One-factor model of liquidity risk

Osadchiy, Maksim

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Maksim Osadchiy[1]

Abstract

Credit and liquidity risks at the bank level depend on idiosyncratic and systematic (market) risks at the firm level. Portfolio effect transforms idiosyncratic risk into expected factor and leaves only systematic risk. Dependence only on market risk allows evaluating credit and liquidity risk using one-factor models. Since market risk is common to both credit risk and liquidity risk, it is useful to evaluate their joint distribution in a closed form.

The one-factor Vasicek model was designed to evaluate credit risk – the probability distribution of the portfolio loss. The one-factor model proposed in the paper is designed to evaluate liquidity risk. Combination of credit risk and liquidity risk models is used to evaluate the joint distribution of credit and liquidity risks.

Keywords: liquidity risk; credit risk; Vasicek model; barrier option; IRB

Introduction

At the heart of modern banking risk management is the Internal ratings-based approach. The IRB approach underlies Basel II and Basel III. In turn, the IRB approach is based on the Vasicek distribution, which evaluates credit risk taking into account the correlation of borrowing firms' assets. The Vasicek model takes into account the peculiarities of the collective behavior of borrowers associated with the common dependence of their business on market risk.

The Vasicek model was created in 1987, 35 years ago. It was used in Basel II in 2005, and since than no substantial progress has been made in applying the results of modern financial mathematics to the regulation of the banking sector. Besides, the Vasicek model has a significant drawback: it does not take into account premature defaults.

The liquidity risk of a bank is also significantly dependent on similar features of the collective behavior of firms, since deposit outflow is just as strongly dependent on the overall market risk as probability of their default. Meanwhile, liquidity risk evaluations used in modern risk management do not take into account this correlation, which leads to an underestimation of liquidity risk.

It is especially important to evaluate joint distribution of credit and liquidity risks. Meanwhile, in modern banking risk management, this issue is not considered.

Thus, the main purpose of the paper is to apply the Vasicek approach to evaluation of liquidity risk and to joint evaluation of credit risk and liquidity risk.

The paper is organized as follows. Section 1 reviews the related literature. Section 2 develops the model for assessing liquidity risk taking into account the correlation effect in the style of the Vasicek model, and premature defaults are taken into account. The dependence of the default probability on parameters is considered. A numerical example is also considered. Section 3 is devoted to joint evaluation of credit risk and liquidity risk. Section 3 concludes.

[1] CFB Bank (Russia, Moscow). Email: OsadchyMS@cfb.ru

1. Literature Review

On the base of the Black-Scholes model Robert Merton (Merton, 1974) proposed the first structural credit risk model for assessing the default probability of the firm and valuation of the debt. Merton modeled the firm's equity as a European vanilla call option written on its assets.

Oldrich Vasicek (Vasicek 1987) created the model of assessing risk of loan portfolio on the base of the Merton model. Vasicek modeled the loan portfolio as a portfolio of binary cash-or-nothing call-options written on the assets of borrowing firms, while the size of the firm's debt is the strike price. The model considers two sources of credit risk: the idiosyncratic risk and the systematic risk. The portfolio effect completely eliminates the idiosyncratic risk, while the systematic risk remains.

The MtM credit risk model KMV Portfolio ManagerTM was constructed on the base of the Vasicek approach. This commercial model was used in the AIRB approach (BIS 2005).

From the study of the simplest options - European vanilla options, researchers quickly moved on to the study of more complex exotic options, and, in particular, barrier options. Barrier options were first evaluated by Merton (1973). And the first results were immediately applied to credit risk modeling. Black and Cox (1976) calculated the default probability of a firm, taking into account the possibility of a premature default.

However, the generalization of the Vasicek model to the case of premature defaults is still not possible, since there is no closed form solution to the corresponding problem.

2. Model of liquidity risk valuation

Assume the size of the deposit is proportional to the value of the firm's assets. Since assets obey geometric Brownian motion, hence the firm's deposit also obeys the geometric Brownian motion. The Wiener process that governs this motion consists of two components - idiosyncratic and systematic (market) risks. The portfolio effect eliminates idiosyncratic risk. The portfolio of deposits also obeys the geometric Brownian motion, leaving only systematic risk. The outflow of deposits reduces liquidity. If the outflow reaches a critical level, then a default occurs. The probability of default is calculated according to the formula known from the theory of barrier options.

Consider a bank portfolio of deposits. Let the firm k has only one deposit, $k = 1, \dots, n$. Size of the deposit is proportional to the value of the assets of the firm

$$D_k(t) = \beta_k V_k(t)$$

If assets grow, the firm increases the deposit. If assets decline, then the firm reduces the deposit.

