A Combinatorial Topology Approach to Arrow's Impossibility Theorem

Rajsbaum, Sergio and Raventós-Pujol, Armajac

Instituto de Matemáticas, UNAM, Mexico, Departamento de Análisis Económico: Economía Cuantitativa, UAM, Spain

2022

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SERGIO RAJSBAUM, Instituto de Matemáticas, UNAM, Mexico

ARMAJAC RAVENTÓS-PUJOL, Departamento de Análisis Económico: Economía Cuantitativa, UAM & Departamento de Estadística, Informática y Matemáticas, UPNA, Spain

Baryshnikov presented a remarkable algebraic topology proof of Arrow's impossibility theorem trying to understand the underlying reason behind the numerous proofs of this fundamental result of social choice theory. We present a combinatorial topology proof that does not use advance mathematics, and gives a very intuitive geometric reason for Arrow's impossibility.

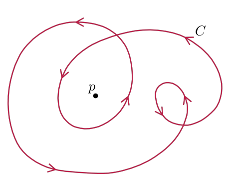
The geometric proof for the basis case of two voters, $n = 2$, and three alternatives, $|X| = 3$, is based on the index lemma, that counts the absolute number of times that a closed curve in the plane travels around a point. This yields a characterization of the domain restrictions that allow non-dictatorial aggregation functions. It also exposes the geometry behind prior pivotal arguments to Arrow's impossibility. We explain why the basis case of two voters, is where this interesting geometry happens, by giving a simple proof that this case implies Arrow's impossibility for any $|X| \geq 3$ and any finite $n \geq 2$.

1 INTRODUCTION

Social choice theory is a highly developed field of interest to economics and political science, and more recently to computer science [13]. The modern field of social choice theory took off with Kenneth Arrow's remarkable 1950 result [3] for the basic problem of democracy: it is impossible to aggregate the individual preferences into a single social preference, under some reasonable-looking axioms. Soon after the publication of Arrow's result alternative proofs began to emerge; starting with Inada [34] in 1954, numerous other proofs followed, and continue to be proposed until recently, e.g. [22, 29]. For an overview, including the importance of Arrow's result, see introductory books such as [28], or more advanced such as [21].

Motivation. Trying to understand the underlying reason behind the many proofs of Arrow's theorem, Baryshnikov [10] presented in 1993 a remarkable different approach, a topological impossibility proof. However, the goal of providing intuition about the nature of the problem of social choice is hindered by the relatively advanced algebraic topology tools used by Baryshnikov (several attempts at explaining the proof have been made [11, 15, 46]).

Our goal here is to further advance the program of Baryshnikov, while making it accessible to an audience not familiar with algebraic topology. Furthermore, we aim at understanding the gap between the literature on topological social choice [38] and combinatorial proofs, which have developed largely independently. We do so by moving from algebraic topology to combinatorial topology, and in doing so discover (and benefit from) remarkable connections with distributed computing [32].



C has winding number 2 around p.

Contributions. First, we provide new geometric proofs of Arrow's impossibility that do not require any acquaintance with algebraic topology. The proofs give a new insight for the reason of the impossibility, a combinatorial topology result called the *index lemma*, a generalization of Sperner's lemma (which is equivalent to Brouwer's fixed point theorem), used to compute winding numbers. The *winding number* of a closed curve in the plane around a point is the number of times that the curve passes counterclockwise around the point minus the number of times it passes clockwise. It is important in topology, calculus, analysis, physics, etc.

The geometric argument shows that the basis case of two individuals and three alternatives is somehow special, explaining an intriguing phenomenon, appearing several times in the literature. Some papers simply treat this case only e.g. [2, 17, 46, 51]. More interestingly, some papers hint at the idea that this is the case where the interesting things happen. Baryshnikov [10, Section 7.1] explains that only the arguments of his proof for triples of alternatives are in fact used, and one could concentrate only on the 2-skeleton of the simplicial complex using one-dimensional (co)homology.

We show the usefulness of the combinatorial topology approach by providing a characterization of the domain restrictions of the basis case, for which there is a non-dictatorial aggregation function. A very simple geometric argument for Arrow's impossibility based on a domain restriction is presented. The domain restriction analysis we present shows that contractibility of the space of preference profiles is not the reason for Arrow's impossibility, as conjectured in topological social choice [38].

We present a combinatorial topology perspective of the recent pivotal arguments to prove Arrow's impossibility by Geanakoplos [29] and Yu [56] that have received much attention e.g. [55]. Notice that Baryshnikov [10] does not try to explain the relation of his topology proof with previous proofs.

Finally, we present a simple proof showing that Arrow's impossibility result for the basis case of two individuals and three alternatives implies the general case. This result has been shown before under the restriction of finite number of alternatives by Tang and Lin [52] and partially by Akashhi [2], but our proof seems, in addition to be more general, more direct.

New intuition behind Arrow's impossibility and the connection with distributed computing. Very roughly, the intuition behind our approach, for the base case of two voters and three alternatives A, B, C is the following (in Section 6 we present the generalization from the base case by a simple inductive argument). The first step is to represent the set of possible preferences of the voters, N_I, as well as the set of possible social preferences, N_O, as geometric objects built from triangles. These objects are called 2-dimensional simplicial complexes; an introduction to combinatorial topology is in Section 2.2.

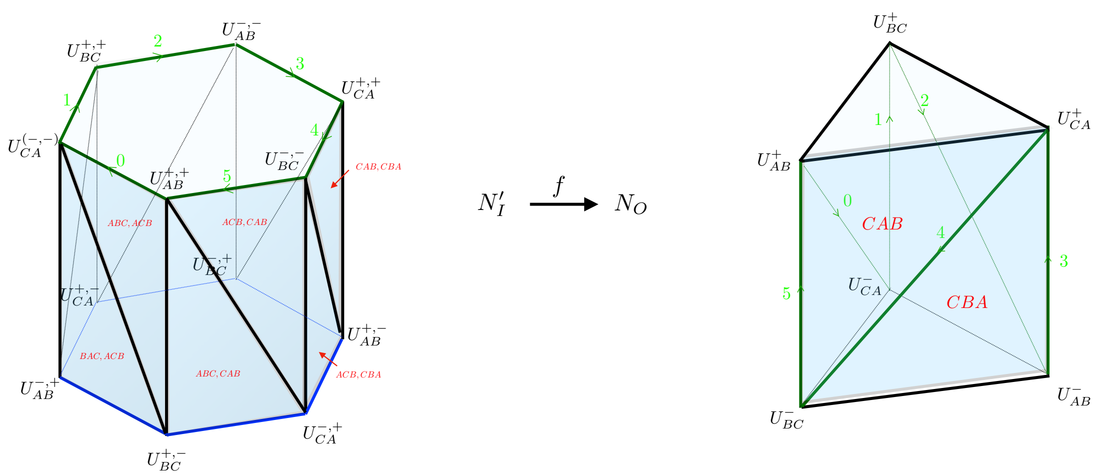


Fig. 1. Two triangulated cylinders: on the left N'_I a domain restriction of N_I, on the right N_O.

The notation N_I, N_O stands for "input" and "output" complexes, following the notion of a *task* in distributed computing. We present an introduction of the relation with distributed computing [32] in a proceedings version of this paper [47] and in Appendix A.

The insight is that either a social profile or social preference is defined by a triangle of three vertices, each one specifying preferences on two alternatives. Let $\mathcal{P} = \{AB, BC, AC\}$, called also *ids*. The vertices of each triangle are labeled with distinct process ids from $\mathcal{P}$. Additionally, the vertices of N_I are also labeled with an element from $\{++, --, +-, -+\}$, while the vertices of N_O are labeled with an element from $\{+, -\}$.

See Figure 1, where a triangulated cylinder N'_I is depicted, a domain restriction of N_I consisting of social profiles where the two voters disagree on one or on two pairs of alternatives (as opposed to triangles with either total agreement or total disagreement). In the figure N_O is also depicted. For clarity, only triangles on the "front" of the cylinders are labeled with the corresponding social profile or social decision.

The output complex N_O consists of all triangles, with each vertex labeled with a unique value from $\{+, -\}$, except for the two triangles labeled with the same value. Consider for example the triangle CBA of N_O depicted in the figure. It is determined by the vertex U^-_{AB}, meaning that B is preferred over A, the vertex U^-_{BC}, meaning that C is preferred over B, and the vertex U^+_{CA}, meaning that C is preferred over A. Notice that CBA is the only social preference satisfying these three preferences.

Similarly, consider for example the social profile BAC, ACB of N_I. It is determined by the vertex $U^{(-,-)}_{CA}$, meaning that both voters prefer A over C, the vertex $U^{(-,+)}_{AB}$, meaning that the first voter prefers B over A and the second voter prefers A over B, and the vertex $U^{(+,-)}_{BC}$, meaning that the first voter prefers B over C and the second voter prefers C over B. Notice that BAC, ACB is the only social profile satisfying these three preferences.

The second step is to observe that the aggregation map F that decides the social output, induces a simplicial map f from N_I to N_O, which is chromatic (preserves vertex ids). In Section 2.4 we reformulate Arrow's problem: we seek a chromatic simplicial map f from N_I to N_O, that sends vertices with input $++$ to vertices with output $+$ and vertices with input $--$ to vertices with output $-$. This comes from the unanimity requirement that if both voters prefer an alternative x over y, then the social preference should prefer x over y. Arrow's impossibility reformulation Theorem 2.1 says that such an aggregation map f must be a dictatorship.

The geometric reason is illustrated in Figure 1, the green cycle of N'_I must wrap around once on the green cycle of N_O. The index lemma can be used to computes the winding number of the boundary triangles of N'_I (the blue and the green cycles) on the boundary triangles of N_O. As we shall see, this number is 0 and implies that f is a projection (on the preferences of one of the two voters, the dictator), assuming that f satisfies unanimity. The mathematics used is elementary: essentially only basic parity counting operations are needed. Interestingly, the index lemma is also behind the distributed computing impossibilities related to weak symmetry breaking e.g. [14, 30].

Organization. First we present the statement of Arrow's theorem, an introduction to combinatorial topology, and how to model Arrow's theorem using combinatorial topology, in Section 2. The topology approach is suitable for studying restricted domains of preferences, as discussed in Section 3. A domain restriction is used in Section 3.1 to prove Arrow's impossibility with a very simple intuitive geometric argument illustrated in Figure 1. In Section 3.2 we present the characterization of non-dictatorial domain restrictions. In Section 3.3 a domain restriction is described that does allow for a non-dictatorial aggregation, in spite of having a non-contractible restriction N''_I. We provide two proofs of Arrow's theorem ($n = 2, |X| = 3$), using combinatorial topology, one in Section 4 using the index lemma, and one based on pivotal arguments in Section 5. We present a simple argument to generalize Arrow's theorem from the basis case of $n = 2, |X| = 3$ in Section 6. In Section 7 we present the conclusions. At the end of the paper an Appendix includes technical details about the proofs and the connection to distributed computing.

2 ARROW'S IMPOSSIBILITY THEOREM STATEMENT: CLASSIC AND GEOMETRIC FORMULATIONS

We start by recalling Arrow's theorem in Section 2.1, we then present a quick introduction to combinatorial topology in Section 2.2, used for the overview of how to use it for Arrow's setting in Section 2.3, and finally the combinatorial topology restatement of Arrow's theorem in Section 2.4.

2.1 Classic formulation

Let X be a set of alternatives, $|X| \geq 3$. The set of all strict total orders of X is denoted by W. Let $n \geq 2$ denote the (finite) number of voters, and W^n be the set of profiles of preferences. Thus, $\mathbf{R} = (R_1, \ldots, R_n) \in W^n$ is a profile, where each R_i is the order on X preferred by the i-th voter, $R_i \in W$. An aggregation map F is a function from W^n to W that maps each profile of W^n to a unique order in W. For example, if $X = \{A, B, C\}$, $R_i = A \succ B \succ C \in W$ denotes that the i-th voter prefers A over B, and B over C. This is also denoted as $A\, R_i\, B\, R_i\, C$, or when no confusion arises, simply by ABC.

A classic form of Arrow's impossibility theorem states that whenever the set X of possible alternatives has at least 3 elements, there is no aggregation map F from W^n to W satisfying the following axioms:

(1) Unanimity. If alternative, a, is ranked strictly higher than b for all orderings $R_1, \ldots, R_n$, then a is ranked strictly higher than b by $F(R_1, \ldots, R_n)$.
(2) Non-dictatorship. There is no individual k whose strict preferences always prevail. That is, there is no $k \in \{1, \ldots, n\}$ such that for all $\mathbf{R} \in W^n$, a ranked strictly higher than b by R_k implies a ranked strictly higher than b by $F(R_1, \ldots, R_n)$, for all a and b.
(3) Independence of irrelevant alternatives. For two preference profiles $\mathbf{R}$ and $\mathbf{S}$ such that for all individuals i, alternatives a and b have the same order in R_i as in S_i, alternatives a and b have the same order in $F(R_1, \ldots, R_n)$ as in $F(S_1, \ldots, S_n)$.

