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Wendner, Ronald

University of Graz, Department of Economics

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Ronald Wendner*

Department of Economics, University of Graz, Austria
Universitaetsstrasse 15, RESOWI-F4,
A-8010 Graz, Austria

E-mail	ronald.wendner@uni–graz.at
Web	http://www.uni-graz.at/ronald.wendner
Phone	+43 316 380 3458

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Abstract. This paper investigates the effects of ("keeping up with the Joneses" and "learning-by-investing") externalities, when labor productivity decreases with age. Within the framework of a continuous time overlapping generations model, the effects of the consumption externality on the propensity to consume, capital level and individual consumption growth rates are ambiguous and depend on the presence (absence) and sign of the "generation replacement effect" (GRE). The sign of the GRE is determined by the rate at which labor productivity declines.
Both externalities generate distortions — even with exogenous labor supply. Depending on the sign of the GRE, in case of a production externality, the consumption externality may raise efficiency by introducing an additional distortion. For a specific rate of labor productivity decline the GRE vanishes. In this case, externalities display the same effects in both a representative agent and the overlapping generations model.

Keywords and Phrases: Externality, labor productivity, overlapping generations, perpetual youth, distortion, growth.

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1 Introduction

This paper investigates the impact of consumption and production externalities on consumption, savings and capital levels. In contrast to the previous literature, this paper considers a continuous time overlapping generations model (perpetual youth model), in which individual labor productivity decreases with age. The effects of the externalities (if any) critically depend on the rate of labor productivity decline.

In the previous literature, there is strong evidence of the *existence* of consumption and production externalities. Important contributions include Brekke and Howarth (2002), Frank (1999), Johansson-Stenman et al. (2002, 2006), Luttmer (2005), and Solnick and Hemenway (1998, 2005).[1] The theoretical implications of externalities on savings and growth, however, are less clear than one would hope. It seems reasonable to assume that a negative consumption externality gives rise to a higher consumption level, as demonstrated, in a static framework, by Dupor and Liu (2003). In a dynamic context, however, a higher consumption level in the present is associated with dissaving and a lower consumption level in the future, thereby requiring a household to trade off the benefit from a higher consumption level today with the cost from lower consumption levels (also from lower consumption levels *relative to others*, ceteris paribus) tomorrow. If the said benefit falls short of the cost, the negative consumption externality will induce a household to shift consumption from the present towards the future, thereby lowering the propensity to consume. In case the benefit equals the cost, a household will not react to the consumption externality at all. Only if the benefit exceeds the cost will households raise the propensity to consume, thereby ending up with lower consumption and capital levels.

Two questions suggest themselves. First, under which conditions do (consumption) externalities change household behavior in a market equilibrium, and does it raise or lower the propensity to consume? Second, if households change behavior, will the

[1] They present significant evidence for negative consumption externalities. Further important references include Alpizar et al. (2005), Carlsson et al. (2007), Easterlin (1995), Ferrer-i-Carbonell (2005), McBride (2001), Neumark et al. (1998). Production externalities are discussed by Caballero and Lyons (1990, 1992), and Lindstrom (2000).

response be distortionary? This paper addresses these questions in the context of a model of perpetual youth (PY model, in the following) in which labor supply is exogenous. The consumption externality considered is a "keeping up with the Joneses" externality, and the production externality is a "learning-by-investing" externality.

If the (human and accumulated) wealth of a new generation deviates from average wealth, the PY model will exhibit a generation replacement effect (GRE), as a consequence of the continuous inflow of new generations. The presence of a GRE opens a channel for the externality to have an impact on consumption and capital, as the steady state capital stock is not determined anymore by the Keynes-Ramsey rule, but by a modified rule which takes the GRE into account. This observation is important for explaining the results of this paper. In particular, in the absence of a GRE, a consumption externality has no effects on behavior at all. In this case, the marginal benefit of increasing consumption today will exactly equal the marginal cost of a lower (relative) consumption tomorrow. In the presence of a GRE, however, the consumption externality affects consumption and savings. In particular, if the wealth of new generations is below-average, the consumption externality leads to a rise in the propensity to consume, thereby to lower steady state levels of average consumption and capital. While this argument might be considered as intuitive, the opposite holds, if the wealth of new generations is above-average — a possibility opened up by a high rate of labor productivity decline. In this case, a household's wealth declines with age, and the consumption externality induces it to lower the propensity to consume. In addition, the paper offers three other main results.

First, the production externality always raises consumption and savings. However, it introduces a distortion, as market prices fail to reflect the social value of capital accumulation, thus capital rises by less than optimal. As a consequence, if the consumption externality tends to raise capital — in case labor productivity declines with age — the consumption externality raises efficiency by introducing an additional distortion.

Second, the impact of a consumption externality on the individual (steady state) consumption growth rate is ambiguous as well. If the generation replacement effect is absent, the consumption externality (alone) has no effect on individual consumption growth rates. Otherwise, the impact of a consumption externality depends on whether or not the rate at which human capital depreciates is smaller than the rate at which accumulated capital depreciates (the difference of which introduces a wealth effect that is not present in a representative agent model).

Third, all effects displayed in the analysis depend on the perpetual inflow of new generations rather than on their finite lifetime. That is, the results also hold in the setting of overlapping families of infinitely-lived agents, as in Weil (1989).

These results are related as follows to the prior literature. Rauscher (1997) and Brekke and Howarth (2002) discuss the impact of consumption externalities on consumption choice and growth and show that, with exogenous labor supply, a consumption externality has no steady state effects in a standard Ramsey model. Arrow and Dasgupta (2007) identify the class of preferences for which, in a representative agent economy, a consumption externality does *not* introduce a distortion,[2] and conclude that "the intuition that conspicuous consumption inevitably leads to a problem of 'the commons' is mistaken." (Arrow and Dasgupta 2007, p.2) Along the same vein, Liu and Turnovsky (2005) and Turnovsky and Monteiro (2007) demonstrate that in a representative agent economy with exogenous labor supply, without a concurrent production externality, a consumption externality alone does *not* introduce a distortion. In addition, a consumption externality has no effect on the steady state equilibrium. In their setting, the (steady state) capital stock is determined by the Keynes-Ramsey rule in both the market economy and the command optimum. Consequently the market economy's capital stock assumes its optimal level, and so must — in spite of a consumption externality — the market's consumption level. In contrast to these papers, Abel (2005) considers an overlapping generations economy, in which each generation is alive for two periods. He shows that in his framework, the equilibrium rate of return depends on the consumption externality. Similarly, Fisher and Heijdra

[2]This class includes, among others, CRRA utility functions with both subtractive and multiplicative specifications of reference consumption.

(2008) consider consumption externalities and show that steady state consumption and capital levels decline as a consequence of initial overconsumption due to the consumption externality. This result is in contrast with the observations reported for representative agent economies, above.[3]

I will argue below that the results discussed above critically depend on the nature of the GRE, thereby on the rate at which labor productivity declines.

Section 2 of the paper presents the economy's structure of both the market economy framework and a command optimum. Section 3 considers the steady state effects of both consumption and production externalities on the propensity to consume, consumption and capital levels and on individual consumption growth rates. Section 4 concludes the paper. The appendix contains a number of derivations that were distracting when placed in the main text.

2 The Economy's Structure

In this section, we extend the continuous time overlapping generations model by consideration of consumption and production externalities. In light of the evidence referred to in the introduction, we focus on a "keeping (catching) up with the Joneses" consumption externality, and a "learning-by-investing" production externality.

Population. An individual born at time v ("vintage") is uncertain about the length of his or her life. As in Blanchard (1985), both the instantaneous probability of death of a cohort (the death or mortality rate), d, and the birth rate, b, are age-independent and constant over time and cohorts. Birth and death rates are independent of each other, and they may be different.

At time t, the population size is $L(t)$. At each instant of time, a new cohort is born, the size of which is $b\,L(t)$. Also, at time t, the mass of people who die is $d\,L(t)$.

[3] In this paper, technology is represented by a neoclassical production function. In an endogenous growth framework, Carroll et al. (1997) and Corneo and Jeanne (1997) demonstrate that the steady state growth rates of consumption, capital and output increase in the consumption externality.

Accordingly, for a large population size, the rate of population growth is

$$n = b - d\,. \tag{1}$$

Population at some date t_1 is given by: $L(t_1) = L(t_0)\,e^{n\,(t1-t0)}$. Without loss of generality, $L(0) = 1$. Consequently, $L(t) = e^{n\,t}$.

Denote the size of a vintage-v cohort at time t by $L(v,t)$. Under this population structure $L(v,t) = L(v,v)\,e^{-d\,(t-v)} = b\,L(v)\,e^{-d\,(t-v)} = b\,e^{n\,v}\,e^{-d\,(t-v)} = b\,e^{b\,v-d\,t}$. Similarly, the share of a vintage-v cohort in total population at time t is:

$$l(v,t) \equiv L(v,t)/L(t) = [b\,e^{b\,v}\,e^{-d\,t}]/e^{(b-d)\,t} = b\,e^{-b\,(t-v)}\,. \tag{2}$$

The expected remaining lifetime of any agent is: d^{-1}. As a special case, the representative-agent model emerges from the PY model by setting $b = 0$, and $d = -n$.

In the following, we focus on the case without population growth:

$$n = 0 \Leftrightarrow b = d \Rightarrow L(t) = 1\,. \tag{A.1}$$

As we conceptually distinguish the birth rate from the death rate, we will be able to distinguish their respective effects. In particular, we will clarify which of the results are driven by the constant inflow of cohorts ($b > 0$) and which are driven by the finiteness of lifetime ($d > 0$).

Households. Time-t utility of a vintage-v household is a function $u(.)$ of consumption $c(v,t)$. The first argument in $c(.)$ refers to the birth date ("vintage"), and the second argument refers to time. At time t, an individual household not only cares about its *own* consumption, but also about how own consumption compares to some reference consumption level $x(t)$. Instantaneous utility is given by $u(c(v,t), x(t))$.

In this paper, we consider the standard case of a CRRA utility function. We follow

Dupor and Liu (2003) in specifying the felicity function[4] as:

$$u(c(v,t),x(t)) = \frac{\left[c(v,t)^{\frac{1}{1-\gamma}}\,x(t)^{-\frac{\gamma}{1-\gamma}}\right]^{1-\sigma}-1}{1-\sigma} = \frac{\left[c(v,t)\,\left(\frac{c(v,t)}{x(t)}\right)^{\frac{\gamma}{1-\gamma}}\right]^{1-\sigma}-1}{1-\sigma}\,, \quad (3)$$

where γ is called the "reference parameter." It, if strictly positive, introduces a "keeping up with the Joneses" externality. That is, it measures the importance of the reference consumption level. In particular, the reference parameter can be interpreted as "marginal degree of positionality."[5] If $\gamma = 0$, then the model reduces to the usual model with interpersonally separable utility.

Parameter σ governs the intertemporal elasticity of substitution. If $\gamma = 0$, the intertemporal elasticity of substitution is given by σ^{-1}. If, however, $\gamma > 0$, both parameters, σ and γ, determine the elasticity of substitution between consumption at any two points in time. As is overwhelmingly suggested by the empirical literature, σ is considered to be larger than unity.