The value of assets $V_k(t)$ of the firm k obeys the geometric Brownian motion with the risk-free rate r and the volatility σ:

$$dV_k(t) = rV_k(t)dt + \sigma V_k(t)dW_k(t)$$

where $W_k(t)$ is the Wiener process. Assume

$$W_k(t) = \sqrt{1-\rho}\,w_k(t) + \sqrt{\rho}\,w(t)$$

where $w_k(t)$ is the idiosyncratic (entity specific) Wiener process, $w(t)$ is the systematic (market) Wiener process; processes $w_k(t)$, k=1,…,n and $w(t)$ are independent; and ρ is the correlation coefficient.

The size of deposit portfolio obeys the geometric Brownian motion

$$D(t) = D(0)e^{\tilde{\nu}t+\tilde{\sigma}w(t)}$$

with deposit portfolio volatility

$$\tilde{\sigma} = \sqrt{\rho}\sigma$$

and deposit portfolio drift

$$\tilde{\nu} = r - \frac{\tilde{\sigma}^2}{2}$$

For the proof see Appendix 1.

Outflow of deposits during period t equals

$$D(0) - D(t)$$

Let the critical level of liquidity outflow is C. Accordingly the barrier

$$Y = D(0) - C$$

is the critical level of the deposit portfolio. If $D(t)$ falls below the barrier Y, then there is a shortage of liquidity, and a default occurs.

Since the size of the deposit portfolio obeys geometric Brownian motion, the well-known formula of the theory of barrier options can be used:

Probability of default during period t equals

$$\mathbb{P}(M(t) \leq Y) = \Phi\left(\frac{y - \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) + e^{\alpha y}\Phi\left(\frac{y + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

where the minimum to date for the size of deposit portfolio

$$M(t) = \min_{s\in[0,t]} D(s)$$

$$y = ln\left(Y/D(0)\right)$$

$$\alpha = \frac{r - \frac{\tilde{\sigma}^2}{2}}{\frac{\tilde{\sigma}^2}{2}}$$

the “direct” distance to default

$$\frac{y - \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}$$

the “image” distance to default

$$\frac{y + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}$$

$\Phi(.)$ is the standard normal CDF. For the proof see Appendix 2. The first term of the sum is the probability of a “mature” default, and the second term is the adjustment for a “premature” default.

If the regulator limits the default probability $\mathbb{P}(M(t) \leq Y) \leq \beta$, then the bank can choose the ratio of the size of liquidity C and the initial size of deposit portfolio $D(0)$ so as to comply with this limitation on liquidity risk.

The equation $\mathbb{P}(M(t) \leq Y) = \beta$ allows us to determine the minimum allowable value of the parameter

$$y = ln\big(Y/D(0)\big) = ln\big(1 - C/D(0)\big)$$

Dependence of default probability on parameters

Easy to check that the intuitively obvious inequalities hold:

$$\frac{\partial}{\partial \rho}\mathbb{P}(M(t) \leq Y) \geq 0$$
$$\frac{\partial}{\partial \sigma}\mathbb{P}(M(t) \leq Y) \geq 0$$
$$\frac{\partial}{\partial y}\mathbb{P}(M(t) \leq Y) \geq 0$$
$$\frac{\partial}{\partial t}\mathbb{P}(M(t) \leq Y) \geq 0$$
$$\frac{\partial}{\partial r}\mathbb{P}(M(t) \leq Y) \leq 0$$

If volatility $\tilde{\sigma} \to 0$ ($\sigma \to 0$ or $\rho \to 0$) then default is impossible:

$$\mathbb{P}(M(t) \leq Y) = 0$$

If

$$y = 0 \Longrightarrow Y = D(0) \Longrightarrow C = 0$$

then default is inevitable:

$$\mathbb{P}(M(t) \leq Y) = \Phi\left(\frac{-\tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) + \Phi\left(\frac{\tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) = 1$$

Numerical example

Let

$$r = 0.02$$
$$\sigma = 0.2$$
$$t = 1$$
$$\frac{C}{D(0)} = 0.1$$

The table shows the dependence of the default probability $\mathbb{P}(M(t) \leq Y)$ on the correlation ρ:

ρ	$\mathbb{P}(M(t) \leq Y)$
20%	19.2%
50%	43.2%
100%	59.8%

3. Joint distribution of credit and liquidity risks

Since both credit risk and liquidity risk depend only on market risk, it is possible to calculate the joint distribution of credit and liquidity risks also using a formula known from the theory of barrier options.