Some formulations of Arrow's impossibility theorem allow ties in the rankings (e.g. [4, 23, 57]). In this sense, it could seem that the framework we present here is not as general as it might be. However, this is not the case e.g. [10, Lemma 1], and indeed previous proofs e.g. [10, 38, 46] of Arrow's impossibility often assume, as we do, strict orders.

2.2 Introduction to combinatorial topology

Algebraic topology is a deep and highly developed branch of mathematics, studying algebraic invariants of topological spaces, such as homology groups. When the spaces are composed of individual cells attached to each other in a simple way, we have combinatorial topology, which has been gaining importance more recently as more and more applications are discovered, and the fact that such invariants can be computable. Here we use only elementary notions that can be found in books such as [31, 32], for more advanced treatments see [36, 50].

2.2.1 Simplicial complex. A *simplicial complex* is a family of sets that is closed under taking subsets, that is, every subset of a set in the family is also in the family. The elements of the sets are called *vertices*. A set of the simplicial complex is called a *simplex*, and its *dimension* is d if it has $d + 1$ elements; we say it is a d-simplex. In this paper we consider only simplicial complexes of dimension 2, meaning that each simplex contains at most 3 elements.

A simplicial complex is a purely combinatorial object, it can be seen as a generalization of a graph; in our case, in addition to edges consisting of pairs of vertices, we allow also triangles consisting of triples of vertices. As in graph theory, it is sometimes useful to embed a simplicial complex in Euclidean space. A simplicial complex can represent a discretization of a geometric object, in the case of dimension 2, a triangulation. We may think of the simplices of size 3 as triangles, the simplices of size 2 as edges, and simplices of size 1 as points, as illustrated in Figure 4.

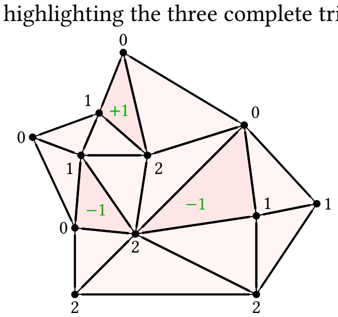
A subset of a simplex is called a *face*. Notice that if a triangle is in the complex, so are its three 1-dimensional faces (edges), and its three 0-dimensional faces (vertices), because a complex is closed under containment.

2.2.2 Simplicial map. A *simplicial map* is a function from the vertices of one simplicial complex K to the vertices of another simplicial complex, K', that preserves simplices: it sends sets of vertices that belong to a simplex of K, to sets of vertices that belong to a simplex of K'; thus, it respects the simplicial structure. A simplicial map is a discrete version of a continuous map.

2.2.3 Index lemma. Quoting from Henle [31],

> *"The combinatorial method is used not only to construct complicated figures from simple ones but also to deduce properties of the complicated from the simple. In combinatorial topology it is remarkable that the only machinery needed to make these deductions is the elementary process of counting!"*

The index lemma illustrates this point. Here we describe the basic version of [31]. Consider the following simplicial complex, K, consisting of a polygon of any number of sides, triangulated. The vertices are labeled arbitrarily, with labels 0, 1, 2. The *content* C is the number of triangles labelled 0, 1, 2, counted by orientation: it counts +1 if its labels read 012 in a counterclockwise direction around the triangle, and counts −1 if they clockwise around the triangle. The *index* I is the number of edges labeled 01 around the boundary of the polygon counted by orientation: and edge counts +1 if it reads 01 counterclockwise around the polygon, and −1 if it reads 01 clockwise. In the figure, $I = C = -1$. The index lemma says that this is always the case, $I = C$. This simplicial complex from [31] illustrates the index lemma, highlighting the three complete triangles.



The miracle of the index lemma is that the proof is a very simple parity counting argument (see Theorem B.2), despite the fact of being at the core of the study vector fields and other areas [31]. Furthermore, it implies Sperner's lemma (which is equivalent to Brouwer's fixed point theorem). For a general formulation of the index lemma see [20].

The see the geometric interpretation, we think of the coloring of the vertices of K as a simplicial map f from K to the complex K', that consists of a single 2-dimensional simplex $\{0, 1, 2\}$, together with all its faces. The index lemma counts the number of times the boundary of K is wrapped around the boundary of K'.

In Section 4 we need a simple generalization to prove Arrow's theorem: while the boundary of the complex consists of exterior edges belonging to a single triangle, each interior edge belongs to an even number of triangles (at least 2). As opposed to Sperner's lemma, the index lemma requires the complex to be orientable (Definition B.1). An example of a such an orientable complex, is the triangulated torus. After removing one triangle, say cef, the boundary consists of the edges of this triangle. An example a complex that is not orientable is a triangulation of the Möbius strip. See Figure 2.

2.3 Representing W^2 and W for three alternatives using combinatorial topology

We use the simplicial complexes N_I and N_O to represent W^2 and W following Baryshnikov [10]. The intuition behind these complexes is as follows, for three alternatives and two voters.

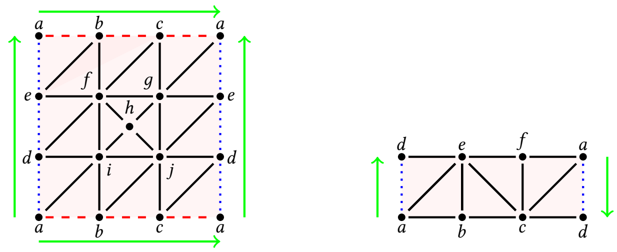


Fig. 2. Arrows indicate edges that are identified. The triangulated torus on the left has 10 vertices. The triangulated Möbius strip on the right has only 4 vertices, the boundary consists of a cycle of 6 edges: ab, bc, cd, de, ef and fa.

On the right side of Figure 3, two triangles of N_O are depicted, labeled BAC and BCA. The label BAC means that society prefers B over A and A over C. The two triangles share an edge because BAC and BCA agree on two pairwise preferences. The first is represented by the vertex U^-_{AB}, namely B is preferred over A, and the second by the vertex U^+_{BC}, namely B is preferred over C. Now, on the left part of Figure 3, four triangles of N_I are depicted, each one labeled with

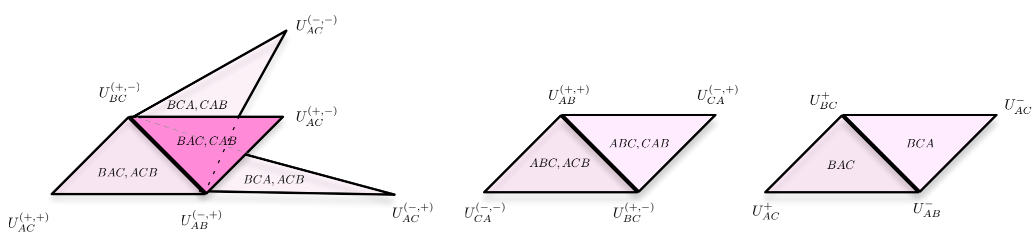


Fig. 3. Four triangles of N_I, then two, and finally two of N_O, intersecting in an edge, because they agree on two pairwise preferences, AB and BC.

a profile of 2 voters. An edge is contained in the four triangles, representing four different profiles, all sharing their pairwise preferences for A, B and B, C (in all 4 triangles, the first voter prefers BA and BC, while the second prefers AB and CB).

An edge in the *boundary* of a complex is contained in a single triangle. The pair of vertices of the edge determine the third one by transitivity. Consider for example the two pairwise preferences of ABC given by the vertices U^+_{AB} and U^+_{BC} in the right side of Figure 4. The edge $\{U^+_{AB}, U^+_{BC}\}$ belongs to a single triangle, ABC, since the vertices together determine the order ABC. Cycles (empty triangles) defined by boundary edges turn out to be important.

The triangles of N_I are defined by using the preferences of two voters. For example, the vertex $U^{(+,-)}_{AB}$ of N_I means that the first voter prefers A over B, and the second voter prefers B over A.

Consider the edge representing that both voters prefer AB and both prefer BC, given by the vertices $U^{(+,+)}_{AB}$ and $U^{(+,+)}_{BC}$ (see Figure 4). This is an edge in the boundary because it is contained in the unique triangle where both prefer ABC. Three such edges (connecting the two former vertices with $U^{(+,+)}_{CA}$) form a hollow triangle, because a Condorcet cycle is created if also both of them prefer CA.

There are internal edges of N_I contained in four triangles, as illustrated on the left of Figure 3, and there are internal edges contained in two triangles, in the center of the figure. The edge $\{U_{AB}^{(+,+)}, U_{BC}^{(+,-)}\}$ when both prefer AB, while the first voter prefers BC and the second prefers CB. This edge is contained in two triangles; in the figure, the left triangle correspond to the first voter's preference ABC, and the second voter's preference ACB. The other triangle in the figure corresponds to the preferences ABC and CAB.

2.4 Combinatorial topology form of Arrow's theorem

Here we state Arrow's theorem in the combinatorial topology framework, based on the two simplicial complexes, N_I, N_O, for a finite set of alternatives, $|X| \geq 3$ and $n \geq 2$ voters. The structure of these complexes is analyzed in Section 2.5, see also [15, 39, 46].

For a fixed n, a vertex $U_{\alpha\beta}^{\boldsymbol{\sigma}}$, with $\alpha, \beta \in X$ and $\boldsymbol{\sigma} \in \{+,-\}^n$ means that for each one of the n voters, i, α is ranked higher than β if $\boldsymbol{\sigma}(i) = +$, and otherwise, β is ranked higher than α.

Now, both N_I and N_O are defined on vertices of the form $U_{\alpha\beta}^{\boldsymbol{\sigma}}$, taking $n = 2$ for N_I, and $n = 1$ for N_O. In both cases, a set of vertices forms a simplex if there is a profile respecting the preferences defined by all its vertices. We will explain in detail these complexes in Sections 2.5.1, 2.5.2.

The remarkable insight is that if the aggregation map F satisfies independence of irrelevant alternatives then the corresponding map f from N_I to N_O is simplicial: it sends triangles of N_I to triangles of N_O, and if two triangles share a vertex (edge) in N_I then f must send them to two triangles in N_O that also share a vertex (edge).

If F satisfies unanimity, it sends profiles where everybody prefers α over β to a social preference where α is preferred over β. Then f sends vertices where everybody prefers α over β, denoted $U_{\alpha\beta}^{(+,\cdots,+)}$, to vertices where α is preferred over β in the social choice, denoted $U_{\alpha\beta}^{+}$. Thus, we say that the simplicial map f *satisfies unanimity* if it is such that for all vertices $U_{\alpha\beta}^{(+,\cdots,+)}$ of N_I, it holds that $f(U_{\alpha\beta}^{(+,\cdots,+)}) = U_{\alpha\beta}^{+}$.

Finally, there is a dictator if f is a projection on some coordinate k, namely, if f always selects the preference of voter k.

Theorem 2.1 (Arrow's impossibility). *Let $|X| \geq 3$ and $n \geq 2$. If $f : N_I \to N_O$ is a simplicial map that satisfies unanimity then f is a projection.*

Intuitively, this theorem says that Arrow's impossibility can be viewed as stating that a continuous map from N_I to N_O preserving unanimity must be a projection. That f is a projection means that there is a dictator k, such that, f returns the preferences of the k-th voter. That is, for all vertices $U_{\alpha\beta}^{\boldsymbol{\sigma}}$ of N_I,

$$f(U_{\alpha\beta}^{\boldsymbol{\sigma}}) = U_{\alpha\beta}^{\boldsymbol{\sigma}(k)},$$

where $\boldsymbol{\sigma}(k) \in \{+,-\}$ denotes the k-th sign of the vector of n signs $\boldsymbol{\sigma}$.

In more detail, an aggregation simplicial map $f : N_I \to N_O$ is defined from an aggregation map $F : W^n \to W$. Since F satisfies the independence of irrelevant alternatives property and $U_{\alpha\beta}^{\boldsymbol{\sigma}}$ represents a subset of profiles in W^n defined purely by the orderings between α and β, $f(U_{\alpha\beta}^{\boldsymbol{\sigma}})$ can be defined to be the vertex $U_{\alpha\beta}^{\sigma}$ with the sign σ determined by the ordering of α and β on the social aggregation of any of the profiles in $U_{\alpha\beta}^{\boldsymbol{\sigma}}$[1].

The images of the higher dimensional simplices of N_I can be defined by extension. We only need such simplices to be in N_O. However, this is immediate because a simplex in N_I exists whenever the intersection of their vertices

[1]If we assume independence of irrelevant alternatives together with unanimity, it can be defined as $f(U_{\alpha\beta}^{\sigma}) = \{F(\mathbf{R}) \in W : \mathbf{R} \in U_{\alpha\beta}^{\sigma}\}$.

contains at least one profile. The image of such a profile must belong to the intersection of the images of those vertices, since the image of a profile is determined by the ordering of pairs of alternatives.

Finally, we get the statement of Theorem 2.1. The independence of irrelevant alternatives property implies that f is a simplicial map from N_I to N_O. Moreover, the unanimity of f determines the image of the vertices formulated as $U_{\alpha\beta}^{(+,\cdots,+)}$ or $U_{\alpha\beta}^{(-,\cdots,-)}$.

