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Abstract

We consider a neoclassical growth model with quasi-hyperbolic discounting under Kantian optimization: each temporal self acts in a way that they would like every future self to act. We introduce the notion of a Kantian policy as an outcome of Kantian optimization in a given class of policies. We derive and characterize a Kantian policy in the class of policies with a constant saving rate for an economy with log-utility and Cobb–Douglas production technology and an economy with isoelastic utility and linear production technology. In all cases, the Kantian saving rate is higher than the saving rate of sophisticated agents, and a Kantian path Pareto dominates a sophisticated path.

Keywords: Quasi-hyperbolic discounting; Time inconsistency; Kantian equilibrium; Sophisticated agents; Saving rate; Welfare

JEL Classification: C70, D15, D91, E21, O40

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1 Introduction

The aim of this paper is to propose and study Kantian optimization in the Ramsey model with quasi-hyperbolic discounting. An agent is interpreted as a sequence of temporal selves who choose policies in the corresponding period (that is, determine the future capital stock, given the current capital stock). There are two standard assumptions about optimizing behavior: an agent is assumed either *naive*; that is, their temporal selves are unaware of each other's existence, or *sophisticated*; that is, their temporal selves play a Nash equilibrium. In this paper, we consider another way in which an agent optimizes and, following Roemer (2019), assume that the agent is Kantian. The main question is: What is the strategy I would like all others to play? Each temporal self acts according to Kant's categorical imperative and chooses a policy that they would like every future self to adopt in a given class of policies. We formally define a Kantian policy in a given class of policies and illustrate Kantian policies in the class of policies with a constant saving rate for, respectively, an economy with log-utility and Cobb–Douglas production technology and an economy with isoelastic utility and linear production technology. We show that the resulting Kantian path always Pareto dominates the sophisticated path. Unless the intertemporal elasticity of substitution is very high, the Kantian path also Pareto dominates the naive path.

People cooperate in many (economic) settings. Paying taxes, tipping, or donating to charities are examples of cooperation between people that are not adequately explained by altruism or other-regarding preferences. As a microfoundation for cooperative behavior in the interpersonal context, Roemer (2015; 2019) proposes Kantian optimization which implies that agents choose the best strategy for themselves assuming that everyone else behaves as they do. In many relevant settings, for instance, in the presence of public goods and externalities, a resulting Kantian equilibrium, unlike a Nash equilibrium, is Pareto-optimal, which explains why people often achieve better results through cooperation than through competition. We argue that cooperative behavior is even more justifiable in the intrapersonal context — a very natural tendency to cooperate is even more pronounced when people bargain with themselves rather than with other people. In this paper, we study Kantian optimization in the model with quasi-hyperbolic discounting where decision makers are in fact different temporal selves of a single agent which motivates the use of Kant's categorical imperative as a principle of rationality.

A standard approach to study decision making in the Ramsey model with quasi-hyperbolic discounting is to assume that an agent is a sequence of temporal selves with conflicting preferences (see, e.g., Laibson, 1997). Each self decides about consumption and savings in the corresponding period. The resulting path of consumption and capital, which is an outcome of the interaction between different selves, essentially depends on their expectations of future behavior. There are two popular assumptions about the anticipation of other selves' behaviour: agents are assumed either naive or sophisticated.

For a naive agent, the current self chooses their best policy expecting that in the future the best path from their perspective will be followed. However, due to dynamic inconsistency, when a future self calculates their best policy, a different savings decision is obtained. The resulting naive path is thus not time-consistent, since a naive agent cannot correctly anticipate future decisions and repeatedly recalculates their path (see, e.g., Barro, 1999; Borissov et al., 2021). For a sophisticated agent, the current self chooses their best policy given that all future policies are chosen in the same manner. The resulting sophisticated path is a symmetric subgame perfect Nash equilibrium in the game played among an agent's different selves and is time-consistent, since a sophisticated agent correctly anticipates future decisions (see, e.g., Krusell et al., 2002; Sorger, 2004). However, a sophisticated path is not Pareto-optimal, as is often the case with Nash equilibria, because the current self is effectively constrained by the decisions of their future selves.

In this paper, we study a different approach to model agent's expectations of future behavior and assume that the agent adopts Kantian optimization. Each temporal self makes decisions according to the Kant's categorical imperative; that is, acts in a way they would like everyone else to act. The current self chooses their best policy from a given class of policies expecting that all future selves will act as the current self does. If each future self using the same procedure obtains the same policy, then we call it a *Kantian policy in a given class*. The resulting Kantian path is time-consistent, since the expectations of each self are correct.

Note that a Kantian path essentially differs from both naive and sophisticated paths, as the latter paths are obtained under different behavioral assumptions. A Kantian agent differs from a naive agent in that the former recognizes their time inconsistency. At the same time, a Kantian agent differs from a sophisticated agent in two respects. First, a Kantian agent does not take policies of future selves as given but instead acts cooperatively and chooses the best policy for themselves under the assumption that it will also be adopted by all future selves.[1] Stated

[1]It should be emphasized that Kantian agents are not altruistic: they choose a common (adopted by all) policy which is best for themselves according to their own preferences.

differently, on a sophisticated path no self has an incentive to unilaterally deviate from a chosen policy, while on a Kantian path no self has an incentive to deviate from a chosen policy under the assumption that their deviation implies that all other selves will deviate in the same way. Second, a Kantian agent is constrained not by the decisions of future selves but only by the class of available policies. Kantian optimization, in our framework, involves comparison with feasible policy alternatives from a given class, and deviation is allowed only with respect to a policy from that given class of policies.[2]

We derive Kantian policies in the class of policies with a constant saving rate for, respectively, an economy with log-utility and Cobb–Douglas production technology and an economy with isoelastic utility and linear production technology. The main results about naive and sophisticated paths in the existing literature are also obtained in these two cases (see, e.g., Phelps and Pollak, 1968; Drugeon and Wigniolle, 2019). In both cases, naive and sophisticated policies belong to the class of policies with a constant saving rate, which allows us to compare Kantian, sophisticated and naive paths in terms of saving rates and welfare. We show that the Kantian saving rate is always higher than the sophisticated saving rate and that the Kantian path Pareto dominates the sophisticated path, in the sense that a Kantian agent obtains a higher intertemporal utility for each temporal self. At the same time, the comparison of Kantian and naive paths is ambiguous. With log-utility and Cobb–Douglas production technology, the Kantian saving rate is higher than the naive saving rate and the Kantian path Pareto dominates the naive path. The same is true with isoelastic utility and linear production technology when the intertemporal elasticity of substitution is sufficiently low. Otherwise, Kantian and naive paths do not Pareto dominate each other.

The paper is organized as follows. Section 2 defines a Kantian policy in a given class of policies. Sections 3 and 4 study Kantian policies in the class of policies with a constant saving rate for, respectively, an economy with log-utility and Cobb–Douglas production technology and an economy with isoelastic utility and linear production technology. Section 5 concludes. Proofs and some derivations are provided in the Appendix.

