{x_1}^{\bar{v}} (y + \delta g(x)) dF(y) \right]. \end{aligned}$$

Note that for any $g, g' \in \mathbf{C}(\mathbf{V}^n)$ such that $g' > g$, we have

$$\begin{aligned} [\hat{T}_n(g' - g)](x) &= \delta q \left[\sum_{j=0}^{n-1} \int_{x_{n-j+1}}^{x_{n-j}} [g' - g](x_2, \dots, x_{n-j}, y, x_{n-j+1}, \dots, x_n) dF(y) \right. \\ &\quad \left. + (1 - F(x_1))(g'(x) - g(x)) \right] > 0. \end{aligned}$$

Furthermore, for any $g \in \mathbf{C}(\mathbf{V}^n)$ and any $\alpha \in \mathbb{R}_{++}$,

$$[\hat{T}_n(g + \alpha)](x) = [\hat{T}_n(g)](x) + \delta q \alpha.$$

Since $\delta q < 1$, Blackwell's monotonicity and discounting conditions are satisfied. Thus, Blackwell's Contraction Lemma and the Banach Fixed Point Theorem imply that \hat{T}_n has a unique fixed point W_n such that

$$(A.8) \quad \begin{aligned} W_n(\hat{y}) &= (1 - q)(\hat{y}_1 + \delta W_{n-1}(\hat{y}_{-1})) \\ &+ q \left[\sum_{j=0}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} (\hat{y}_1 + \delta W_n(\hat{y}_2, \dots, \hat{y}_{n-j}, v', \hat{y}_{n-j+1}, \dots, \hat{y}_n)) dF(v') \right. \\ &\quad \left. + \int_{\hat{y}_1}^{\bar{v}} (v' + \delta W_n(\hat{y})) dF(v') \right]. \end{aligned}$$

Differentiating this expression implicitly with respect to \hat{y}_1 yields

$$W_n^{(1)}(\hat{y}) = (1 - q) + qF(\hat{y}_1) + \delta q(1 - F(\hat{y}_1))W_n^{(1)}(\hat{y}) = \delta^{-1}\lambda(\hat{y}_1).$$

Note that this expression is independent of n and of \hat{y}_j for $j \neq 1$, implying that $W_n^{(1,j)}$ is identically zero for all $j \neq 1$.

¹¹Notice that this implies that all of the cross-derivatives of W_{n-1} are identically zero.

Similarly, implicit differentiation with respect to \hat{y}_2 yields

$$W_n^{(2)}(\hat{y}) = \delta(1-q)W_{n-1}^{(1)}(\hat{y}_{-1}) + \delta q \left[\sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} W_n^{(1)}(\hat{y}_{-1}(v')) dF(v') \right. \\ \left. + \int_{\hat{y}_2}^{\hat{y}_1} W_n^{(2)}(v', \hat{y}) dF(v') + (1-F(\hat{y}_1))W_n^{(2)}(\hat{y}) \right],$$

where $\hat{y}(v')$ is the re-ordering of \hat{y} and v' . Since $W_n^{(1,j)}$ is identically zero,

$$\sum_{j=0}^{n-2} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} W_n^{(1)}(\hat{y}_{-1}(v')) dF(v') = \delta^{-1}\lambda(\hat{y}_2)F(\hat{y}_2).$$

Furthermore, $W_n^{(2,1)} = 0$ implies that

$$\int_{\hat{y}_2}^{\hat{y}_1} W_n^{(2)}(v', \hat{y}) dF(v') + (1-F(\hat{y}_1))W_n^{(2)}(\hat{y}) = (1-F(\hat{y}_2))W_n^{(2)}(\hat{y}).$$

Thus, making use of the fact that $W_{n-1}^{(1)}(\hat{y}_{-1}) = \delta^{-1}\lambda(\hat{y}_2)$, we may conclude that

$$(A.9) \quad W_n^{(2)} = (1-q)\lambda(\hat{y}_2) + qF(\hat{y}_2)\lambda(\hat{y}_2) + \delta q(1-F(\hat{y}_2))W_n^{(2)}(\hat{y}) = \delta^{-1}\lambda^2(\hat{y}_2).$$

Once again, note that this expression is independent of n and of \hat{y}_j for $j \neq 2$, implying that $W_n^{2,j}$ is identically zero for all $j \neq 2$.