2.5 The structure of the complexes N_I and N_O

We now describe the two complexes more formally and in more detail, the complex N_O in Section 2.5.1 and the complex N_I in Section 2.5.2.

We illustrate the whole set of triangles of N_I and N_O in Figure 4 (for N_I only schematically). Each triangle of N_I represents a social profile, and it is mapped by the aggregation map to a triangle of N_O representing the corresponding social choice. The aggregation map f maps (hollow) boundary triangles to (hollow) boundary triangles. Notice that N_O

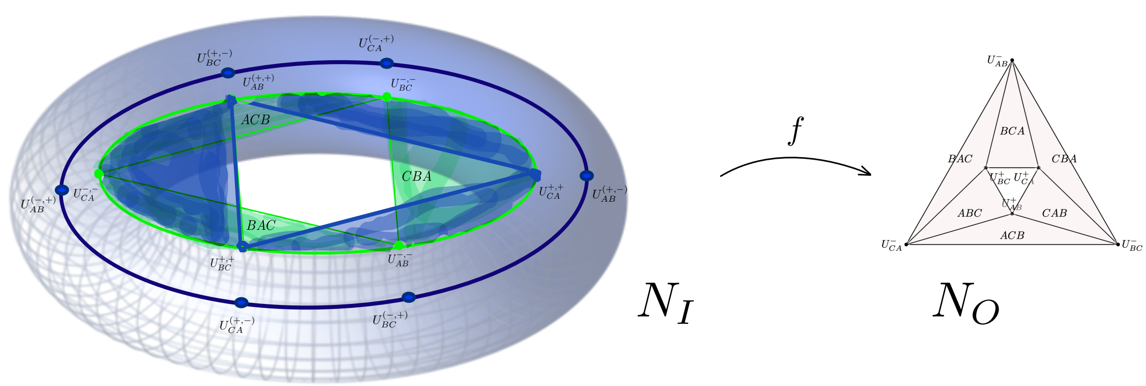


Fig. 4. On the left, N_I is a torus with 12 additional triangles that form four boundary hollow triangles. Here only 6 of them are shown together with their 2 hollow triangles (attached to the green cycle); the other 6 triangles are omitted for clarity, they are attached to the blue cycle. Instead, N_O is homeomorphic to a cylinder with two hollow boundary triangles.

is a triangulation of a cylinder with two boundary triangles, while N_I is a kind of product of two cylinders and has 4 boundary triangles. The index lemma computes the winding number. As we shall see, this number is 0 and implies that f is a projection (on the preferences of one of the two voters, the dictator), assuming that f satisfies unanimity.

2.5.1 The output complex N_O. The *output complex* N_O is defined as follows. Consider the notation $U_{\alpha\beta}^{\sigma}$, for $\alpha, \beta \in X$ with $\alpha \neq \beta$ and $\sigma \in \{+,-\}$. Then, $U_{\alpha\beta}^{\sigma}$ denotes the subset of W, of all strict orderings on X such that α is ranked higher than β if $\sigma = +$, and otherwise, β is ranked higher than α. Notice that $U_{\alpha\beta}^{+}$ denotes the same set as $U_{\beta\alpha}^{-}$. The set of vertices V of the output complex N_O consists of all such subsets of W, each one identified by one $U_{\alpha\beta}^{\sigma}$. A set of vertices of V forms a simplex of N_O iff their intersection is nonempty. This family of sets forms a simplicial complex, as it is closed under containment.

As mentioned earlier, for the purposes of this article, it is sufficient to consider X of size 3. Then, the complex N_O is depicted in Figure 4 taking $X = \{A, B, C\}$. We remark that our discussion holds for any finite X.

In the case of $|X| = 3$, N_O is of dimension 2. A facet is a 2-simplex $\{U_{\alpha_0\beta_0}^{\sigma_0}, U_{\alpha_1\beta_1}^{\sigma_1}, U_{\alpha_2\beta_2}^{\sigma_2}\}$, which represents the strict order that is compatible with its three vertices, that is, the strict order contained in $U_{\alpha_0\beta_0}^{\sigma_0} \cap U_{\alpha_1\beta_1}^{\sigma_1} \cap U_{\alpha_2\beta_2}^{\sigma_2}$.

Consider for example the triangle ABC, and its two vertices U^+_{AB} and U^+_{BC}. Notice that $U^+_{AB} = \{ABC, ACB, CAB\}$, and $U^+_{BC} = \{ABC, BAC, BCA\}$. These two vertices form an edge of N_O because their intersection is not empty. Moreover, it belongs to a single triangle, because the third vertex is unique, $U^-_{CA} = \{ABC, ACB, BAC\}$. Indeed, the three vertices intersect in a unique order, ABC.

There are exactly two triangles that are empty, that do not form a simplex, the external one requiring that $A \succ B, B \succ C, C \succ A$, and the central one, requiring that $A \succ C, C \succ B, B \succ A$. Furthermore, the boundary edges that belong to a single triangle are those that by transitivity uniquely imply the third vertex, e.g. the edge $\{U^+_{AB}, U^+_{BC}\}$ implies the third vertex, U^-_{CA}. Similarly, a partial order defined by an edge, e.g. $\{U^+_{AB}, U^+_{AC}\}$, is compatible with the two vertices that resolve the incomparability of B and C, namely, U^-_{BC} and U^+_{BC}.

The complex N_O is the space of output preferences because each one of its facets represents a possible social preference. Such a social preference is decided by an aggregation rule f, applied to a set of individual preferences of W^n, represented by the complex N_I.

Remark 2.1. For simplicity, we always denote the six vertices of N_O by the representatives U^+_{AB}, U^-_{AB}, U^+_{BC}, U^-_{BC}, U^+_{CA} and U^-_{CA}, as in the figure. In Section 4 we will need all vertices in the same boundary to share the same sign.

Remark 2.2. Consider two *adjacent* 2-simplices, intersecting in an edge. The strict order associated with one simplex and the one associated with the other simplex are equal, modulo permuting two consecutive elements in the strict order. For example, the facet corresponding to ABC and the one corresponding to ACB are adjacent: they are equal modulo the permutation of B and C. This fact will be used in the proof of Section 5.

2.5.2 The input complex N_I. We define the sets $U^{\boldsymbol{\sigma}}_{\alpha\beta}$, with $\alpha, \beta \in X$ and $\boldsymbol{\sigma} \in \{+, -\}^n$ as the subset of profiles of W^n where for each voter i, α is ranked higher than β if $\boldsymbol{\sigma}(i) = +$, and otherwise, β is ranked higher than α. As before, $U^{\boldsymbol{\sigma}}_{\alpha\beta}$ defines the same set of social preferences as $U^{-\boldsymbol{\sigma}}_{\beta\alpha}$. The set of vertices of the input complex N_I consists of all such subsets of W^n. As in the previous section, a set of vertices is a simplex of N_I iff their intersection is nonempty.

The complex N_I is much bigger than N_O. Whereas N_O has $|X|(|X|-1)$ vertices and its dimension is $(|X|+1)(|X|-2)/2$, N_I has $|X|(|X|-1)2^{n-1}$ vertices, but it has the same dimension as N_O (see [10]). So, in contrast to N_O, the complex N_I cannot be drawn in the plane even when $|X| = 3$, but a schematic representation is in Figure 4 and Figure 5. Notice, in the remark below, that analogous observations to the ones we made for N_O hold for N_I as well.

Remark 2.3. First, whereas each 2-simplex of N_O is a preference, in N_I each 2-simplex is represented by two individual preferences. Second, consider two adjacent 2-simplices (intersecting in an edge) of N_I. The individual preferences associated with one simplex and those associated with the other simplex are equal, modulo permuting the preference of two alternatives, x, y, of one or two voters, without changing the preferences of other alternatives. For example, in Figure 3, the triangles BAC, ACB and BCA, CAB are adjacent, because the preferences of both voters over A and C are exchanged, and only over A and C. This fact will be a keystone of the proof of Section 5.

As an example, consider the inner cylinder on the left of the Figure 5. The front triangle has vertices $U^{(-,-)}_{CA}, U^{(+,+)}_{AB}, U^{(+,+)}_{BC}$. This represents that both voters prefer ABC. The vertex $U^{(+,+)}_{AB}$ is also contained in its right triangle where both prefer CAB. The green edge of this triangle, $\{U^{(+,+)}_{AB}, U^{(-,-)}_{BC}\}$, is contained in the triangle (in the torus on the right side of the figure) that also contains $U^{(+,-)}_{CA}$, representing that the first voter prefers CAB but the second prefers ACB. Figure 4 illustrates how N_I consists of a torus, where two "parallel" cycles, a green one and a blue one are identified with some additional triangles (in the figure only the triangles identified with the green one are drawn, for clarity). In the green cycle, 6 vertices are used to add 6 triangles, as "flaps" of the torus (same for the blue cycle).

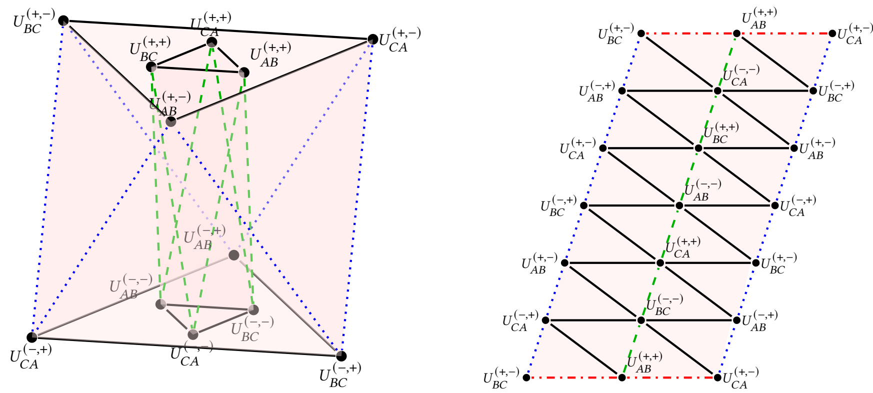


Fig. 5. When $|X| = 3$ and $n = 2$, the complex N_I can be built using two (cylindrical) copies of N_O placed one inside the other (on the left side of the figure). The outer cylinder are the unanimous profiles, whereas the inner one are the profiles where the voters have opposite preferences. Additionally, both cylinders are joined through the torus in the right (the torus is folded by identifying vertices according to the coloured edges), so the total number of vertices of N_I is 12.

3 APPLYING THE COMBINATORIAL TOPOLOGY APPROACH TO DOMAIN RESTRICTIONS

Arrow's impossibility applies to *universal domains*, where all possible individual preferences are considered. There is an extensive literature on the subject of domain restrictions, going back at least to Black [12], Arrow [4] and their famous *single-peaked* domain restriction, where the alternatives to be ranked lie on a one-dimensional axis and voters prefer values that are close to their favorite value. The research area is still very active today, some recent surveys are [9, 19]. Researchers have proved that it is possible to avoid Arrow's impossibility on various non-universal domains, including generalizations of single-peakedness, see, e.g. [27, 40] and the previous surveys for many examples. However, there is no general rule characterizing the domains in which aggregation is possible.

We illustrate here how the combinatorial topology approach can shed light on this topic. We present a very intuitive proof of Arrow's impossibility using domain restrictions in Section 3.1. We provide a characterization of the domain restrictions of the basis case in which non-dictatorial aggregation is possible in Section 3.2. We also discuss the role of contractibility of the restricted domain, showing it is not what determines the possibility of avoiding Arrow's impossibility, in Section 3.3.

Remarkably, considering task solvability under restricted domains has been thoroughly studied in distributed computing since [43].

3.1 Arrow's impossibility using domain restrictions

We start with a domain restriction that exposes clearly a geometric reason for Arrow's impossibility, related to winding numbers, already discussed in the Introduction using Figure 1, providing a proof of Theorem 2.1 for $|X| = 3$ and $n = 2$. It is the basis of the characterization of the domain restrictions in which non-dictatorial aggregation is possible of Section 3.2.

Recall the torus on the right of Figure 5. It consists of all the social profiles of N_I where the two voters disagree in either 1 or 2 of their pairwise preferences. The torus is depicted again in Figure 6, where in the top-left triangle, the profile is ABC, ACB, and there is disagreement in only one pairwise preference, BC, since the first voter prefers B over

C and the second prefers C over B. In the following triangle on the left, the profile is BAC, ACB, with two pairwise disagreements, on BC and on AB. In fact, the torus is made of two triangulated cylinders, joined by the blue dashed circle and by the green dashed circle. The left cylinder is called C_1 and the right one is C_2. They are symmetric, if one exchanges the voter 1 and voter 2 in C_1 one gets C_2. Namely, the top-left triangle of C_1 is ABC, ACB, and the symmetric triangle in C_2 is ACB, ABC. Similarly for the next triangle of C_1, BAC, ACB, its symmetric triangle on C_2 is ACB, BAC.

