Three Remarks On Asset Pricing

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Abstract

We consider well-known consumption-based asset pricing theory and regard the choice of the time interval Δ used for averaging the market price time-series as the key factor of asset pricing. We show that the explicit usage of the averaging interval Δ allows expand investor's utility into Taylor series and derive successive approximations of the basic asset pricing equation. For linear and quadratic Taylor series approximations of the basic pricing equation we derive new expressions of the mean price, mean payoff, their volatilities, skewness and amount of asset ξ_{max} that delivers max to investor's utility. The treatment of the market price as a coefficient between the trade value and volume prohibits independent definition of the trade value, volume and price probabilities. We introduce price *n-th* statistical moments $p(t;n)$ as generalization of the well-known definition of volume weighted average price (VWAP). We demonstrate that usage of VWAP causes zero correlations between price and trade volume. Usage of price *n-th* statistical moments causes zero correlations between *n-th* power of price p^n and trade volume U^n, but don't causes statistical independence. As example, we derive expression for correlation between price p and squares of trade volume U^2. Any predictions of the market-based price probability at horizon T should match forecasts of finite number of *n-th* statistical moments of the trade value $C(t;n)$ and volume $U(t;n)$ at the same horizon T. The new definition of the market-based asset price probability emphasizes its direct dependence on random properties of the market trade.

Keywords : asset pricing, volatility, price probability, market trades

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1. Introduction

Predictions of the asset prices define the main desires of investors, traders and all the participants of financial markets. Last decades give a great progress in the asset price valuation and setting. Starting with Hall and Hitch (1939) many researchers investigate the price theory (Friedman, 1990; Heaton and Lucas, 2000) and the factors those impact markets (Fama, 1965), equilibrium economy (Sharpe, 1964), fluctuations (Mackey, 1989) macroeconomics (Cochrane and Hansen, 1992) and business cycles (Mills, 1946; Campbell, 1998). Muth (1961) initiated studies on the dependence of asset pricing on the expectations and numerous scholars developed his ideas further (Lucas, 1972; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014). Many researchers describe the price dynamics and references (Goldsmith and Lipsey, 1963; Campbell, 2000; Cochrane and Culp, 2003; Borovička and Hansen, 2012; Weyl, 2019) give only a small part of them.

Asset pricing depends on price fluctuations and volatility. The mean price trends and the price volatility are the most important issues that impact investors' expectation. Description of volatility is inseparable from price modeling (Hall and Hitch, 1939; Fama, 1965; Stigler and Kindahl, 1970; Tauchen and Pitts, 1983; Schwert, 1988; Mankiw, Romer and Shapiro, 1991; Brock and LeBaron, 1995; Bernanke and Gertler, 1999; Andersen et.al., 2001; Poon and Granger, 2003; Andersen et.al., 2005). The list of references can be continued as hundreds and hundreds of articles describe different faces of the price-volatility puzzle.

Simple and practical advises on the price modeling and forecasting among the most demanded by investors. Different price models were developed to satisfy and saturate investors' desires. We refer only some pricing models (Ferson et.al., 1999; Fama and French, 2015) and studies on Capital Asset Pricing Model (CAPM) (Sharpe, 1964; Merton, 1973; Cochrane, 2001; Perold, 2004). Cochrane (2001) shows that CAPM includes different versions of asset pricing as ICAPM and consumption-based pricing model (Campbell, 2002) are CAPM variations. Further we consider Cochrane (2001) as clear and consistent presentation of CAPM basis, problems and achievements. Resent study (Cochrane, 2021) complements the rigorous asset price description with deep and justified general considerations of the nature, problems and possible directions for further research.

Despite the fact that asset pricing, risk, uncertainties and financial markets were studied with a great accuracy and solidity there are still "some" problems left. We assume that the core economic difficulties and the fundamental economic relations may still impede further significant development of the price theory. To explain the nature of the existing economic obstacles that may hamper price forecasting we consider three interrelated remarks that impact asset pricing.

We outline that any averaging of economic and financial variables presented by time-series is performed during some time interval *Δ*. The choice of *Δ* allows derive Taylor series of the basic pricing equation for variables averaged during *Δ*. Linear and quadratic approximations by price and payoff variations during *Δ* give simple equations on mean price and payoff, their volatilities, skewness and other factors that define CAPM. The core factor of any asset price model is the price probability measure. We present the reasons those support substitution of the conventional price probability *P(p)* proportional to the frequency of trades at price *p* during an interval *Δ* by a different price probability measure entirely determined by the probability measures of the market trade value and volume time-series during *Δ*. Indeed, price $p(t_i)$ of any particular trade at moment t_i is a coefficient between the trade value $C(t_i)$ and the trade volume $U(t_i)$

$$C(t_i) = p(t_i)U(t_i) \tag{1.1}$$

Time-series of market trade value $C(t_i)$ and volume $U(t_i)$ during *Δ* determine the market value and volume probabilities proportional to the frequency of trades at particular value and volume. It impossible independently define probabilities of three variable - trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ – those match equation (1.1). For any given *Δ* and probabilities of the trade value and volume, price probability must be result of equation (1.1). We define a market-based price probability measure that match (1.1) as function of probability measures of the market trade value and volume time-series during *Δ*. The market-based price probability measure reflects randomness of the market trades and prediction of the price probability at horizon *T* equals forecasting the market probability measures at same horizon.

