# An endogenous profit rate cycle

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# AN ENDOGENOUS PROFIT RATE CYCLE

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## 1. THE EQUATIONS

Consider the following system:

$$K' = I \quad (1)$$

$$I' = a\left(\frac{S}{K} - R\right) S \quad (2)$$

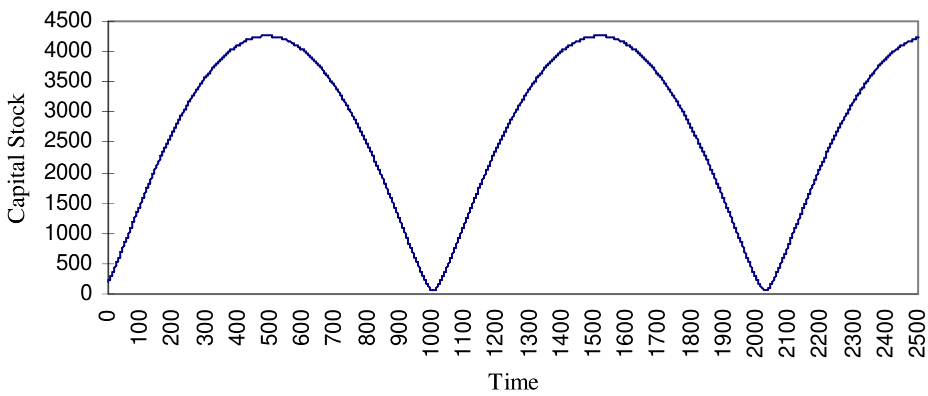
The variable $K$ represents the money value of capital stock; the variable $I$ is the money value of the rate of investment; $S$, a constant, is the money value of profits per period. $R$ is a 'target' profit rate and $a$ a parameter. $S/K$ is thus the average profit rate which we will call $r$. Negative $I$ signifies net disinvestment.

As we will show, this system exhibits stable periodic cycles in both $I$ and $K$ except at the stable fixed point $K = S/R$. $I$ oscillates between equal negative and positive extremes, while $K$ oscillates asymetrically between a minimum lower than $S/R$ but greater than zero, and a maximum higher than $S/R$ but bounded. This holds for any initial values of $I$ and any positive initial value of $K$ and the parameters $S$,$R$ and $a$.

Equation (1) states that capital stock grows or shrinks at a rate equal to the rate of investment. Equation (2) states that if the average profit rate falls below the target rate, entrepreneurs in aggregate reduce investment; the greater the gap, the more rapid the reduction. Conversely if the profit rate rises above this target rate, investment rises in the same manner. The change in investment is here represented as a proportion $a(r - R)$ of the mass of profit $S$.

Charts 1a, 1b and 1c show the graph of $K$, $I$ and $r$ against time and Charts 2a and 2b the orbits of $K$ and $r$ plotted against $I$, for initial values $K_0 = 200$, $I_0 = 10$, and for $S = 50$, $R = 0.05$ and $a = 0.02$.