[2] A strategy profile dominating all feasible alternative profiles can be naturally called a Kantian equilibrium. Roemer (2015, 2019) proposes definitions of a *multiplicative Kantian equilibrium* where feasible alternatives are profiles rescaled by a non-negative constant and an *additive Kantian equilibrium* where feasible alternatives are profiles translated by a constant. Our Kantian policy in a given class is a common strategy in what might be called a *constrained Kantian equilibrium* where feasible alternatives are symmetric profiles whose common strategy belongs to a given class of policies.

2 The model

Consider the Ramsey model with quasi-hyperbolic discounting. An agent is modeled as a sequence of temporal selves, each self deciding about consumption and savings in the corresponding period. The intertemporal utility function of self τ acting in period $\tau = 0, 1, \ldots$ is given by

$$U_\tau \;=\; u(c_\tau) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t) \;=\; u(c_\tau) + \beta\delta u(c_{\tau+1}) + \beta\delta^2 u(c_{\tau+2}) + \ldots\,, \tag{1}$$

where c denotes consumption, $u(c)$ is the instantaneous utility function, β is the present bias parameter and δ is the long-run discount factor. We assume that $0 < \beta < 1$, $0 < \delta < 1$, and $u(c)$ is strictly concave.

Technology is given by a concave production function $f(k)$, where k denotes the stock of capital. The resource constraint is given by $c_t + k_{t+1} = f(k_t)$. Therefore, the intertemporal utility function of each self can be rewritten as depending on a sequence of capital stocks $\{k_t\}_{t=0}^{\infty}$:

$$J[\{k_t\}_{t=0}^{\infty}] \;=\; u(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \delta^t u(f(k_t) - k_{t+1})\,. \tag{2}$$

A sequence $\{k_t\}_{t=0}^{\infty}$ is called a *feasible trajectory* starting from k_0 if $0 \le k_{t+1} \le f(k_t)$ for all $t \ge 0$.

A resulting trajectory is an outcome of the interaction between different selves, and hence depends on the expectations that each current self has about the behavior of future selves. There are two standard assumptions about how an agent's selves anticipate the future behaviour of others: the agent is assumed either naive or sophisticated. We define the behavior of each agent's type in terms of the corresponding policy adopted by each current self.

A *feasible policy* is a function $k \mapsto h(k)$ such that $0 \le h(k) \le f(k)$ for all k. We define the iterates of the function h by

$$h^{(0)}(k) = k, \qquad h^{(1)}(k) = h(k), \qquad h^{(2)}(k) = h(h(k)), \qquad \ldots,$$

so that in general $h^{(q+1)}(k) = h \circ h^{(q)}(k)$ for all $q \ge 0$.

A naive agent is unaware of conflicting preferences of future selves. When choosing the best policy for the current period, self τ maximizes their intertemporal utility over the set of all feasible trajectories and expects that the resulting trajectory will be followed in the future. Let $\{k, k'(k), k''(k), \ldots\}$ be a trajectory that maximizes (2) over the set of all feasible trajectories starting from k. Formally, *a naive policy h^N is a feasible policy such that for all k it holds that $h^N(k) = k'(k)$.* Due to conflicting preferences, a naive policy is obtained under incorrect expectations about future policies. Self τ implicitly expects that starting from $\tau + 1$ all future selves would adopt the policy function implied by the standard Ramsey model with discount factor δ, but these expectations turn out to be wrong, as self $\tau + 1$ recalculates their policy and also chooses the naive policy h^N. A resulting *naive path* starting from $k_0^N = k_0$ is a sequence $\{c_\tau^N, k_{\tau+1}^N\}_{\tau=0}^\infty$ such that for all $\tau \geq 0$, $k_{\tau+1}^N = h^N(k_\tau^N)$ and $c_\tau^N = f(k_\tau^N) - h^N(k_\tau^N)$. The naive path is thus not time-consistent, since a naive agent cannot correctly anticipate future decisions and repeatedly recalculates their trajectory.

A sophisticated agent recognizes their time inconsistency and acts strategically. The current self expects that all future selves adopt the same policy and takes their decisions as given. A sophisticated policy is defined as the best policy for the current period provided that all future policies are chosen in the same manner. Formally, *a sophisticated policy h^S is a policy such that for all k it holds that $h^S(k)$ maximizes the function $x \mapsto J[\{k, x, h^S(x), h^{S(2)}(x), \ldots\}]$ defined over the set $0 \leq x \leq f(k)$.* A *sophisticated path* starting from $k_0^S = k_0$ is a sequence $\{c_\tau^S, k_{\tau+1}^S\}_{\tau=0}^\infty$ such that for all $\tau \geq 0$, $k_{\tau+1}^S = h^S(k_\tau^S)$ and $c_\tau^S = f(k_\tau^S) - h^S(k_\tau^S)$. A sophisticated path is time-consistent, as a sophisticated agent correctly foresees future policies and has no incentives to change their decisions.

There are two cases where naive and sophisticated policies attract particular attention: the case of log-utility and a Cobb–Douglas production technology and the case of isoelastic utility and a linear production technology. In both cases, naive and sophisticated policies belong to the class of policies with a constant saving rate.

We now turn to the principal contribution of this paper. We introduce the notion of a Kantian agent. The idea is that the current self, instead of ignoring conflicting preferences of future selves or playing a non-cooperative game with future selves, acts cooperatively and chooses the best policy from some class of policies that this current self would like every future self to choose. If each future self, acting in the same cooperative manner and being constrained by the same class of available policies, obtains the same policy, this policy is a Kantian policy in a given class. Loosely speaking, a Kantian policy is the optimal time-consistent policy in a given class of policies.

Formally, we define a Kantian policy in a given class in two steps. Let H be a given class of feasible policies. The best state-k policy in H is defined as the preferred policy in this class obtained under the assumption that this policy is adopted in the future, provided the initial state is k.

Definition 1. *The best state-k policy in the class H is a policy $h \in H$ such that, given k, it holds that h maximizes the functional $h \mapsto J[\{k, h(k), h^{(2)}(k), h^{(3)}(k), \ldots\}]$ over the set H.*

Note that the best state-k policy may not be the best state-$h(k)$ policy; that is, the preferred policy may be different for different temporal selves. A Kantian policy in the class H is defined as a policy such that for any k, it is the best state-k policy, the best state-$h(k)$ policy, the best state-$h^{(2)}(k)$ policy, and so on.