It is convenient consider asset pricing having the single reference that describes almost all extensions and model variations within the uniform frame. We propose that readers are familiar with Cochrane (2001) and refer this monograph for any notions and clarifications.

In Sec.2 we remind main CAPM notions. In Sec.3 we consider remarks on the time scales and introduce the averaging interval *Δ* of the market trade and price time-series. In Sec.4 we discuss remarks on Taylor series generated by the averaging interval *Δ*. We expand the utility functions by Taylor series and in linear and quadratic approximations by the price and payoff variations we consider the idiosyncratic risk, the utility max conditions and the impact of price-volume correlations. In Sec.5 we introduce the new market-based price probability measure and briefly consider its implications on asset pricing. Sec.7 – Conclusion. In App.A we collect some calculations that define maximum of investor's utility. In App.B we present simple approximations of the price characteristic function.

Equation (4.5) means equation 5 in the Sec. 4 and (A.2) – notes equation 2 in Appendix A. We assume that readers are familiar with basic notions of probability, statistical moments, characteristic functions and etc.

2. Brief CAPM Assumptions

The general frame that determines all CAPM versions and extensions states: "All asset pricing comes down to one central idea: the value of an asset is equal to its expected discounted payoff" (Cochrane, 2001; Cochrane and Culp, 2003; Hördahl and Packer, 2007; Cochrane, 2021). Let's follow (Cochrane, 2001) and briefly consider CAPM notions and assumptions. The basic consumption-based equation has form:

$$p = E[m\, x] \tag{2.1}$$

In (2.1) p denotes the asset price at date t, $x=p_{t+1}+d_{t+1}$ – payoff, p_{t+1} - price and d_{t+1} - dividends at date $t+1$, m - the stochastic discount factor and E – mathematical expectation at day $t+1$ made by the forecast under the information available at date t. Cochrane (2001) considers relations (2.1) in various forms to show that almost all models of asset pricing united by the title CAPM can be described by the similar equations. We shall consider (2.1) and refer (Cochrane, 2001) for all CAPM extensions. For convenience we briefly reproduce consumption-based derivation of (2.1). Cochrane "models investors by a utility function defined over current c_t and future c_{t+1} values at date t of consumption. c_t and c_{t+1} denote consumption at date t and $t+1$."

$$U(c_t;\, c_{t+1}) = u(c_t) + \beta E[u(c_{t+1})] \tag{2.2}$$

$$c_t = e_t - p\xi \quad ; \quad c_{t+1} = e_{t+1} + x\xi \tag{2.3}$$

$$x = p_{t+1} + d_{t+1} \tag{2.4}$$

Here (2.3) e_t and e_{t+1} "denotes original consumption level (if the investor bought none of the asset), and ξ denotes the amount of the asset he chooses to buy" (Cochrane, 2001). A payoff x (2.4) is determined by a price p_{t+1} and a dividend d_{t+1} of asset at date $t+1$. Cochrane calls β as "subjective discount factor that captures impatience of future consumption". *E[...]* in (2.2) denotes mathematical expectation of the random utility due to the random payoff x (2.4) made at date $t+1$ by forecast on base of information available at date t. The first-order maximum condition for (2.2) by amount of asset ξ is fulfilled by putting derivative of (2.2) by ξ equals zero (Cochrane, 2001):

$$\max_\xi U(c_t;\, c_{t+1}) \leftrightarrow \frac{\partial}{\partial \xi} U(c_t;\, c_{t+1}) = 0 \tag{2.5}$$

From (2.2-2.5) it is obvious that:

$$p = \beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}x\right] = E[mx] \quad ; \quad m = \beta\frac{u'(c_{t+1})}{u'(c_t)} \quad ; \quad \frac{d}{dc}u(c) = u'(c) \qquad (2.6)$$

and (2.6) reproduces (2.1) for *m* (2.6). This completes the brief derivation of the basic equation (2.1; 2.6) and we refer Cochrane (2001) for any further details.

3. Remarks on Time Scales

We start with simple remarks on averaging procedure and time scales. Any averaging of the market trade time-series delivers the mean values during some time interval Δ. The averaging procedure can be different but any such procedure aggregates the time-series during an interval Δ. The choice of the averaging interval Δ defines the *internal* time scale of the problem under consideration. Prediction of asset price at time-horizon T at "the next day" $t+1 = t+T$ defines the *external* time scale of the problem. Relations between the *internal* Δ and *external* T scales determine evolution of the averaged variables, sustainability and accuracy of the model description. Financial variables – price, volatility, beta – averaged during the interval Δ can behave irregular or randomly on time scales T for $T \gg \Delta$. This effect mentioned, for example, by Cochrane (2021): "Another great puzzle is how little we know about betas. In continuous-time diffusion theory, 10 seconds of millisecond data should be enough to measure betas with nearly infinite precision. In fact, betas are hard to measure and unstable over time". It's clear, that averaging of time-series during the interval Δ smooth perturbations with scales less than Δ. If factors that disturb the market have a time scale $d > \Delta$ then variables averaged during Δ will demonstrate irregular or random properties during the term T. It is clear that the price, payoff and discount factor are under impact of the disturbing factors with different time scales. Eventually, the choice of the averaging interval Δ is important for asset pricing, but sadly it is not the main trouble.