Consider a portfolio of n loans, each with a face value 1 and a maturity t. Portfolio loss is equal to the share of defaulted firms

$$Loss = \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{V_k(t)\le L_k}$$

where

$$V_k(t) = V_k(0)e^{\nu t+\sigma\left(\sqrt{1-\rho}w_k(t)+\sqrt{\rho}w(t)\right)}$$

$$\mathbb{I}_A = \begin{cases} 1, & if\ A\ is\ true \\ 0, & o/w \end{cases}$$

L_k is the size of liabilities of the firm k.

Condition of default

$$V_k(t) \le L_k$$

can be written as

$$V_k(0)e^{\nu t+\sigma\left(\sqrt{1-\rho}w_k(t)+\sqrt{\rho}w(t)\right)} \le L_k$$

Assume

$$\frac{L_k}{V_k(0)} = \frac{L}{V(0)}$$

for each $k = 1 \dots n$, where

$$L = \sum_{k=1}^{n} L_k$$

Hence the default condition

$$\nu t + \sigma\left(\sqrt{1-\rho}w_k(t) + \sqrt{\rho}w(t)\right) \le \ln(L/V(0))$$

$$w_k(t) \le \frac{\ln(L/V(0)) - \nu t - \sigma\sqrt{\rho}w(t)}{\sigma\sqrt{1-\rho}}$$

$$Z_k(t) \le \frac{\ln(L/V(0)) - \nu t - \sigma\sqrt{\rho}w(t)}{\sigma\sqrt{t}\sqrt{1-\rho}}$$

Hence the loss of portfolio conditional on the market shock $Z(t)$ equals

$$Loss(Z(t)) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{Z_k(t)\le\frac{\ln(L/V(0))-\nu t-\sigma\sqrt{\rho}w(t)}{\sigma\sqrt{t}\sqrt{1-\rho}}}$$

Due to the law of large numbers

$$Loss(Z(t)) = \Phi\left(\frac{\ln(L/V(0)) - \nu t - \sigma\sqrt{\rho}w(t)}{\sigma\sqrt{t}\sqrt{1-\rho}}\right)$$

Hence the portfolio loss $Loss(Z(t))$ doesn't obey the geometric Brownian motion.

If

$$Loss(Z(t)) \ge \lambda$$

where λ – some critical level of portfolio loss, then

$$\tilde{\sigma}w(t) + \tilde{\nu}t \le x(t)$$

where

$$x(t) = \ln(L/V(0)) + \frac{\hat{\sigma}^2}{2}t - \hat{\sigma}\sqrt{t}\Phi^{-1}(\lambda)$$

$$\hat{\sigma} = \sigma\sqrt{1-\rho}$$

Hence the probability of default due to both loss of liquidity and loss of capital is

$$\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s) + \frac{\tilde{\nu}}{\tilde{\sigma}}s\right) \le \frac{y}{\tilde{\sigma}}, w(t) + \frac{\tilde{\nu}}{\tilde{\sigma}}t \le \frac{x(t)}{\tilde{\sigma}}\right)$$

$$= \Phi\left(\frac{y - \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) + e^{\alpha y}\Phi\left(\frac{y + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) - e^{\alpha y}\Phi\left(\frac{2y - x(t) + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

if

$$y \le x(t)$$

For the proof see Appendix 2.

Hence

$$\mathbb{P}(M(t) \le Y, Loss(Z(t)) \ge \lambda) = \Phi\left(\frac{y - \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) + e^{\alpha y}\Phi\left(\frac{y + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) - e^{\alpha y}\Phi\left(\frac{2y - x(t) + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

Since

$$\mathbb{P}(M(t) \le Y) = \Phi\left(\frac{y - \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) + e^{\alpha y}\Phi\left(\frac{y + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

the amendment

$$e^{\alpha y}\Phi\left(\frac{2y - x(t) + \tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

is the default probability due to loss of liquidity, provided that there is no default due to loss of capital.

The disadvantage of using the Vasicek model and the liquidity risk valuation model together is that the Vasicek model takes into account only "mature" defaults, while the liquidity risk valuation model also takes into account "premature" defaults.

4. Conclusion

The paper builds a one-factor liquidity risk model that takes into account premature defaults. The main result of the article is the calculation of the probability of default in the event of a liquidity outflow. The joint distribution of credit risk and liquidity risk was also calculated. Calculations are based on the theory of barrier options.