Consider C_1 as a domain restriction of N_I, in Figure 6. It is obtained by removing the cylinder C_2 from the torus on the right of Figure 5, and removing also both of the concentric cylinders on the left of the figure, corresponding to unanimous profiles and those where the voters have opposite preferences. In Figure 6 all the triangles of C_2 are removed from the torus: from top to bottom, the triangles CAB, ABC, ACB, ABC, etc. Only the triangles on the left remain, which form the cylinder C_1. Notice that N_O is also a cylinder, except that the cylinder C_1 is subdivided into 12 triangles while N_O consists of 6 triangles. Denote by N_I' the resulting restricted domain, and recall the different drawing in Figure 1.

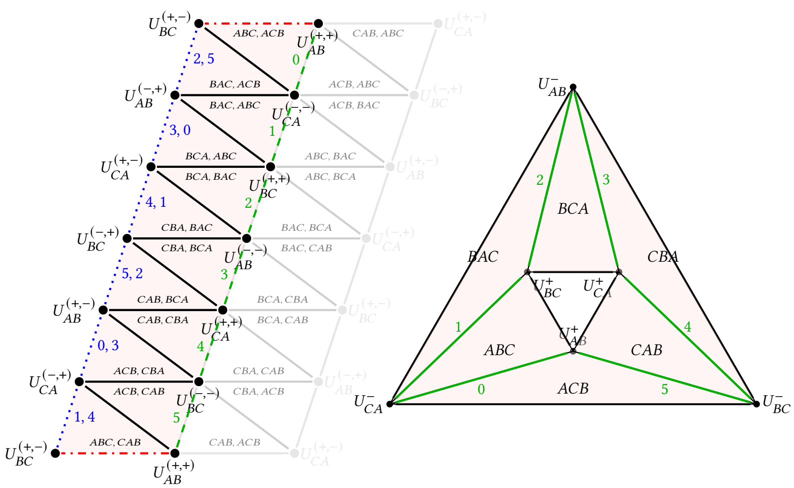


Fig. 6. On the left is N_I', a domain restriction on N_I, resulting in a cylinder and how the green cycle is mapped to N_O. Inside of each triangle of N_I' is the corresponding individual preference; the top triangle is ABC, ACB, the next one BAC, ACB, and so on. The blue cycle has two labels on each of its edges; the first one is the social choice where the first voter is the dictator, from top to bottom, 2, 3, 4, 5, 0, 1. With the second labels, the second voter is the dictator.

Now, Arrow's geometric impossibility becomes clear: C_1 is wrapped once around N_O, and the wrapping is determined by the green-dashed cycle in C_1, due to unanimity. In Figure 1 the image of the green-dashed cycle in C_1 on N_O is shown. This implies that the blue-dotted cycle, which is parallel to the green-dashed cycle, also has to wrap once around the cylinder, going in the same direction. There are two options for the aggregation function, labeled on the blue edges; to map the first (from top to bottom) blue edge to the edge 2 or to 5, the next one to 3 or 0 in N_O, and so on. In the first option the first voter is the dictator, in the second option the second voter is (in either case, the blue cycle goes on top of the green cycle of N_O).

3.2 The non-dictatorial domain restrictions

The profiles on the cylinders C_1 and C_2 are the basis of the characterization of subdomains of N_I allowing unanimous and non-dictatorial aggregation maps when $|X| = 3$ and $n = 2$.

We are interested in triangles that contain an edge in the blue cycle, and a vertex in the green cycle. Consider a profile $\mathbf{R}$ that corresponds to such a triangle, that we call a *critical profile*. An example of a critical profile is the top one on the left, (BAC, ACB). Notice that the two voters disagree on their preferences of the pair AB and the pair BC, but they agree on the pair CA. In general, for each critical profile, $\mathbf{R}$, there exists an edge defined by two pairs of alternatives xy and $x'y'$, such that the two voters disagree on them, but agree on the third pair of alternatives, $x''y''$. Namely, $\mathbf{R}$ is defined by the edge $\{U_{xy}^{(+,-)}, U_{x'y'}^{(-,+)}\}$, together with the vertex $U_{x''y''}^{(s,s)}$, $s \in \{+, -\}$.

We now define the main notion of a *pair of critical profiles*. It is a pair of critical profiles, $(\mathbf{R_1}, \mathbf{R_2})$, $\mathbf{R_1}$ on C_1 and $\mathbf{R_2}$ on C_2, such that they do not share a blue edge. That is, if the blue edge of $\mathbf{R_1}$, is $\{U_{xy}^{(+,-)}, U_{x'y'}^{(-,+)}\}$, then this edge does not belong to $\mathbf{R_2}$.

We are interested in characterizing domain restrictions $D \subseteq N_I$ that contain all vertices of N_I, for two voters and three alternatives. This assumption is not new in the literature: it is equivalent to requiring that every pair of alternatives is free (see [27, 40]). A pair is free if, for every ordering over such pair, there is a profile whose restriction on the pair is identically ordered[2]. As a summary, we will study the domains in which every pair of alternatives are comparable. The theorem below characterizes such domains.

Theorem 3.1 (Domain Restriction Characterization). *A domain restriction D that contains all vertices of N_I allows for a unanimous, non-dictatorial aggregation map if and only if D does not contain at least one critical pair of profiles.*

Proof. To prove the "$\Rightarrow$" direction of the theorem, assume there is a unanimous, non-dictatorial aggregation map f. We show by contradiction, that if D does not omit any critical pair, then f must be dictatorial. Two scenarios may occur: one of the cylinders restricted to D (i.e. $C_1 \cap D$ or $C_2 \cap D$) contains all of its triangles with blue edges, or both of the cylinders lack one triangle with a blue edge and both triangles, $\mathbf{R_1}$ and $\mathbf{R_2}$, share a blue edge. We will see that in both cases, f can be extended to a simplicial map on one of the cylinders (and we are back in the situation of Section 3.1).

We start with the first case. Suppose without loss of generality that $C_1 \cap D$ contains all the critical triangles with blue edges from C_1. We denote $C_1 \cap D$ as D_- and f_- as the restriction of f in D_-. In case D_- is not C_1, we can extend f_- to C_1 because the image of the green edges of C_1 are determined by unanimity. Since they are not mapped to the boundary of N_O, the image of the triangles with a green edge are well-defined by the image of their vertices. That is, f_- has been extended to a unanimous simplicial map f_+ defined on C_1. Using the argument in Section 3.1, we conclude that f_+ must be dictatorial. This is a contradiction because being dictatorial is determined by the images of the vertices, and f and f_+ have the same twelve vertices with the same images.

In the second case, we will see first that f can be extended over one of the missing triangles $\mathbf{R_1}, \mathbf{R_2}$ with a common blue edge. First, if the blue edge is mapped to an interior edge of N_O, then f can be extended in both triangles $\mathbf{R_1}$ and $\mathbf{R_2}$ using the image of their third vertex (the third vertex of both $\mathbf{R_1}$ and $\mathbf{R_2}$ are on the green cycle). Second, if the blue edge is mapped on the boundary, then f can be extended on one of the triangles, since the third vertices of $\mathbf{R_1}, \mathbf{R_2}$ are of the form $U_{\alpha\beta}^{(+,+)}$ and $U_{\alpha\beta}^{(-,-)}$, and hence the image of the blue edge will form a triangle of N_O together with the image of

[2]There are numerous works in social choice that escape from this framework and assume that there is some structural incapacity to compare some alternatives [24] or only allowing non-complete social rankings, but complete individual preferences [25, 54]

one of these two vertices. Once f has been extended, we are in the first case. Following the same arguments, we arrive to a contradiction.

To prove the "$\Leftarrow$" direction, assume a domain not containing the critical pair $(\mathbf{R_1}, \mathbf{R_2})$. Without loss of generality, we can suppose that the first profile is $\mathbf{R_1} = (BAC, ACB) \in C_1$ and the second one, $\mathbf{R_2}$, can be any triangle in C_2 but (BCA, CAB). We define the following aggregation maps for the five cases in Figure 7. It can be checked that they are all well-defined and non-dictatorial. The algorithm used to find these maps is in Appendix D. $\square$

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{+}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{-}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{-}

(a) $\mathbf{R_2} = (BAC, CAB)$

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{+}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{-}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(b) $\mathbf{R_2} = (ABC, BCA)$

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{-}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{-}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(c) $\mathbf{R_2} = (ACB, BAC)$

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{-}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{+}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(d) $\mathbf{R_2} = (CAB, ABC)$

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{-}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{+}
$U_{CA}^{(+,-)}$	U_{CA}^{-}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(e) $\mathbf{R_2} = (CBA, ACB)$

Fig. 7. This figure contains the definition of the five aggregations maps depending on $\mathbf{R_2}$. Their definition relies on the image of the vertices. We do not include the images of the unanimous vertices since they are determined by the unanimity axiom.

The maps in Figure 7 may seem somewhat opaque. However, for example, the aggregation map for $\mathbf{R_2} = (CAB, ABC)$ can be expressed as:

$$AF(\mathbf{R})B \Leftrightarrow AR_1B \text{ and } AR_2B \qquad BF(\mathbf{R})C \Leftrightarrow BR_2C \qquad AF(\mathbf{R})C \Leftrightarrow AR_1C \text{ and } AR_2C$$

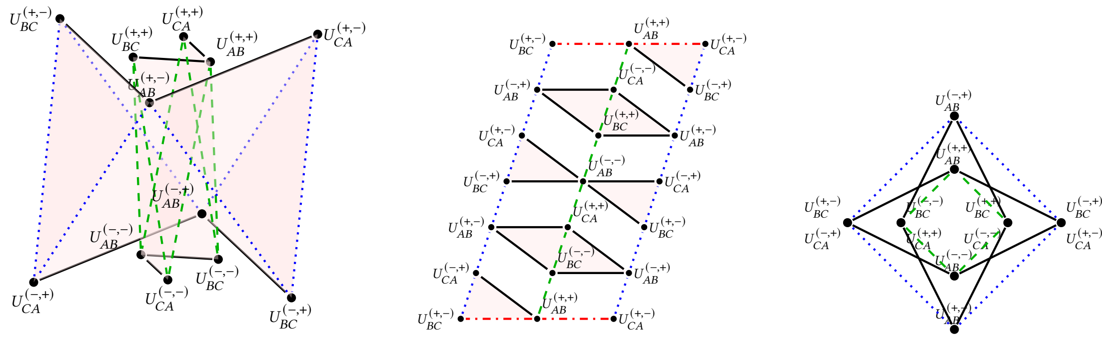
Using the expression above, we can see that the map is composed of a local dictator (the social choice between B and C) and two almost constant decisions (the social choice between A and B and between A and C).

This simplicity is mainly due to two factors: First, we are working with the simplest basis case (three alternatives and two voters). Second, as it is explained in Appendix D, these maps are deduced from the domains in which the unique removed profiles is a single critical pair. Moreover, in such domains, these maps are the unique ones that are not dictatorial. But the more profiles are removed, the more aggregation maps are compatible with the axioms. The next Section 3.3 is devoted to a domain restriction with a political interpretation, that allows more sophisticated aggregation maps.

3.3 Eluding Arrow's impossibility while preserving non-contractibility

It has been argued that the existence of a rule that permits aggregation is related to contractibility of a topological space. For the existence case in the continuous setting (which is different from our Arrovian setting), Chichilnisky and Heal [16], and a 1954 topology theorem by Eckmann [18] show that, for a general class of domains, contractibility is necessary and sufficient. Building on this result and on Baryshnikov [10], for weak orders, Tanaka [51] shows a connection with Brower's fixed point theorem, in the case of $n = 2$ and $|X| = 3$. Baryshnikov [10] and other authors such as Lauwers [38] and Baigent [46] hypothesised in subsequent publications that the aggregation on non-universal domains could be equivalent to the contractibility of the induced input simplicial complex. That is, the aggregation á la Arrow on a domain $D \subseteq W^n$ would be possible iff the induced complex N_I' is contractible. Moreover, they added that in the well-known case of single-peaked preferences (in which aggregation is possible) contractibility is satisfied.

Next, we present a domain of preferences that proves that Baryshnikov's hypothesis above is not true. That is, the domain N_I'' represented in Figure 8 is not contractible and it allows non-dictatorial aggregation maps.



(a) (b) (c)

Fig. 8. The restricted domain N_I'' is the union of the simplicial complexes represented in (a) and (b) according the identifications defined by vertices' labeling and colours. The simplicial complex N^* is represented in (c). The colours of the edges (resp. the labelings of the vertices) show where the edges (resp. the vertices) of N_I'' have been compressed in N^*.

This restricted domain N_I'' corresponds to a polarised society where political parties are classified as left-wing and right-wing parties. Assume that every left-wing voter will prefer all left-wing parties over all right-wing parties (vice-versa for right-wing voters). *A priori* we do not know if a voter is right-wing or left-wing. The polarized preferences in this section are a particular case of group-separable preferences (see. e.g. [19]).

We focus on the case in which there are two right-wing parties $\{A, B\}$ and one left-wing party C and two voters ($n = 2$). This way, N_I'' can be compared with the previous examples and proofs on this article.