As we note, the averaging interval Δ defines the *internal* time scale of the problem. In simplest case averaging of the price time-series during the interval Δ that equals 1 min, 1 hour, 1 day, 1 week establish the least time divisions of "the Clocks" of the problem under consideration that equal 1 min, 1 hour, 1 day, 1 week and moments of time $t(k)$ of the model after averaging during Δ take form

$$t(k) = t(0) + k\,\Delta \quad ; \quad k = 0, \pm 1, \pm 2, \ldots \qquad (3.1)$$

Here t_0 can be the moment "to-day". It is reasonable use the same time scale divisions "to-day" at moment t and the "next-day" at $t+1$. Indeed, time scale divisions can't be measured "to-day" in hours and "next-day" in weeks. The utility (2.2) "to-day" at moment t and the "next-day" at $t+1$ should have the same time divisions. Averaging of any time-series at the "next-day" at $t+1$ during the interval Δ undoubtedly implies averaging "to-day" at date t during same time interval Δ and vise-versa. Thus, if the utility (2.2) is averaged at $t+1$ during the interval Δ, then the utility (2.2) also should be averaged at date t during Δ and take form:

$$U(c_t;\ c_{t+1}) = E_t[u(c_t)] + \beta E[u(c_{t+1})] \tag{3.2}$$

We denote $E_t[..]$ in (3.2) as mathematical expectation "to-day" at date t during Δ. It does not matter how one considers the price time-series "to-day" – as random or as irregular. Mathematical expectation $E_t[..]$ performs smoothing of the random or irregular time-series via aggregating data during Δ under particular probability measure. Mathematical expectations $E_t[..]$ and $E[..]$ within identical averaging intervals Δ establish identical time division of the problem at dates t and $t+1$ in (3.2). Hence, relations similar to (2.5; 2.6) should derive modification of the basic pricing equation in the form:

$$E_t[p\, u'(c_t)] = \beta E\ [x\, u'(c_{t+1})] \tag{3.3}$$

Cochrane (2001) takes "subjective discount factor" β as non-random and we follow the same assumption. Mathematical expectation in the left side $E_t[...]$ assesses mean price p at moment t during Δ. In the right side $E[xu'(c_{t+1})]$ forecasts the average of $xu'(c_{t+1})$ at date $t+1$ within the same averaging interval Δ on base of data available at date t.

Brief resume 1. The choice of the averaging time interval Δ, transition to time divisions (3.1) and analysis of dependence of mean variables on duration of Δ establish the key problems of any economic or financial model that describes relations between averaged variables.

4. Remarks on Taylor series

Relation (2.5) presents first-order condition at point ξ_{max} that delivers maximum to investor's utility (2.2) or (3.2). Let us choose the averaging interval Δ and take the price p at date t during the interval Δ and the payoff x at date $t+1$ during the interval Δ as:

$$p\ =\ p_0 + \delta p\ ; \quad x = x_0 + \delta x \tag{4.1}$$

$$E_t[p] = p_0\ ;\ E[x] = x_0\ ;\ E_t[\delta p] = E[\delta x] = 0\ ;\ \sigma^2(p) = E_t[\delta^2 p]\ \ ;\ \ \sigma^2(x) = E[\delta^2 x] \tag{4.2}$$

The relations (4.1; 4.2) give the average price p_0 and its volatility $\sigma^2(p)$ at date t and the average payoff x_0 its volatility $\sigma^2(x)$ at date $t+1$. We underline, that we consider averaging during Δ as averaging of a random or as smoothing of an irregular variable. Thus $E_t[p]$ – at date t smooth random or irregular price p during Δ and $E[x]$ – averages the random payoff x during Δ at date $t+1$. We call both procedures as mathematical expectations. We present the derivatives of utility functions in (3.3) by Taylor series in a linear approximation by δp and δx during Δ:

$$u'(c_t) = u'(c_{t;0}) - \xi u''(c_{t;0})\delta p \quad ; \quad u'(c_{t+1}) = u'(c_{t+1;0}) + \xi u''(c_{t+1;0})\delta x \tag{4.3}$$

$$c_{t;0} = e_t - p_0\xi \quad ; \quad c_{t+1;0} = e_{t+1} + x_0\xi$$

Now substitute (4.3) into (3.3) and obtain equation (4.4):

$$u'(c_{t;0})p_0 - \xi u''(c_{t;0})\sigma^2(p) = \beta u'(c_{t+1;0})x_0 + \beta\xi u''(c_{t+1;0})\sigma^2(x) \tag{4.4}$$

Taylor series are simplest entry-level mathematical tools like as ordinary derivatives and we see no sense refer any studies those also use Taylor or ordinary derivatives in asset pricing. However, Cochrane (2001) uses Taylor expansions. We underline: Taylor series and (4.1-4.4) are determined by the duration of the averaging interval Δ. The change of Δ implies change of the mean price p_0, the mean payoff x_0 and their volatilities $\sigma^2(p)$, $\sigma^2(x)$ (4.2). Equation (4.4) is a linear approximation by the price and payoff fluctuations of the first-order max conditions (2.5) and assesses the root ξ_{max} that delivers maximum to the utility $U(c_t;c_{t+1})$ (3.2)

$$\xi_{max} = \frac{u'(c_{t;0})p_0 - \beta u'(c_{t+1;0})x_0}{u''(c_{t;0})\sigma^2(p) + \beta u''(c_{t+1;0})\,\sigma^2(x)} \tag{4.5}$$

We note that (4.5) is not an "exact" equation on ξ_{max} as utilities u' and u'' also depend on ξ_{max} as it follows from (4.3). However, (4.5) gives an assessment of ξ_{max} in a linear approximation by Taylor series δp and δx averaged during Δ. Let underline that the ξ_{max} (4.5) depends on the price volatility $\sigma^2(p)$ at date t and on the payoff volatility $\sigma^2(x)$ at date $t+1$ (4.2).