Appendix 1

The solution of the stochastic differential equation

$$dV_k(t) = rV_k(t)dt + \sigma V_k(t)dW_k(t)$$

is

$$V_k(t) = V_k(0)e^{\nu t + \sigma W_k(t)}$$

where drift

$$\nu = r - \frac{\sigma^2}{2}$$

Since size of deposit of firm k

$$D_k(t) = \beta_k V_k(t)$$

then

$$D_k(t) = D_k(0)e^{\nu t + \sigma W_k(t)}$$

Since

$$W_k(t) = \sqrt{1-\rho}w_k(t) + \sqrt{\rho}w(t)$$

then

$$D_k(t) = D_k(0)e^{\nu t + \sigma\left(\sqrt{1-\rho}w_k(t) + \sqrt{\rho}w(t)\right)}$$

The size of deposit portfolio

$$D(t) = \sum_{k=1}^{n} D_k(t)$$

weight of the deposit in deposit portfolio

$$c_{k,n} = D_k(0)/D(0)$$

Due to the Kolmogorov's strong law of large numbers if

$$\sum_{k=1}^{n}\left(\frac{c_{k,n}}{k}\right)^2 < \infty$$

then

$$\begin{aligned} D(t) = \sum_{k=1}^{n} D_k(t) &= D(0)e^{\nu t + \sigma\sqrt{\rho}w(t)}\sum_{k=1}^{n} c_{k,n}e^{\sigma\sqrt{1-\rho}w_k(t)} \\ &= D(0)e^{\nu t + \sigma\sqrt{\rho}w(t)}\sum_{k=1}^{n} c_{k,n}e^{\sigma\sqrt{1-\rho}\sqrt{t}Z^{(k)}(t)} = D(0)e^{\nu t + \sigma\sqrt{\rho}w(t)}\mathbb{E}_x\left(e^{\sigma\sqrt{1-\rho}\sqrt{t}x}\right) \\ &= D(0)e^{\nu t + \sigma\sqrt{\rho}w(t)}e^{(1-\rho)\frac{\sigma^2}{2}t} = D(0)e^{\tilde{\nu}t + \tilde{\sigma}w(t)} \end{aligned}$$

where

$$w_k(t) = \sqrt{t}Z^{(k)}(t)$$

is the idiosyncratic Wiener process,

$$Z^{(k)}(t) \sim N(0,1), 1 \le k \le n$$

are i.i.d. r.v.,

$$x \sim N(0,1)$$

Example: $c_{k,n} = 1/n$ (all deposits have the same size).

Hence the size of the deposit portfolio obeys the geometric Brownian motion

$$dD(t) = rD(t)dt + \tilde{\sigma}D(t)dw(t)$$

with the deposit portfolio volatility $\tilde{\sigma} = \sqrt{\rho}\sigma$.

Due to this circumstance, it is possible (for the proof see Appendix 2) to calculate the probability of premature default in a closed form in the case of liquidity risk, which is impossible in the case of credit risk.

Appendix 2

Let's prove that if

$$D(t) = D(0)e^{\tilde{\nu}t+\tilde{\sigma}w(t)}$$

then

$$\mathbb{P}(M(t) \leq Y) = \Phi\left(\frac{y-\tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right) + e^{\alpha y}\Phi\left(\frac{y+\tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

This is a known result. This paper provides a proof based on the paper (Kostadinov, 2008).

Probability of default

$$\mathbb{P}(M(t) \leq Y) = \mathbb{P}\left(\min_{s\in[0,t]} D(0)e^{\tilde{\nu}s+\tilde{\sigma}w(s)} \leq Y\right) = \mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{\nu}}{\tilde{\sigma}}s\right) \leq \frac{y}{\tilde{\sigma}}\right)$$

where

$$y = ln\big(Y/D(0)\big)$$

To get rid of the trend we use the new measure with the Brownian motion

$$\widetilde{w}(t) = w(t) + \frac{\tilde{\nu}}{\tilde{\sigma}}t$$

The event

$$A = \left(\min_{s\in[0,t]} \widetilde{w}(s) \leq \frac{y}{\tilde{\sigma}}, \widetilde{w}(t) \geq \frac{x}{\tilde{\sigma}}\right)$$

where $y \leq x$.

Girsanov's theorem allows to get

$$\mathbb{P}(A) = \mathbb{E}(\mathbb{I}_A) = \widetilde{\mathbb{E}}\left(e^{-\frac{1}{2}\left(\frac{y}{\tilde{\sigma}}\right)^2 t+\frac{\tilde{\nu}}{\tilde{\sigma}}\widetilde{w}(t)}\mathbb{I}_A\right)$$

using the Radon-Nikodym derivative.