Definition 2. *A Kantian policy in the class H, $h^K \in H$, is a policy such that for any given k, h^K is the best state-$h^{K(q)}(k)$ policy in the class H for all $q \geq 0$.*

A *Kantian path* starting from $k_0^K = k_0$ is a sequence $\{c_\tau^K, k_{\tau+1}^K\}_{\tau=0}^{\infty}$ such that for all $\tau \geq 0$, $c_\tau^K = f(k_\tau^K) - h^K(k_\tau^K)$ and $k_{\tau+1}^K = h^K(k_\tau^K)$. By construction, a Kantian path is time-consistent, because expectations of each self about future policies are correct.

Two remarks about this definition are in order. First, the notion of a Kantian policy in a given class generalizes the usual notion of an optimal policy in the standard Ramsey model.

Remark 1. *When $\beta = 1$, the Kantian policy in the class of all feasible policies is the policy function in the standard Ramsey model with discount factor δ.*

Second, a Kantian policy in the class H crucially depends on the given class of policies H. In general, for a sufficiently large class of policies, it is unreasonable to expect that a Kantian policy in this class exists. This fact is a manifestation of dynamic inconsistency arising from non-exponential discounting.

Remark 2. *When $\beta < 1$, there is no Kantian policy in the class of all feasible policies. Moreover, there is no Kantian policy in the class of all monotone policies. If $h(k)$ is such a policy, then for some k_0 it should be both the best state-k_0 policy and the best state-$h(k_0)$ policy. However, this is impossible, since, because of present bias, at state $h(k_0)$ an agent prefers to consume more than it has been planned for state $h(k_0)$ at state k_0.*

For a sufficiently small class of policies, a unique Kantian policy in this class exists. For example, if H consists of only three policies $h_1(k) = 0$, $h_2(k) = f(k)/2$, $h_3(k) = f(k)$, then it is evident that the Kantian policy in the class H is $h_2(k)$. While this example is trivial, there are Kantian policies in a significantly broader class of policies. For illustration, we employ the two cases, most commonly discussed in the literature: (i) log-utility and Cobb–Douglas technology; (ii) isoelastic utility and linear production technology. In both cases, there exist Kantian policies in the class of policies with a constant saving rate.

In what follows, we consider the class of policies $\mathcal{H}$ defined by a constant saving rate:

$$\mathcal{H} \;=\; \{h(k) \;\mid\; h(k) = \sigma f(k), \text{ where } 0 < \sigma < 1\}\,. \tag{3}$$

We are studying not only the properties of Kantian policies in the class $\mathcal{H}$ but also their respective welfare implications. In particular, we are interested in welfare comparisons between different paths of consumption and capital followed by naive, sophisticated and Kantian agents.

Let $\{c_\tau, k_{\tau+1}\}_{\tau=0}^{\infty}$ be a feasible path starting from k_0, where $\{k_\tau\}_{\tau=0}^{\infty}$ is a feasible trajectory starting from k_0, and $c_\tau = f(k_\tau) - k_{\tau+1}$ for all $\tau \geq 0$. Note that each feasible path is associated with a sequence of utilities of different temporal selves $\{U_\tau\}_{\tau=0}^{\infty}$, where U_τ is given by (1). To evaluate welfare, we use the following Pareto criterion.

Definition 3. *A feasible path $\{c_\tau, k_{\tau+1}\}_{\tau=0}^{\infty}$ starting from k_0 Pareto dominates another feasible path $\{\tilde{c}_\tau, \tilde{k}_{\tau+1}\}_{\tau=0}^{\infty}$ starting from the same $\tilde{k}_0 = k_0$ if $U_\tau \geq \tilde{U}_\tau$ for all $\tau \geq 0$, and at least for one τ this inequality is strict.*

The Pareto criterion takes into account utilities of all temporal selves, not just the long run outcome.[3]

3 Log-utility and Cobb–Douglas technology

Consider $u(c) = \ln c$ and $f(k) = k^\alpha$. Combining (2) and (3), it is easily seen that the best state-k policy in the class $\mathcal{H}$ is determined by the saving rate σ which is a solution to the following problem:

$$\max_{0<\sigma<1}\; \ln((1-\sigma)k^\alpha) + \beta\delta V(\sigma, \sigma k^\alpha)\,, \tag{4}$$

where $V(\sigma, k)$ is the intertemporal utility in the standard Ramsey model with log-utility and Cobb–Douglas production technology obtained under a constant saving rate σ and the initial capital stock k.

[3]Because of dynamic inconsistency, there is no agreed-upon welfare criterion in models with quasi-hyperbolic discounting. Another popular welfare criterion is “long-run perspective” which compares different paths according to intertemporal utility of the form (1) with $\beta = 1$. While these two criteria often lead to the same conclusions, Kang (2015) shows that Pareto dominance implies long-run perspective dominance but not vice versa.

Problem (4) can be interpreted as follows. The current self is constrained by the requirement that the saving rate is constant. The current self chooses the best saving rate among those that this current self would like every future self to adopt. The cooperative nature of this procedure reflects the fact that different temporal selves are, after all, the same person.

Taking into account the closed-form expression for $V(\sigma, k)$ (see Eq. (A.1) in Appendix A), the objective function in (4) can be written as

$$U(\sigma, k) = \frac{1-\delta+\beta\delta}{1-\delta}\ln(1-\sigma) + \frac{\alpha\beta\delta}{(1-\delta)(1-\alpha\delta)}\ln\sigma + \alpha\frac{1-\alpha\delta+\alpha\beta\delta}{1-\alpha\delta}\ln k\,. \quad (5)$$

The first-order condition for problem (4) is given by

$$\frac{1-\delta+\beta\delta}{1-\delta}\frac{1}{1-\sigma} = \frac{\alpha\beta\delta}{(1-\delta)(1-\alpha\delta)}\frac{1}{\sigma}\,.$$

Hence the best state-k policy in the class $\mathcal{H}$ is determined by the saving rate σ^K given by

$$\sigma^K = \frac{\alpha\beta\delta}{(1-\delta+\beta\delta)-\alpha\delta(1-\beta)(1-\delta)}\,. \quad (6)$$

Since σ^K does not depend on k, it determines the Kantian policy in the class $\mathcal{H}$, and we call σ^K the *Kantian saving rate*.[4]

The following properties of the Kantian saving rate are worth noting. First, the Kantian saving rate is monotonically increasing in β: the lower is the degree of present bias, the higher is the saving rate. Second, when $\beta = 1$, the Kantian saving rate coincides with the optimal saving rate in the standard Ramsey model, $\sigma^R = \alpha\delta$.

Third, in the considered case the naive and sophisticated policies coincide and belong to the same class $\mathcal{H}$ (see Appendix B). The naive policy has the form $h^N(k) = \sigma^N k^\alpha$, and the sophisticated policy has the form $h^S(k) = \sigma^S k^\alpha$, where the constant saving rates are the same and are given by[5]

$$\sigma^N \;=\; \sigma^S \;=\; \frac{\alpha\beta\delta}{1-\alpha\delta+\alpha\beta\delta}\,. \quad (7)$$

[4]The expression for σ^K already appeared in the proof of Proposition 3 in Krusell et al. (2002), though in a different context.