It is clear that sequential iterations may give more accurate approximations of ξ_{max}. Nevertheless, our approach and (4.5) give a new look on the basic equation (2.6; 3.3). If one follows the standard derivation of (2.6) (Cochrane, 2001) and neglects the averaging at date t in the left side (3.3), then (2.6; 4.5) give

$$\xi_{max} = \frac{u'(c_t)p - \beta u'(c_{t+1;0})x_0}{\beta u''(c_{t+1;0})\sigma^2(x)} \tag{4.6}$$

Relations (4.6) show that even the standard form of the basic equation (2.6) hides dependence of ξ_{max} on the payoff volatility $\sigma^2(x)$ at date $t+1$. If one has the independent assessment of ξ_{max} then can use it to present (4.6) in a way alike to the basic equation (2.6):

$$p = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta x_0 + \xi_{max}\frac{u''(c_{t+1;0})}{u'(c_t)}\beta\sigma^2(x) \tag{4.7}$$

One can transform (4.7) alike to (2.6):

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) \tag{4.8}$$

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_t)}\beta \ ; \ m_1 = \frac{u''(c_{t+1;0})}{u'(c_t)}\beta \tag{4.9}$$

For the given ξ_{max} equation (4.8) in a linear approximation by Taylor series describes dependence of the price p at date t (3.1) on the mean discount factors m_0 and m_1 (4.9), the mean payoff x_0 (4.1) and the payoff volatility $\sigma^2(x)$ during Δ. Let underline that while the mean discount factor $m_0>0$, the mean discount factor $m_1<0$ because the utility $u'(c_t)>0$ and $u''(c_t)<0$ for all t. Hence, for (4.8) valid:

$$p < m_0 x_0 \quad ; \quad \xi_{max} m_1 \sigma^2(x) < 0$$

We underline that (4.6-4.9) have sense for the given value of ξ_{max}. Equation (4.8) in a linear approximation by Taylor series δx during the interval Δ describes the modified CAPM statement: *the value of an asset is equal the mean payoff x_0 discounted by the mean factor m_0 minus payoff volatility $\sigma^2(x)$ discounted by factor $|m_1|$ and multiplied by the amount of asset ξ_{max} that delivers maximum to the investor's utility (2.2).* As the price p in (4.8) should be positive hence ξ_{max} should obey inequality (4.10):

$$0 < \xi_{max} < -\frac{u'(c_{t+1;0})}{u''(c_{t+1;0})} \frac{x_0}{\sigma^2(x)} \tag{4.10}$$

Taking into account (4.3) it is easy to show for (4.10) that for the conventional power utility (Cochrane, 2001) (A.2):

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \quad ; \quad \frac{u'(c)}{u''(c)} = -\frac{c}{\alpha} \quad ; \quad 0 < \alpha \leq 1$$

inequality (4.10) valid always if

$$\alpha\, \sigma^2(x) < x_0^2$$

For this approximation (4.10) limits the value of ξ_{max}. If one takes (4.5) then obtains equations similar to (4.8; 4.9):

$$m_0 = \frac{u'(c_{t+1;0})}{u'(c_{t;0})} \beta > 0 \, ; \;\; m_1 = \frac{u''(c_{t+1;0})}{u'(c_{t;0})} \beta < 0 \, ; \;\; m_2 = \frac{u''(c_{t;0})}{u'(c_{t;0})} < 0 \tag{4.11}$$

$$p_0 = m_0 x_0 + \xi_{max}[m_1 \sigma^2(x) + m_2 \sigma^2(p)] \tag{4.12}$$

We use the same notions m_0, m_1 to denote the discount factors taking into account replacement of $u'(c_t)$ in (4.9) by $u'(c_{t;0})$ in (4.11; 4.12). Modified basic equation (4.12) at date t describes dependence of the price p_0 on the price volatility $\sigma^2(p)$ at date t, the mean payoff x_0 and payoff volatility $\sigma^2(x)$ at date $t+1$ averaged during same interval Δ.

Equation (4.15) illustrates well-known practice that high volatility $\sigma^2(p)$ of the price at date t and high forecast of payoff volatility $\sigma^2(x)$ at date $t+1$ may cause decline of the mean price p_0 at date t. We leave the detailed analysis of (4.5-4.12) for the future.