The polarised domain restriction deletes the profiles in which a voter has C as the middle preferred party. For example, no voter will have the preference ACB because it prefer the right-wing party A over the left-wing party C and C over the right-wing party B. Formally, applying this restriction means deleting from Figure 5 the edges of the form $\{U_{CA}^{(+,\cdot)}, U_{BC}^{(+,\cdot)}\}$, $\{U_{CA}^{(-,\cdot)}, U_{BC}^{(-,\cdot)}\}$, $\{U_{CA}^{(\cdot,+)}, U_{BC}^{(\cdot,+)}\}$ and $\{U_{CA}^{(\cdot,-)}, U_{BC}^{(\cdot,-)}\}$ and all triangles containing them, and we obtain the simplicial complex N_I'' represented in Figure 8.

There are non-dictatorial aggregation rules for N_I''. One of these rules is defined by two local dictators. The first voter is a local dictator between the right-wing parties A and B, whereas the second voter is a local dictator between a right-wing party and the left wing-party C. Formally, this aggregation map F is defined for every profile $\mathbf{R}$ in the domain as:

$$AF(\mathbf{R})B \Leftrightarrow AR_1B, \qquad AF(\mathbf{R})C \Leftrightarrow AR_2C, \qquad BF(\mathbf{R})C \Leftrightarrow BR_2C.$$

Using the fact that $AF(\mathbf{R})C \Leftrightarrow BF(\mathbf{R})C$, it is straightforward to check that F is well defined (i.e. $F(\mathbf{R})$ is transitive and complete for every $\mathbf{R}$). Additionally, F is unanimous, non-dictatorial and satisfies the independence of irrelevant alternatives.

It remains to check that N_I'' is not contractible. In Figure 8, N_I'' has been drawn deleting a triangle on each of the concentric cylinders of N_I, and from the torus they only remain four pairs of triangles that join both cylinders. To see that N_I'' is not contractible, we apply contractions to N_I'' obtaining a new topological space N^* (that is non-contractible). This contractions consist on contracting first the eight triangles placed in the former torus (Figure 8b) to eight edges (black edges in Figure 8c). Second, we contract both cylinders (Figure 8a) into two concentric circles (green and blue edges in Figure 8c).

4 IMPOSSIBILITY PROOF BASED ON THE INDEX LEMMA

We present the first of the topological proofs of Theorem 2.1, for $|X| = 3, n = 2$, using the index lemma. The classic form of the index lemma is in Appendix 2.2. We use a simple generalization, Theorem B.2 described in Appendix B, where in addition to orientability, we assume that each interior edge belongs to an even number of triangles (at least 2). Let K be an oriented simplicial complex of dimension 2 with each vertex labeled with a color from $\{0, 1, 2\}$. The *content* C of K is the number of tricoloured triangles in K counted +1 if the order of the labeling agrees with the orientation and -1 otherwise. The *index* I of K is the number of edges $\overrightarrow{01}$ on the boundary (contained in exactly one triangle) counted +1 if the order of the vertices agrees with the orientation and -1 otherwise. The index lemma states that $I = C$.

Assuming N_I is orientable and we can use the index lemma (we defer the proof to Section B), we present our first proof of Theorem 2.1 here, for the case $|X| = 3, n = 2$.

Let $f : N_I \longrightarrow N_O$ be a simplicial map such that for all vertices $U_{\alpha\beta}^{(+,\cdots,+)}$ of N_I, it holds that $f(U_{\alpha\beta}^{(+,\cdots,+)}) = U_{\alpha\beta}^{+}$. We use f to define a coloring of the vertices of N_I with colors $\{0, 1, 2\}$, and then use the index lemma (Theorem B.2) to show that f is a projection.

In order to define the coloring of the vertices of N_I, first we colour them with $\{+1, -1\}$ according to the image of every vertex by f. That is, we label $U_{\alpha\beta}^{\boldsymbol{\sigma}}$ with +1 iff $f(U_{\alpha\beta}^{\boldsymbol{\sigma}}) \in N_O$ has the superindex +, and otherwise with -1. We call it the sign of $U_{\alpha\beta}^{\boldsymbol{\sigma}}$ and it is denoted by $s(U_{\alpha\beta}^{\boldsymbol{\sigma}})$.

Second, we color every vertex of N_I with one colour $p \in \{0, 1, 2\}$ following the rule:

$$p(U_{\alpha\beta}^{\boldsymbol{\sigma}}) = ID(U_{\alpha\beta}^{\boldsymbol{\sigma}}) + s(U_{\alpha\beta}^{\boldsymbol{\sigma}}) \qquad (mod\ 3) \tag{1}$$

where $ID(U_{AB}^{\boldsymbol{\sigma}}) = 0$, $ID(U_{BC}^{\boldsymbol{\sigma}}) = 1$ and $ID(U_{CA}^{\boldsymbol{\sigma}}) = 2$ (for every $\boldsymbol{\sigma} \in \{+, -\}^n$).

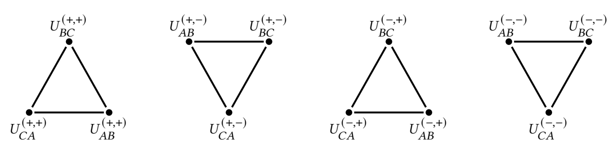


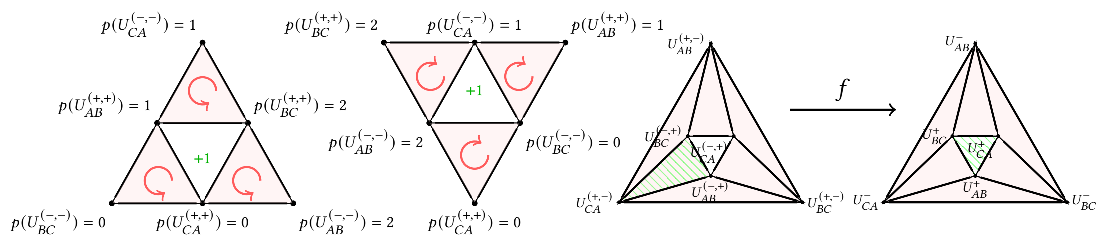
Fig. 9. N_I has four boundary components generated by Condorcet cycles. A single triangle intersects each boundary edge since each pair of vertices determines the third one by transitivity.

Notice that a cycle of three vertices is 3-coloured if and only if the sign of all of them is the same. This implies that the content $C = 0$ because no 2-simplex in N_I can be mapped to one of the holes in N_O.

We conclude from the index lemma that $I = 0$, on the boundary of N_I, which consists of 4 combinations of Condorcet cycles (see Figure 9). The contribution to the index from the unanimity cycles is +2 (see Figure 10a).

Since the contribution of the unanimity cycles is +2 and $I = 0$, the two remaining contributions to I have to be -1 for each one of the remaining boundary components. So, we can conclude that both have to be tricoloured and mapped by f to the boundary of N_O.

Both of these boundary components cannot be mapped to the same boundary of N_O because if it were the case the simplex $\{U_{AB}^{(-,+)}, U_{BC}^{(-,+)}, U_{CA}^{(+,-)}\}$ would be mapped to one of the holes of N_O (see Figure 10b).



(a) (b)

Fig. 10. (a) The contribution of these two boundary components to the index is +2 according with the orientation exposed in Proposition B.3 . (b) If these two boundary components of N_I are mapped to the same boundary component of N_O, then $U_{AB}^{(-,+)}$, $U_{BC}^{(-,+)}$ and $U_{CA}^{(+,-)}$ are also mapped to the same boundary component. That is, a hole.

Finally, we have all the information we need about the images of the 12 vertices of N_I to state that f is a projection. Recall that the images of the first and the fourth boundaries in Figure 9 are determined by the unanimity. If the second boundary is mapped to the inner boundary of N_O (and the third in the outer), it is straightforward to check that f is the projection over the first component. In contrast, if the second boundary is mapped to the outer boundary of N_O (and the third on the inner), then f is the projection over the second component.

5 IMPOSSIBILITY PROOF WITH PIVOTAL VOTERS

The second proof of Theorem 2.1 for $|X| = 3, n = 2$ exposes the geometry behind the combinatorial proofs by Geanakoplos [29] and Yu [56], using pivotal voters, that have received much attention e.g. [55].

5.1 Paths and pivotal voters

We say that a sequence of triangles in either N_I or N_O is a *path*, if each two consecutive triangles are adjacent (share an edge). Let $R = R_0, \ldots, R_m$ be a sequence of preferences in W such that every R_i can be obtained from R_{i-1} by a permutation of the preference of two alternatives (see Remark 2.2). This sequence induces a path in N_O.

Similarly, a sequence of profiles $\mathbf{R} = \mathbf{R}_0, \ldots, \mathbf{R}_m$ in W^2 defines a path in N_I, if $\mathbf{R}_i$ can be obtained from $\mathbf{R}_{i-1}$ by a permutation of the preference of two alternatives of at least one of the voters (see Remark 2.3). We will consider here only paths in N_I where $\mathbf{R}_i$ is obtained from $\mathbf{R}_{i-1}$ by a permutation of the preference of two alternatives of exactly one of the voters.

Notice that since the aggregation map f is a simplicial map, it sends triangles to triangles, and the image of a path in N_I is a path in N_O.

We will consider paths in N_I starting and ending in unanimous profiles. Additionally, such that all triangles in the path share a vertex $U_{xy}^{\boldsymbol{\sigma}}$, $x, y \in X$, for $\boldsymbol{\sigma}$ consisting of the same sign, either + or −. Notice that since all the triangles share vertex $U_{xy}^{\boldsymbol{\sigma}}$, then all the triangles of the path in N_O of the image under f share the vertex U_{xy}^{σ}, where σ is equal to the single sign in $\boldsymbol{\sigma}$.

An example is the path $\boldsymbol{R} = \boldsymbol{R}_0, \dots, \boldsymbol{R}_4$, in N_I, defined on the left of Figure 11. All the triangles in this path contain the vertex $U^{(+,+)}_{BC}$, since both voters prefer B over C. Additionally, the path starts in the unanimous profile ABC, ABC and ends in the unanimous profile BCA, BCA. In the figure there is another example, the path $\boldsymbol{R'} = \boldsymbol{R'_0}, \dots, \boldsymbol{R'_4}$ starting in the triangle BAC, BAC, ending in the triangle ACB, ACB, and around the vertex $U^{(-,-)}_{CA}$.

Consider the path $\mathbf{R}$ of Figure 11, and its depiction in Figure 12. We call such a path *bivalent* because the social choice has to move from $f(\mathbf{R}_0) = ABC$ to $f(\mathbf{R}_4) = BCA$, by the unanimity axiom. The notion of pivotal voter arises in such bivalent paths. The social choice has to exchange the preferences of the pair A, B and also A, C, because it starts in the edge $\{U^{(+,+)}_{AB}, U^{(-,-)}_{CA}\}$ and ends in the edge $\{U^{(-,-)}_{AB}, U^{(+,+)}_{CA}\}$. It does not change preferences over B, C, since the path keeps fixed the vertex $U^{(+,+)}_{BC}$.

Consider a sequence of profiles in which the first profile unanimously prefers an alternative x over another y, we change at each step the preference of a *single* individual from x over y to y over x until we arrive at the unanimous profile in which everyone prefers y over x. By unanimity, the first profile socially prefers x over y, whereas the last one y over x. Barberà [8] named the first voter who produces the change on the social preference from x over y to y over x, the *pivotal voter* of y over x. Denote this voter by k_{yx}.

In Section 5.2, we will use these paths to prove Theorem 2.1. Whereas in Section C we will compare this topological proof based on pivotal voters with the combinatorial ones by Geanakoplos [29] and Yu [56].

5.2 The proof based on pivotal voters

Following Geanakoplos [29] and Yu [56], we will first prove that all pivotal voters are the same, and then apply a simple argument to show that this pivotal voter is, in fact, a dictator.

Step 1: all pivotal voters are the same. Consider the path $\mathbf{R}$ of Figure 11 and its depiction in Figure 12. Notice that indeed all the triangles of the path share the vertex $U^{(+,+)}_{BC}$, and it starts in the edge $\{U^{(+,+)}_{AB}, U^{(-,-)}_{CA}\}$ and ends in the edge $\{U^{(-,-)}_{AB}, U^{(+,+)}_{CA}\}$. Traversing the path, we see that voter 1 changes its preferences twice, first from $\boldsymbol{R}_0$ to $\boldsymbol{R}_1$ (AB to BA) and then from $\boldsymbol{R}_1$ to $\boldsymbol{R}_2$ (AC to CA). The next two changes of preferences are by voter 2, from $\boldsymbol{R}_2$ to $\boldsymbol{R}_3$ (AB to BA) and then from $\boldsymbol{R}_3$ to $\boldsymbol{R}_4$ (AC to CA). We are interested in comparing k_{CA} with k_{BA}.