**Chart 1a: Capital Stock (K) against time**



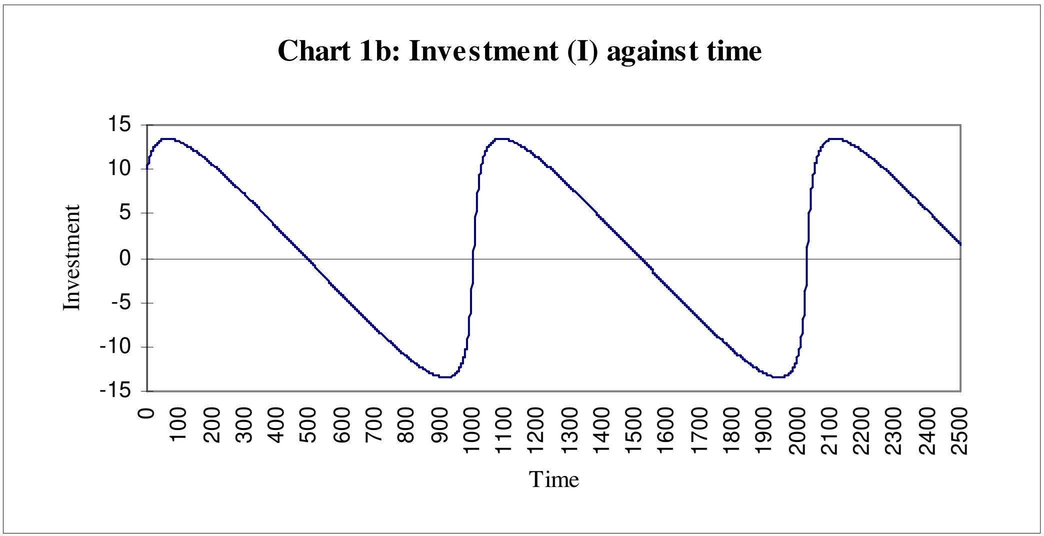

The qualitative 'mechanism' is as follows; begin at a 'typical point' (A) when the actual profit rate exceeds the target and $I$ is positive. $K$ rises, decreasing the profit rate until the threshold $K = S/R$ or $r = R$ is reached. Once $r$ exceeds $R$, investment begins to fall but remains positive until point B, the maximum of $K$ and the minumum of $r$ determined by $a$, $R$ and the initial conditions. Once investment falls below zero (disinvestment), $K$ begins to fall so that $r$ is now rising, though $I$ is still falling. When $K$ again reaches the threshold $S/R$ at point C, investment is still negative but $I'$ becomes positive. $K$ undershoots until the minimum D, also determined by the parameters and initial conditions; $I$ now becomes positive again and capital stock begins to rise, resuming the cycle.

## 2. ECONOMIC SIGNIFICANCE

It is known that endogenous business cycles can be generated or simulated using second-order linear equation systems, the best-known early such example being the Samuelson (1939) Multiplier-Accelerator model.

A generally accepted criticism of linear models is that only one value of the control parameter produces self-sustaining stable cycles. The problem therefore remains of accounting for two of the most salient observed features of a market economy, namely the persistence of business cycles and the lack of a stable equilibrium.

A nonlinear accelerator and other subsequent models developed by Goodwin (1951, 1992) and his co-workers brought home the fact that persistent cycles arise only if there are nonlinear terms in the equations of motion. But in most such models, the rate of profit as a determinant of investment behaviour has been either incidental or absent, and the adjustment process has focussed either on the interaction between investment and consumption or output, or on the interaction between employment and wage levels.




Neither neoclassical nor Marxist thinkers have, to my knowledge, constructed formal models in which the rate of profit itself exercises the predominant influence on investment behaviour, notwithstanding the importance which the rate of profit assumes in Marxist and classical theory, and notwithstanding the significant empirical evidence uncovered, by authors in both schools, of profit rate variations during the course of the cycle.