4.1 The Idiosyncratic Risk

Here we briefly consider the case of the idiosyncratic risk for which the payoff x in (2.6) is not correlated with the discount factor m at moment $t+1$ (Cochrane, 2001):

$$cov(m, x) = 0 \tag{4.13}$$

In this case equation (2.6) takes form:

$$p = E[mx] = E[m]E[x] + cov(m,x) = E[m]x_0 = \frac{x_0}{R_f} \quad (4.14)$$

The risk-free rate R^f in (4.14) is known ahead (Cochrane, 2001). Taking into account (4.3) in a linear approximation by δx Taylor series for derivative of the utility $u'(c_{t+1})$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x \quad (4.15)$$

Hence, the discount factor m (2.6) takes form:

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x]$$

$$E[m] = \bar{m} = \beta\frac{u'(c_{t+1;0})}{u'(c_t)}$$

$$\beta E\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]x_0 = \frac{x_0}{R_f} \quad ; \quad E[u'(c_{t+1})x] = 0$$

and

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}u''(c_{t+1;0})\xi\delta x$$

Hence, (4.13) implies:

$$cov(m,x) = E[\delta m\delta x] = \beta\frac{u''(c_{t+1;0})}{u'(c_t)}\xi_{max}\sigma^2(x) = 0 \quad (4.16)$$

That causes zero payoff volatility $\sigma^2(x)=0$. Of course zero payoff volatility does not model market reality but (4.16) reflects restrictions of the linear approximation (4.15). To overcome this discrepancy let take into account Taylor series up to the second degree by $\delta^2 x$:

$$u'(c_{t+1}) = u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\delta^2 x \quad (4.17)$$

$$m = \beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\delta^2 x] \quad (4.18)$$

For this case the mean discount factor $E[m]$ takes form:

$$E[m] = \bar{m} = \frac{\beta}{u'(c_t)}[u'(c_{t+1;0}) + u'''(c_{t+1;0})\xi^2\sigma^2(x)] \quad (4.19)$$

and variations of the discount factor δm:

$$\delta m = m - \bar{m} = \frac{\beta}{u'(c_t)}\ [u''(c_{t+1;0})\xi\delta x + u'''(c_{t+1;0})\xi^2\{\delta^2 x - \sigma^2(x)\}$$

In this case

$$cov(m,x) = E[\delta m\delta x] = \left[u''(c_{t+1;0})\xi\sigma^2(x) + u'''(c_{t+1;0})\xi^2\,\gamma^3(x)\,\right] = 0 \quad (4.20)$$

$$\gamma^3(x) = E[\delta^3 x] \quad ; \quad Sk(x) = \frac{\gamma^3(x)}{\sigma^3(x)} \quad (4.21)$$

Sk(x) – denotes normalized payoff skewness at date $t+1$ treated as the measure of asymmetry of the probability distribution during Δ. For approximation (4.18) from (4.20; 4.21) obtain relations on the skewness $Sk(x)$ and ξ_{max}:

$$\xi_{max}\, Sk(x)\sigma(x) = -\frac{u''(c_{t+1;0})}{u'''(c_{t+1;0})} \tag{4.22}$$

For the conventional power utility (A.2) and (4.3) relations (4.22) take form

$$\xi_{max} = \frac{e_{t+1}}{(1+\alpha)Sk(x)\sigma(x)-x_0} \tag{4.23}$$

It is assumed that second derivative of utility $u''(c_{t+1})<0$ always negative and third derivative $u'''(c_{t+1})>0$ is positive and hence the right side in (4.22) is positive. Hence to get positive ξ_{max} for (4.23) for the power utility (A.2) the payoff skewness $Sk(x)$ should obey inequality (4.24) that defines the lower limit of the payoff skewness $Sk(x)$:

$$Sk(x) > \frac{x_0}{(1+\alpha)\sigma(x)} \tag{4.24}$$

In (4.14) R_f denotes known risk-free rate. Hence, (4.19; 4.22; 4.24) define relations:

$$\frac{\beta}{u'(c_t)}\left[u'(c_{t+1;0}) + u'''(c_{t+1;0}){\xi_{max}}^2\sigma^2(x)\right] = \frac{1}{R_f}$$

$${\xi_{max}}^2\sigma^2(x) = \frac{1}{\beta R_f}\frac{u'(c_t)}{u'''(c_{t+1;0})} - \frac{u'(c_{t+1;0})}{u'''(c_{t+1;0})}$$

$$Sk^2(x) = \frac{R_f}{1-m_0R_f}\frac{m_1^2}{m_3} > \frac{x_0^2}{(1+\alpha)^2\sigma^2(x)} \quad ; \quad m_0 < 1/R_f$$

$$\frac{\sigma^2(x)}{x_0^2} > \frac{m_3}{m_1^2}\frac{1-m_0R_f}{(1+\alpha)^2R_f} \tag{4.25}$$

Inequality (4.25) establishes the lower limit on the payoff volatility $\sigma^2(x)$ normalized by the square of the mean payoff x_0^2. The lower limit in the right side of (4.25) is determined by the discount factors (4.26), the risk-free rate R_f and the conventional power utility factor α (A.2).

$$m_0 = \beta\frac{u'(c_{t+1;0})}{u'(c_t)} \; ; \; m_1 = \beta\frac{u''(c_{t+1;0})}{u'(c_t)} \quad ; \quad m_3 = \beta\frac{u'''(c_{t+1;0})}{u'(c_t)} \tag{4.26}$$

The coefficients in (4.26) differ a little from (4.1) as (4.26) takes the denominator $u'(c_t)$ instead of $u'(c_{t;0})$ in (4.11) but we use the same letters to avoid extra notations. The similar calculations for (3.2; 3.3) describe both the price volatility $\sigma^2(p)$ and the skewness $Sk(p)$ at date t and the payoff volatility $\sigma^2(x)$ and the skewness $Sk(x)$ at date $t+1$. Further approximations by Taylor series of the utility derivative $u'(c_t)$ up to δ^3p and $u'(c_{t+1})$ up to δ^3x similar to (4.17) give assessments of kurtosis of the price probability at date t and kurtosis of the payoff probability at date $t+1$ estimated during interval Δ. We leave these exercises for future.