Let use the reflection principle. After replacement $\widetilde{w}(t)$ with $2\frac{y}{\tilde{\sigma}} - \widetilde{w}(t)$ we get

$$\mathbb{P}(A) = \widetilde{\mathbb{E}}\left(e^{-\frac{1}{2}\left(\frac{y}{\tilde{\sigma}}\right)^2 t+\frac{\tilde{\nu}}{\tilde{\sigma}}\left(2\frac{y}{\tilde{\sigma}}-\widetilde{w}(t)\right)}\mathbb{I}_{2\frac{y}{\tilde{\sigma}}-\widetilde{w}(t)\geq\frac{x}{\tilde{\sigma}}}\right) = e^{2\frac{y\tilde{\nu}}{\tilde{\sigma}^2}}\widetilde{\mathbb{E}}\left(e^{-\frac{1}{2}\left(\frac{y}{\tilde{\sigma}}\right)^2 t-\frac{\tilde{\nu}}{\tilde{\sigma}}\widetilde{w}(t)}\mathbb{I}_{\widetilde{w}(t)\leq\frac{2y-x}{\tilde{\sigma}}}\right)$$

To get rid of the Radon-Nikodym derivative we use the new measure with the Brownian motion

$$\breve{w}(t) = \widetilde{w}(t) + \frac{\tilde{\nu}}{\tilde{\sigma}}t$$

Girsanov's theorem allows to get

$$\widetilde{\mathbb{E}}\left(e^{-\frac{1}{2}\left(\frac{y}{\tilde{\sigma}}\right)^2 t-\frac{\tilde{\nu}}{\tilde{\sigma}}\widetilde{w}(t)}\mathbb{I}_{\widetilde{w}(t)\leq\frac{2y-x}{\tilde{\sigma}}}\right) = \breve{\mathbb{P}}\left(\breve{w}(t) \leq \frac{2y-x+\tilde{\nu}t}{\tilde{\sigma}}\right) = \Phi\left(\frac{2y-x+\tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

Hence the joint distribution

$$\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{\nu}}{\tilde{\sigma}}s\right) \leq \frac{y}{\tilde{\sigma}}, w(t) \geq \frac{x}{\tilde{\sigma}}\right) = e^{2\frac{y\tilde{\nu}}{\tilde{\sigma}^2}}\Phi\left(\frac{2y-x+\tilde{\nu}t}{\tilde{\sigma}\sqrt{t}}\right)$$

Hence

$$\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{v}}{\tilde{\sigma}}s\right)\leq\frac{y}{\tilde{\sigma}}\right)$$
$$=\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{v}}{\tilde{\sigma}}s\right)\leq\frac{y}{\tilde{\sigma}},w(t)\geq\frac{x}{\tilde{\sigma}}\right)$$
$$+\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{v}}{\tilde{\sigma}}s\right)\leq\frac{y}{\tilde{\sigma}},w(t)<\frac{x}{\tilde{\sigma}}\right)$$
$$\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{v}}{\tilde{\sigma}}s\right)\leq\frac{y}{\tilde{\sigma}},w(t)<\frac{y}{\tilde{\sigma}}\right)=\mathbb{P}\left(w(t)<\frac{y}{\tilde{\sigma}}\right)$$

$$\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{v}}{\tilde{\sigma}}s\right)\leq\frac{y}{\tilde{\sigma}}\right)=\mathbb{P}\left(\min_{s\in[0,t]}\left(w(s)+\frac{\tilde{v}}{\tilde{\sigma}}s\right)\leq\frac{y}{\tilde{\sigma}},w(t)\geq\frac{y}{\tilde{\sigma}}\right)+\mathbb{P}\left(w(t)\leq\frac{y}{\tilde{\sigma}}\right)$$

Hence

$$\mathbb{P}(M(t)\leq Y)=\Phi\left(\frac{y-\tilde{v}t}{\tilde{\sigma}\sqrt{t}}\right)+e^{\frac{2\tilde{v}y}{\tilde{\sigma}^2}}\Phi\left(\frac{y+\tilde{v}t}{\tilde{\sigma}\sqrt{t}}\right)=\Phi\left(\frac{y-\tilde{v}t}{\tilde{\sigma}\sqrt{t}}\right)+e^{\alpha y}\Phi\left(\frac{y+\tilde{v}t}{\tilde{\sigma}\sqrt{t}}\right)$$

Q.E.D.

Declarations of Interest

The author reports no conflicts of interest. The author alone is responsible for the content and writing of the paper.

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