[5]With log-utility and Cobb–Douglas technology, the naive saving rate σ^N is also derived in Borissov et al. (2021). Krusell et al. (2002) and Drugeon and Wigniolle (2019) also study a "sophisticated equilibrium saving rate" $\frac{\alpha\beta\delta}{1-\delta+\beta\delta}$, which corresponds to a sophisticated equilibrium path on which an agent in each period consumes a constant share of their total expected wealth.

By comparing (6) and (7), it is easily seen that

$$\sigma^R \; > \; \sigma^K \; > \; \sigma^S \; = \; \sigma^N \,,$$

that is, the Kantian saving rate is lower than the optimal saving rate without present bias but is always higher than the sophisticated and naive saving rates. Note that while both the Kantian policy in the class $\mathcal{H}$ and the sophisticated policy are time-consistent (each current self correctly anticipates future saving rates), there is an important difference between them. The Kantian current self chooses the best saving rate together for themselves and for future selves, while the sophisticated current self chooses the best saving rate for themselves taking future saving rates as given. In the considered case, even if the sophisticated current self expects that all future selves adopt σ^K, their best response is still σ^S, not σ^K (see Appendix B).

Finally, we compare Kantian, sophisticated and naive paths in terms of welfare. In all cases, intertemporal utility $U(\sigma, k)$ is given by (5) and depends only on the respective constant saving rate σ and the initial capital stock k. It is easily seen from (5) that $U(\sigma, k)$ is increasing in σ for $\sigma \leq \sigma^K$ and for all k. Thus, for paths starting from the same initial capital stock, the higher is the saving rate, the higher is intertemporal utility. It follows that the ordering of Kantian, sophisticated and naive paths starting from the same initial capital stock k_0 in terms of welfare from the period-0 perspective coincides with their ordering in terms of saving rates. Moreover, it is evident that $U(\sigma, k)$ is increasing in k for all k and σ. Since on each path $k_{t+1} = \sigma f(k_t)$, the higher is the saving rate, the higher is the capital stock in each period. Therefore, each Kantian temporal self obtains a strictly higher utility than the respective sophisticated and naive temporal selves, and hence a Kantian path Pareto dominates both sophisticated and naive paths.

The following theorem summarizes the above discussion.

Theorem 1. *Consider the Ramsey model with quasi-hyperbolic discounting, log-utility and Cobb–Douglas production technology. A Kantian policy in the class of policies* (3) *exists and is characterized by the saving rate* σ^K *given by* (6)*. The Kantian saving rate* σ^K *is increasing in* β*, and is higher than the sophisticated and naive saving rates. The Kantian path Pareto dominates both sophisticated and naive paths.*

4 Isoelastic utility and linear technology

Suppose that $u(c) = \frac{c^{1-\rho}}{1-\rho}$, where $\rho > 0$,[6] and $f(k) = Ak$, with $A > 0$. Again, combining (2) and (3), it is easily seen that the best state-k policy in the class $\mathcal{H}$ is determined by a saving rate σ which is a solution to the following problem:

$$\max_{0<\sigma<1} \frac{((1-\sigma)Ak)^{1-\rho}}{1-\rho} + \beta\delta W(\sigma, \sigma Ak)\,, \tag{8}$$

where $W(\sigma, k)$ is the intertemporal utility in the standard Ramsey model with isoelastic utility and linear production technology obtained under a constant saving rate σ and an initial capital stock k. Recall that the optimal path in the latter model is characterized by a constant saving rate $\sigma^R = (\delta A^{1-\rho})^{\frac{1}{\rho}}$, and exists only when $\delta A^{1-\rho} < 1$, which is assumed throughout.

Taking into account the closed-form expression for $W(\sigma, k)$ (see Eq. (A.2) in Appendix A), the objective function in (8) can be written as

$$U(\sigma, k) \;=\; \frac{(Ak)^{1-\rho}}{1-\rho}(1-\sigma)^{1-\rho}\left(1 + \frac{\beta\delta(\sigma A)^{1-\rho}}{1-\delta(\sigma A)^{1-\rho}}\right)\,, \tag{9}$$

provided that σ is such that $\delta(\sigma A)^{1-\rho} < 1$.[7]

The first-order condition for problem (8) yields:

$$(1-\sigma)^{-\rho}\left(1 + \frac{\beta\delta(\sigma A)^{1-\rho}}{1-\delta(\sigma A)^{1-\rho}}\right) = (1-\sigma)^{1-\rho}\left(\frac{\beta\delta A^{1-\rho}(\sigma)^{-\rho}}{(1-\delta(\sigma A)^{1-\rho})^2}\right)\,.$$

Thus, the best state-k policy in the class $\mathcal{H}$ is determined by the saving rate σ^K which is a solution to the following equation in σ:

$$\beta\delta A^{1-\rho}\frac{1-\sigma}{\sigma^\rho} \;=\; \left(1-\delta(\sigma A)^{1-\rho}\right)\left(1-\delta(\sigma A)^{1-\rho} + \beta\delta(\sigma A)^{1-\rho}\right)\,. \tag{10}$$

Since σ^K does not depend on k, it determines the Kantian policy in the class $\mathcal{H}$, and we call σ^K the *Kantian saving rate*.

It can be shown that the Kantian saving rate given by a solution to Eq. (10) always exists. While for $\rho \geq 1$ this solution is unique, for small enough ρ Eq. (10) has multiple solutions, and it is unclear, which of its solutions is the Kantian saving rate. We provide a simple sufficient condition for uniqueness of the Kantian saving rate determined by Eq. (10).

[6] With the usual convention that $\rho = 1$ refers to the logarithmic case $u(c) = \ln c$.

[7] When $0 < \rho < 1$, this inequality is satisfied since $\delta A^{1-\rho} < 1$. When $\rho > 1$, this inequality is satisfied for $\sigma > (\delta A^{1-\rho})^{\frac{1}{\rho-1}} \equiv \underline{\sigma}$.

Lemma 1. *For any $\rho > 0$, there exists a Kantian saving rate. When $\rho > 1-\sqrt{\beta}$, the Kantian saving rate is given by the unique solution to Eq.* (10).

Proof. See Appendix C. ■

In what follows, we assume that $\rho > 1-\sqrt{\beta}$. The Kantian saving rate exhibits the following properties. First, the Kantian saving rate is monotonically increasing in β. Second, it is evident from (10) that when $\beta = 1$, the Kantian saving rate coincides with the optimal saving rate in the standard Ramsey model $\sigma^R = (\delta A^{1-\rho})^{\frac{1}{\rho}}$. Also, when utility is logarithmic ($\rho = 1$), it follows from (10) that

$$\sigma^K = \frac{\beta\delta}{\beta\delta + (1-\delta)(1-\delta+\beta\delta)},$$

which coincides with the limit of (6) when $\alpha \to 1$.