4.2 The Utility Maximum

The relations (2.5) define the first-order condition that determines the amount of asset ξ_{max} that delivers the max to the utility $U(c_t; c_{t+1})$ (2.2; 3.2). To confirm that function $U(c_t; c_{t+1})$ has max at ξ_{max} , the first order condition (2.5) must be supplemented by condition:

$$\frac{\partial^2}{\partial \xi^2} U(c_t;\ c_{t+1}) < 0 \tag{4.27}$$

Usage of (4.27) gives interesting consequences. From (2.2–2.4) and (4.27) obtain:

$$p^2 > -\frac{\beta}{u''(c_t)} E[x^2\, u''(c_{t+1})\,] \tag{4.28}$$

Take the linear Taylor series expansion of the second derivative of the utility $u''(c_{t+1})$ by δx

$$u''(c_{t+1}) = u''\big(c_{t+1;0}\big) + u'''\big(c_{t+1;0}\big)\xi\delta x$$

Then (4.28) takes form:

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [{x_0}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\, [2x_0\sigma^2(x) + \gamma^3(x)] \tag{4.29}$$

For the power utility (A.2) simple calculations (see App.A) give relations on (4.27; 4.29). If the payoff volatility $\sigma^2(x)$ multiplied by factor $(1+2\alpha)$ is less then mean payoff ${x_0}^2$ (4.30; A.5)

$$(1+2\alpha)\sigma^2(x) < {x_0}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \tag{4.30}$$

Then (4.29) is always valid. If payoff volatility $\sigma^2(x)$ is high (A.6)

$$(1+2\alpha)\sigma^2(x) > \ {x_0}^2$$

Then (4.29) valid only for ξ_{max} (A.6):

$$\xi_{max} < \frac{e_{t+1}[{x_0}^2 + \sigma^2(x)]}{x_0\ [(1+2\alpha)\sigma^2(x) - {x_0}^2]}$$

However, this upper limit for ξ_{max} can be high enough. The same but more complex considerations can be presented for (3.2).

$$E_t[p^2 u''(c_t)] < -E[\beta x^2\, u''(c_{t+1})\,]$$

Brief resume 2. Different asset pricing, financial and economic models describe relations between averaged or smoothed variables. Similar usage of Taylor series determined by the averaging interval Δ can help develop successive approximations determined by linear, quadratic, etc., perturbations of the random variables and their further averaging during Δ.

5. Remarks on the Price Probability Measure

As usual the problems that are the most common and "obvious" hide the most difficulties. The price probability measure is exactly the case of such hidden complexity.

All asset pricing models assume that it is possible forecast the probability of random price p and payoff x at horizon T. Let us consider the choice and forecasting of the price probability measure as most interesting and complex problem of finance.

The usual treatment of the price p probability "is based on the probabilistic approach and using A. N. Kolmogorov's axiomatic of probability theory, which is generally accepted now" (Shiryaev, 1999). The conventional definition of the price probability is based on the frequency of trades at a price *p* during the averaging interval *Δ*. The economic ground of such choice is simple: it is assumed that each trade of *N* trades during *Δ* has equal probability ~ *1/N*. If there are *m(p)* trades at the price *p* then probability *P(p)* of the price *p* during *Δ* equals *m(p)/N*. The frequency of the particular event is absolutely correct, general and conventional approach to the probability definition. The conventional frequency-based approach to the price probability uses different assumptions on form of the price probability measure and checks how almost all standard probability measures (Walck, 2007; Forbes et.al., 2011) fit the market random price. Parameters that define standard probabilities permit calibrate each in a manner that increase the plausibility and consistency with the observed market price time-series. For different assets, options and markets different standard probabilities are tested and applied to fit and predict the random price dynamics as well as possible.

However, one may ask a simple question: does the conventional frequency-based approach to the price probability fit the random market pricing? The asset price is a result of the market trade and it seems reasonable that the market trade randomness conducts the price stochasticity. We propose the new market-based price probability measure that is different from the conventional frequency-based probability and is entirely determined by the probability measures of the market trades values and volumes.