The fact that the image of this path in N_O has to exchange the preferences of the pair A, B and also A, C, means that the path in N_O has to cross the triangle BAC. The figure shows why it has to cross first the edge adjacent to U^{-}_{CA}, and then the one adjacent to the vertex U^{-}_{AB}, both of this edges incident on U^{+}_{BC}. The social preference has to change to B over A before it changes C over A, and given that in the path $\mathbf{R}$ the first changes are by voter 1, followed by the changes by voter 2, we conclude that that $k_{BA} \leq k_{CA}$.

This argument can be repeated using any path analogous to $\mathbf{R}$ around the green cycle in Figure 12, even in the opposite direction, such as $\mathbf{R'}$. That is, taking any two of the three unanimous green triangles labeled ABC, BCA or CAB, and the corresponding bivalent path connecting them clockwisely (that preserves along the path the vertex in the intersection of the two selected triangles). This proves three inequalities $k_{yx} \leq k_{zx}$, for the corresponding $x, y, z \in X$. Conversely, taking the three unanimous blue triangles labeled BAC, CBA and ACB and the corresponding bivariant paths connecting them counterclockwisely (as $\mathbf{R'}$), we obtain three additional inequalities $k_{xy} \leq k_{xz}$ for some $x, y, z \in X$. Joining the six inequalities we obtain that $k_{BA} \leq k_{CA} \leq k_{CB} \leq k_{AB} \leq k_{AC} \leq k_{BC} \leq k_{BA}$. So, there is a unique pivotal voter.

	1	2		1	2
	A	A		B	B
$\mathbf{R}_0$	B	B	$\mathbf{R}'_0$	A	A
	C	C		C	C
	B	A		A	B
$\mathbf{R}_1$	A	B	$\mathbf{R}'_1$	B	A
	C	C		C	C
	B	A		A	B
$\mathbf{R}_2$	C	B	$\mathbf{R}'_2$	C	A
	A	C		B	C
	B	B		A	A
$\mathbf{R}_3$	C	A	$\mathbf{R}'_3$	C	B
	A	C		B	C
	B	B		A	A
$\mathbf{R}_4$	C	C	$\mathbf{R}'_4$	C	C
	A	A		B	B

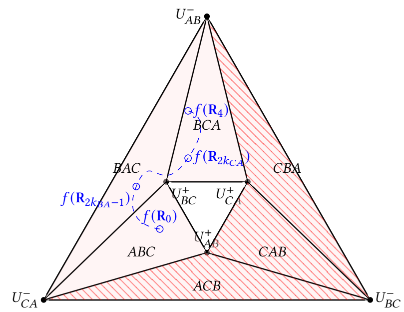


Fig. 11. On the left side, the sequences $\mathbf{R}$ and $\mathbf{R}'$ are defined. Writing an alternative on the top on another means that the one on top is preferred to the one in the bottom. On the right side there is a graphical representation of the paths defined by $f(\mathbf{R})$.

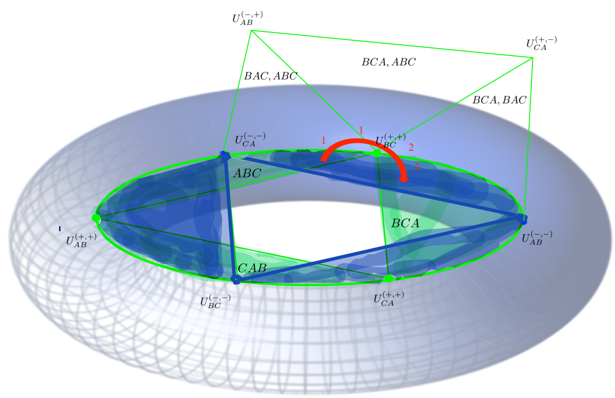


Fig. 12. The sequence $\mathbf{R} = \mathbf{R}_0, \ldots, \mathbf{R}_4$ in the complex N_I. The red curved arrow shows the order in which these triangles appear in $\mathbf{R}$, and it indicates that voter 1 changes its preference twice and then voter 2 changes its preference twice. For clarity, the triangle $\mathbf{R}_0$ is labeled with ACB, and the triangle $\mathbf{R}_4$ is labeled with CBA.

Surprisingly, as we will see on Section 3.1, the triangles conforming these six bivariant paths constitute a minimal subsimplex N'_I of N_I (see N'_I in Figure 6) that causes an impossibility. That is, the cylinder N'_I contained in the torus is sufficient to connect the unanimity vertices and the vertices with opposite pairwise preferences leading to an impossibility result. Whereas we use here 6 paths going across the 12 triangles of N'_I, in Section 3.1 they have been joined together in a single closed path. Using this closed path we will describe a geometric argument for the impossibility. Cutting this closed path into 6 paths, we have connected the geometrical arguments with the classical pivotal argument.

Thus, the domain does not need to contain all preferences and, consequentially, the whole complex N_I, to apply the arguments contained in this section.

Step 2: the pivotal voter is a dictator. It remains to prove that f is a projection over the k-th component. That is, $f(U^{\sigma}_{xy}) = U^{\sigma(k)}_{xy}$. However, this is immediate to see taking the definition of pivotal voter (for $n = 2$). When there are two voters, being a pivotal voter and a dictator is equivalent. The Figure 13 shows, as an example, how to use the definition of a pivotal voter to compute $f(U^{(+,-)}_{xy})$ when $k = 1$ and $k = 2$.

	S_0		S_1		S_2			S'_0		S'_1		S'_2	
Case $k=1$	y	y	x	y	x	x	Case $k=2$	y	y	x	y	x	x
	x	x	y	x	y	y		x	x	y	x	y	y
Social pref.	yx		xy		xy		Social pref.	yx		yx		xy	

Fig. 13. The table on the left represents a sequence of profiles $S = S_0, S_1, S_2$ starting from unanimity profile of y over x to x over y in which the pivotal voter is $k = 1$. Since $k = 1$ is the pivotal voter, the social preference changes in the first step, so $f(U_{xy}^{(+,-)}) = U_{xy}^{+}$. The table on the right represents the converse situation, when $k = 2$.

In Appendix C, we further discuss the correspondence of pivotal with the simplicial complex setting.

6 REDUCTION TO THE CASE OF $n = 2$ AND $|X| = 3$

We have proved Arrow's impossibility Theorem 2.1 for $|X| = 3, n = 2$. The proof of Theorem 2.1 for $|X| \geq 3, n \geq 2$ follows directly from Lemma 6.1 and 6.2, given that the case $|X| = 3, n = 2$ has been proved.

There are several works in which the proof of Arrow's theorem is only for $|X| = 3$ and/or $n = 2$ (e.g. [2, 17, 46, 51]). Using Lemma 6.1 and 6.2, all these proofs are extended to $|X| \geq 3$ and/or $n \geq 2$.

A few works have used inductive arguments over the number of voters or alternatives. In the fifties, Weldon [53] proved an impossibility theorem under a set of non-Arrovian axioms. Unlike our case, he could set the initial case of his inductive argument on the trivial case $n = 1$ (instead of $n = 2$). More recent works [2, 52] use inductive arguments using the base case $|X| = 3, n = 2$, as we do. However, our proof is more general. That is, whereas the results of Akashi [2, Lemma 1] and Tang and Lin [52, Lemma 1] are constrained to finite sets of alternatives, Lemma 6.1 works also for infinite X. In addition, the inductive step in [52, Lemma 2] is proved by contradiction using a large family of maps, while Lemma 6.2 uses only two, and using an explicit map that helps to understand the inductive step.

Lemma 6.1. *Let the number of voters be any $n \geq 2$. Arrow's impossibility theorem for $|X| = 3$ implies it for $|X| \geq 3$.*

Proof. Suppose that Arrow's theorem is true when $|X| = 3$. We prove that for any X (with $|X| \geq 3$) and any $F : W^n \to W$ satisfying unanimity and independence of irrelevant alternatives, F is dictatorial.

Given F, choose three distinct alternatives $x, y, z \in X$ and denote $\overline{W}_0$ the set of all strict orders over these three alternatives. Define an aggregation map $\overline{F} : \overline{W_0}^n \to \overline{W}_0$ as follows. The image of a profile $(\overline{R}_1, \ldots, \overline{R}_n) \in \overline{W_0}^n$ by $\overline{F}$ is the restriction of the ordering $F(R_1, \ldots, R_n) \in W$ on the set $\{x, y, z\} \subseteq X$, where for each i, R_i is any extension of $\overline{R}_i$ from $\overline{W}_0$ to W. Notice that the definition of $\overline{F}$ does not depend on the chosen extension because of the independence of irrelevant alternatives of F. Moreover, it is easy to check that $\overline{F}$ satisfies unanimity as well as independence of irrelevant alternatives. So, it follows that $\overline{F}$ is dictatorial because we have supposed that Arrow's theorem is true when $|X| = 3$. It remains to prove that F is also dictatorial.

If k is the dictator of $\overline{F}$, we will prove that it is also a dictator for F. Consider a profile $\mathbf{R} = (R_1, \ldots R_n) \in W^n$ where aR_kb for some $a, b \in X$. Then take a profile $\mathbf{R}' = (R'_1, \ldots R'_n) \in W^n$ satisfying that, for every i, $xR'_ibR'_iaR'_iy$ if bR_ia, and $aR'_iyR'_ixR'_ib$ if aR_ib.

Since k is a dictator of $\overline{F}$ and yR'_kx (k prefers a over b in R_k), we know that the image by $\overline{F}$ of the restriction of $\mathbf{R}'$ over $\overline{W}_0^n$ prefers y over x, hence $F(\mathbf{R}')$ also prefers y over x. Moreover, by unanimity, it holds that $aF(\mathbf{R}')y$ and $xF(\mathbf{R}')b$. Then, we obtain that $aF(\mathbf{R}')b$ from the relations $aF(\mathbf{R}')yF(\mathbf{R}')xF(\mathbf{R}')b$ using the transitivity. Finally, using the independence of irrelevant alternatives, we obtain that $aF(\mathbf{R})b$. Since this happens for every pair $a, b \in X$, k must be the dictator of F. □

The proof of the previous lemma, contrary to the ones in [2, 52], is not inductive. This fact enables us to reduce the cases of any cardinality of X to $|X| = 3$ in a single step.

Lemma 6.2. *Let the number of alternatives be any $|X| \geq 3$. If Arrow's impossibility theorem is true for $n = 2$ then it is true for $n > 2$.*

Proof. The proof is by induction on n. By hypothesis, the theorem is true when $n = 2$. Suppose that it is true for $n - 1$ and we will prove it for n.

Let $F^n : W^n \to W$ an aggregation map satisfying unanimity and independence of irrelevant alternatives. We will prove that F^n is dictatorial in three steps:

Step 1: We define the aggregation map on W^{n-1}, $F_1^{n-1}(R_1, \dots, R_{n-1}) := F^n(R_1, \dots, R_{n-1}, R_1)$. Since F_1^{n-1} satisfies unanimity and independence of irrelevant alternatives, the induction hypothesis guarantees that it has a dictator k_1. We will prove that if $k_1 \neq 1$, then k_1 is also a dictator for F^n.

Suppose $\mathbf{R} \in W^n$ and $xR_{k_1}y$. If the ordering of R_1 and R_n coincides on $\{x, y\}$, then $xF^n(\mathbf{R})y$ because F_1^{n-1} has k_1 as a dictator. Otherwise, we can suppose without loss of generality that xR_1y, yR_nx. Then, let $z \in X$ be an auxiliary alternative and let $\mathbf{R}' \in W^n$ be a profile which coincides with $\mathbf{R}$ over $\{x, y\}$, $xR'_{k_1}zR'_{k_1}y$ and z is below x and y for the remaining voters.

Since R'_1 and R'_n agrees on $\{y, z\}$ and k_1 is a dictator for F_1^{n-1}, we have that $zF^n(\mathbf{R}')y$. Moreover, $xF^n(\mathbf{R}')z$ because of the unanimity. Using the transitivity we obtain that $xF^n(\mathbf{R}')y$, and applying the independence of irrelevant alternatives we obtain that $xF^n(\mathbf{R})y$. So, k_1 is a dictator of F^n (if $k_1 \neq 1$).

Step 2: We define $F_2^{n-1}(R_1, \dots, R_{n-1}) := F^n(R_1, \dots, R_{n-1}, R_2)$. Using the inductive hypothesis, F_2^{n-1} has a dictator k_2. If $k_2 \neq 2$, apply a symmetric reasoning to the one in step 1 to deduce that k_2 is the dictator of F^n (if $k_2 \neq 2$).

Step 3: If $k_1 = 1$ and $k_2 = 2$, we show that n is the dictator of F^n. Let $\mathbf{R} \in W^n$ be a profile with xR_ny. Consider $z \in X$, and $\mathbf{R}' \in W^n$ coinciding with $\mathbf{R}$ over $\{x, y\}$, $xR'_nzR'_ny$, xR'_1z and zR'_2y. Using that 1 (resp. 2) is the dictator of F_1^{n-1} (resp. F_2^{n-1}) and the independence of irrelevant alternatives, we obtain that $xF^n(\mathbf{R})z$ (resp. $zF^n(\mathbf{R})y$). So, using transitivity, we obtain that $xF^n(\mathbf{R})y$. Finally we conclude that n is the dictator of F^n (if $k_1 = 1$ and $k_2 = 2$). □

The reader may wonder why Lemma 6.2 is inductive, instead of applying some direct argument extending from $n = 2$ to any number of voters (as we have done in Lemma 6.1). If such argument existed, it would allow to extend the theorem to an infinite number of voters. However, this is not possible because Arrow's impossibility is not true when n is infinite [23].