Third, in the considered case both naive and sophisticated policies belong to the same class $\mathcal{H}$ (see Appendix D). The naive policy has the form $h^N(k) = \sigma^N Ak$, where the naive saving rate σ^N is given by

$$\sigma^N = \frac{(\beta\delta A^{1-\rho})^{\frac{1}{\rho}}}{1-(\delta A^{1-\rho})^{\frac{1}{\rho}} + (\beta\delta A^{1-\rho})^{\frac{1}{\rho}}}. \tag{11}$$

The sophisticated policy has the form $h^S(k) = \sigma^S Ak$, where the sophisticated saving rate σ^S satisfies the following equation:[8]

$$\sigma^S = \frac{(\beta\delta A^{1-\rho})^{\frac{1}{\rho}}}{(1-\delta(\sigma^S A)^{1-\rho} + \beta\delta(\sigma^S A)^{1-\rho})^{\frac{1}{\rho}}}. \tag{12}$$

Comparing saving rates obtained under different assumptions, we find that the Kantian saving rate is lower than the optimal saving rate without present bias but is always higher than the sophisticated saving rate. Again, note that even if the sophisticated current self expects all future selves to adopt saving rate σ^K, their best response does not coincide with σ^K (see Appendix D), which highlights that the Kantian policy in a given class is not a non-cooperative Nash equilibrium. The following proposition summarizes the above properties.

[8]With isoelastic utility and linear technology, the naive saving rate σ^N and the sophisticated saving rate σ^S are derived and compared in Phelps and Pollak (1968). The sophisticated saving rate is also calculated in Sorger (2004). The naive saving rate coincides with the "naive equilibrium saving rate" corresponding to the naive equilibrium path on which an agent in each period consumes a constant share of their total expected wealth (see Drugeon and Wigniolle, 2019).

Proposition 1. *Consider the Ramsey model with quasi-hyperbolic discounting, isoelastic utility and linear technology. Suppose that $\rho > 1 - \sqrt{\beta}$. A Kantian policy in the class of policies* (3) *exists, is unique, and is characterized by the saving rate σ^K which is the unique solution to Eq.* (10)*. The Kantian saving rate σ^K is increasing in β, and is such that*

$$\sigma^R \ > \ \sigma^K \ > \ \sigma^S.$$

Proof. See Appendix E. ■

The ordering of Kantian and naive saving rates is ambiguous in this case. When $\rho \geq 1$, the sophisticated saving rate is always higher than the naive saving rate, and hence we have $\sigma^R > \sigma^K > \sigma^S > \sigma^N$, which corresponds to the intuition that naivety leads to less saving. However, when $1-\sqrt{\beta} < \rho < 1$, the naive saving rate is always higher than the sophisticated saving rate, and for small enough values of ρ, the naive saving rate may even be higher than the Kantian saving rate. In order to compare naive and Kantian saving rates, one has to solve a non-linear equation, which, in general, is not analytically tractable (see Eq. (E.1) in Appendix E). However, a large number of simulations with different values of ρ, β and δ provides the following observations.

For any given δ, there is a unique value $\bar{\beta}$ such that for any $\beta > \bar{\beta}$, there exists a unique value $\bar{\rho}$, satisfying $1 - \sqrt{\beta} < \bar{\rho} < 1$, such that for these parameters the naive saving rate coincides with the unique Kantian saving rate ($\sigma^N = \sigma^K$). Thus, for a sufficiently small ρ, there are knife-edge cases where a naive agent is observationally equivalent to a Kantian agent. In other words, observing only the path of consumption and capital, one cannot determine whether the agent is Kantian and time-consistent or the agent is naive and time-inconsistent. Furthermore, for any $\beta > \bar{\beta}$ and for all ρ such that $1 - \sqrt{\beta} < \rho < \bar{\rho}$, the naive saving rate is higher than the Kantian saving rate: $\sigma^R > \sigma^N > \sigma^K > \sigma^S$. That is, a naive agent tends to over-accumulate capital (compared to a Kantian agent). At the same time, when $\beta > \bar{\beta}$ and $\bar{\rho} < \rho < 1$ or when $\beta < \bar{\beta}$ and $1 - \sqrt{\beta} < \rho < 1$, the Kantian saving rate is higher than the naive saving rate: $\sigma^R > \sigma^K > \sigma^N > \sigma^S$.

Finally, consider the welfare ordering of Kantian, sophisticated and naive paths. It can be shown that intertemporal utility $U(\sigma, k)$ given by (9) is increasing in σ for $\sigma \leq \sigma^K$ and is decreasing in σ for $\sigma \geq \sigma^K$. Moreover, it is evident that $U(\sigma, k)$ is increasing in k. Therefore, when $\sigma^K > \max\{\sigma^N, \sigma^S\}$, the welfare ordering of Kantian, sophisticated and naive paths (starting from the same initial capital stock) is the same in each period and coincides with their ordering in terms of saving rates. Thus, in the general case (and in particular when $\rho \geq 1$), all Kantian temporal selves obtain a strictly higher utility than sophisticated and naive temporal selves, and thereby a Kantian path Pareto dominates both sophisticated and naive paths.[9]

However, for sufficiently small ρ, there are cases where the naive saving rate is higher than the Kantian saving rate. While even in these cases the Kantian path Pareto dominates the sophisticated path, the Kantian and naive paths do not Pareto dominate each other.

The following theorem summarizes the above discussion.

Theorem 2. *Consider the Ramsey model with quasi-hyperbolic discounting, isoelastic utility and linear technology. When $\rho \geq 1$, the Kantian path Pareto dominates both the sophisticated and naive paths. When $1 - \sqrt{\beta} < \rho < 1$, two cases are possible. If $\sigma^K > \sigma^N$, the Kantian path Pareto dominates both the sophisticated and naive paths. If $\sigma^N > \sigma^K$, the Kantian and naive paths Pareto dominate the sophisticated path, but they do not Pareto dominate each other.*

Proof. See Appendix E. ■

5 Concluding remarks

In this paper, we study the Ramsey model with quasi-hyperbolic discounting and propose the notion of a Kantian policy in a given class of policies. An agent is modeled as a sequence of temporal selves with conflicting preferences. Each member of the sequence of temporal selves adopts Kant's categorical imperative as a principle of rationality and makes only those decisions that they would like everyone else to make. A Kantian policy determined by the current self is the best time-consistent policy in a given class of policies obtained under the assumption that everybody behaves as the current self does.

While dynamic inconsistency arising from quasi-hyperbolic discounting leads to the non-existence of a Kantian policy in the class of *all* feasible policies, we argue that there are sufficiently large classes of policies for which Kantian policies exist. We derive and characterize Kantian policies in the class of policies with a constant saving rate for an economy with log-utility and Cobb–Douglas production technology and for an economy with isoelastic utility and linear production technology. We show that in all cases a path of consumption and

[9]See also a recent contribution by Fishman (2020) who compares sophisticated and naive paths in terms of saving rates and welfare in a continuous time model.

capital obtained under a Kantian policy Pareto dominates a sophisticated path and is never Pareto dominated by a naive path.