Sign Restrictions. We impose the following sign restrictions:

$$\sigma > 1\,, \quad (A.2)$$

$$0 \leq \gamma < 1\,. \quad (A.3)$$

In the taxonomy of Dupor and Liu (2003), felicity function (3) exhibits the "keeping up with the Joneses" property, when $u_{cx}(.) > 0 \Leftrightarrow \gamma(\sigma-1) > 0$, which is implied by (A.2) and (A.3), for any $\gamma > 0$. Positivity of marginal utility of a proportionate increase in both individual consumption and the reference stock is always satisfied for felicity function (3). Moreover, to keep the economy from explosive growth, marginal

[4]As demonstrated by Dupor and Liu (2003), felicity function (3) represents a special case of the more general specification: $u(c(v,t),x(t)) = \frac{1}{1-\sigma}\left\{\left[\frac{c(v,t)^{\varepsilon}-\gamma\,x(t)^{\varepsilon}}{1-\gamma}\right]^{\frac{1}{\varepsilon}}\right\}^{1-\sigma}$. In the limit, as ε approaches zero, felicity function (3) emerges. This case is often referred to as the "multiplicative case" (Galí, 1994). Some authors consider an alternative special case: the limit, as ε approaches unity. This case is referred to as the linear, or subtractive specification (Campbell and Cochrane 1999, Ljungqvist and Uhlig 2000).

[5]According to Johansson-Stenman et al. (2002), the marginal degree of positionality is the fraction of marginal utility of consumption stemming from a rise in own consumption relative to the reference stock. Define $r = c/x$, and $\tilde{u}(c,r) \equiv u(c,x)$. Then $\gamma = [\partial\,\tilde{u}/\partial\,r\;\partial\,r/\partial\,c]/[(\partial\,\tilde{u}/\partial\,c)+(\partial\,\tilde{u}/\partial\,r\;\partial\,r/\partial\,c)]$.

utility of a proportionate increase of both individual consumption and the reference stock is required to decline: $u_{cc}(.) + u_{cx}|_{x=c} < 0 \Leftrightarrow -\sigma(1-\gamma) < 0$, which is ensured by the fact that $\gamma < 1$ in (A.3).

It is worth noting the impact of the reference parameter on the consumption elasticity of marginal utility.

Lemma 1 *Holding $x(t)$ fixed, a rise in γ raises the elasticity of marginal utility with respect to consumption, $c(v,t)$.*

Proof. The (absolute) elasticity of marginal utility with respect to consumption is defined by: $-u_{cc}(.)\, c/u_c(.) = (\sigma-\gamma)/(1-\gamma)$. Clearly, $\partial\,[(\sigma-\gamma)/(1-\gamma)]/(\partial\,\gamma) = (\sigma-1)/(1-\gamma)^2$, which is strictly positive by (A.2) and (A.3). ||

To interpret Lemma 1, consider felicity function (3), and let $c(v,t)^{1/(1-\gamma)}\, x(t)^{-\gamma/(1-\gamma)}$ be *effective consumption*. The elasticity of effective consumption with respect to $c(v,t)$ (i) exceeds unity, and (ii) rises in γ. That is, given $x(t)$, an increase in $c(v,t)$ by one percentage point raises effective consumption by more than one percentage point, as a household not only enjoys the higher level of own consumption but also the higher ratio of its individual consumption level relative to the reference level. Furthermore, an increase in $c(v,t)$ raises effective consumption by the more the higher is γ. Therefore, a given increase in $c(v,t)$ by one percentage point lowers marginal utility of consumption the more the higher is γ. Consequently, the (absolute) elasticity of marginal utility with respect to consumption rises in γ.

Lemma 1 holds not only for the "multiplicative" specification, employed in (3), but also for the more general specification suggested in footnote 4.

At time t, expected remaining lifetime utility of a cohort born at date v is:

$$U(v,t) = \int_t^\infty u(c(v,\tau), x(\tau))\, e^{-(\rho+d)\,(\tau-t)}\, d\tau\,, \tag{4}$$

where ρ is the household's pure rate of time preference. The possibility of death ($d > 0$) leads to a subjective discount rate $(\rho + d)$ higher than the pure rate of time preference.

Reference Consumption. Reference consumption is either equal to average consumption or is affected by average consumption over time.[6] Average consumption is defined by:

$$c(t) \equiv \int_{-\infty}^{t} l(v,t)\, c(v,t)\, dv = b \int_{-\infty}^{t} e^{-b\,(t-v)}\, c(v,t)\, dv\,. \tag{5}$$

The reference consumption stock develops according to:

$$\dot{x}(t) = \varphi\, \left(c(t) - x(t)\right)\,, \quad x(0) = x_0\,, \tag{6}$$

with x_0 exogenously given. The reference level $x(t)$ is a weighted average of past *average* consumptions. The parameter $\varphi \geq 0$ determines the weight of average consumption at different times. The larger is φ, the more important is average consumption in the *recent* past. As $\varphi \to \infty$, $x(t) \to c(t)$, in which case the reference level is given by current average consumption.

Production. There is a large number of competitive, identical firms. Firm i produces a homogeneous output, Y_i, according to

$$Y_i(t) = A\, K_i(t)^{\alpha}\, (E(t)\, N_i(t))^{1-\alpha}\,, \quad A > 0,\, 0 < \alpha < 1\,, \tag{7}$$

where K_i is capital, N_i are "effective" labor services of firm i, A is total factor productivity, and E — which is considered exogenously given by individual firms — represents a measure for "aggregate labor productivity."

Define the (aggregate) capital stock $K(t) = \sum_i K_i(t)$. Labor productivity is given by:

$$E(t) = K(t)^{p}\,, \quad -\alpha/(1-\alpha) < p < 1\,, \tag{8}$$

where, p is the learning parameter. If $p > 0$, a higher level of K raises labor productivity.[7] As in Romer (1986), the aggregate capital stock serves as a proxy for knowledge,

[6]Consider an aggregate variable, $Z(t)$. Generally, we define an arbitrary *average* variable $z(t) \equiv Z(t)/L(t)$ by: $z(t) \equiv \int_{-\infty}^{t} l(v,t)\, z(v,t)\, dv = b \int_{-\infty}^{t} e^{-b\,(t-v)}\, z(v,t)\, dv$.

[7]More generally, $E(t) = [K(t)\, L(t)^{\psi-1}]^{p}$, where ψ is the scale parameter. If $\psi = 0$, labor productivity rises in the economy-wide capital labor ratio. In this case, no scale effects are present, and neither the productivity level nor the growth rates depend on the size of the labor force. Here, we consider $\psi = 1$ (full scale).

which gives rise to a learning-by-investing externality. The magnitude of the production externality is determined by p. The inequalities in (8) ensure positivity of the social marginal product of capital (in case p is negative) and diminishing marginal products of capital.

Any *individual* laborer's productivity depends on her age. In particular, age-dependent productivity, $\pi(t-v)$, develops according to

$$\pi(t-v) = e^{-\lambda\,(t-v)}\,, \quad 0 \leq \lambda < \rho\,\frac{1-\gamma}{\sigma-\gamma}\,, \tag{9}$$

where λ measures the rate at which individual labor productivity declines with age. This formulation can be extended to accommodate more complex paths of productivity profiles (e.g., an inverse U-shaped pattern). In the following, however, the main purpose of considering nonconstant productivity paths is to make the present value of future wage payments dependent on the profile of individual labor productivity. The simplest formulation of which is given by (9).[8]

Accordingly, labor supply, $L(t)$, and effective labor supply, $N(t)$, are related as follows:

$$N(v,t) = \pi(t-v)L(v,t) \Rightarrow N(t) = \frac{b}{b+\lambda}\,L(t)\,, \tag{10}$$

that is, aggregate effective labor supply declines in λ.

Defining aggregate production $Y(t) \equiv \sum_i Y_i(t) = A\,K(t)^{\alpha+p(1-\alpha)}\,[\frac{b}{b+\lambda}\,L(t)]^{1-\alpha}$, average products $y(t) \equiv Y(t)/L(t)$, $y_i(t) \equiv Y_i(t)/L_i(t)$, and capital-labor ratios $k_i(t) \equiv K_i(t)/L_i(t)$, $k(t) \equiv K(t)/L(t)$, individual and economy-wide average products are:

$$\begin{aligned} y_i(t) &= A\,k_i(t)^{\alpha}\,K(t)^{p(1-\alpha)}\,\left[\frac{b}{b+\lambda}\right]^{1-\alpha} \\ &= A\,k_i(t)^{\alpha}\,k(t)^{p(1-\alpha)}\,\left[\frac{b}{\lambda+b}\right]^{1-\alpha}\,, \end{aligned} \tag{11}$$

$$y(t) = A\,k(t)^{\alpha+p(1-\alpha)}\,\left[\frac{b}{\lambda+b}\right]^{1-\alpha}\,, \tag{12}$$

where we use the fact that in equilibrium, $k_i = k$, and $L_i = L = 1$.

[8] An alternative interpretation is that λ represents the "retirement rate." According to this interpretation, λ serves as a measure for exogenous gradual retirement. The upper bound on λ is a necessary condition for the existence of a nontrivial steady state equilibrium, as discussed below.

The learning parameter p, if strictly positive, introduces a (positive) learning-by-investing externality.

Firms maximize profits and hire factors from households on competitive factor markets:

$$r(t) + \delta = \alpha\, A\; k(t)^{(\alpha-1)(1-p)} \left[\frac{b}{b+\lambda}\right]^{1-\alpha}, \tag{13}$$

$$w(v,t) = (1-\alpha)\, A\, k(t)^{\alpha+p(1-\alpha)} \left[\frac{b}{b+\lambda}\right]^{-\alpha} e^{-\lambda\,(t-v)} = w(t)\, e^{-\lambda(t-v)}\,, \tag{14}$$

where $r(t)$ is the rate of interest, $w(v,t)$ is the wage rate, $w(t)$ is the wage rate per "efficiency unit" of labor, and δ is the rate of depreciation of capital. On an aggregate (economy-wide) level, the interest rate is obtained by:

$$\tilde{r}(t) = [\alpha + p(1-\alpha)]\, A \left[\frac{b}{b+\lambda}\right]^{1-\alpha} k(t)^{(\alpha-1)(1-p)} - \delta\,. \tag{15}$$

The rate $\tilde{r}(t)$ corresponds to the *optimal* rate of interest a social planner — taking into account the production externality — considers. With $p = 0$, the market interest rate coincides with the optimal one. With $p > 0$, however, the optimal rate of interest exceeds the market one, for a given k.

According to the resource constraint, the average stock of capital evolves according to:

$$\dot{k}(t) = y(t) - c(t) - \delta\, k(t)\,, \tag{16}$$

where $y(t)$ is negatively affected by the productivity parameter λ, as seen in (12).

2.1 The Market Economy

Households do not have a bequest motive. However, they can buy fair life annuity contracts from life insurance companies, for which they pay or receive the annuity rate of interest $r^A(t)$. The contracts are canceled upon death of an individual. Actuarial fairness requires $r^A(t) = r(t) + d$. Thus, the annuity interest factor is given by: $R^A(t,\,\tau) \equiv \int_t^\tau [r(s) + d]\, d\, s$.