7 CONCLUSIONS

We have given new proofs of Arrow's theorem consisting of two parts. The first part deals with the base case of two voters and three alternatives, and we presented three different versions: using the index lemma, using pivotal voters, and using domain restrictions. The second part proves the general case by a simple reduction to the base case.

The first part shows that any aggregation function is dictatorial, because in essence it is mapping a torus onto a cylinder, in a continuous way, respecting unanimity. The argument sheds light on the remarkable algebraic topology proof of Baryshnikov [10], and makes it accessible to a wider audience. Also, it connects it to standard proofs of Arrow's theorem based on pivotal arguments, by explaining how the paths of such arguments move along the torus and the cylinder. Furthermore, it provided a guide on how to characterize the domain restrictions that allow non-dictatorial maps.

The conformation of our proofs, in two parts, suggests that the interesting geometry happens in the base case. We have considered domain restrictions on the base case, showing that there is a domain restriction where Arrow's impossibility is derived from the geometry in an intuitive way, and there is another domain restriction where it does not hold, yet it is not contractible.

We hope that bringing in combinatorial topology to social choice problems opens interesting opportunities for future work. These tools have been encountering many applications recently. Some examples are in concurrency [1], image processing [7], political structures [42], data analysis [35] and wireless networks [48].

In particular, combinatorial topology has been very useful in distributed computing [32]. We described some analogies that are worth exploring, since computing processes that communicate with each other need to agree on one of their inputs in many applications. Remarkably, while Sperner's lemma is the key to the impossibilities of tasks where processes need to reach agreement such as *consensus*, *set agreement* [5], *vector consensus* [45] and *interactive consistency* [26] (where domain restrictions are studied), for Arrow's impossibility, the key is the index lemma, as it is for tasks related to renaming and weak symmetry breaking [14, 30]. Here we studied only Arrow's setting, where the aggregation map is defined directly on the input complex; it would be interesting to explore the case where the agents can communicate with each other and subdivisions of the input complex arise. Notice that the index lemma is preserved under subdivisions e.g. [30, Corollary 4]. However, we are not aware of a distributed task where the impossibility is proved in dimension 2, and then extended easily to any dimension.

Acknowledgments. The authors would like to thank Isaac Lara for his valuable comments.

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A TASKS AND DISTRIBUTED COMPUTING

There are many good books about distributed computing e.g. [6, 49]. Here we give a very brief introduction to the notion of a task, and its representation using simplicial complexes, following the overview of the topology approach to distributed computing [32], and provide more details about the analogy with Arrow's theorem.

A task is a specification of a concurrent problem, namely, a problem to be solved by a set of individual computing processes communicating with each other. Each process runs its own sequential program code, that includes instructions to communicate with other processes. Typical ways of communicating is by sending messages or by writing and reading a shared memory. A task is a distributed version of a function. When there is a single computing process, the function f specifies, for each possible initial input x, the value $f(x)$ that the process should compute. In a distributed system composed of several processes, each one gets only part of the input x. Thus, we may think of x as a vector $(x_1, x_2, \ldots, x_n)$, for n processes, where initially process i gets as input x_i, and does not know what the inputs of the the other processes are. Then, the processes run their individual programs, communicating with each other, and eventually produce individual local output values, defining a vector $(y_1, y_2, \ldots, y_n)$, where y_i is the output value of process i. The task defines an input output relation Δ, that specifies, for each possible input vector x, a set of legal output vectors y. A classic example is binary *consensus*, where the possible inputs x_i are taken from the set $\{0, 1\}$, and there are only two possible output vectors: either everybody decides 0 or everybody decides 1. Then, $\Delta(x)$ states that if everybody starts with the same input, then everybody decides that input, else, it is valid to decide either of the two output vectors.

A task can be defined in terms of simplicial complexes [32]. For a set of processes $\{id_1, \ldots, id_k\}$, a set $\sigma = \{(id_1, x_1), \ldots, (id_k, x_k)\}$ is used to denote the input values, or output values, where x_i denotes the value of the process with identity id_i, either an input value or an output value. The elements of σ are pairs, called *vertices*. And they are said to be colored by the identities id_i's. A set σ as above is called a *chromatic simplex*, because the vertices are colored with distinct ids. If the values are input values, it is an *input simplex*, if they are output values, it is an *output simplex*. An *input vertex* $v = (id_i, x_i)$ represents the initial state of process id_i, while an *output vertex* represents its decision. The *dimension* of a simplex σ, denoted $\dim(\sigma)$, is $|\sigma| - 1$, and it is *full* if it contains n vertices, one for each process. A subset of a simplex, which is a simplex as well, is called a *face*. The set of possible input simplexes forms a *complex* because its sets are closed under containment. Similarly, the set of possible output simplexes also form a complex.

The dimension of a complex K is the largest dimension of its simplexes, and K is *pure* of dimension k if each of its simplexes is face of a k-dimensional simplex. In distributed computing, the simplexes (and complexes) are *chromatic*, since each vertex v of a simplex is labeled with a distinct process identity, and we usually get pure complexes. The set of processes identities in an input or output simplex σ is denoted $ID(\sigma)$.

A *task* T for n processes is a triple $(\mathcal{I}, \mathcal{O}, \Delta)$ where $\mathcal{I}$ and $\mathcal{O}$ are pure chromatic $(n-1)$-dimensional complexes, and Δ maps each simplex σ from $\mathcal{I}$ to a subcomplex $\Delta(\sigma)$ of $\mathcal{O}$, satisfying:

(1) $\Delta(\sigma)$ is pure of dimension $\dim(\sigma)$,
(2) For every τ in $\Delta(\sigma)$ of dimension $\dim(\sigma)$, $ID(\tau) = ID(\sigma)$,

(3) If σ, σ' are two simplexes in $\mathcal{I}$ with $\sigma' \subset \sigma$ then $\Delta(\sigma') \subset \Delta(\sigma)$.

We say that Δ is a *carrier map* from the input complex $\mathcal{I}$ to the output complex $\mathcal{O}$.

Thus, each input simplex $\sigma \in \mathcal{I}$ defines an initial configuration of the distributed system. After the processes run their local algorithms and communicate with each other, they eventually stop, and end up in a final configuration σ'. The simplex τ' is of the same form of the input and output simplexes, except that in a pair (id_i, x_i), x_i denotes the final local state of process id_i. This local state x_i determines the output value decided by the process id_i, and is denoted by $\delta(id_i, x_i)$.

Actually, there may be many possible runs all starting on input σ, because of possible failures, different speed of execution of the processes, etc, The set of all possible final configurations can also be represented as a chromatic complex, denoted $\mathcal{P}(\sigma)$. The *protocol complex*, $\mathcal{P}$, is the union of $\mathcal{P}(\sigma)$, over all $\sigma \in \mathcal{I}$. The task is solved, if there exists a chromatic simplicial map δ from $\mathcal{P}$ to $\mathcal{O}$ respecting Δ, such that $\delta(\mathcal{P}(\sigma))$ is contained in $\Delta(\sigma)$. The simplicial map δ is chromatic in the sense that it sends vertices to vertices preserving ids.

This approach to the theory of distributed computing is so successful, because the solvability of a task depends on the topological properties of the protocol complex, and how they relate to the topological properties of the task. Furthermore, the protocol complex preserves topological properties of the input complex. How well this topological properties are preserved, depends on the specific assumptions about the distributed system model: how many processes can fail, what types of failures are possible, how the processes communicate with each other, and their relative speed of execution. Many different models have been analyzed, and the topological properties preserved by their protocol complexes identified [32].

Remarkably, in the most basic model, called *wait-free*, if we denote the protocol complex after t rounds of communication by $\mathcal{P}_t$, then $\mathcal{P}_{t+1}$ is a chromatic subdivision of $\mathcal{P}_t$. The main theorem [33] is that a protocol in the wait-free model solves a task $(\mathcal{I}, \mathcal{O}, \Delta)$, if and only if there exists a chromatic subdivision of $\mathcal{I}$ and a chromatic simplicial map from the subdivision to $\mathcal{O}$ respecting Δ.

Notice that the protocol complex $\mathcal{P}_t$ is equal to the input complex $\mathcal{I}$, when $t = 0$, before any communication takes place. This is precisely the situation corresponding to Arrow's setting. In this case, a 0-round protocol solves a task if and only if there exists a chromatic simplicial map f from $\mathcal{I}$ to $\mathcal{O}$ respecting Δ. This explains the analogy of distributed computing with Arrow's impossibility theorem, in the form of Theorem 2.1, where the input/output relation is requiring only that $f(U_{\alpha\beta}^{(+,\cdots,+)}) = U_{\alpha\beta}^{+}$.

We present the relation with distributed computing in a conference version of this paper [47], the idea is that the processes of the task correspond to the pairs of alternatives $\mathcal{P} = \{AB, BC, AC\}$, called also *ids*. Thus, we consider *chromatic* simplicial complexes, where the vertices of each triangle are labeled with distinct process ids from $\mathcal{P}$. There are four possible individual inputs $\{++, --, +-, -+\}$, while the possible individual outputs are $\{+, -\}$. The output complex N_O consists of all chromatic triangles, with each vertex labeled with an output value from $\{+, -\}$, except for the two triangles labeled with the same value. Thus, N_O is the output complex of the *weak symmetry breaking* task e.g. [14, 37, 41]. Similarly, N_I consists of all chromatic triangles whose vertices are labeled with input values from $\{++, --, +-, -+\}$, except the 16 triangles whose vertices have the same sign in the first or in the second component. Thus, N_I includes the (torus) complex of the renaming task [44], where every triangle is labeled with distinct values from $\{++, --, +-, -+\}$, plus 12 additional triangles where one value repeats twice, illustrated in Figure 4 (for N_I only schematically).

B INDEX LEMMA AND THE COMPLEX N_I

Here we present the generalized version of the index lemma, and show that it holds on N_I.

Definition B.1. Let K be a simplicial complex of dimension 2 satisfying that every simplex of dimension 1 has a single or an even number of 2-simplices containing it. An *orientation on* K is an orientation on every 2-simplex satisfying that the induced orientations on the 1-simplices by the 2-simplices have to be opposite by pairs.

Fig. 14. The simplicial complex on the left is oriented because the induced orientations on the inner edge are opposite. However, the right one is not because it has three orientations in one direction and one on the opposite direction.

As in the original framework, let K be an oriented simplicial complex of dimension 2 with each vertex labeled with a color from $\{0, 1, 2\}$. The content C of K is the number of tricoloured triangles in K counted $+1$ if the order of the labeling agrees with the orientation (see the right side of Figure 15) and -1 otherwise. The index I of K is the number of edges $\overrightarrow{01}$ on the boundary counted $+1$ if the order of the vertices agrees with the orientation and -1 otherwise. Now, we can state and proof the index lemma for oriented simplicial 2-complexes.

Theorem B.2 (Index lemma). *Let K be a 3-colored oriented simplicial complex of dimension 2. Then, the index I is equal to the content C.*

Proof. Let S be the number of edges $\overrightarrow{01}$ counted according to the orientation. We will prove that $I = S$ and $C = S$. First, we will see that the contribution of every interior edge $\overrightarrow{01}$ is equal to 0. Since every interior edge has an even number of incident 2-simplices, by definition of being oriented, their contribution is 0. Then $I = S$.

For every triangle in the complex, the contribution is only non-zero if the triangle is tricoloured. If it is not tricoloured, it is 0 because, in case it has at least one 0 and one 1, the third vertex has to be coloured by 0 or 1, then one edge compensates the other. Otherwise, if it is tricoloured, its contribution is the same as the content's contribution (see Figure 15).

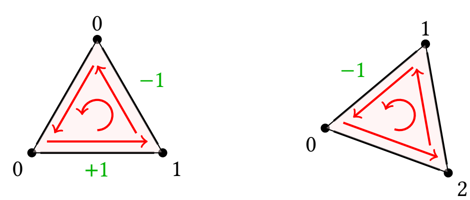

Fig. 15. On the left, the contribution of the simplex is 0 because the two edges $\overrightarrow{01}$ compensate each other. On the right, the contribution of the tricolored triangle is -1.

□

Now we provide N_I with an orientation. Recall that we assume that the number of alternatives is $|X| = 3$ and the number of voters is $n = 2$.