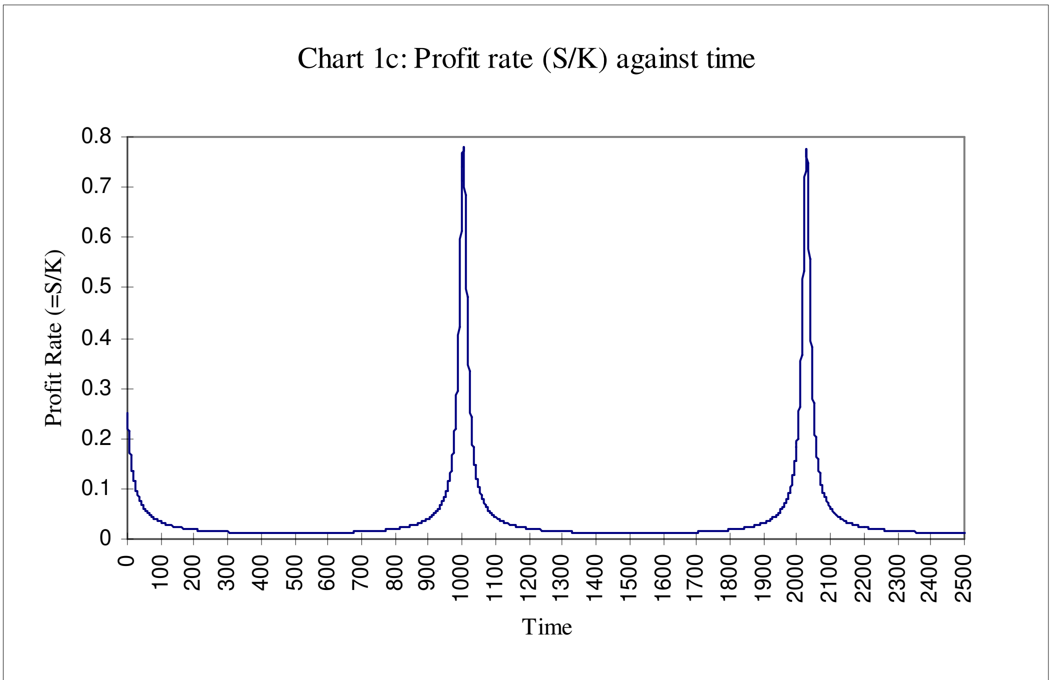



**Chart 1 time plots for $K_0 = 200$, $I_0 = 10$, $S = 50$, $R = 0.05$, $a = 0.02$**

Neoclassical and Post-Keynesian studies have focussed on pricing behaviour, on expectations or the relation between investment and output or consumption. Marxist interest in cycles in the early part of the century generally focussed, from Rosa Luxemburg onwards (see Day 1981), on the instability of the proportions of reproduction *per se* whilst cross-dual models such as those of Semmler, Flaschel and their co-workers (see Wegberg 1990) occupy an intermediate position, inasmuch as the Marxist concern with intersectoral proportions is maintained but the price system plays a genuinely dynamic role.

I devised this equation system to test whether stable cycles could be generated purely on the basis of changes in the rate of profit. The system therefore abstracts from all variations of investment behaviour which might be influenced or determined by changes in the labour market, the price level or by variations in quantities consumed or produced. For this reason *S*, the share of the money surplus available either for investment or private capitalist consumption, is held constant and only capital stock and investment may vary; investment itself is affected only by the rate of profit.

These simplifying features do not purport to represent a real economy. It is an exercise in thought to show that from a theoretical point of view the influence of the rate of profit in the business cycle must be seriously reexamined. Since the model exhibits persistent cyclic fluctuations on the basis of no other variation than the rate of profit, and since empirically the variation in the rate of profit is a key variable in the actual course of business cycles, there are, it seems to me, strong grounds for much more theoretical study of this neglected variable.