Every household inelastically supplies labor services and chooses consumption at all $t \geq v$ such as to maximize expected lifetime utility (4) subject to its intertemporal budget constraint:[9]

$$a(v,t) + h(v,t) - \int_t^\infty c(v,\tau)\, e^{-R^A(t,\tau)}\, d\tau = 0\,, \tag{17}$$

where $a(v,t)$ stands for time-t assets (accumulated wealth) of a vintage-v household, and human wealth $h(v,t) \equiv \int_t^\infty w(v,\tau)\, e^{-R^A(t,\tau)} d\tau$ is the discounted integral of future wage payments. In the market framework, a household does *not* consider the impact of its individual consumption on average consumption or on the consumption reference level.

Individual consumption levels are derived by applying Pontryagin's maximum principle, in the appendix. Define:[10]

$$\Delta(t) \equiv \int_t^\infty e^{(\sigma-\gamma)^{-1} \int_t^\tau [(r(s)-\rho)(1-\gamma)+\gamma\,(\sigma-1) g_x(s)]\, ds - R^A(t,\tau)} d\,\tau\,. \tag{18}$$

Then:

$$c(v,t) = \Delta^{-1}\, [a(v,t) + h(v,t)]\,, \quad c(t,t) = \Delta^{-1}\, h(t,t)\,, \tag{19}$$

where the second expression follows from the fact that there is no operative bequest motive: $a(t,t) = 0$. Consumption levels are proportional to (accumulated and human) wealth, with the age-independent factor of proportionality given by: $\Delta^{-1}(t)$, which can be interpreted as the propensity to consume out of total wealth. Notice that consumption levels are *not* equal across cohorts, as demonstrated below.

Individual consumption growth rates are given by:

$$g_v(t) \equiv \frac{\dot{c}(v,t)}{c(v,t)} = \frac{[r(t)-\rho](1-\gamma)}{(\sigma-\gamma)} + \gamma\, \frac{\sigma-1}{(\sigma-\gamma)}\, \frac{\dot{x}(t)}{x(t)}\,. \tag{20}$$

The derivation is shown in the appendix. Consumption growth rates are equal across cohorts. Individual consumption growth rates do not only depend on the rate of interest and the pure rate of time preference, as suggested by the standard Keynes-Ramsey rule, but they also depend *positively* on the growth rate of the consumption reference stock.

[9] The transversality condition required to prevent households from running Ponzi schemes is: $\lim_{\tau\to\infty} e^{-R^A(t,\tau)}\, a(v,\tau) = 0$, or, equivalently, $\lim_{\tau\to\infty} \mu_a(\tau)\, e^{-(\rho+d)\tau}\, a(v,\tau) = 0$, where μ_a is the shadow price of accumulated wealth. Budget constraint (17) follows from combining the flow budget constraint, $\dot{a}(v,t) = r^A(t)\, a(v,t) + w(v,t) - c(v,t)$, with the transversality condition.

[10] Throughout the paper, g_z denotes the growth rate of some variable z.

Average accumulated wealth, $a(t)$, is given by $a(t) \equiv \int_{-\infty}^{t} l(v,t)\, a(v,t)\, dv$. Capital market clearing requires:

$$a(t) = k(t)\,. \tag{21}$$

Finally, average human wealth, $h(t)$, is:

$$h(t) = \int_{-\infty}^{t} l(v,t)\, h(v,t)\, dv = \frac{b}{b+\lambda}\, h(t,t) = \frac{b}{b+\lambda} \int_{t}^{\infty} w(\tau)\, e^{-R^A(t,\tau)}\, e^{-\lambda(\tau-t)}\, d\tau\,. \tag{22}$$

We are now ready to represent a perfect foresight equilibrium by a series of four differential equations in the variables c, x, k, Δ:

$$\frac{\dot{c}(t)}{c(t)} = \frac{[r(t)-\rho](1-\gamma)}{(\sigma-\gamma)} + \gamma\, \frac{\sigma-1}{(\sigma-\gamma)}\, \frac{\dot{x}(t)}{x(t)} + \lambda - (b+\lambda)\, \Delta^{-1}(t)\, \frac{k(t)}{c(t)}\,, \tag{M.1}$$

$$\dot{x}(t) = \varphi[c(t) - x(t)]\,, \tag{M.2}$$

$$\dot{k}(t) = A\left[\frac{b}{b+\lambda}\right]^{1-\alpha} k(t)^{\alpha+p(1-\alpha)} - c(t) - \delta\, k(t)\,, \tag{M.3}$$

$$\dot{\Delta}(t) = -1 + \Delta(t)\left[\zeta(t) - \gamma\, \frac{(\sigma-1)}{\sigma-\gamma}\, \frac{\dot{x}(t)}{x(t)}\right]\,, \tag{M.4}$$

where $\zeta(t) \equiv r(t) + d - \frac{(r(t)-\rho)(1-\gamma)}{\sigma-\gamma}$, and $r(t) = \alpha\, A\, \left[\frac{b}{b+\lambda}\right]^{1-\alpha} k(t)^{(\alpha-1)(1-p)} - \delta$. Equation (M.1) is derived by differentiation of average consumption, as given in (5), with respect to time, and using (20).[11] Equation (M.3) restates the resource constraint. Equation (M.4) follows right from differentiation of (18).

If $b = \lambda = p = d = 0$, the standard Ramsey model emerges. In a steady state – if $\dot{c} = 0$ – equation (M.1) represents the Keynes-Ramsey rule: $r = \rho$. If, however, $(b, \lambda) \gg 0$, there is a continuous inflow of new cohorts without accumulated wealth ($b > 0$), and there is a continuous decay of human wealth of any existing cohort over time ($\lambda > 0$). As a consequence, the (total) wealth of a new cohort may differ from the average wealth, which gives rise to the generation replacement effect.

Generation Replacement Effect. The Generation Replacement Effect (GRE) refers

[11] Denote individual consumption growth by $g_v(t)$. Here, we consider the fact $\dot{c}(t) = b\, c(t,t) - b\, c(t) + g_v(t)\, c(t)$, where $[b\, c(t,t) - b\, c(t)] = b\, \Delta^{-1}\, h(t,t) - b\, c(t) = b\, \Delta^{-1}\, (b+\lambda)/b\, h(t) - b\, c(t)$. Considering both $\Delta^{-1}\, h(t) = [c(t) - \Delta^{-1}\, k(t)]$ and $\Delta^{-1}(t) = c(t)/[k(t)+h(t)]$, yields: $b\, c(t,t) - b\, c(t) = \lambda\, c(t) - (b+\lambda)\, c(t)\, k(t)/[k(t)+h(t)]$. Equation (M.1) follows.

to the difference between average and individual consumption growth rates, due to $c(v,t) \neq c(t)$. Analytically, the GRE is determined by:

Definition 1 $\Gamma(t) \equiv \lambda - (b+\lambda)\,\frac{k(t)}{k(t)+h(t)} = \lambda - (b+\lambda)\,\Delta^{-1}(t)\,\frac{k(t)}{c(t)}$.

Employing Definition 1,

$$g_c(t) \equiv \frac{\dot{c}(t)}{c(t)} = g_v(t) + \Gamma(t)\,. \tag{23}$$

In particular, $g_c(t) \gtreqless g_v(t) \Leftrightarrow \Gamma(t) \gtreqless 0$.

Lemma 2 (Generation Replacement Effect)
(i) $h(t,t) = a(t) + h(t)$ *or* $b = 0 \Leftrightarrow \Gamma(t) = 0$.
(ii) $b > 0$ *and* $\lambda = 0 \Rightarrow \Gamma(t) < 0 \Leftrightarrow g_c(t) < g_v(t)$.
(iii) Suppose $b > 0$. *Then there exists* $\tilde{\lambda}(t) > 0$*, such that* $\Gamma(t) = 0$*, where* $\tilde{\lambda}(t) = b/[h(t)/k(t)]$. *For* $\lambda \lesseqgtr \tilde{\lambda}(t) \Leftrightarrow \Gamma(t) \lesseqgtr 0$.

Proof. (i) Restate the average consumption growth rate as:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{c}(v,t)}{c(v,t)} - b\,\frac{c(t)-c(t,t)}{c(t)}\,,$$

where we consider the fact $\dot{c}(t) = b\,c(t,t) - b\,c(t) + g_v(t)\,c(t)$. Taking (23) into account: $\Gamma(t) = b\,[c(t,t) - c(t)]/c(t)$, that is, $\Gamma(t) = 0 \Leftarrow c(t,t) = c(t) \Leftrightarrow \Delta^{-1}\,h(t,t) = \Delta^{-1}\,[h(t) + a(t)] \Leftrightarrow h(t,t) = h(t) + a(t)$. Also, $\Gamma(t) = 0 \Leftarrow b = 0$. |
(ii) If $b > 0$ and $\lambda = 0$, $h(t,t) = h(t) < h(t) + a(t) \Leftrightarrow c(t,t) < c(t) \Leftrightarrow \Gamma(t) < 0$, based on the fact that $h(t,t) = (b+\lambda)/b\,h(t)$. |
(iii) From the definition of $\Gamma(t)$, it directly follows that $\tilde{\lambda}(t) = b/[h(t)/k(t)]$. Ceteris paribus, $\partial\,\Gamma(t)/(\partial\,\lambda) = h(t)/[h(t) + k(t)] > 0$. As shown below, λ raises Γ in a steady state. ||

Lemma 1 shows that in the PY economy, the average consumption growth rate may differ from the individual consumption growth rate for two reasons. First, with $b > 0$, newborn cohorts *without accumulated wealth* continuously enter the economy. Second, with $b > 0$ and $\lambda > \tilde{\lambda} > 0$, the newborn cohorts enter the economy with an above-average level of human wealth: $h(t,t) = h(t)\,(b+\lambda)/b > h(t)$. Thus a newborn cohort's total (accumulated and human) wealth may be smaller than or larger than average total wealth. Henceforth, $c(t,t) < c(t)$ in the former case, and $c(t,t) > c(t)$ in the latter case.

At the same time, individual consumption growth rates are independent of age and equal among all cohorts at any given point in time. By the very fact that newborn cohorts — with consumption levels different from the average consumption levels — continuously enter the economy, the average consumption growth rate differs from the individual consumption growth rate. Only in the special case in which $h(t,t) = h(t) + a(t)$, the two opposing effects discussed above exactly cancel each other, and no GRE occurs: $\Gamma(t) = 0$, and $g_c(t) = g_v(t)$.

Except for the special case in which $h(t,t) = h(t)+a(t)$, $b > 0$ is not only necessary but also sufficient for the GRE to occur. In particular, the GRE does not occur in a representative agent model where $b = 0$. However, the GRE occurs even in the setting of a PY model with infinitely-lived agents, that is, with $b > 0$, $d = 0$, as in Weil (1989). If, however, $b > 0$, a positive death rate reinforces the GRE. A rise in d raises the propensity to consume out of total wealth, Δ^{-1}, thereby raising the difference between a newborn cohort's consumption and the average consumption level. The latter, in turn, raises the difference between individual and average consumption growth rates.