Proposition B.3. *The complex N_I is orientable.*

Proof. We will define an orientation on N_I as follows. For every 2-simplex $\Delta = \{U_{AB}^{\sigma_1}, U_{BC}^{\sigma_2}, U_{CA}^{\sigma_3}\}$ we define its parity as the product of all the signs of σ_1, σ_2 and σ_3. For instance, if $\sigma_1 = (+,+)$, $\sigma_2 = (+,-)$ and $\sigma_3 = (-,-)$, the parity is -1 (see Figure 16a). We define the orientation of this 2-simplex as clockwise ($AB \to CA \to BC \to AB$) if its parity is -1 and ($AB \to BC \to CA \to AB$) if its parity is 1.

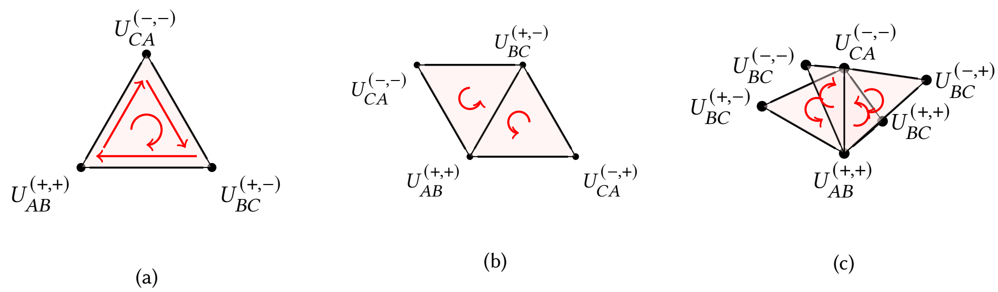

Fig. 16. (a) Since the parity of the triangle is negative, the orientation is $U_{AB}^{(+,+)} \leftarrow U_{BC}^{(+,-)} \leftarrow U_{CA}^{(-,-)}$. (b) Two triangles sharing the edge $\{U_{AB}^{(+,+)}, U_{BC}^{(+,-)}\}$. (c) Four triangles sharing the edge $\{U_{AB}^{(+,+)}, U_{CA}^{(-,-)}\}$

This is an orientation because for every non-boundary edge, there are an even number of 2-simplices containing it, and they are paired by their opposite induced orientations. For example, consider the edge $\{U_{AB}^{\sigma_1}, U_{BC}^{\sigma_2}\}$, this edge only can be completed with a vertex indexed as $U_{CA}^{\sigma_3}$ for some compatible $\sigma_3 \in \{+,-\}^n$ constrained by the transitivity property. That is, for every component $i \in \{1,\dots n\}$, if $\sigma_1(i) = \sigma_2(i) = +$ (resp. $\sigma_1(i) = \sigma_2(i) = -$, then $\sigma_3(i) = +$ (resp. $\sigma_3(i) = +$). However, if $\sigma_1(i)$ and $\sigma_2(i)$ have different signs, both signs are compatible in $\sigma_3(i)$. We can conclude that the admissible σ_3 are exactly 2^k (where k is equal to the number of voters i on the third situation). And, since by hypothesis $\{U_{AB}^{\sigma_1}, U_{BC}^{\sigma_2}\}$ is not in the boundary, $k > 0$.

Second, we can pair these 2^k 2-simplices saying that $\{U_{AB}^{\sigma_1}, U_{BC}^{\sigma_2}, U_{CA}^{\sigma_3}\}$ and $\{U_{AB}^{\sigma_1}, U_{BC}^{\sigma_2}, U_{CA}^{\sigma_3'}\}$ are paired if σ_3 and σ_3' are equal on each component but one. Then the parity associated to every triangle of a pair is opposite to the other member, so, their contribution on the edge $\{U_{AB}^{\sigma_1}, U_{BC}^{\sigma_2}\}$ determined by the induced orientations is also opposite. □

C PIVOTAL VOTERS AND PATHS IN N_I

In this section, we further discuss the correspondence of the pivotal setting with the simplicial complex setting of Section 5.

To discuss the role of pivotal voters and the paths defined by sequences, consider as an example the path **R** defined in Figure 11. This path starts and ends in the inner cylinder of N_I, that is, the unanimity simplices (see Figure 5). Obviously, this cylinder is identified with N_O because of the unanimity property of the aggregation map f. The remaining simplices $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$ of the path link the inner cylinder with the outer one (see the complex at the right of Figure 17).

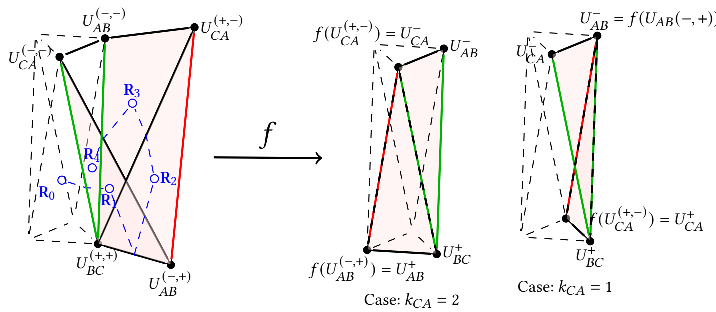

Fig. 17. The figure in the right represents the simplices $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$ linking the inner cylinder of N_I (green edges) with the outer cylinder (red edge) and the path $\mathbf{R}$. The figure in the middle represents the folding of the hinges and the inner cylinder when $k_{CA} = 2$; the one on the left, when $k_{CA} = 1$.

When the aggregation map f is applied, the inner cylinder remains invariant because we have identified it with N_O, but the outer cylinder and the links (the torus joining both cylinders) are compressed into the inner cylinder. We have to imagine the simplices between the cylinders (from Figure 5), the ones linking the cylinders, playing the role of "hinges", folding into each other so that the two cylinders fit together.

In Figure 17 we can see that the *hinge* $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$ can fold two ways. It folds one way or another depending on the value of k_{CA}. Notice that its folding also determines the value of $f(U_{AB}^{(-,+)})$, and this determination of the folding is the geometrical representation of the inequality $k_{CA} \leq k_{BA}$, proved in Section 5.2. Moreover, the simplex $\mathbf{R}_3$ also belongs to another hinge, which at the same time will represent an inequality. So, all hinges are connected and they constrain each other foldings. Consequently, there are only two possible ways to fold and fit both cylinders together: the two projections.

D SCHEMA OF HOW OBTAIN AGGREGATION MAPS ON RESTRICTED DOMAINS

Here we give an overview of the procedure we have followed to obtain the maps of Figure 7 in Section 3.2.

First, we have studied the scenario in which only a critical pair has been removed from N_I. Notice that if a non-dictatorial map f exists in a domain D like this, then in every subdomain $D' \subseteq D$ we will have as a non-dictatorial map $f_{|D'}$. This assertion is true because D and D' have the same vertices.

As in the proof of Theorem 3.1, we focus on the domain D obtained by removing the critical pair $(\mathbf{R_1}, \mathbf{R_2})$ being $\mathbf{R_1} = (BAC, ACB)$. We will define a non-dictatorial map f, but it has to satisfy certain conditions. First, the boundary of $\mathbf{R_1}$ would have to be mapped on the boundary of N_O, otherwise the f could be extended to another map defined on C_1 and, using the arguments in Section 3.1, we conclude it would be dictatorial. Using that $f(U_{CA}^{(-,-)}) = U_{CA}^{-}$, we state that $f(U_{BC}^{(+,-)}) = U_{BC}^{-}$ and $f(U_{AB}^{(-,+)}) = U_{AB}^{-}$. However, we find clearer using the same nomenclature as in Section 3.1, working with the green and blue cycles. Using this approach, the edge $\{U_{BC}^{(+,-)}, U_{AB}^{(-,+)}\}$ is mapped to the edge $\alpha = \{U_{BC}^{-}, U_{AB}^{-}\} \in N_O$ (see Figure 18c).

Following the same argument as above, we would conclude that the boundary of $\mathbf{R_2}$ should be also mapped on the boundary of N_O. However, instead of developing an argument for each feasible $\mathbf{R_2}$, we will start our argument uniquely considering the image of the boundary of $\mathbf{R_1}$ fixed (i.e. $f(\{U_{BC}^{(+,-)}, U_{AB}^{(-,+)}\}) = \alpha$). Moreover, we will use the same argument to propose the five candidates to aggregation maps (one for each triangle $\mathbf{R_2}$).

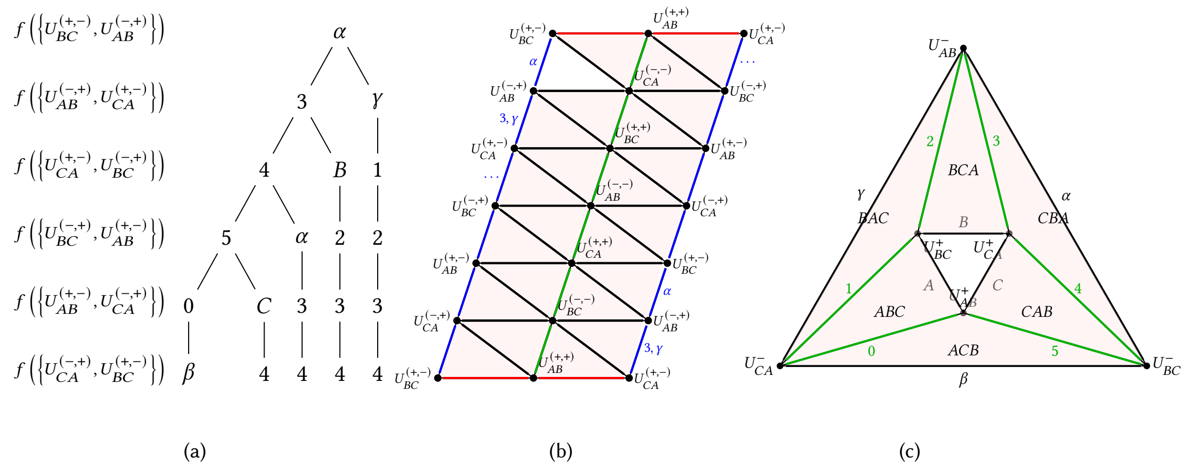

Fig. 18. (a) The tree representing the admissible mappings of the blue edges when $\mathbf{R_1} = \{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}, U_{CA}^{(-,-)}\}$. The first row of the three represents the admissible image of the edge $\{U_{BC}^{(+,-)}, U_{AB}^{(-,+)}\}$, the second row the admissible images of $\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\}$, and successively until the edge $\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\}$. So, a tupple represents an admissible mapping of the blue cycle. For instance, the tuple $(\alpha, 3, B, 2, 3, 4)$ represents a map in which the first blue edge is mapped to α, the second to 3 and the sixth to 4. (b) The torus without the triangle $\mathbf{R_1} = \{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}, U_{CA}^{(-,-)}\}$ and some admissible mappings of the edges represented. (c) The N_O complex with their edges labeled.

Our strategy will be the following: We will determine all possible images of the blue path (i.e. the antiunanimity vertices), using exclusively the simplicial properties of C_1 and the unanimity axiom. For instance, taking into account that $f(U_{AB}^{(-,+)}) = U_{AB}^{-}$ and $f(U_{BC}^{(+,+)}) = U_{BC}^{+}$, the image of $U_{CA}^{(+,-)}$ is a priori not determined. In other words, the edge $\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\}$ can be mapped either in 3 or in γ (second row of the tree in Figure 18a). If it were mapped to γ, using the same reasoning, we conclude that the next edge $\{U_{CA}^{(+,-)}, U_{BC}^{(-,+)}\}$ must be mapped in 1. Otherwise, if $f(\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\}) = 3$, then $f(\{U_{CA}^{(+,-)}, U_{BC}^{(-,+)}\})$ could be 4 of B (third row in Figure 18a).

We repeat the same types of arguments until we have mapped all possible images for the blue cycle. In Figure 18a each branch corresponds to a candidate for the mapping. Starting with α as the image of $\{U_{BC}^{(+,-)}, U_{AB}^{(-,+)}\}$ and finishing with 4 or β as the image of $\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\}$.

We have five candidates for the image of the blue cycle, equivalently, five candidates for an aggregation map. By the definition of f, we know that these maps are simplicial in C_1, but we need to verify that these candidates are simplicial in the whole domain $N_I \setminus \{(\mathbf{R_1}, \mathbf{R_2})\}$ (for a suitable $\mathbf{R_2}$).

It turns out that the unique obstacle for each of these five maps to be simplicial is overcomed by removing a single triangle from C_2. That is, for each critical pair $(\mathbf{R_1}, \mathbf{R_2})$ (being $\mathbf{R_1} = (BAC, ACB)$), we obtain a unique non dictatorial aggregation map.

Given a triangle $\mathbf{R_2} \in C_2$, as we have argued before, it has to be mapped to the boundary of N_O, then the unique map compatible, is the one which maps the blue edge of $\mathbf{R_2}$ in the boundary of $\mathbf{R_2}$. For example, if $\mathbf{R_2} = (ABC, BCA)$, the unique candidate to be simplicial is the map which maps $\left\{U_{AB}^{(+,-)}, U_{CA}^{(-,+)}\right\}$ to C. That is, the map represented by the tupple $(\alpha, 3, 4, 5, C, 4)$.