Proceeding inductively, consider the derivative of W_n with respect to its k -th argument, where $k \leq n$. We have

$$W_n^{(k)}(\hat{y}) = (1-q)\delta W_{n-1}^{(k-1)}(\hat{y}_{-1}) + \delta q \left[\sum_{j=0}^{n-k} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} W_n^{(k-1)}(\hat{y}_{-1}(v')) dF(v') \right. \\ \left. + \sum_{j=n-k+1}^{n-1} \int_{\hat{y}_{n-j+1}}^{\hat{y}_{n-j}} W_n^{(k)}(\hat{y}_1(v')) dF(v') + (1-F(\hat{y}_1))W_n^{(k)}(\hat{y}) \right].$$

Applying the same simplifications as above, along with our inductive hypothesis that $W_{n-1}^{(k-1)}(\hat{y}_{-1}) = \delta^{-1}\lambda^{k-1}(\hat{y}_k)$, we have

$$(A.10) \quad W_n^{(k)}(\hat{y}) = (1-q)W_{n-1}^{(k-1)}(\hat{y}_{-1}) + qF(\hat{y}_k)W_{n-1}^{(k-1)}(\hat{y}_{-1}) + \delta q(1-F(\hat{y}_k))W_n^{(k)}(\hat{y}) \\ = \delta^{-1}\lambda^k(\hat{y}_k).$$

Finally, note that $W_n(0, \dots, 0) = W_0$ since, as with a single buyer with value zero, “null” agents provide no social benefit. By induction on n , we may then conclude that

$$(A.11) \quad W_n(\hat{y}) = W_0 + \delta^{-1} \sum_{j=1}^n \int_0^{\hat{y}_j} \lambda^j(v') dv'$$

for all $n \in \mathbb{N}$ and all $\hat{y} \in \mathbf{V}^n$. □

APPENDIX B. SOCIALLY EFFICIENT POLICY

In this appendix, we demonstrate that the socially efficient policy in the setting with random object availability is that which allocates an object to the highest-valued buyer currently present on the market whenever an object is available. The argument for the case in which objects arrive in every period is subsumed by the present discussion, and therefore is not explicitly considered.

B.1. Preliminaries

Denote the period- t state space of the planner's problem by $\Omega := \Omega_1 \times \Omega_2$, where

$$(B.1) \quad \Omega_1 := \{0, 1\} \text{ and } \Omega_2 := \{S \subset \mathbb{Z} \times \mathbf{V} : |S| < \infty, (0, 0) \in S\}.$$

The interpretation of these states is straightforward: an object is available for assignment in period t if, and only if, $\omega_1^t = 1$; and we denote by $\omega_2^t \in \Omega_2$ the set of buyers and their valuations present on the market in period t . Note that we require $|\omega_2^t| < \infty$ since there may only be a finite number of buyers present at any time. Thus, buyer $i \in \mathbb{Z}$, with value $v_i \in \mathbf{V}$, is present on the market in period t if, and only if, $(i, v_i) \in \omega_2^t$. Note that we will be indexing buyers by their period of arrival, so that buyer t is the buyer who may arrive in period t . Buyers that are present on the market in the initial period, however, will be denoted by negative indices.¹² Finally, we will always assume that a “null” agent with value zero, denoted by $(0, 0)$, is always present—this null agent will represent the act of “discarding” an object and not assigning it to a buyer.

Given a state ω^t , the planner must choose an action $a^t \in A(\omega^t)$, where

$$A(\omega^t) := \begin{cases} \omega_2^t, & \text{if } \omega_1^t = 1; \\ \{(0, 0)\}, & \text{if } \omega_1^t = 0. \end{cases}$$

Thus, if an object is available in period t (when $\omega_1^t = 1$), the planner may choose to allocate the object to any of the available agents or to the null agent. On the other hand, if $\omega_1^t = 0$, there is no object available, leaving the planner with no possible action but to “allocate” to the null agent.

Recall that the arrival processes of both buyers and objects are history independent. In particular, the seller arrivals are independently and identically distributed according to

$$\Pr(\omega_1^{t+1} = 1) := p = 1 - \Pr(\omega_1^{t+1} = 0) \text{ for all } t.$$

On the other hand, the buyers available in the next period depends on both the current period allocation and the action taken. Letting $\zeta^t \subset \Xi := \mathbb{N} \times \mathbb{R}_+$ denote the realization of the buyer arrival process in period t , we have

$$\zeta^t := \begin{cases} \{(0, 0), (t, v_t)\}, & \text{with probability } q; \\ \{(0, 0)\}, & \text{with probability } 1 - q, \end{cases}$$

where v_t is drawn independently from the distribution F and is unknown to the planner until buyer t is present on the market. Note that the arrival process always includes the “null” agent

¹²For instance, if 2 buyers are present at the beginning of the game (in period 0), then they will be referred to as buyers -2 and -1 , with values v_{-2} and v_{-1} , respectively.