2.2 The Planner's Objective

To be able to address the distortionary impact of externalities, we define a command optimum here. In analogy to Calvo and Obstfeld (1988), the time-consistent utilitarian social welfare function must take the form:

$$W(t) = \int_{-\infty}^{t} L(v,t)\, U(v,t)\, e^{-\rho(t-v)}\, e^{-\tilde{\rho}(v-t)}\, d\,v \;+ \int_{t}^{\infty} L(v,v)\, U(v,v)\, e^{-\tilde{\rho}(v-t)}\, d\,v \ .$$

The planner's objective, at time t, is the sum of two components. First, the integral of the expected remaining lifetime utilities of all cohorts alive at time t, measured from the perspective of his and her birthdate.[12] Second, the integral of the lifetime expected

[12] The factor $e^{-\rho(t-v)}$ discounts time-t-lifetime utility back to the birthdate. Clearly, once a cohort has survived up to time t, the appropriate discount rate is ρ, not $(\rho + d)$. As shown by Calvo and Obstfeld (1988), discounting back to birthdates ensures time-consistency.

utilities of each of the generations to be born, as measured from the respective time of birth. The planner's discount rate, $\tilde{\rho}$, needs not equal an individual's pure rate of time preference. In the following analysis, we assume both discount rates to coincide.

Consider $L(v,t) = b\, e^{b\,v - d\,t}$, $L(v,v) = b\, e^{n\,v}$, and change the order of integration:

$$W(t) = \int_t^\infty \left\{ \int_{-\infty}^\tau b\, u[c(v,\tau), x(\tau)]\, e^{[-(\rho+d)+(\tilde{\rho}-n)](\tau-v)+n\,t}\, d\,v \right\} e^{-(\tilde{\rho}-n)(\tau-t)}\, d\,\tau\,.$$

Finally, consider $\tilde{\rho} = \rho$, and $n = b - d = 0$. The social welfare function then becomes:

$$W(t) = \int_t^\infty \left\{ \int_{-\infty}^\tau b\, u[c(v,\tau), x(\tau)]\, e^{-b\,(\tau-v)}\, d\,v \right\} e^{-\rho\,(\tau-t)}\, d\,\tau\,. \tag{24}$$

The planning problem consists of maximizing (24) by choosing $c(v,\tau)$ and $c(\tau)$ subject to:

(i) $c(\tau) = b\int_{-\infty}^\tau e^{-b\,(\tau-v)}\, c(v,\tau)\, d\,v\,,$
(ii) $\dot{x}(\tau) = \varphi[c(\tau) - x(\tau)],$
(iii) $\dot{k}(\tau) = A\,\left[\frac{b}{b+\lambda}\right]^{1-\alpha} k(\tau)^{\alpha+p(1-\alpha)} - c(\tau) - \delta\, k(\tau)\,,$

where the planner considers the impact of consumption on the reference level. The costate variables associated with x and k are: μ_x and μ_k respectively. Define $\mu \equiv \mu_x/\mu_k$. The (constrained) optimal control problem, which is discussed in the appendix, yields four differential equations in the variables c, x, k, μ that represent aggregate behavior in the command optimum:

$$\frac{\dot{c}(t)}{c(t)} = \frac{(\tilde{r}(t)-\rho)(1-\gamma)}{(\sigma-\gamma)}\,\frac{1}{\mu(t)}\,\left[\frac{1-\mu(t)\,\varphi}{\mu(t)}\right]^{-1} + \gamma\,\frac{\sigma-1}{(\sigma-\gamma)}\,\frac{\dot{x}(t)}{x(t)}$$

$$+\,\frac{1-\gamma}{\sigma-\gamma}\,\varphi\left[-(\tilde{r}-\rho)+\frac{\dot{\mu}(t)}{\mu(t)}\right]\,\left[\frac{1-\mu(t)\,\varphi}{\mu(t)}\right]^{-1}, \tag{SO.1}$$

$$\dot{x}(t) = \varphi[c(t) - x(t)]\,, \tag{SO.2}$$

$$\dot{k}(t) = A\,\left[\frac{b}{b+\lambda}\right]^{1-\alpha} k(t)^{\alpha+p(1-\alpha)} - c(t) - \delta\, k(t)\,, \tag{SO.3}$$

$$\dot{\mu}(t) = \mu(t)\,\left[\tilde{r}(t) + \varphi + \gamma\,\frac{c(t)}{x(t)}\,\frac{1-\mu(t)\,\varphi}{\mu(t)}\right]\,. \tag{SO.4}$$

Two observations are of particular importance. First, from (29) in the appendix, for any given τ, $\partial\, H_c/\partial\, c(v,\tau) = 0$ yields $u_c(.) = \mu_c(\tau)$, thus:

$$u_{cc}[c(v,\tau), x(\tau)]\, c_v(v,\tau) = \partial\,[\mu_c(\tau)]/\partial\, v = 0\,. \tag{25}$$

It follows that $c_v(v,\tau) = 0$. This result is very intuitive. An allocation of aggregate consumption is optimal, if there is no incentive to shift consumption between cohorts at any time τ. As private and social discount rates are equal, it is optimal to implement an egalitarian plan under which all cohorts receive the *same* consumption level at a given point in time.[13] As a consequence, $c(v,t) = c(v',t)$, $v, v' \leq t$, and the average consumption level coincides with any cohort's consumption level: $c(v,t) = c(v',t) = c(t)$. Consequently, in the command optimum, both individual consumption levels and (individual) consumption growth rates are, according to (SO.1), independent of age.

Second, in contrast to the market equilibrium, *optimal* consumption levels and growth rates are independent of both the birth and the death rate.

3 The Effects of Externalities

In the following section, we consider the effects of consumption and production externalities on balanced growth paths (steady state equilibria). For steady state values, we omit the time indexes as of here.[14] In the market economy, a steady state equilibrium is given by $\dot{c} = \dot{x} = \dot{k} = \dot{\Delta} = 0$:

$$0 = \frac{r(k) - \rho}{\tilde{\sigma}} + \lambda - (b + \lambda)\,\Delta^{-1}\,\frac{k}{c}\,, \tag{M.SS.1}$$

$$x = c\,, \tag{M.SS.2}$$

$$0 = A\left[\frac{b}{b+\lambda}\right]^{1-\alpha} k^{\alpha + p(1-\alpha)} - c - \delta\,k\,, \tag{M.SS.3}$$

$$\Delta^{-1} = [r(k) + d] - \frac{r(k) - \rho}{\tilde{\sigma}}\,, \tag{M.SS.4}$$

$$\tilde{\sigma} \equiv \frac{\sigma - \gamma}{1 - \gamma}\,.$$

From (M.SS.1)–(M.SS.4) it follows that a steady state equilibrium can be represented by the two variables c, k, as shown in Figure 1.[15]

[13] If the social planner's discount rate is larger (smaller) than the pure rate of time preference, optimal consumption increases (decreases) with age.

[14] That is, c is average consumption. Individual consumption will still be denoted by $c(v,t)$, as it is generally not constant in a steady state (as discussed below).

[15] It can readily be verified that the equilibrium is a saddle point.

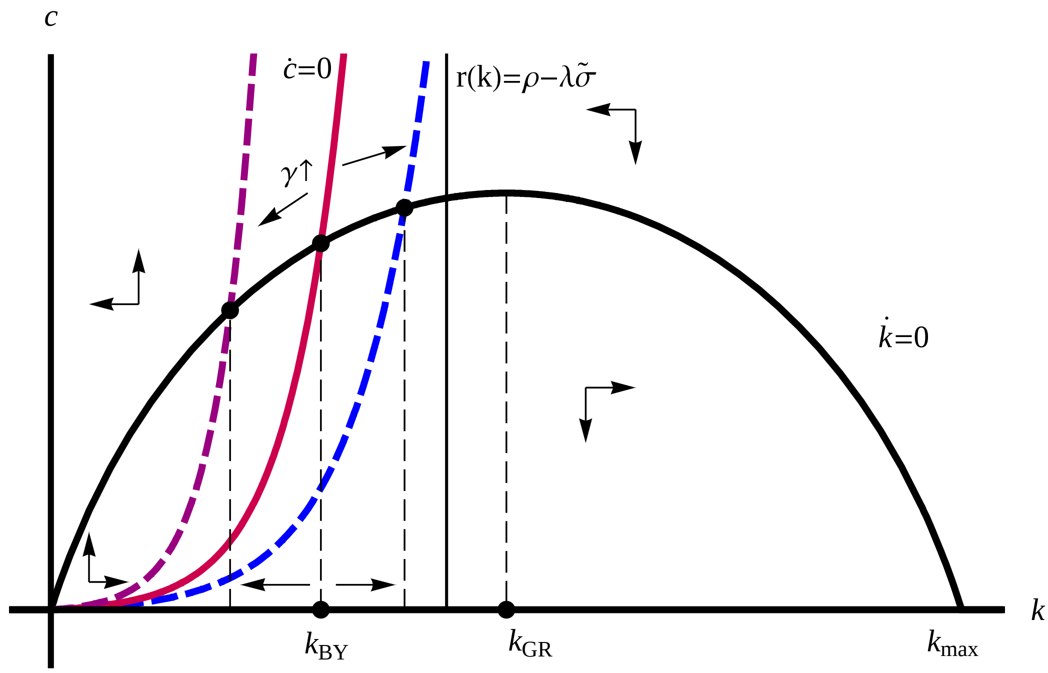

Figure 1: Impact of the Consumption Externality

Figure 1 displays the $\dot{k} = 0$- and $\dot{c} = 0$-lines in (k, c) space. The point of intersection (with $k > 0$, $c > 0$) shows the nontrivial steady state equilibrium. The properties of the $\dot{k} = 0$- and $\dot{c} = 0$-lines, as displayed in the figure, are discussed in the appendix. Arrows indicate the movements of trajectories.

Figure 1 also shows an asymptote for the $\dot{c} = 0$-line at: $r(k) = \rho - \lambda\,\tilde{\sigma}$. The asymptote implies an important fact. At a steady state equilibrium, $r(k) > \rho - \lambda\,\tilde{\sigma}$. That is, if $\lambda > 0$, $r(k)$ is allowed to assume a value lower than ρ (possibly negative) in a steady state. In particular, we note:

Lemma 3 *In a steady state equilibrium,*

(i) $[r(k) - \rho] \gtreqless 0 \Leftrightarrow \Gamma \lesseqgtr 0\,,$

(ii)there exists $\tilde{\lambda} > 0$ *such that* $\Gamma(\tilde{\lambda}) = 0$,

where $\tilde{\lambda} = [\alpha\, b(d + \rho)]/[(1 - \alpha)(\delta + \rho) - \alpha\, b] > 0\,,$

(iii) $\Gamma(\lambda) \gtrless 0 \Leftrightarrow \lambda \gtrless \tilde{\lambda}\,.$

Proof. (i) Consider (M.SS.1) and Definition 1. Then, $0 = \frac{r(k)-\rho}{\tilde{\sigma}} + \Gamma$, and (i) directly follows.