The use of Kantian optimization in a model with quasi-hyperbolic discounting gives rise to further research regarding both applications in economic models with time-inconsistent decision making and generalizations to such models with non-constant saving rates. We believe that our proposed notion of Kantian policy in a given class of policies will turn out to be useful for studying time-inconsistent decision making beyond the notions of naive and sophisticated agents.

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Appendix

A Intertemporal utility for a constant saving rate

Intertemporal utility $V(\sigma, k)$ in the standard Ramsey model with log-utility and Cobb–Douglas production technology obtained under a constant saving rate σ and initial capital stock k is given by

$$V(\sigma,k)=\sum_{t=0}^{\infty}\delta^t \ln c_t, \quad \text{where} \quad c_t=(1-\sigma)k_t^{\alpha}, \quad k_{t+1}=\sigma k_t^{\alpha}, \quad k_0=k.$$

Therefore, we have $c_0=(1-\sigma)k^{\alpha}$, $c_1=(1-\sigma)(\sigma k^{\alpha})^{\alpha}$, and in general

$$c_t=(1-\sigma)\sigma^{\alpha+\alpha^2+\ldots+\alpha^t}k^{\alpha^{t+1}}\,.$$

It follows that

$$\begin{aligned}V(\sigma,k)&=\sum_{t=0}^{\infty}\delta^t\ln(1-\sigma)+\sum_{t=0}^{\infty}\delta^t\alpha^{t+1}\ln k+\sum_{t=1}^{\infty}\delta^t(\alpha+\alpha^2+\ldots+\alpha^t)\ln\sigma\\ &=\frac{1}{1-\delta}\ln(1-\sigma)+\frac{\alpha}{1-\alpha\delta}\ln k+\sum_{t=1}^{\infty}\alpha\delta^t\frac{1-\alpha^t}{1-\alpha}\ln\sigma\\ &=\frac{1}{1-\delta}\ln(1-\sigma)+\frac{\alpha}{1-\alpha\delta}\ln k+\frac{\alpha}{1-\alpha}\left(\frac{\delta}{(1-\delta)}-\frac{\alpha\delta}{1-\alpha\delta}\right)\ln\sigma,\end{aligned}$$

and hence

$$V(\sigma,k) \;=\; \frac{1}{1-\delta}\ln(1-\sigma)+\frac{\alpha\delta}{(1-\delta)(1-\alpha\delta)}\ln\sigma+\frac{\alpha}{1-\alpha\delta}\ln k\,. \tag{A.1}$$

Intertemporal utility $W(\sigma, k)$ in the standard Ramsey model with isoelastic utility and linear production technology obtained under a constant saving rate σ and initial capital stock k is given by

$$W(\sigma, k) = \sum_{t=0}^{\infty} \delta^t \frac{(c_t)^{1-\rho}}{1-\rho}, \quad \text{where} \quad c_t = (1-\sigma)Ak_t, \quad k_{t+1} = \sigma Ak_t, \quad k_0 = k\,.$$

It follows that $c_0 = (1-\sigma)Ak$, $c_1 = (1-\sigma)(\sigma A)Ak$, and hence $c_t = (1-\sigma)(\sigma A)^t Ak$. Therefore, provided that $\delta(\sigma A)^{1-\rho} < 1$, we have

$$W(\sigma, k) \;=\; \frac{(Ak)^{1-\rho}}{1-\rho}(1-\sigma)^{1-\rho} \sum_{t=0}^{\infty} \delta^t (\sigma A)^{t(1-\rho)} = \frac{(Ak)^{1-\rho}}{1-\rho} \frac{(1-\sigma)^{1-\rho}}{1-\delta(\sigma A)^{1-\rho}}\,. \tag{A.2}$$

B Naive and sophisticated policies with log-utility and Cobb–Douglas technology

Given the capital stock k_τ, self τ solves the following problem:

$$\max_{c_\tau, k_{\tau+1}} \ln c_\tau + \beta\delta V(\sigma, k_{\tau+1}), \qquad \text{s. t.} \quad c_\tau + k_{\tau+1} = k_\tau^\alpha,$$

where $V(\sigma, k)$ is given by (A.1), though the assumptions about the future saving rates σ are different for naive and sophisticated selves. The first-order condition for this problem yields

$$\frac{1}{c_\tau} = \beta\delta V_k'(\sigma, k_\tau^\alpha - c_\tau) = \frac{\alpha\beta\delta}{1-\alpha\delta} \frac{1}{k_\tau^\alpha - c_\tau}.$$

Let $\tilde{\sigma} = 1 - \frac{c_\tau}{k_\tau^\alpha}$ be the period-τ saving rate. Then

$$\frac{1-\tilde{\sigma}}{\tilde{\sigma}} \;=\; \frac{1-\alpha\delta}{\alpha\beta\delta},$$

so that $\tilde{\sigma}$ is constant and does not depend on the future saving rate σ. Therefore, the optimal saving rate for both naive and sophisticated agents is σ^N $(= \sigma^S)$ and given by (7).

C Proof of Lemma 1

Let $L(\sigma) = \beta\delta A^{1-\rho} \frac{1-\sigma}{\sigma^\rho}$ and $R(\sigma) = (1 - \delta(\sigma A)^{1-\rho})(1 - \delta(\sigma A)^{1-\rho} + \beta\delta(\sigma A)^{1-\rho})$. Then Eq. (10) can be written as $L(\sigma) = R(\sigma)$.

It is clear that $L(0) = +\infty$, and $L(1) = 0$. Moreover, for all $\sigma \in (0,1)$,

$$L'(\sigma) = \beta\delta A^{1-\rho}\sigma^{-\rho}(-1 - \rho(1-\sigma)/\sigma) < 0\,, \tag{C.1}$$

and $L''(\sigma) = \beta\delta A^{1-\rho}\rho\sigma^{-1-\rho}(1 + 1/\sigma + \rho(1-\sigma)/\sigma) > 0$. Thus, $L(\sigma)$ monotonically decreases and is strictly convex. It is also easily seen that

$$R'(\sigma) = (1-\rho)\delta A^{1-\rho}\sigma^{-\rho}\left(2\delta(\sigma A)^{1-\rho}(1-\beta) - (2-\beta)\right)\,. \tag{C.2}$$

Since for all ρ and σ, $\delta(\sigma A)^{1-\rho} < 1 < (2-\beta)/(2-2\beta)$, $R(\sigma)$ monotonically increases when $\rho > 1$ and monotonically decreases when $0 < \rho < 1$.