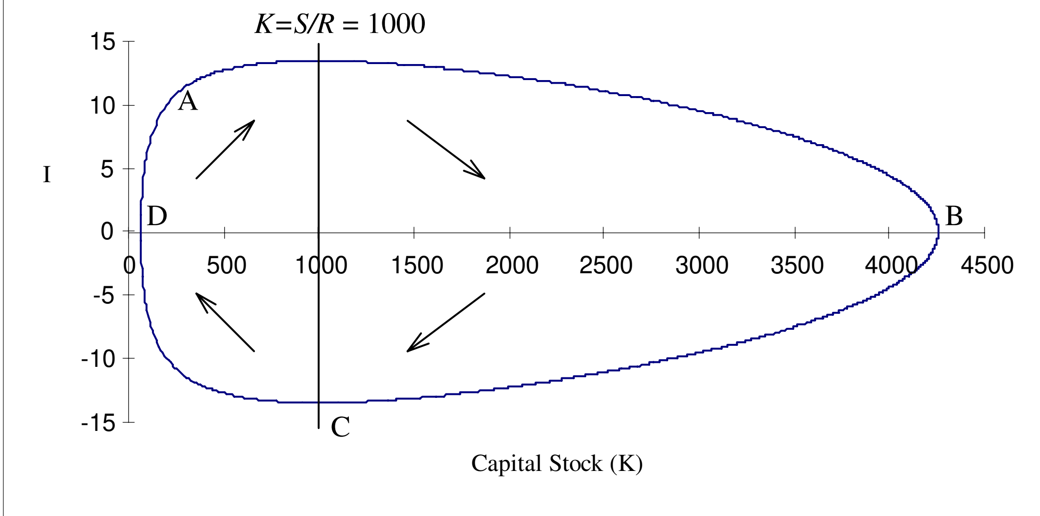



**Chart 2: Investment (I) against Capital Stock (K)**

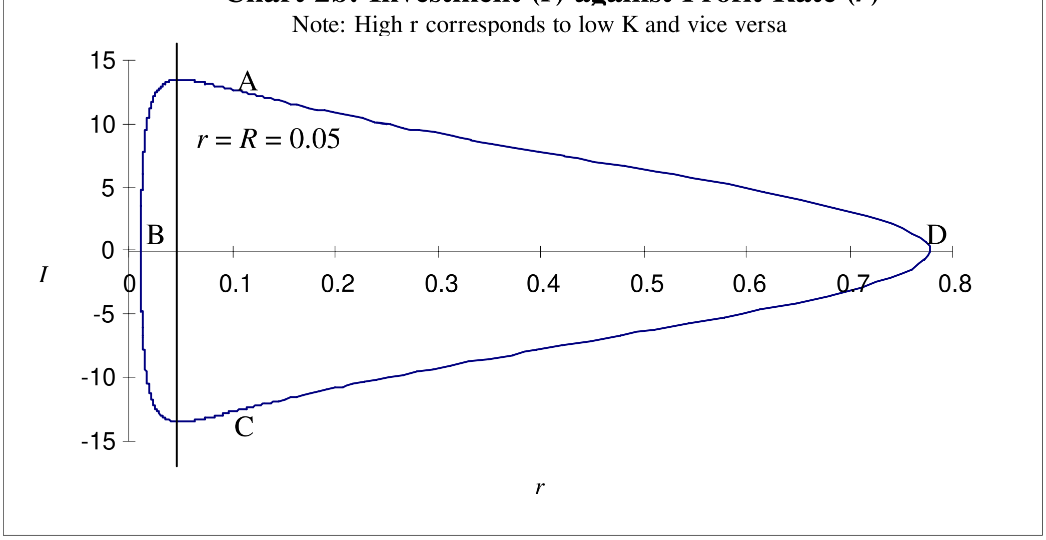



**Chart 2b: Investment (*I*) against Profit Rate (*r*)**

**Chart 2 orbits for $K_0 = 200$, $I_0 = 10$, S = 50, R = 0.05, a = 0.02**

However the special assumptions required for the model are very few. *K* and *I* refer to the money values and the first equation is therefore an identity. Moreover *I* is a residuum. It does not therefore represent all profits but merely that portion not privately consumed. By the same token it includes 'unintentional' investment. The equations do not therefore rule out changes either in the real or money wage, nor in the physical makeup of either capital stock or investment and hence the 'capital-output ratio'. What they do is abstract from such changes. They isolate the impact of the average profit rate which is here a summary variable representing the combined effect of all the changes listed above. However as the results show, it is a very powerful summary variable.

Its key property is that though other potential explanatory variables such as real wage may or may not change, the behaviour of the system is *independent* of whether they do so. This casts some doubt on how wise it is to treat them as causal factors of cyclic variation, and re-opens the neglected question of the role of the profit rate itself in the cycle.