(ii) From (13) and (14), $(w/k) = (1-\alpha)/\alpha\,(b+\lambda)/b\,[r(k)+\delta]$. Thus:

$$\frac{h}{k} = \frac{w/k}{r+d+\lambda} = \frac{(1-\alpha)}{\alpha}\,\frac{r+\delta}{r+d+\lambda}\,. \tag{26}$$

As $\Gamma(\lambda) = \lambda - (b+\lambda)/(1+h/k)$:

$$\Gamma(\lambda) = \frac{(1-\alpha)(r+\delta)\,\lambda - \alpha\, b(d+r+\lambda)}{r+\delta(1-\alpha)+\alpha(d+\lambda)}\,.$$

From (i), $\Gamma = 0 \Leftrightarrow r = \rho$. It can easily be verified that $\Gamma(\tilde{\lambda}) = 0$. Strict positivity of $\tilde{\lambda}$ follows from: $\tilde{\lambda} = b/(h/k) > 0$.

(iii) Considering Figure 1, a rise in λ tilts both the $\dot{k} = 0$- and $\dot{c} = 0$-lines clockwise (down), as proven in the appendix (in connection with Figure 1). Consequently, $\partial\, k/(\partial\,\lambda) > 0$. As $\Gamma = -(r(k)-\rho)/\tilde{\sigma}$ it follows that $\partial\,\Gamma/(\partial\,\lambda) = -r'(k)/\tilde{\sigma}\,\partial\, k/(\partial\,\lambda) > 0$. ||

The fact that a rise in λ raises k is intuitive. As labor income declines with age (in contrast to $\lambda = 0$), savings rise, and so does the steady state average capital stock.

Clearly, $\Gamma > 0$ does *not* imply dynamic inefficiency. Noticing that capital increases in λ, we define $\bar{\lambda}$ to be the value of λ, which implies the golden rule capital stock: $r(k(\bar{\lambda})) = 0$. From above, $r(k(\tilde{\lambda})) = \rho > 0$. Thus, $\bar{\lambda} > \tilde{\lambda}$. Dynamic efficiency occurs for all $\lambda \in [0, \bar{\lambda}]$. If, in addition $\lambda \in [\tilde{\lambda}, \bar{\lambda}]$, then $\Gamma > 0$. Dynamic inefficiency is possible only for $\lambda > \bar{\lambda}$.[16]

Making use of the Lemmas, we can now analyze the impact of externalities on steady state equilibria. Figure 1 indicates that (a rise in the strength of) the consumption externality tilts the $\dot{c} = 0$-line either anticlockwise or clockwise. The effects of the consumption externality are ambiguous, depending on the sign of Γ.

Proposition 1 (Effects of the Consumption Externality) *Suppose $p \geq 0$, $b > 0$, and $\lambda \leq \bar{\lambda}$. In the market economy, the consumption externality has an ambiguous*

[16]The transversality conditions require $a(v,t)$ to increase at a rate higher than $r+d$, in a steady state. They are also satisfied for $\Gamma > 0$, in which case $a(v,t)$ grows at a negative rate.

impact on the steady state allocation. In particular:

$$\Gamma \lesseqgtr 0 \Leftrightarrow \frac{\partial\, c}{\partial\, \gamma} \lesseqgtr 0\,, \quad \frac{\partial\, k}{\partial\, \gamma} \lesseqgtr 0\,, \quad \frac{\partial\, (c/k)}{\partial\, \gamma} \gtreqless 0\,.$$

Proof. We use the properties of the $\dot{k} = 0$- and $\dot{c} = 0$-lines. A change in γ has no impact on the strictly concave $\dot{k} = 0$-curve (see the appendix). The $\dot{c} = 0$-line can be expressed by:

$$c = k\, \frac{(b + \lambda)[(r + d)\, \tilde{\sigma} - (r - \rho)]}{r - [\rho - \lambda\, \tilde{\sigma}]}\,.$$

To the left of the asymptote (for all positive $k : r(k) - [\rho - \lambda\, \tilde{\sigma}] > 0$) consumption rises (declines) at points above (below) the $\dot{c} = 0$-line. Hold k fixed:

$$\frac{\partial\, c}{\partial\, \gamma}|_{\{\dot{c}=0,\ k\,\mathrm{fixed}\}} = (r - \rho)\, \frac{k\, (b + \lambda)(d + r + \lambda)\tilde{\sigma}_\gamma}{[r - \rho + \lambda\, \tilde{\sigma}]^2}\,. \tag{27}$$

As $\tilde{\sigma}_\gamma = (\sigma - 1)/(1 - \gamma)^2 > 0$, the sign of (27) depends on the sign of $[r(k) - \rho]$.
If $\Gamma < 0$, $[r(k) - \rho] > 0$ by Lemma 3. From (27) it follows that $\frac{\partial c}{\partial \gamma}|_{\dot{c}=0} > 0$, for a fixed k. That is, a rise in γ induces the $\dot{c} = 0$-line to tilt up and to the left. As the slope of the $\dot{k} = 0$-line is positive and smaller than the slope of the $\dot{c} = 0$-line, $\partial\, k/\partial\, \gamma < 0$.
As $\lambda < \bar{\lambda}$, $k_{BY} < k_{GR}$. Thus, $0 < r(k_{BY}) = \partial\, [y(k) - \delta\, k]/(\partial\, k) = \partial\, c/(\partial\, k) \Rightarrow \partial\, c/\partial\, \gamma = (\partial\, c/\partial\, k)\, (\partial\, k/\partial\, \gamma) < 0$.
$c/k = y/k - \delta \Rightarrow \partial\, (c/k)/(\partial\, k) = \partial\, (y/k)/(\partial\, k) < 0$, as the production function is strictly concave. As k declines, c/k rises.
Similar reasoning applies to the other cases, in which $[r(k) - \rho] < 0$ and $[r(k) - \rho] = 0$, respectively. ||

According to Lemma 2, a *low* value of $\lambda < \tilde{\lambda}$ implies: $\Gamma < 0$. For this case, Proposition 1 shows that the keeping up with the Joneses externality raises the steady state propensity to consume out of accumulated wealth, c/k. Intuitively, consumption is a positional good. A household not only derives utility from its own consumption level but also from the consumption-to-reference level ratio (roughly, from above-average consumption). Thus, the consumption externality provides an incentive to raise individual consumption (relative to the reference level).

Initially, for given levels of x and k, a rise in γ raises individual consumption levels and lowers the individual consumption growth rate. This reaction is seen by restating the steady state version (with $\dot{x} = 0$) of Euler equation (20) as follows:

$$r(k) = \rho + \left[\frac{\sigma - \gamma}{1 - \gamma}\right] g_v \,. \tag{28}$$

The term in square brackets represents the (absolute) consumption elasticity of marginal utility. A rise in γ raises the elasticity of marginal utility, as demonstrated by Lemma 1. That is, for a given *positive* growth rate of individual consumption, a rise in γ induces the marginal utility to decline too strongly over time, and households will aim to smooth their consumption paths. Consequently, households will bring some future consumption forward to the present and, according to (28), lower the consumption growth rate.

As, initially, every individual household raises its consumption level, average consumption rises as well. This reaction, in turn, implies a lowering in aggregate savings. Subsequently the capital stock declines, and the new steady state is characterized by a lower level of k. Consequently, average consumption declines as well, as shown in the proof. As the production function is strictly concave, the average product of capital declines in k. That is, $\partial\,(y/k)/(\partial\,k) = \partial\,(c/k)/(\partial\,k) < 0$. Therefore, the average steady state propensity to consume out of accumulated wealth, c/k, rises.

Lemma 3(iii) shows that given $\lambda > \tilde{\lambda}$, $\Gamma > 0$. In this case, Proposition 1 demonstrates that the keeping up with the Joneses externality *lowers* the propensity to consume out of accumulated wealth, c/k, and it raises average steady state consumption and capital levels. At first sight, this result does not square well with intuition. To gain insight, it is important to note that one's relative consumption $(c(v,t)/x(t))$ not only matters today but also in the future. Consuming more today rises one's relative consumption, ceteris paribus. This rise comes at a cost, however. Consuming more today lowers tomorrow's consumption level, thereby tomorrow's relative consumption, ceteris paribus. In contrast, lowering one's consumption today allows for a higher level of consumption (and a better relative position, other things being equal) in the future. If $\Gamma > 0$, households prefer the latter option.

If $\Gamma > 0$, it follows from Lemma 3 that $r(k) < \rho$. Consequently, the individual consumption growth rate is negative, as seen in the Euler equation. That is, an individual's consumption level declines, and its marginal utility of consumption rises over time. According to Lemma 1, an increase in γ raises the elasticity of marginal utility, as for a given g_v, the absolute growth rate of effective consumption, $c(v,t)^{1/(1-\gamma)}\, x(t)^{-\gamma/(1-\gamma)}$, rises in γ.[17] As a consequence, for a given g_v, effective consumption declines at too big a rate, and marginal utility increases too strongly over time.

Initially, for given levels of x and k, Euler equation (28) requires the individual consumption growth rate to increase (g_v to become less negative) as of a rise in γ. This rise is achieved by initially lowering the individual consumption levels. As, initially, every individual household lowers its consumption level, average consumption declines as well. This lowering implies a rise in average savings. Subsequently the capital stock and average consumption rise. In the new steady state, the propensity to consume, c/k, is lower (by strict concavity of the production function).

Proposition 1 also shows a third case: $\Gamma = 0$. In this special case, the optimal consumption growth rate is zero. While a rise in γ still raises the elasticity of marginal utility, Euler equation (28) shows that the rise in the reference parameter induces no change in g_v. Consequently, there is neither an initial nor a steady state response to the change in γ. In this case, the consumption externality does not have an impact on the steady state equilibrium. It is important to note, however, that this special case can only occur if $\lambda > 0$. In case individual labor productivity is constant over lifetime ($\lambda = 0$), equation (M.SS.1) implies that $r(k) > \rho$ in a steady state.

According to Proposition 1, a consumption externality alone — even with exogenous labor supply and without a production externality ($p = 0$) — does have a steady state impact on consumption and capital, given $\Gamma \neq 0$. This result is in contrast with the previous literature (Liu and Turnovsky 2005, Turnovsky and Monteiro 2007) that shows that consumption externalities without a concurrent production externality do

[17] Effective consumption, $c(v,t)^{1/(1-\gamma)}\, x(t)^{-\gamma/(1-\gamma)}$, grows at the rate: $g_v/(1-\gamma)$.

not have an impact on the steady state allocation *in representative agent models.* This seeming contradiction can be rectified, however. The representative agent model is a special case of the present framework, with $b = d = 0$. By Lemma 2(i), $\Gamma = 0$ in the representative agent model. Whenever $\Gamma = 0$, Lemma 3 implies $r(k) = \rho$, in which case the consumption externality does not have an impact on the steady state equilibrium, as seen in Euler equation (28).