$(0,0)$, reflecting the planner's option in each period to not allocate an object. We may therefore define a state transition function $\tau : \Omega \times A(\Omega) \times \Xi \rightarrow \Omega_2$ by

$$\omega_2^{t+1} = \tau(\omega^t, a^t, \xi^{t+1}) := (\omega_2^t \setminus \{a^t\}) \cup \xi^{t+1}.$$

Thus, given an initial state $\omega^0 = (\omega_1^0, \omega_2^0)$, the planner's problem may be written as

$$(B.2) \quad \max_{\{a^t\}_{t=1}^{\infty}} \left\{ \mathbb{E}_{\xi} \left[\sum_{t=1}^{\infty} \delta^{t-1} \omega_1^t a^t \right] \right\} \text{ subject to } a^t \in A(\omega^t), \omega_2^{t+1} = \tau(\omega^t, a^t, \xi^{t+1}) \text{ for all } t \in \mathbb{N}.$$

B.2. Solution

We define the set of period t arrival histories as

$$\mathcal{H}^0 := \{(\omega_1^0, \omega_2^0)\}, \text{ and } \mathcal{H}^t := \mathcal{H}^{t-1} \times (\Omega_1 \times \Xi) \text{ for all } t \in \mathbb{N}.$$

The set of all possible histories is then $\mathcal{H} := \bigcup_{t=0}^{\infty} \mathcal{H}^t$. We will say that a history $h \in \mathcal{H}$ precedes a history $h' \in \mathcal{H}$ if h is a prefix of h' , and will denote this by $h \rightarrow h'$.

Note that given a history $h^t = ((\omega_1^0, \omega_2^0), (\omega_2^1, \xi^1), \dots, (\omega_2^t, \xi^t))$ and a sequence of feasible actions $\mathbf{a}^{t-1} = (a^1, a^2, \dots, a^{t-1})$, we may reconstruct the resultant state ω^t by repeated application of the state transition function τ . We will use the notational shorthand $\hat{\omega}(h^t, \mathbf{a}^{t-1})$ for this state. Thus, we may define an allocation policy as a function $\mathbf{a} : \mathcal{H} \rightarrow A(\Omega)$ such that, for all $t \in \mathbb{N}$ and $h^t = (h^{t-1}, (\xi^t, \omega_2^t))$,

$$\mathbf{a}(h^t) \in A\left(\hat{\omega}\left(h^t, \mathbf{a}^{t-1}(h^{t-1})\right)\right),$$

where $\mathbf{a}^{t-1}(h^{t-1})$ is the sequence of allocation decisions taken earlier in the policy.

LEMMA 2 (Socially Efficient Policy).

The policy \mathbf{a}^ which always allocates an object (when available) to the highest-valued buyer present maximizes the social planner's objective function.*¹³

Proof. Fix any policy $\mathbf{a}_0 \neq \mathbf{a}^*$ such that \mathbf{a}_0 yields a the planner a strictly higher payoff than the policy \mathbf{a}^* , and define

$$(B.3) \quad \hat{\mathcal{H}}_0 := \{h \in \mathcal{H} : \mathbf{a}_0(h) \neq \mathbf{a}^*(h) \text{ and } \mathbf{a}_0(h') = \mathbf{a}^*(h') \text{ for all } h' \rightarrow h\}.$$

Note that $\hat{\mathcal{H}}_0$ is the set of all histories h such that \mathbf{a}^* and \mathbf{a}_0 disagree at h but agree on all its prefixes—it is the set of “first disagreements” between \mathbf{a}^* and \mathbf{a}_0 . Since \mathbf{a}_0 does strictly better than \mathbf{a}^* , this set must have nonzero measure (with respect to the measure induced by the arrival process ξ), as otherwise the two policies would agree almost everywhere.

For each $h \in \hat{\mathcal{H}}_0$, define

$$(B.4) \quad i_0(h) := \mathbf{a}^*(h) \text{ and } \mathcal{I}_0(h) := \{h' \in \mathcal{H} : h \rightarrow h' \text{ and } \mathbf{a}_0(h') = i_0(h)\}.$$

¹³Thanks are due to Larry Samuelson for suggesting the method of proof used below.