## 3. FORMAL PROPERTIES OF THE SYSTEM

We can transform equations (1) and (2) to a canonical form, which makes it easier to study its fundamental properties, by using co-ordinates $k$, $i$ in which $k$, the transform of $K$, is 0 when $K = S/R$ and –1 when $K = 0$, $i$ being rescaled to preserve the relation $k' = i$. The equations of this transformation are

$$k = K\frac{R}{S} - \frac{R}{S}$$

$$\text{hence } K = (k+1)\frac{S}{R} \text{ and } K' = x'\frac{S}{R}$$

$$i = I\frac{R}{S}$$

$$\text{hence } I = I\frac{S}{R} \text{ and } I' = i'\frac{S}{R}$$

This leads to

$$k' = i$$

$$i' = aS\left(\frac{S}{1+k}\,\frac{R}{S} - R\right) = aSR\left(\frac{1}{1+k} - 1\right) = -aSR\,\frac{k}{1+k}$$

Writing the constant $B$ in place of $aSR$ gives

$$k' = i \tag{3}$$

$$i' = -B\frac{k}{1+k} \tag{4}$$

This is a plane autonomous system, meaning that time plays no explicit role. This makes it possible to derive the equation of the orbit (the relation between $k$ and $i$, or $K$ and $I$) directly by dividing $k'$ by $i'$ to give

$$\frac{di}{dk} = -B\frac{k}{i(1+k)} \tag{5}$$

Separation of variables leads to

$$idi = -B\frac{k}{1+k}dk = B\left(\frac{1}{1+k} - 1\right)$$

whence integration yields

$$i^2 = B\{\log(1+k) - k\} + C \tag{6}$$

where $C$ is the constant of integration. This gives a set of closed curves in the $k$-$i$ plane, corresponding to periodic oscillations of $K$ and $I$.

Some insight can be gained by noting that if $i$ is eliminated to give the second-order differential equation

$$k'' = -B\frac{k}{1+k} \tag{7}$$

which may be compared with the classical linear oscillator (and the linearisation of this system)

$$k'' = -Bk \tag{8}$$

in which $Bk$ is a restoring force proportional to the distance $k$ from equilibrium. This oscillates indefinitely in a fixed orbit unless either damped or forced. If the restoring force is not proportional to distance $k$, the system may be compared either to a stiffening spring (force increases 'faster' than $Bk$) or a 'softening' spring (force increases 'slower' than $Bk$).

The denominator $1/(1 + k)$ modifies this behaviour asymmetrically; for positive $k$ it lessens the 'restoring force' with distance from $k = 0$, behaving like a softening spring whilst for negative $k$ ($-1 < k < 0$) it increases it, behaving like a stiffening spring. Hence the phase of the cycle for which investment is decreasing ($k < 0$) is longer than the phase for which investment is increasing ($k > 0$), so that on the one hand the profit rate is above the target for a longer part of the cycle than below it, but on the other the plot of investment against time is a reverse sawtooth, the leading edge in which investment is rising being sharper than the trailing edge in which it falls.

The second result does not accord with the observed pattern of the business cycle, but our intention is not to mimic the cycle. It is to demonstrate that, since cyclic behaviour can be emulated in a system in which all variation due to fluctuations in price and quantity has been abstracted from, it would be rash to conclude that these factors can be the only ones at work. Since cycles can be emulated when the impact of profit rate variations is the only remaining factor, this factor must be taken seriously in further research.

The orbits of the system are given by the equation

$$H(k,i) = i^2 + Bk - B\log(1+k) = C \qquad (9)$$

where $H$, we note in passing, is the Hamiltonian of the system, sometimes interpreted as its 'energy'. It is perhaps worth noting that $i^2$ corresponds to the integral of investment over time, so it might be treated as the 'pump' of the oscillating system.

## 4. Shape and stability

As remarked, the orbit is a closed curve in the $k$-$i$ plane, so the system will perform stable cycles which return it to the same point ($k_0$, $i_0$) from which it started. Since at this point the equations are identical to the starting point, its period must be fixed, though this does not tell us what it is.

To study the orbit's shape a brief look at the curve

$$z = \log(1 + k) - k \qquad (10)$$

shown in Chart 3, is helpful. It has a singularity at $k = -1$, which according to our choice of coordinates corresponds to the point $K = 0$. It is important to establish if $k$ can fall below $-1$, therefore, as this would correspond to a negative capital stock.