Corollary 1 *Suppose $p \geq 0$ and $b > 0$. If $\lambda = 0$:*

$$\frac{\partial\,(c/k)}{\partial\,\gamma} > 0\,, \quad \frac{\partial\,k}{\partial\,\gamma} < 0\,, \quad \frac{\partial\,c}{\partial\,\gamma} < 0\,.$$

Proof. $\lambda = 0 \Rightarrow \Gamma < 0$, by Lemma 2(ii). ||

In case individual labor productivity does not decline over lifetime ($\lambda = 0$), individual total (accumulated and human) wealth rises over time and so does individual consumption. The consumption smoothing effect of a rise in γ, initially leads to an increase in average consumption, and to a rise in the propensity to consume out of accumulated wealth in the steady state.

The effect of the consumption externality is quite different when $\lambda > \tilde{\lambda}$. In this case, total individual wealth declines over time and so does individual consumption. The consumption smoothing effect of the consumption externality, by initially raising the individual consumption growth rate, lowers the steady state propensity to consume out of accumulated wealth. Thus, the individual labor productivity parameter, λ, plays a key role in explaining the impact of the consumption externality on individual behavior.

Individual consumption growth. We now turn to the fact that — even in a steady state — the individual consumption growth rate is (regularly) different from zero in the PY model, while the average consumption growth rate is zero. The steady state impact of the consumption externality on the individual consumption growth rate is ambiguous.

Proposition 2 (Individual Consumption Growth)

Suppose $p \geq 0$, and $b > 0$. In the market economy, the impact of a rise in γ on the individual consumption growth rate is ambiguous. In particular:

$$\frac{\partial g_v}{\partial \gamma} \gtreqless 0 \Leftrightarrow [r(k) - \rho]\,[\delta - (d + \lambda)] \gtreqless 0\,.$$

Depending on the GRE and the individual labor productivity parameter, the keeping-up-with-the-Joneses externality may either raise or lower the individual consumption growth rate.

Proof. From $g_v = -\Gamma = -\lambda + (b + \lambda)/(1 + h/k)$ it follows:

$$\frac{\partial g_v}{\partial \gamma} = -\frac{b + \lambda}{(1 + h/k)^2}\,\frac{\partial\,(h/k)}{\partial\,k}\,\frac{\partial\,k}{\partial \gamma}\,.$$

From the definition of human wealth, in a steady state, we know: $h/k = [b/(b + \lambda)]\,(w/k)/[r(k) + d + \lambda]$. Considering (13) and (14), $(w/k) = (1-\alpha)/\alpha\,(b+\lambda)/b\,[r(k) + \delta]$. From (26) and as $r'(k) < 0$, it follows: $\mathrm{sgn}[\partial\,(h/k)/\partial\,k] = \mathrm{sgn}[\delta - (d + \lambda)]$. Since $\mathrm{sgn}\,k_\gamma = \mathrm{sgn}[-(r(k) - \rho)]$, $\mathrm{sgn}\,[(\partial\,g_v)/(\partial\,\gamma)] = \mathrm{sgn}\,\{[\delta - (d + \lambda)]\,[r(k) - \rho]\}$. ||

Proposition 2 shows that the impact of the consumption externality on the individual consumption growth rate depends on its impact on the GRE, which, in turn, depends on the externality's impact on human wealth *relative to* accumulated wealth (h/k). The consumption externality exerts three effects on Γ. First, it changes accumulated wealth, k, as discussed in Proposition 1. Second, it changes the present value of average human wealth, $h = w(k)/[r(k) + d + \lambda]\,b/(b + \lambda)$, by a change in the steady state wage stream. Third, it changes h also by a change in the (effective) discount rate, $[r(k) + d + \lambda]$, which we will refer to as the *discount rate effect.* Clearly, if $\Gamma < 0 \Leftrightarrow r(k) > \rho$, the consumption externality lowers average accumulated capital, and thereby — due to a lower steady state wage, and a higher discount rate — also human capital. Thus, whether the externality raises or lowers h/k is ambiguous. The ambiguity can be clarified by expressing h/k as in (26). If the discount rate effect dominates the other effects, that is, if the denominator in (26) is smaller than the numerator, then h/k rises in k (as the discount rate declines in k), otherwise h/k

decreases in k. The discount rate effect dominates the other effects if $d + \lambda$ (the rate at which human capital effectively depreciates) is smaller than δ (the rate at which accumulated capital depreciates). Therefore, the impact of the consumption externality on the individual consumption growth rate depends on the GRE (sign of Γ) and on whether or not the discount rate effect dominates the other effects (sign of $[\delta - (d + \lambda)]$).

It is important to notice that a wealth effect like the discount rate effect discussed above is absent in standard two period OLG models, in which all labor income accrues at the beginning of life. The discount rate effect is also absent in representative agent models. Once $b = 0$, the Keynes-Ramsey rule determines the steady level of state capital, which is independent of the present value of human wealth. In the PY model, if $d = \lambda = 0$, the discount rate effect *always* dominates the other effects. We therefore have:

Corollary 2 *Suppose* $p \geq 0$ *and* $b > 0$. *If* $\delta > d = \lambda = 0$, *the impact of a rise in* γ *on the individual consumption growth rate depends on the GRE only. In particular,*

$$\frac{\partial g_v}{\partial \gamma} \gtreqless 0 \Leftrightarrow \Gamma \lesseqgtr 0 \, .$$

In Weil's (1989) model with overlapping families of infinitely-lived agents, the discount rate effect always dominates the effect on h/k. That is, if $[r(k) > \rho]$, a rise in γ always lowers h/k. As a consequence, the consumption externality raises both individual consumption growth rates *and* the average propensity to consume out of total wealth, c/k.

Three remarks are in order. First, Proposition 2 shows that the initial responses to a rise in γ, as discussed in Proposition 1, do not necessarily carry over to the steady state. That is, for given values of the stocks k and x, a rise in γ initially induces a consumption smoothing effect. Subsequently, k changes, and so does $r(k)$ and — according to Euler equation (28) — g_v. Whether or not the initial consumption smoothing effect carries over to the steady state depends on the discount rate effect — that is, on the sign of $[\delta - (d + \lambda)]$.

Second, if $\Gamma = 0$ the consumption externality has *no* steady state impact on the individual consumption behavior, and if $\delta = d+\lambda$, the consumption externality has no steady state impact on the individual consumption growth rate. As discussed above, in an intertemporal context, a household may raise consumption — thereby relative consumption, ceteris paribus — today, at the cost of a lower consumption level in the future. Alternatively, a household may lower (relative) consumption today, but enjoy a higher level of consumption (relative consumption, ceteris paribus) in the future. If $\Gamma = 0$ or $\delta = d + \lambda$, the net benefits of both strategies are zero, and there is no gain from changing one's consumption pattern.

Third, if $\Gamma > 0$, the consumption externality was shown to rise the average steady state consumption level. However, the rise in the consumption level does not necessarily imply a lower growth rate of individual consumption. If the discount rate effect is *not* dominating in this case, the consumption externality raises both the growth rate of individual consumption and the average consumption level — a result that is not appreciated in the previous literature.

Corollary 3 *(Consumption Level versus Growth Rate)*
(i) Suppose $\lambda \in (\tilde{\lambda}, \bar{\lambda})$, and $[\delta - (d + \lambda)] < 0$. Then a rise in γ increases both the average steady state consumption level and the individual steady state consumption growth rate.

(ii) If $\lambda = 0$, a rise in γ always lowers the average steady state consumption level, while it either raises (if $\delta > d$) or lowers (if $\delta < d$) the individual steady state consumption growth rate.

Proof. Part (i) follows from Propositions 1, 2 and the fact that $\Gamma > 0 \Leftrightarrow [r(k) < \rho]$. The restriction $\lambda < \bar{\lambda}$ ensures dynamic efficiency. |
(ii) Lemma 2(ii) shows that $\Gamma < 0 \Leftrightarrow [r(k) > \rho]$. Propositions 1 and 2 then directly imply the result.||

A necessary condition for (i) to hold is $\lambda > \tilde{\lambda} > 0$. Only in this case, it is possible that $\Gamma > 0$. Otherwise $\Gamma < 0$, and c and g_v cannot both rise due to an increase in γ.

Production Externality. We now turn to the effects of the learning-by-investing externality. In particular, we consider the effects of a rise in p on the steady state allocations in both the market economy and the command optimum.

Proposition 3 (Production Externality)
(i) Suppose $b > 0$ and $p > 0$. In the market economy, the production externality has no steady state impact on y/k, c/k, r.
(ii) For $\lambda < \bar{\lambda}$, and $k > 1$, the production externality unambiguously raises both the levels of average capital and consumption.

Proof. Consider $r+\delta = \alpha\, A\, k^{(\alpha-1)(1-p)}\, [b/(b+\lambda)]^{1-\alpha}$. In a steady state, from (M.SS.1) and (M.SS.3) it follows:

$$\frac{r + (1-\alpha)\,\delta}{\alpha} = (b+\lambda)\,\left[\frac{(r+d)\,\tilde{\sigma} - (r-\rho)}{(r-\rho) + \lambda\,\tilde{\sigma}}\right]\,,$$

which is an implicit equation for the market rate of interest: $r = r(b, d, \alpha, \gamma, \delta, \lambda, \rho, \sigma)$. That is, the market rate of interest is independent of p in a steady state. |
As r is independent of p, and $(r+\delta) = \alpha\,(y/k)$, and $(y/k) = (c/k) + \delta$, it follows that also (y/k), thus (c/k), are independent of p in a steady state. |
As $\partial\, \ln(y/k)/\partial\, k = 0$, it follows that $\partial\, k/\partial\, p = (\ln k)\, k/(1-p)$, which is positive if $k > 1$. Thus, if $k > 1$, $\partial\, c/\partial\, p = (\partial\, c/\partial\, k)\,\partial\, k/\partial\, p = (\partial\, y/\partial\, k - \delta)\,\partial\, k/\partial\, p > 0$ (by dynamic efficiecy), and it follows that $\partial\, k/\partial\, p > 0 \Rightarrow \partial\, c/\partial\, p > 0$. ||

The production externality does not have an impact on the steady state rate of interest. This result is very similar to what is observed in a representative agent model, in which the steady state rate of interest is independent of the productivity parameter and is given by the Keynes-Ramsey rule. If $b = \lambda = 0$, the same applies here. If, however $b > 0$, the rate of interest also depends on other parameters — $r = r(b, d, \alpha, \gamma, \delta, \lambda, \rho, \sigma)$ — but it is independent of p.

Consider $k > 1$. Intuitively, a rise in p, initially (for given k), raises $r(k)$. Euler equation (28) requires households to raise the individual consumption growth rate, thereby to reduce consumption levels. Consequently, savings rise, initially. Subsequently, the capital stock starts to increase and $r(k)$ starts do decrease. Eventually, $r(k)$ returns to its pre-shock value.

Proposition 3 displays that the effects of a production externality are very different from those of a consumption externality. The consumption externality affects the rate of interest and the ratios c/k and y/k, none of which is influenced by the production externality. If $\Gamma < 0$ (if $\Gamma > 0$), the production and consumption externalities have opposing effects (the same effects) on the average consumption and capital levels.