Thus, $\mathcal{I}_0(h)$ is the set of histories (possibly empty) at which policy \mathbf{a}_0 eventually allocates an object to the buyer who had the highest value at h . Letting

$$\hat{\mathcal{I}}_0 := \bigcup_{h \in \hat{\mathcal{H}}_0} \mathcal{I}_0(h),$$

we may define, for each $h \in \mathcal{I}_0$, $\hat{h}_0(h)$ to be the element of $\hat{\mathcal{H}}_0$ such that $\hat{h}_0(h) \rightarrow h$.

With these definitions in mind, we may define a new allocation policy

$$(B.5) \quad \mathbf{a}_1(h) := \begin{cases} i_0(h), & \text{if } h \in \hat{\mathcal{H}}_0; \\ \mathbf{a}_0(\hat{h}_0(h)), & \text{if } h \in \hat{\mathcal{I}}_0; \\ \mathbf{a}_0(h), & \text{otherwise.} \end{cases}$$

Thus, \mathbf{a}_1 is identical to \mathbf{a}_0 except that it “swaps” the allocation decisions at histories $h \in \hat{\mathcal{H}}_0$ with those at histories $\mathcal{I}_0(h)$. Since the value of the agent associated with $i_0(h)$ is greater than that of the agent associated with $\mathbf{a}_0(h)$ for all $h \in \hat{\mathcal{H}}_0$, this implies that \mathbf{a}_1 yields the planner a strictly greater payoff than \mathbf{a}_0 . To see this, consider any $v > v'$, and $t < t'$. Since $\delta < 1$,

$$(\delta^t v + \delta^{t'} v') - (\delta^t v' + \delta^{t'} v) = (\delta^t - \delta^{t'})(v - v') > 0.$$

Such a gain is realized for every history in $\hat{\mathcal{H}}_0$. Since this set has positive measure, it must be the case that the planner’s payoff increases.

Notice that if \mathbf{a}_1 yields the planner a payoff less than or equal to that of \mathbf{a}^* , transitivity of the planner’s payoffs leads to a contradiction, implying that there does not exist a policy \mathbf{a}_0 such that \mathbf{a}_0 does strictly better than \mathbf{a}^* , and hence that \mathbf{a}^* is optimal. So suppose, on the other hand, that \mathbf{a}_1 does provide a strictly higher payoff than \mathbf{a}^* . We may then define

$$(B.6) \quad \hat{\mathcal{H}}_1 := \{h \in \mathcal{H} : \mathbf{a}_1(h) \neq \mathbf{a}^*(h) \text{ and } \mathbf{a}_1(h') = \mathbf{a}^*(h') \text{ for all } h' \rightarrow h\}.$$

Since \mathbf{a}_1 is better than \mathbf{a}^* , $\hat{\mathcal{H}}_1$ must have nonzero measure. For each $h \in \hat{\mathcal{H}}_1$, define

$$i_1(h) := \mathbf{a}^*(h), \mathcal{I}_1(h) := \{h' \in \mathcal{H} : h \rightarrow h' \text{ and } \mathbf{a}_1(h') = i_1(h)\}, \text{ and } \hat{\mathcal{I}}_1 := \bigcup_{h \in \hat{\mathcal{H}}_1} \mathcal{I}_1(h).$$

For each $h \in \mathcal{I}_1$, let $\hat{h}_1(h)$ be the element of $\hat{\mathcal{H}}_1$ such that $\hat{h}_1(h) \rightarrow h$. Define

$$(B.7) \quad \mathbf{a}_2(h) := \begin{cases} i_1(h), & \text{if } h \in \hat{\mathcal{H}}_1; \\ \mathbf{a}_1(\hat{h}_1(h)), & \text{if } h \in \hat{\mathcal{I}}_1; \\ \mathbf{a}_1(h), & \text{otherwise.} \end{cases}$$

As before, \mathbf{a}_2 is identical to \mathbf{a}_1 except that it “swaps” the allocation decisions at histories $h \in \hat{\mathcal{H}}_1$ with those at histories $\mathcal{I}_1(h)$, leading to a gain along the path of each history. Such a gain is realized at every history in $\hat{\mathcal{H}}_1$, and since this set has positive measure, it must be the planner realizes a payoff gain by switching from \mathbf{a}_1 to \mathbf{a}_2 .

If \mathbf{a}_2 yields the planner a payoff less than or equal to that of \mathbf{a}^* , transitivity of the planner’s payoffs leads to a contradiction, implying that there does not exist a policy \mathbf{a}_0 such that \mathbf{a}_0 does strictly better than \mathbf{a}^* , and hence that \mathbf{a}^* is optimal.