The case $\rho = 1$ is trivial. Consider the case $\rho > 1$. Note that $R(\underline{\sigma}) = 0$ and $R(1) > 0$. Since $L(\sigma)$ monotonically decreases and $R(\sigma)$ monotonically increases, there is a unique solution to the equation $L(\sigma) = R(\sigma)$, which defines a unique Kantian saving rate $\sigma^K \in (\underline{\sigma}, 1)$.

Consider the case $0 < \rho < 1$. We have $R(0) = 1$, $R(1) > 0$, and hence there is at least one solution to the equation $L(\sigma) = R(\sigma)$. However, since both $L(\sigma)$ and $R(\sigma)$ decrease, in general the solution is not unique. Let us rewrite Eq. (10) as $\sigma^\rho L(\sigma) = \sigma^\rho R(\sigma)$. Its solution will be unique if for all $\sigma \in (0,1)$ we have $(\sigma^\rho(L(\sigma) - R(\sigma)))' < 0$, or

$$\rho(L(\sigma) - R(\sigma)) < -\sigma(L'(\sigma) - R'(\sigma)). \tag{C.3}$$

Using the definition of $L(\sigma)$ and Eq. (C.1), we get $\rho L(\sigma) + \sigma L'(\sigma) = -\beta\delta(\sigma A)^{1-\rho}$. Therefore, taking into account (C.2), we can rewrite the above inequality as

$$(1-\rho)x\,(2(1-\beta)x - (2-\beta)) + \rho(1-x)(1-x+\beta x) \;>\; -\beta x\,,$$

where we have denoted $x \equiv \delta(\sigma A)^{1-\rho}$. Simplifying the above inequality, we get

$$x^2 - \frac{2}{2-\rho}x + \frac{\rho}{(2-\rho)(1-\beta)} \;>\; 0\,.$$

For this inequality to hold for all $x \in (0,1)$, the corresponding quadratic equation should have a non-positive discriminant, which yields the condition $1-\beta < \rho(2-\rho)$. Therefore, a sufficient condition for the uniqueness of a solution to Eq. (10) is $\beta > (1-\rho)^2$, or $\rho > 1 - \sqrt{\beta}$.

D Naive and sophisticated policies with isoelastic utility and linear technology

Given the capital stock k_τ, self τ solves the following problem:

$$\max_{c_\tau, k_{\tau+1}} \frac{(c_\tau)^{1-\rho}}{1-\rho} + \beta\delta W(\sigma, k_{\tau+1}), \qquad \text{s. t.} \quad c_\tau + k_{\tau+1} = Ak_\tau \,,$$

where $W(\sigma, k)$ is given by (A.2). Again, the assumptions about the future saving rates σ are different for naive and sophisticated selves. The first-order condition for the above problem yields

$$(c_\tau)^{-\rho} = \beta\delta W'_k(\sigma, Ak_\tau - c_\tau) = \beta\delta A^{1-\rho} \frac{(1-\sigma)^{1-\rho}}{1-\delta(\sigma A)^{1-\rho}} (Ak_\tau - c_\tau)^{-\rho} \,,$$

which can be rewritten in terms of the corresponding period-τ saving rate $\tilde{\sigma} = 1 - \frac{c_\tau}{Ak_\tau}$ as follows:

$$\left(\frac{1-\tilde{\sigma}}{\tilde{\sigma}}\right)^\rho = \frac{1-\delta(\sigma A)^{1-\rho}}{\beta\delta A^{1-\rho}(1-\sigma)^{1-\rho}} \,. \tag{D.1}$$

Now, let us specify the assumptions about the future saving rates for different types of agents. A naive agent believes that starting from $\tau + 1$ the optimal path in the standard Ramsey model with isoelastic utility and linear production technology will be followed. Therefore, the naive saving rate σ^N is given by $\tilde{\sigma}$ for $\sigma = \sigma^R = (\delta A^{1-\rho})^{\frac{1}{\rho}}$. Using (D.1), we have

$$\left(\frac{1-\sigma^N}{\sigma^N}\right)^\rho = \frac{1-\delta(\delta A)^{\frac{1-\rho}{\rho}}}{\beta\delta A^{1-\rho}(1-(\delta A^{1-\rho})^{\frac{1}{\rho}})^{1-\rho}} = \frac{\left(1-(\delta A^{1-\rho})^{\frac{1}{\rho}}\right)^\rho}{\beta\delta A^{1-\rho}} \,,$$

and hence a naive saving rate σ^N is a solution to the following equation in σ:

$$\beta\delta A^{1-\rho}\frac{1-\sigma}{\sigma^\rho} = (1-\sigma)^{1-\rho}\left(1-\sigma^R\right)^\rho \,. \tag{D.2}$$

This equation has a unique solution given by

$$\sigma^N = \frac{(\beta\delta A^{1-\rho})^{\frac{1}{\rho}}}{1-(\delta A^{1-\rho})^{\frac{1}{\rho}} + (\beta\delta A^{1-\rho})^{\frac{1}{\rho}}} = \frac{\beta^{\frac{1}{\rho}}\sigma^R}{1-(1-\beta^{\frac{1}{\rho}})\sigma^R} \,. \tag{D.3}$$

It is immediately clear that $\sigma^N < \sigma^R$.

A sophisticated agent assumes that all future selves use some constant saving rate σ, and a sophisticated saving rate σ^S is a fixed point of the mapping $\sigma \mapsto \tilde{\sigma}$.

Therefore, one should have $\tilde{\sigma} = \sigma$, and it follows from (D.1) that

$$\left(\frac{1-\sigma^S}{\sigma^S}\right)^\rho = \frac{1-\delta(\sigma^S A)^{1-\rho}}{\beta\delta A^{1-\rho}(1-\sigma^S)^{1-\rho}}\,.$$

Thus, a sophisticated saving rate σ^S is a solution to the following equation in σ:

$$\beta\delta A^{1-\rho}\frac{1-\sigma}{\sigma^\rho} = 1-\delta(\sigma A)^{1-\rho}\,. \tag{D.4}$$

E Proof of Proposition 1 and Theorem 2

Consider intertemporal utility $U(\sigma, k)$ given by (9). It can be seen that

$$U'(\sigma) = \frac{(Ak)^{1-\rho}}{(1-\delta(\sigma A)^{1-\rho})\,(1-\sigma)^\rho}\cdot \left[\frac{1-\sigma}{\sigma^\rho}\frac{\beta\delta A^{1-\rho}}{1-\delta(\sigma A)^{1-\rho}} - \left(1-\delta(\sigma A)^{1-\rho}+\beta\delta(\sigma A)^{1-\rho}\right)\right].$$

It is shown in Lemma 1 that when $\rho > 1-\sqrt{\beta}$, there is a unique value σ^K such that $U'(\sigma^K) = 0$. Moreover, it is easily seen that $U'(\sigma) \to -\infty$ when $\sigma \to 1$, and $U'(\sigma) \to +\infty$ when $\sigma \to 0$ for $1-\sqrt{\beta} < \rho < 1$ or when $\sigma \to \underline{\sigma}$ for $\rho > 1$. Therefore, in both cases $U'(\sigma) > 0$ for all feasible $\sigma < \sigma^K$, and $U'(\sigma) < 0$ for all $\sigma > \sigma^K$, which ensures that σ^K provides the highest welfare for any given k.