Distortionary Effects of the Externalities. We next show that the effects of both production and consumption externalities are distortionary – even with exogenous labor supply. The starting point is a situation without a production externality ($p = 0$). The steady state equilibrium in the command optimum is characterized in the appendix — by the equations (SO.SS.1) – (SO.SS.4). Let $\tilde{k}$ and $\tilde{c}$ be optimal capital and consumption levels. The effects of externalities are considered distortionary if they alter $k/\tilde{k}$.

Proposition 4 (Distortionary Effects of γ) *Suppose $p = 0$ and $b > 0$. In the market economy, if and only if $\Gamma \neq 0$, the consumption externality introduces a distortion. In particular:*

$$\Gamma \frac{\partial \, (k/\tilde{k})}{\partial \, \gamma} > 0 \, , \quad \Gamma \frac{\partial \, (c/\tilde{c})}{\partial \, \gamma} > 0 \, , \quad \Gamma \frac{\partial \, [(c/k)/(\widetilde{c/k})]}{\partial \, \gamma} < 0 \, .$$

Proof. From (M.SS.1) and (SO.SS.1), $r(\tilde{k}) = \rho$ in the command optimum, and $r(k) = \rho - \tilde{\sigma} \, \Gamma$ in the market economy. Thus, $\Gamma = 0 \Rightarrow k = \tilde{k}$, and $\partial \, (\tilde{k}/k)/\partial \, \gamma = 0$. | $\Gamma \neq 0 \Rightarrow \partial \, k/\partial \, \gamma \neq 0$, by Proposition 1. As $r(\tilde{k}) = \rho$, $\partial \, \tilde{k}/\partial \, \gamma = 0$. Thus, $\partial \, (\tilde{k}/k)/\partial \, \gamma \neq 0$, and the consumption externality introduces a distortion. |
Proposition 1 implies $\Gamma \, \partial \, k/(\partial \, \gamma) > 0$, $\Gamma \, \partial \, c/(\partial \, \gamma) > 0$, and $\Gamma \, \partial \, (c/k)/(\partial \, \gamma) < 0$. The proposition follows from the fact that $\tilde{k}$ does not change in γ. ||

The previous literature considers representative agent economies and shows that without a concurrent production externality, the consumption externality does *not* have a distortionary impact on the steady state allocation. In contrast, Proposition 4 demonstrates that in overlapping generation economies, when there is a GRE ($\Gamma \neq 0$), the consumption externality always introduces a distortion. Clearly, in a representative agent economy $b = 0 \Rightarrow \Gamma = 0$, by Lemma 2.

Households do not take into account that individual consumption causes interpersonal externalities. Thereby, consumption behavior gives rise to economic distortions. Suppose $\Gamma < 0$ (suppose $\Gamma > 0$). Then the consumption externality raises (lowers) the propensity to consume out of accumulated wealth of the market economy relative to that of the social optimum.

In the proof, $r(\tilde{k}) = \rho$ implies that $\tilde{k}$ is independent of the consumption externality. The social planner considers the impact of c on x. In a steady state, from the point of view of the social planner,

$$u(c(v), x) = u(c, x) = \frac{\left[c^{\frac{1}{1-\gamma}}\, x^{-\frac{\gamma}{1-\gamma}}\right]^{1-\sigma} - 1}{1-\sigma} = \frac{\left[c^{\frac{1}{1-\gamma}}\, c^{-\frac{\gamma}{1-\gamma}}\right]^{1-\sigma} - 1}{1-\sigma} = \frac{c^{1-\sigma} - 1}{1-\sigma}\,,$$

and the optimal steady state allocation is independent of γ. It is important to emphasize, however, that the result offered by Proposition 4 does not depend on this fact. In contrast, Proposition 4 holds for a wide class of specifications[18], including all (linear and multiplicative) externality specifications included in Dupor and Liu's (2003) general formulation (see footnote 4).

Proposition 5 (Distortionary Effects of p**)** *Suppose* $b > 0$*, and* $p > 0$*. The production externality causes the following (distortionary) effects:*

(i) $\partial\, \tilde{r}/\partial\, p = 0$,

(ii) $\Gamma \leq 0 \Rightarrow \partial\,(k/\tilde{k})/(\partial\, p) < 0$,

(iii) $\partial\,[(\widetilde{y/k})/(y/k)]/(\partial\ p) < 0$*, and* $\partial\,[(\widetilde{c/k})/(c/k)]/(\partial\ p) < 0$.

Proof. In a steady state, $\tilde{r}(k) = \rho$, by (SO.SS.1). Thus, $\tilde{r}(k)$ is independent of p. | In a steady state, $\tilde{r} = \rho = r + \tilde{\sigma}\,\Gamma$, implying that also Γ is independent of p.[19] Thus,

$$\frac{\tilde{r} + \delta}{r + \delta} = 1 + \frac{\tilde{\sigma}\,\Gamma}{r + \delta} \Rightarrow \frac{\partial}{\partial p}\left[\frac{\tilde{r} + \delta}{r + \delta}\right] = 0\,,$$

[18]That is, for other specifications the consumption externality will have an impact on the optimal allocation. However, the impact will be different (thereby distortionary) in the market economy.

[19]This property can also be seen as follows: $\Gamma = \lambda - (b + \lambda)/(1 + h/k)$, $h/k = (1-\alpha)/\alpha\,(r + \delta)/(r + d + \lambda)$. As r is independent of p, it follows that Γ is independent of p in a steady state.

where the right hand side follows from the fact that both Γ and r are independent of p. Considering:

$$\frac{\tilde{r}+\delta}{r+\delta}=\frac{\alpha+p(1-\alpha)}{\alpha}\left[\frac{k}{\tilde{k}}\right]^{(1-\alpha)(1-p)} ,$$

it follows:

$$\frac{\partial\,(k/\tilde{k})}{\partial\,p}\,\frac{\tilde{k}}{k}=-[(1-p)(\alpha+p(1-\alpha))]^{-1}+\ln{(k/\tilde{k})}/(1-p)\,.$$

$\Gamma\leq 0\Leftrightarrow k\leq\tilde{k}\Leftrightarrow\ln{(k/\tilde{k})}\leq 0$. Thus, $\partial\,(k/\tilde{k})/(\partial\,p)<0$. |

Finally,

$$\frac{\tilde{r}+\delta}{r+\delta}=\frac{\alpha+p(1-\alpha)}{\alpha}\left[\frac{\widetilde{y/k}}{y/k}\right].$$

The left hand side does not depend on p. As the first expression on the right hand side rises in p, the second expression essentially declines in p. As $c/k=y/k+\delta$, the proposition follows. ||

Part (i) of the proposition follows from the fact that the rate of interest is determined by the Keynes-Ramsey rule in the social optimum, and it is determined by a "modified" Keynes-Ramsey rule in the PY model, in which $r=r(b,d,\alpha,\gamma,\delta,\lambda,\rho,\sigma)$. A change in p neither changes $\tilde{r}$ nor r.[20]

The production externality, however, has an impact on $k/\tilde{k}$. If $\Gamma<0$, it raises $\tilde{k}$ by more than k. Due to the externality, market prices fail to reflect the social value of capital accumulation, and k rises by less than optimal.[21]

Similarly, (for any value of Γ), $\partial\,[(\widetilde{y/k})/(y/k)]/(\partial\ p)<0$. A rise in p induces the optimal k to increase. By strict concavity of the production function, it induces the optimal y/k to decline. In contrast, in the market economy, $y/k=(r+\delta)/\alpha$, which is unaffected by p, as individual firms and households do not take the social value of capital accumulation into account. So $(\widetilde{y/k})$ declines, while (y/k) is constant in p. The same reasoning holds for c/k, as $c/k=y/k-\delta$.

[20]As a consequence, the production externality also does not affect the individual consumption growth rate in a steady state.

[21]Theoretically, if $\Gamma>0$, and large enough, the production externality could possibly raise $k/\tilde{k}$, given $\ln(k/\tilde{k})>[\alpha+p(1-\alpha)]^{-1}$.

Corollary 4 *Suppose the production externality lowers $k/\tilde{k}$. If $\Gamma > 0$, the distortionary impact of the production externality is weakened by the consumption externality.*

Proof. Define the elasticity of $k/\tilde{k}$ with respect to p by:

$$E_p = \frac{\partial\,(k/\tilde{k})}{\partial\, p} \Bigg/ \frac{k/\tilde{k}}{p} < 0 \;\Rightarrow\; \frac{\partial\,(-E_p)}{\partial\,\gamma} = -\frac{p}{1-p}\,\frac{\tilde{k}}{k}\,\frac{\partial\,(k/\tilde{k})}{\partial\,\gamma} < 0\,,$$

where the last inequality follows from the fact that $k/\tilde{k}$ rises in γ when $\Gamma > 0$. ||

The consumption externality raises k, while the production externality raises $\tilde{k}$ relative to k. In this case, the consumption externality raises efficiency by introducing an additional distortion. This case, however, can only happen if $\lambda > 0$.

4 Conclusions

This paper addresses three questions. First, under which conditions does a consumption externality change household behavior? Second, does a consumption externality raise (as would be expected) or lower the propensity to consume? Third, under which conditions will the responses to externalities be distortionary, and will a consumption externality strengthen or weaken a production externality?

If the wealth of a new generation deviates from average wealth, the PY model exhibits a generation replacement effect (GRE), which opens a channel for the externality to have an impact on consumption and capital. In the presence of a GRE, the steady state capital stock is *not* determined by the Keynes-Ramsey rule, but by a modified rule which takes the GRE into account. Consumption externalities influence the GRE and thus have an impact on behavior whenever there is a GRE.

When the GRE is absent, a consumption externality does not change behavior. The GRE is absent in two situations: (i) in a representative agent economy, in which there is no inflow of new generations, (ii) when the wealth of new generations equals average wealth. The latter, however, can happen only when labor productivity declines with age.

Wether or not a consumption externality raises or lowers the propensity to consume depends on the sign of the GRE. If the GRE is negative (low productivity decline), household raise their propensity to consume due to the consumption externality. If the GRE is positive, however, households lower their propensity to consume. In this case, they benefit more from higher consumption levels in the future as compared to a higher consumption level in the present. In the absence of a GRE, households have no incentive to alter their consumption stream due to the consumption externality. This fact reconciles the results derived from representative agent models with those from an overlapping generations model.

Along the same line, the steady state effects of a consumption externality were shown to be distortionary only in the presence of a GRE. The nature of the distortion, however, depends on the sign of the GRE. If the rate at which labor productivity declines is low, the consumption and production externalities reinforce each other. If, however, the rate at which labor productivity declines is high, the consumption externality was shown to raise efficiency by fostering capital accumulation.

These results have optimal fiscal policy implications. The results support arguments in favor of tax reform based on the argument that consumption externalities introduce distortions. Given the ambiguous nature of the effects of consumption externalities, however, it is less than clear whether or not the tax base should be shifted towards or away from consumption. That is, given the ambiguity of the results, it turns out to be an important matter to assess, empirically, the sign of the GRE.

We note the following limitations in this analysis. Households are homogeneous with respect to initial endowments and preferences. Real life consumption externalities, however, are likely to be related to comparisons within age-groups (as opposed to economy-wide comparisons) or to comparisons within groups of consumption goods (as opposed to considering the average consumption level only). These limitations call for the introduction of heterogeneity in a non-trivial way.