The curve has a maximum at $k = 0$ ($K = S/R$), at which point $z$ *is* equal to zero. This immediately yields the extrema of $i$ from equation (6), namely $i = \pm\sqrt{C}$. This also highlights the result that there are no real orbits for $C < 0$.

We turn before establishing the extrema of $k$ to the dependence of the orbit on the initial conditions and the parameter $a$, which between them determine the value of the constant $B$.

Substituting initial values $k_0$, $i_0$ leads to

$$C = i_0^2 - B\{\log(1+ k_0) - k_0\}$$

whence (6) becomes

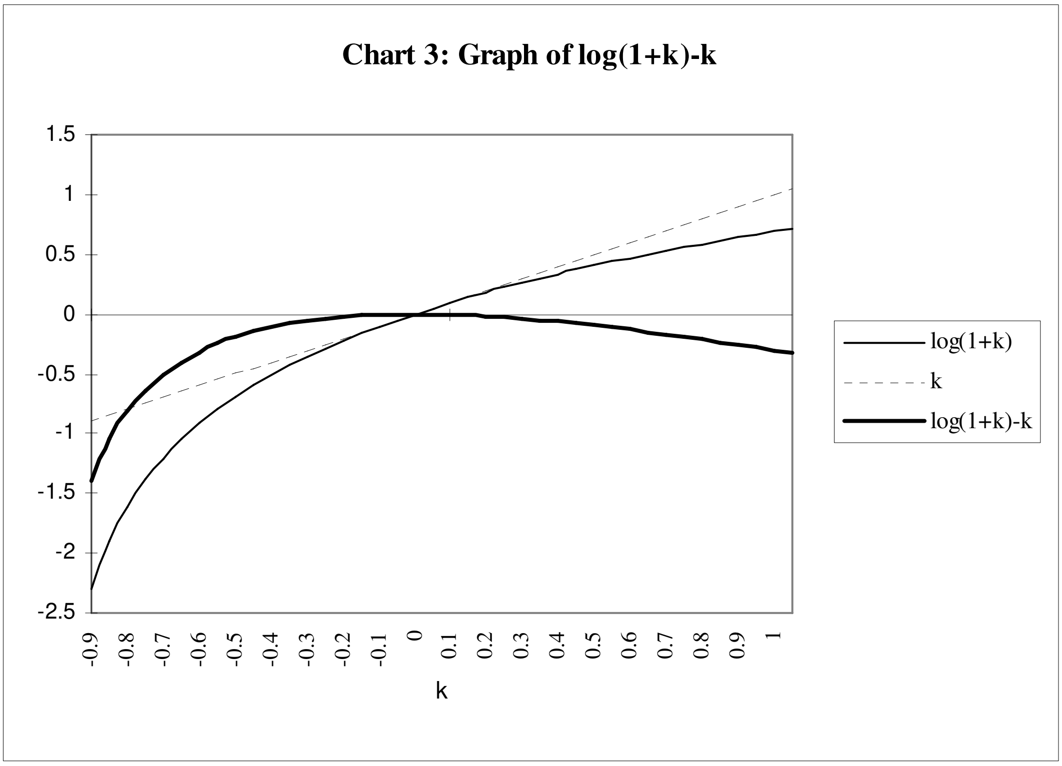
$$i^2 - i_0^2 = B\{\log(1 + k) - \log(1 + k_0) - (k - k_0)\} \qquad (11)$$

The extrema of $k$ are given by writing $di/dk = 0$, which from (5) leads to $i = 0$ and hence

$$k - k_0 - \log(1 + k) + \log(1 + k_0) = i_0^2$$

$$\log (1 + k) - k = \log(1 + k_0) - k_0 - i_0^2 \qquad (12)$$

As noted, if $k$ tends to $-1$ or $\infty$, $\log(1 + k) - k$ tends to $-\infty$, so that (since the orbit is a closed curve) if these extrema exist they must lie in the range $-1 < k_0 < \infty$. The extrema will not exist if the righthand side of (12) is above the maximum of the lefthand side, namely $k = 0$, which means that the orbit exists and yields values of $k$ between $-1$ and $\infty$ for any values of $i_0$ and hence any real initial value of $I$ provided $k_0$ is also between $-1$ and $\infty$, that is, for any real positive initial value $K_0$ and any positive $B$.