Let note that almost 30 years ago the volume weighted average price (VWAP) was introduced and is widely used now (Berkowitz et.al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; Duffie and Dworczak, 2018; CME Group, 2020). Some notations of the current Section can differ from the previous Sections, but we hope that readers adopt both. The definition of the VWAP *p(t;1)* that match (1.1) during the interval *Δ* is follows. Let take that during *Δ* (5.3) there are *N* market trades at moments t_i, $i=1,...N$. Let denote *E[...]* as mathematical expectation. Then the VWAP *p(t;1)* (5.1) that match (1.1) during *Δ* (5.3) at moment *t* equals

$$p(t;1) = E[p(t_i)] = \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i) U(t_i) = \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} \tag{5.1}$$

$$C_\Sigma(t;1) = \sum_{i=1}^{N} C(t_i) = \sum_{i=1}^{N} p(t_i)\, U(t_i) \quad ; \quad U_\Sigma(t;1) = \sum_{i=1}^{N} U(t_i) \tag{5.2}$$

$$\Delta = \left[t - \frac{\Delta}{2}, t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta,\ i = 1, \dots N \tag{5.3}$$

We consider time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ as random variables during the averaging interval *Δ* (5.3). Relations (1.1) at moment t_i define the price $p(t_i)$ of trade value $C(t_i)$ and volume $U(t_i)$. The sum $C_\Sigma(t;1)$ of values $C(t_i)$ (5.2) and sum $U_\Sigma(t;1)$ of volumes $U(t_i)$ (5.2) of *N* trades during *Δ* (5.3) define the VWAP *p(1)* (5.1).

We hope that our readers able distinguish the difference between notations of consumption c_t (2.2; 2.3) and utility U (2.2) in Sections 2-4 and trade value $C(t_i)$ and volume $U(t_i)$ (5.1) in current Section.

It is obvious, that VWAP (5.1) can be equally determined (5.4) by the mean value $C(t;1)$ (5.5) and the mean volume $U(t;1)$ (5.6) of N trades during Δ:

$$C(t;1) = p(t;1)\, U(t;1) \tag{5.4}$$

The mean trade value $C(t;1)$ and volume $U(t;1)$ during Δ (5.3) are determined by the conventional frequency-based probabilities:

$$C(t;1) = E[C(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \tag{5.5}$$

$$U(t;1) = E[U(t_i)] = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{5.6}$$

The mean VWAP price $p(t;1)$ (5.4) is a coefficient between the mean value $C(t;1)$ (5.5) and the mean volume $U(t;1)$ (5.6).

However, it is obvious that probabilities of trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ time-series that match equations (1.1) cannot be determined independently. Forecast of asset price probability that is independent from predictions of the market trade value and volume probabilities has almost no economic sense. Given probabilities of trade value $C(t_i)$ and volume $U(t_i)$ time-series during Δ that match (1.1) should determine the price probability measure. Asset pricing should follow the market trade probabilities of the trade value $C(t_i)$ and volume $U(t_i)$ time-series. However, VWAP $p(t;1)$ and relations (5.1-5.6) are not sufficient to define all random properties of price during Δ (5.3). Probability measures of the market trade value and volume are unknown. To describe properties of the market trade value and volume and price as random variables during Δ (5.3) one could assess *n-th* statistical moments of the trade value $C(t;n)$ and volume $U(t;n)$ and price $p(t;n)$. Statistical moments of the trade value $C(t;n)$ and volume $U(t;n)$ take form:

$$C(t;n) = E[C^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \tag{5.7}$$

$$U(t;1) = E[U^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \tag{5.8}$$

Let us mention that *n-th* power of (1.1) for each particular trade at moment t_i gives:

$$C^n(t_i) = p^n(t_i) U^n(t_i) \ ; \quad n = 1, 2, \dots \tag{5.9}$$

We use (5.7-5.9) to determine *n-th* price statistical moments $p(t;n)$ for $n=1,2,3,\dots$ via *n-th* statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8). That should completely determine price as random variable during Δ (5.3). To do that we extend definition of the VWAP and introduce *n-th* price statistical moment *p(t;n)* as *n-th* power volume averaged:

$$p(t;n) = E[p^n(t_i)] = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i) U^n(t_i) = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} \tag{5.10}$$

$$C_\Sigma(t;n) = \sum_{i=1}^{N} C^n(t_i) = \sum_{i=1}^{N} p^n(t_i) U^n(t_i) \quad ; \quad U_\Sigma(t;n) = \sum_{i=1}^{N} U^n(t_i) \tag{5.11}$$

We underline that definitions (5.10) use relations (5.9) and that results in equal expression of price *n-th* statistical moments *p(t;n)* through *n-th* statistical moments of the market trade value *C(t;n)* and volume *U(t;n)*:

$$C(t;n) = p(t;n)U(t;n) \tag{5.12}$$

Definitions of price *n-th* statistical moments *p(t;n)* (5.10; 5.12) for all *n=1,2,…* match relations for *n-th* power of price $p^n(t_i)$ at time t_i during *Δ* (5.3) determined by (5.9). It is important that price *n-th* statistical moments *p(t;n)* (5.10; 5.12) for all *n=1,2,…* completely determine properties of market price considered as random variable during *Δ* (5.3).

Let us outline important unnoticed consequence of the VWAP *p(t;1)* (5.1) and similar consequences of our definition of price *n-th* statistical moments *p(t;n)* (5.10; 5.12).