Notwithstanding these limitations, I hope this study helps to clarify effects of externalities, and it will contribute to the future debate of the effects of externalities in dynamic frameworks.

Appendix

A.1 Derivation of Individual Consumption. For any individual, consider the current value Hamiltonian:[22]

$$H_c[c(v,\tau),a(v,\tau),\mu_a(\tau),\tau] = u(c(v,\tau),x(\tau)) + \mu_a(\tau)[(r(\tau)+d)\,a(v,\tau)+w(v,\tau)-c(v,\tau)]\,.$$

For all $\tau \geq t$, $c(v,\tau)$ is chosen such as to maximize expected utility (4), subject to:

$$\begin{aligned}
&c(v,\tau) \geq 0\,,\\
&\dot{a}(v,t) = (r(\tau)+d)\,a(v,\tau) + w(v,\tau) - c(v,\tau)\,,\\
&a(v,t) \text{ given }, \quad \lim_{\tau\to\infty} \mu_a(\tau)\,e^{-(\rho+d)\tau}\,a(v,\tau) = 0\,.
\end{aligned}$$

From $\partial\, H_c/\partial\, c(v,\tau) = 0$ and $-\partial\, H_c/\partial\, a(v,\tau) = \dot{\mu}_a - (\rho+d)\,\mu_a$, it follows:

$$g_v(t) \equiv \frac{\dot{c}(v,\tau)}{c(v,\tau)} = \frac{[r(\tau)-\rho](1-\gamma)}{(\sigma-\gamma)} + \gamma\,\frac{\sigma-1}{(\sigma-\gamma)}\,\frac{\dot{x}(\tau)}{x(\tau)}\,,$$

where no individual considers its impact of individual consumption on the reference level. Thus,

$$c(v,\tau) = c(v,t)\,e^{(\sigma-\gamma)^{-1}\int_t^\tau [(r(s)-\rho)(1-\gamma)+\gamma(\sigma-1)\,g_x(s)]\,d\,s}\,.$$

Combining this equation with the intertemporal budget constraint (17), and considering the definition of $\Delta(t)$ in (18) yields: $c(v,t) = \Delta^{-1}(t)\,[a(v,t)+h(t)]\,.$ ||

A.2 Command Optimum. The current value Hamiltonian for the command op-

[22]Notice that $\gamma < \sigma$, which is implied by assumptions (A.2) and (A.3), is sufficient for the Hamiltonian to be concave in the decision and state variables.

timum is given by:

$$
\begin{aligned}
&H_c[c(v,\tau), c(\tau), x(\tau), k(\tau), \mu_x(\tau), \mu_c(\tau), \mu_k(\tau), \tau] = \\
&\int_{-\infty}^{\tau} b\, u[c(v,\tau), x(\tau)]\, e^{-b(\tau-v)}\, d\, v \\
&+ \mu_c(\tau)\, [c(\tau) - b \int_{-\infty}^{\tau} e^{-b\,(\tau-v)}\, c(v,\tau)\, d\, v\,] \\
&+ \mu_k(t) \left[A \left[\frac{b}{b+\lambda}\right]^{1-\alpha} k^{\alpha+p(1-\alpha)} - c(t) - \delta\, k(t) \right] + \mu_x(t)\, \varphi[c(t) - x(t)]\,, \qquad (29)
\end{aligned}
$$

where $c(v,\tau)$ and $c(\tau)$ are decision variables, $x(.)$, $k(.)$ are state variables, and $\mu_x(.)$ and $\mu_k(.)$ are costate variables. Variable $\mu_c(.)$ is the Lagrange multiplier associated with the consumption constraint. In the following, I omit time indexes and denote individual consumption by $c(v)$:

$$
\begin{aligned}
&u_c(.) + \varphi\, \mu_x = \mu_k\,, \\
&\tilde{r} = \rho - \frac{\dot{\mu}_k}{\mu_k}\,, \\
&\frac{u_x(.)}{\mu_x} - \varphi = \rho - \frac{\dot{\mu}_x}{\mu_x}\,, \\
&\lim_{\tau\to\infty} e^{-\rho\,\tau}\, \mu_x\, x = \lim_{\tau\to\infty} e^{-\rho\,\tau}\, \mu_k\, k = 0\,.
\end{aligned}
$$

From (3), we know: $u_x(.)/u_c(.) = -\gamma\, c(v)/x$. Thus, $u_x(.)/\mu_x = -\gamma c(v)/x\,[(1-\varphi\,\mu)/\mu]$, where $\mu \equiv \mu_x/\mu_k$. Together with $\dot{u}_c + \varphi\, \dot{\mu}_x = \dot{\mu}_k$, the differential equations describing behavior in the command optimum follow. ||

A.3 Figure 1. The figure displays two demarcation lines in (k, c) phase space: $\dot{k} = 0$, and $\dot{c} = 0$.

The $\dot{k} = 0$ *line.* We define the graph of $\dot{k} = 0$ by the set $kk \equiv \{(k,c) \in \mathbb{R}^2_+ \mid c = y(k) - \delta\, k\}$. As $y(0) = 0$, $(0,0) \in kk$. Next, $\partial\, c/\partial\, k > 0$ as long as $\tilde{r}(k) > 0$, and $\partial\, c/\partial\, k < 0$ when $\tilde{r}(k) < 0$. As $y(k)$ is strictly concave, and $(-\delta\, k)$ is weakly concave, $y(k) - \delta\, k$, that is, the $\dot{k} = 0$ line, is strictly concave. The capital stock for which $\tilde{r}(k) = 0$ is denoted the "golden rule" capital stock in Figure 1: $r(k_{GR}) = 0$. Finally, for the maximal attainable stock of capital, k_{max}, it holds: $y(k_{max}) = \delta\, k_{max}$.

For any given (k, c), a rise in λ clearly lowers y and tilts the $\dot{k} = 0$ line clockwise down. It lowers both k_{GR} and k_{max}.

The $\dot{c} = 0$ line. From (M.SS.1) it follows:

$$c = k \, \frac{(b + \lambda)[(r + d)\,\tilde{\sigma} - (r - \rho)]}{r - [\rho - \lambda\,\tilde{\sigma}]} \,. \tag{30}$$

We define the graph of $\dot{c} = 0$ by the set: $cc \equiv \{(k, c) \in \mathbb{R}^2_+ \mid c = k \, \frac{(b+\lambda)[(r+d)\,\tilde{\sigma} - (r-\rho)]}{r-\rho+\lambda\,\tilde{\sigma}}\}$. As k approaches zero, $r(k)$ goes to infinity, and $[r(k)\,k]$ approaches zero. We therefore find: $(0, 0) \in cc$.[23]

Next, in (30), we find an asymptote at: $r(k) = \rho - \lambda\,\tilde{\sigma}$. For $(k, c) \gg 0$, I we know $r(k) > \rho - \lambda\,\tilde{\sigma}$. To the left of the asymptote, the slope of the $\dot{c} = 0$ curve is positive:

$$\begin{aligned} \frac{\partial\, c}{\partial\, k}|_{\dot{c}=0} =& \frac{(b + \lambda)[(r + d)\,\tilde{\sigma} - (r - \rho)]}{r - [\rho - \lambda\,\tilde{\sigma}]} \\ & - k \, \frac{(b + \lambda)\,\tilde{\sigma}}{[r - \rho + \lambda\,\tilde{\sigma}]^2} \, [\rho + d - (\sigma - 1)/(1 - \gamma)\,\lambda]\, r'(k) > 0 \,, \end{aligned}$$

where the positive sign follows from two facts. First, $r'(k) < 0$. Second, $[\rho + d - (\sigma - 1)/(1 - \gamma)\,\lambda] > 0$. To see the latter, consider the sign restriction in (9): $\rho + d - (\sigma - 1)/(1 - \gamma)\,\lambda > \rho + d - (\sigma - 1)/(1 - \gamma)\,\rho\,(1 - \gamma)/(\sigma - \gamma) = d + \rho\,(1 - \gamma)/(\sigma - \gamma) > 0$. By the fact that there is an asymptote, the positive slope increases in k for $0 \leq k < r^{-1}(\rho - \lambda\,\tilde{\sigma})$.

We finally discuss the impact of (a change in) λ on the $\dot{c} = 0$ locus. A rise in λ tilts the $\dot{c} = 0$ line clockwise. To see this, consider (30):

$$\frac{\partial\, c}{\partial\, \lambda}|_{\{k \text{ fixed}\}} = k \, \frac{\rho + r(\tilde{\sigma} - 1) + \tilde{\sigma}\, d}{\tilde{\sigma}[r - \rho + \lambda\,\tilde{\sigma}]^2} \, (r - \rho - b\,\tilde{\sigma}) < 0 \,.$$

It remains to show that $(r - \rho - b\,\tilde{\sigma}) < 0$. In a steady state, I claim that $r(k) < \rho + b\,\tilde{\sigma}$. Suppose, to the contrary, $r(k) \geq \rho + b\,\tilde{\sigma}$. Then there exists some $\varepsilon \geq 0$ such that:

[23] As $(0, 0) \in cc$ and $(0, 0) \in kk$, there exists a trivial steady state.

$r(k) = \rho + b(1+\varepsilon)\,\tilde{\sigma}$, or, $[r(k) - \rho]/\tilde{\sigma} = b(1+\varepsilon)$. From (M.SS.1) and (M.SS.3):

$$\begin{aligned} \frac{f(k)}{k} - \delta &= \frac{(b+\lambda)[r + d - (r-\rho)/\tilde{\sigma}]}{\lambda + (r-\rho)/\tilde{\sigma}} \\ &= \frac{b+\lambda}{\lambda + b(1+\varepsilon)}\,[f'(k) - \delta + d - b - \varepsilon\, b] \\ &= \frac{b+\lambda}{\lambda + b(1+\varepsilon)}\,[f'(k) - \delta - \varepsilon\, b] \leq [f'(k) - \delta]\,, \end{aligned}$$

where the third line follows from $b = d$. The inequality, however cannot possibly be true by strict concavity of the production function. Thus, $r(k) < \rho + b\,\tilde{\sigma}$. ||

A.4 Optimal Steady State. We characterize an *optimal* BGP (steady state) by $\dot{c} = \dot{x} = \dot{k} = \dot{\mu} = 0$. From (SO.1)–(SO.4) it follows:

$$\tilde{r} = \rho\,, \tag{SO.SS.1}$$

$$x = c\,, \tag{SO.SS.2}$$

$$0 = A\left[\frac{b}{b+\lambda}\right]^{1-\alpha} k^{\alpha + p(1-\alpha)} - c - \delta\, k\,, \tag{SO.SS.3}$$

$$0 = \left[\tilde{r} + \varphi + \gamma\,\frac{c}{x}\,\frac{1-\mu\,\varphi}{\mu}\right]\,, \tag{SO.SS.4}$$

where $c(v) = c$. According to (SO.SS.1)–(SO.SS.4), the steady state values of c and k do not depend on both x and μ. ||

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