## 5. COMMENTARY AND CONCLUSION

To my knowledge this is the first endogenous model of the business cycle in which the sole explanatory variable is the rate of profit. It is tempting just to offer a model such as this either as toy for mathematical specialists or as a vindication of a neglected emphasis in economics. I believe more is at stake.

It is commonplace that the test of a theory lies in the prediction of observed reality. There is a tendency to reduce this to numerical accuracy alone, econometrics maybe serving as a softening spring. A more exacting empirical touchstone for theory is its ability to explain all the observed facts missing nothing out. This includes qualitative facts. A theory which numerically predicts prices for six years to within five percent, and fails to account for a devaluation, is not empirically valid, not because of size of the numerical deviation – which may be shortlived, leading to good tests of significance – but because of the qualitative importance of devaluations.

Moreover theories generally conquer qualitative peaks before ironing out quantitative valleys. It is the vantage point from which their glacial power depends. They must of course conquer the whole terrain; but a theory which does not scale peaks can never scour valleys. From this point of view the superiority of Galilean astronomy lay not in the accuracy of its detail, but the fact that it could explain a decisive qualitative observation – the moons of Jupiter – which could not be explained otherwise. In introducing a new cycle model is is therefore appropriate to ask what qualitative phenomena an account of business cycles should address.

Goodwin's decisive contribution is surely his insistence that two essential qualitative features of capitalism remain to be explained by theory, namely, the persistence and stability of cycles. These, along with the inequality of people and nations, are perhaps the two most persistently-observed phenomena of a market economy. Their explanation is a Holy Grail of serious economic research though they intrude more rudely on mundane life.

Two points are less obvious, but worth stating. First, a truly general theory has to be based on the most persistent and profound features of a market economy, since business cycles have existed as long as capitalism. Any explanation which resorts to an ephemeral or transient feature of this system is surely open to question, since the rigorous enquirer must ask why cycles still occur when the designated feature is absent. Second as regards these more transient features of a market economy, even if they are not employed as explanatory variables it is important that the model does not rule them out *a priori*. Otherwise it will be impossible to integrate them into a subsequent more complete and therefore more accurate version of the model.

Two features are therefore required of a theory of cyclic behaviour which are difficult to reconcile. On the one hand in its simplest and most abstract form its variables should refer to the most profound, persistent and general features of market economies; but on the other hand these variables should not exclude but should summarise and encompass all the other and many variations which are associated with particular cycles at particular times and places.

Consider the accelerator equation

$$K^* = kY$$

where $K^*$ is desired capital stock, $Y$ is capital, and $k$ the capital-output ratio. This is the basis for Goodwin's (1951) non-linear accelerator model, the dynamic process of adjustment consisting in the way in which investment $K'$ reacts to differences between $K^*$ and $K$, the actual capital stock.

The customary interpretation of $k$ is a technical feature of the economy, representing the physical productivity of capital. Indeed, this is why it is normally treated either as a constant or a consequence of technical change.

Goodwin (1951) has already pointed to some of the limitations of this equation. We draw attention only to the following subtle point: since monetary assets form part of the capital stock $K$ (and compete equally for investors' attention) and since $K$ and $Y$ are money quantities, the assumed technical relation between them does not in fact exist. For, during the actual course of a slump the first and probably main process is not the scrapping of capital stock but the migration of new investment capital from productive to unproductive investment; essentially into monetary assets.

This particular feature of a market economy – the migration of capital between monetary and productive assets – may not be universal but was the object of considerable attention from Keynes for whom the 'adjustment' of capital stock to output may and does take place through a downward adjustment of output, not an upward adjustment of capital stock. But this also revises the capital-output ratio downwards, as is to be expected if assets are transferred from production but remain part of the general stock of capital. A key link in the chain of the accelerator-multiplier construction is thus broken; the supposed technical link between output and capital.