Proceeding inductively in this manner, we either arrive at a contradiction or we construct a sequence of policies $\{\mathbf{a}_t\}_{t=1}^{\infty}$ with strictly increasing payoffs $\{v_t\}_{t=1}^{\infty}$. Note, however, that for all $t \in \mathbb{N}$, each \mathbf{a}_t agrees with \mathbf{a}^* on *at least* all histories $h \in \mathcal{H}^{t-1}$. Since δ^t approaches zero as t becomes increasingly large and values are bounded above, this implies that this

$$(B.8) \quad \lim_{t \rightarrow \infty} v_t \rightarrow v^*,$$

where v^* is the expected payoff to the planner from following allocation policy \mathbf{a}^* . Moreover, since the sequence $\{v_t\}_{t=1}^{\infty}$ is increasing, this implies that

$$(B.9) \quad v^* \geq v_t \text{ for all } t \in \mathbb{N},$$

a contradiction. It must then be the case that there does not exist a policy \mathbf{a}_0 that yields the planner a strictly higher expected payoff than \mathbf{a}^* . Therefore, we may conclude that \mathbf{a}^* is, in fact, a socially optimal policy. \square

Note that at the ex ante stage, other policies may do as well as \mathbf{a}^* in terms of the planner's payoffs; however, using the same "swapping" argument as above, one may show that the set of histories at which such a policy disagrees with \mathbf{a}^* must be of measure zero. This implies that \mathbf{a}^* is the *unique* socially efficient policy when optimality is desired after *every* history.

REFERENCES

- BAJARI, P., AND A. HORTAÇSU (2003): "The Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions," *The RAND Journal of Economics*, 34(2), 329–355.
- BERGEMANN, D., AND J. VÄLIMÄKI (2008): "The Dynamic Pivot Mechanism," Cowles Foundation Discussion Paper 1672, Yale University.
- CAI, G., P. R. WURMAN, AND X. CHAO (2007): "The Non-Existence of Equilibrium in Sequential Auctions when Bids are Revealed," *Journal of Electronic Commerce Research*, 8(2), 141–156.
- CAILLAUD, B., AND C. MEZZETTI (2004): "Equilibrium Reserve Prices in Sequential Ascending Auctions," *Journal of Economic Theory*, 117(1), 78–95.
- CAVALLO, R., D. C. PARKES, AND S. SINGH (2007): "Efficient Online Mechanisms for Persistent, Periodically Inaccessible Self-Interested Agents," Unpublished manuscript, Harvard University.
- FUCHS, W., AND A. SKRZYPACZ (forthcoming): "Bargaining with Arrival of New Traders," *American Economic Review*.
- INDERST, R. (2008): "Dynamic Bilateral Bargaining under Private Information with a Sequence of Potential Buyers," *Review of Economic Dynamics*, 11(1), 220–236.
- JEITSCHKO, T. (1998): "Learning in Sequential Auctions," *Southern Economic Journal*, 65(1), 98–112.
- KITTSTEINER, T., J. NIKUTTA, AND E. WINTER (2004): "Declining Valuations in Sequential Auctions," *International Journal of Game Theory*, 33(1), 89–106.
- MILGROM, P. R., AND R. J. WEBER (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50(5), 1089–1122.
- (2000): "A Theory of Auctions and Competitive Bidding, II," in *The Economic Theory of Auctions*, ed. by P. Klemperer, vol. 2, pp. 179–194. Edward Elgar Publishing, Cheltenham, UK.
- NEKIPELOV, D. (2007): "Entry Deterrence and Learning Prevention on eBay," Unpublished manuscript, Duke University.
- OK, E. A. (2007): *Real Analysis with Economic Applications*. Princeton University Press.
- PETERS, M., AND S. SEVERINOV (2006): "Internet Auctions with Many Traders," *Journal of Economic Theory*, 130(1), 220–245.
- ROTH, A., AND A. OCKENFELS (2002): "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review*, 94(4), 1093–1103.
- SAID, M. (2008): "Stochastic Equivalence in Sequential Auctions with New Buyers," Unpublished manuscript, Yale University.
- SAILER, K. (2006): "Searching the eBay Marketplace," CESifo Working Paper 1848, University of Munich.
- ZEITHAMMER, R. (2006): "Forward-Looking Bidding in Online Auctions," *Journal of Marketing Research*, 43, 462–476.