Consider the value $U'(\sigma^S)$, where σ^S satisfies Eq. (D.4). We have

$$U'(\sigma^S) = \frac{(Ak)^{1-\rho}}{(1-\delta(\sigma^S A)^{1-\rho})\,(1-\sigma^S)^\rho}(1-\beta)\delta(\sigma^S A)^{1-\rho} > 0\,,$$

and hence $\sigma^K > \sigma^S$.

Consider the value $U'(\sigma^R)$, where $\sigma^R = (\delta A^{1-\rho})^{\frac{1}{\rho}}$ is the optimal saving rate in the standard Ramsey model. Since $\delta(\sigma^R A)^{1-\rho} = \sigma^R$, it is easily seen that

$$U'(\sigma^R) = -\frac{(Ak)^{1-\rho}}{(1-\sigma^R)^{1+\rho}}(1-\beta)(1-\sigma^R) < 0\,,$$

and hence $\sigma^R > \sigma^K$.

Let us show that σ^K is increasing in β. By the implicit function theorem,

$$\frac{\partial\sigma^K}{\partial\beta} = -\frac{\frac{\partial L}{\partial\beta}(\sigma^K) - \frac{\partial R}{\partial\beta}(\sigma^K)}{L'(\sigma^K) - R'(\sigma^K)},$$

where $L(\sigma)$ and $R(\sigma)$ are defined in Appendix C. Note that

$$\frac{\partial L}{\partial \beta}(\sigma^K) - \frac{\partial R}{\partial \beta}(\sigma^K) = \delta(\sigma^K A)^{1-\rho}\left(\frac{1-\sigma^K}{\sigma^K} - (1-\delta(\sigma^K A)^{1-\rho})\right) > 0,$$

because, as $U'(\sigma^K) = 0$, we have

$$\frac{1-\sigma^K}{\sigma^K}\frac{1}{1-\delta(\sigma^K A)^{1-\rho}} = 1 + \frac{1-\delta(\sigma^K A)^{1-\rho}}{\beta\delta(\sigma^K A)^{1-\rho}} > 1.$$

When $\rho \geq 1$, $L'(\sigma) < 0$ and $R'(\sigma) > 0$ for all σ. When $0 < \rho < 1$, it follows from (C.3) that $L'(\sigma^K) - R'(\sigma^K) < 0$. Thus, in both cases $\partial\sigma^K/\partial\beta > 0$.

Let us compare the naive saving rate σ^N and the sophisticated saving rate σ^S. Consider Eqs. (D.2) and (D.4). The left-hand sides of these equations are the same and are given by $L(\sigma)$. Let $R_N(\sigma) = (1-\sigma)^{1-\rho}\left(1-\sigma^R\right)^\rho$ be the right-hand side of Eq. (D.2), and let $R_S(\sigma) = 1 - \delta(\sigma A)^{1-\rho}$ be the right-hand side of Eq. (D.4). Note that $R_S(0) > R_N(0)$ when $0 < \rho < 1$, $R_N(0) > R_S(0)$ when $\rho > 1$, and $R_N(\sigma^R) = R_S(\sigma^R)$ for all ρ. Moreover, it is easily seen that

$$R'_N(\sigma) \;=\; (\rho-1)\left(1-\sigma^R\right)^\rho(1-\sigma)^{-\rho}, \qquad \text{and} \qquad R'_S(\sigma) \;=\; (\rho-1)\left(\sigma^R\right)^\rho(\sigma)^{-\rho}.$$

Consider the case $\rho > 1$. We have $R'_S(\sigma) > R'_N(\sigma)$ for all $\sigma < \sigma^R$, and $R'_S(\sigma) < R'_N(\sigma)$ for all $\sigma > \sigma^R$. Then $R_N(\sigma)$ is located above $R_S(\sigma)$ everywhere, and hence crosses $L(\sigma)$ at a smaller value of σ, implying that $\sigma^S > \sigma^N$. Consider the case $0 < \rho < 1$. Then $R'_N(\sigma) > R'_S(\sigma)$ for all $\sigma < \sigma^R$, and $R'_N(\sigma) < R'_S(\sigma)$ for all $\sigma > \sigma^R$. Therefore, $R_N(\sigma)$ is located below $R_S(\sigma)$ everywhere, and hence crosses $L(\sigma)$ at a greater value of σ, implying that $\sigma^N > \sigma^S$. It is also clear from (D.2) and (D.4) that when $\rho = 1$, $\sigma^N = \sigma^S$.

Finally, consider the value $U'(\sigma^N)$, where the naive saving rate σ^N satisfies Eq. (D.2). Using (D.3), we can rewrite equation $U'(\sigma^N) = 0$ as

$$\frac{1-\sigma^R}{\sigma^R} \;=\; \beta^{\frac{1-\rho}{\rho}}\frac{1-x^*}{x^*}\left(1-x^*+\beta x^*\right), \tag{E.1}$$

where we have denoted

$$x^* \;=\; \delta(\sigma^N A)^{1-\rho} \;=\; \beta^{\frac{1-\rho}{\rho}}\sigma^R/(1-\sigma^R+\beta^{\frac{1}{\rho}}\sigma^R)^{1-\rho}.$$

Recall that $U'(\sigma) > 0$ for all $\sigma < \sigma^K$. When $\rho > 1$ we have $\sigma^K > \sigma^S > \sigma^N$, and therefore, the higher is the saving rate, the higher is welfare. Thus, the Kantian path Pareto dominates the sophisticated path, that in turn Pareto dominates the naive path. When $\rho < 1$ and we have $\sigma^K > \sigma^N > \sigma^S$, then again the higher is the saving rate, the higher is welfare. Here the Kantian path Pareto dominates the naive path that in turn Pareto dominates the sophisticated path.

However, when $\rho < 1$ and $\sigma^N > \sigma^K > \sigma^S$, the situation is different. The Kantian path Pareto dominates the sophisticated path, though the Kantian and naive paths do not Pareto dominate each other. Indeed, it follows from (9) that $U(\sigma^K, k_0) > U(\sigma^N, k_0)$, but since $\sigma^N > \sigma^K$, there is a self $\underline{\tau}$ such that for all $\tau \geq \underline{\tau}$,

$$U(\sigma^K, k_\tau^K) \; = \; U(\sigma^K, (\sigma^K A)^\tau k_0) \; < \; U(\sigma^N, (\sigma^N A)^\tau k_0) \; = \; U(\sigma^N, k_\tau^N).$$