The apparent simplicity of multiplier-accelerator models is therefore purchased at a certain cost: the simplification rules out vital explanatory variables. These can be put back, but at the cost of the original simplicity. It is not possible to theorise a purely technical relation between output and production except at the cost of the monetary dimension.

A different problem arises with the assumptions required by Post-Keynesian, cross-dual, or wage-employment adjustment models as well as many other more arcane accounts, all of which introduce some special behavioural assumption governing agents' behaviour – for example, in Goodwin's well-known Lotka-Volterra model it is required that a rise in money wages should be met by a fall in employment and vice versa. This is common, but it is not universal.

It is not fully appreciated that the rate of profit, and the relation between investment and accumulation, is immune to both these difficulties. The classical Harrodian relation

$$K' = I$$

is an accounting identity provided the variable $K$ is extended to include financial assets and provided $I$ is extended to include the effects of depreciation and appreciation. The mass of profit, provided it is extended to include all appropriations from gross income including interest and rent, is likewise related to net output by the accounting identity

$$S = Y - W$$

where $W$ is money wages. The equation for the rate of profit

$$r = \frac{S}{K}$$

is not therefore derived, as is often supposed, from any technical or special assumptions but is an identity deduced from two identities and therefore defines a universal parameter of a market economy. The model we have presented is every bit as simple and skeletal as the multiplier-accelerator model, but is devoid of its special assumptions.

The link between the profit rate and the stock of capital can in fact be stated independently of technical or physical relations if it is stated in money terms and includes monetary assets. It is simply the ratio between total money profits and total money capital. This must hold, regardless of the underlying physical ratios; it is among the most basic facts of a money economy. Equally persistent, and equally independent of all technical relations, is the tendency of capital to seek the highest possible return and it seems to me *prima facie* more plausible to seek the determinants of investment behaviour (as did both Keynes and Marx) in the relation between this investment and the return it yields; or at least to include this relation in the account.

It may be felt that this emphasis excludes the process of technical change itself from the account by focussing exclusively on the apparently superficial appearance of money prices. This would misunderstand what the equation system allows us to say.

What occurs when, through technical change, a more cheaply-produced product replaces a more expensive one? According to much traditional theory, the current value of the existing stock of this product counts in production at its new, replacement value. A growing body of theoretical research (see Freeman and Carchedi) challenges this view with the obvious point that the value of any item of capital stock in the inventory of the investor is not what it costs now to replace it, but its money price at the time of purchase. Depreciation, from this point of view, subsumes a windfall gain for the producer of the newer, cheaper product and a loss for the owner of the older, more expensive stock.

From the point of view of society as a whole, therefore, the money value of capital stock – the sum on which the profit rate is calculated – is equal, not to the replacement cost of this stock but to its historical money cost.

This being so the Harrodian equation $K'=I$ is, in money terms, an exact truth – an identity.

Under this fundamental truth may lie a multiplicity of physical realities. Disinvestment in money terms, for example, may result *either* from physical scrapping or from simply replacing existing, older and more expensive physical stock at the lower price made possible by technical advance, the money surplus being disposed of unproductively. In short, it is sufficient that the process of technical change continue at a reduced rate, without real expansion, for the slump phase of production representable by the above model to apply.

Equally, the bald assertion of a constant money volume of profit by no means precludes substantial variation in the real wage; $S$ is a summary variable which expresses the combined interaction of changes in both productivity in the consumer goods sector and the money wage. In conventional terms it simply expresses a combination of costs which rises in total at the same rate as productivity.

Consequently the simplicity of the profit rate equation belies the complexity of the physical relations it can express. In particular, if through value analysis we can isolate the effects of changes in productivity from ephemeral changes in the aggregate price level, we find that such variables act, like energy in physics, as the basis for establishing the *constants of motion* of an otherwise complex system – perhaps the nearest that economics can get to a global expression of the ancient goal of an essentially simple summary of its complex law of motion.

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