Definition of VWAP *p(t;1)* (5.1) result in zero correlations between time-series of price $p(t_i)$ and market trade volume $U(t_i)$ during *Δ* (5.3). Indeed, from (1.1; 5.1; 5.5; 5.6) obtain:

$$E[C(t_i)] = \frac{1}{N} \sum_{i=1}^{N} C(t_i) \equiv E[p(t_i)U(t_i)] \equiv \frac{1}{N} \sum_{i=1}^{N} p(t_i)U(t_i) \equiv$$

$$\equiv \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N} \sum_{i=1}^{N} U(t_i) \equiv \ E[p(t_i)]E[U(t_i)] \tag{5.13}$$

Hence, from (5.13) obtain for correlation $corr\{p(t_i)U(t_i)\}$ between time-series of price $p(t_i)$ and market trade volume $U(t_i)$ during *Δ* (5.3)

$$corr\{p(t_i)U(t_i)\} = E[p(t_i)U(t_i)] - E[p(t_i)]E[U(t_i)] = 0 \tag{5.14}$$

Zero correlations (5.14) between price-volume time-series impact many results those "observe" positive or negative correlations between price and trading value (Tauchen and Pitts, 1983; Karpoff, 1987; Gallant et.al., 1992; Campbell et.al., 1993; Llorente et.al., 2001; DeFusco et.al., 2017). Actually, above researchers ignore the trivial equation (1.1) that prohibits independent definitions of price probabilities. Assessments of correlations between any time-series should always follow definitions of their averaging procedures. Usage of VWAP states no correlations between trade volume and price and many papers on price-volume relations should be reconsidered.

Our definitions of price *n-th* statistical moments *p(t;n)* (5.7-5.12) for all *n=1,2,3,..* cause zero correlations $corr\{p^n(t_i)U^n(t_i)\}$ between time-series of *n-th* power of price $p^n(t_i)$ and volume $U^n(t_i)$ during *Δ* (5.3). One can easy reproduce (5.13; 5.14) for any n=1,2,3,…:

$$E[C^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \equiv E[p^n(t_i)U^n(t_i)] \equiv \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) \equiv$$

$$\equiv \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \equiv E[p^n(t_i)]E[U^n(t_i)] \quad (5.15)$$

$$corr\{p^n(t_i)U^n(t_i)\} = E[p^n(t_i)U^n(t_i)] - E[p^n(t_i)]E[U^n(t_i)] = 0 \quad (5.16)$$

Thus, our definition of price *n-th* statistical moments *p(t;n)* (5.7-5.12) results in zero correlations between time-series of *n-th* power of price $p^n(t_i)$ and volume $U^n(t_i)$ during Δ (5.3). However, one can easy obtain that it doesn't imply statistical independence between time series of $p(t_i)$ and volume $U(t_i)$ during Δ (5.3). For example we derive correlation $corr\{p(t_i)U^2(t_i)\}$ between time-series of price $p(t_i)$ and squares of trade volume $U^2(t_i)$ during Δ

$$E[p(t_i)U^2(t_i)] \equiv E[C(t_i)U(t_i)] = E[C(t_i)]E[U(t_i)] + corr\{C(t_i)U(t_i)\}$$

$$E[p(t_i)U^2(t_i)] = E[p(t_i)]E[U^2(t_i)] + corr\{p(t_i)U^2(t_i)\}$$

Thus, from above one easy obtains:

$$corr\{p(t_i)U^2(t_i)\} = corr\{C(t_i)U(t_i)\} - p(t;1)\sigma^2(U) \quad (5.17)$$

$$\sigma^2(U) = U(t;2) - U^2(t;1) \quad (5.18)$$

It is obvious that price statistical moments *p(t;n)* (5.10; 5.12) differ from statistical moments *π(t;n)* generated by frequency-based price probability *P(p) ~ m(p)/N* during Δ (5.3):

$$\pi(t;n) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) \quad (5.19)$$

$$\pi(t;n) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{C^n(t_i)}{U^n(t_i)} \neq \frac{\sum_{i=1}^{N} C^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} = p(t;n) \quad (5.20)$$

The difference between the frequency-based *π(t;n)* and market-based *p(t;n)* price statistical moments determine the distinctions between two approaches to definition of the price probability. Only if during Δ (5.3) all trade volumes equal unit $U(t_i)=1$ then price statistical moments *p(t;n)* equal statistical moments of market value *C(t;n)* and for this special case the market trade price stochasticity is described by usual frequency-based probability.

Let us explain economic meaning and importance of *n-th* statistical moments of the trade value *C(t;n)* (5.7), volume *U(t;n)* (5.8) and price *p(t;n)* (5.10; 5.12) for any asset-pricing models. Indeed, *n-th* statistical moments of value *C(t;n)* and volume *U(t;n)* describe average state of market trading taking into account different proportions between high and low values and volumes of market transactions. Actually, at moments t_i during interval Δ agents perform *N* trades at different values $C(t_i)$, volumes $U(t_i)$ and price $p(t_i)$ (1.1). Mean value *C(t;1)* (5.5) and volume *U(t;1)* (5.6) are not the only assessments of "mean" properties of *N* trades. Mean value *C(t;1)* and volume *U(t;1)* are complemented by the set of *n-th* mean values and volumes determined as *n-th* roots of *n-th* statistical moments *C(t;n)* (5.7) and *U(t;n)* (5.8):

$$C(t;1/n) = [C(t;n)]^{1/n} \quad ; \quad U(t;1/n) = [U_m(n)]^{1/n} \quad (5.21)$$

So, one obtains the sequence of the mean values *C(t;1/n)* and volumes *U(t;1/n)* (5.21). With increasing *n* the mean values *C(t;1/n)* and volumes *U(t;1/n)* (5.21) more and more take into account contribution of trades with large values and volumes and decreases impact of trades performed at a low values. So, with increasing *n* relations (5.10; 5.12) describe *n-th* statistical moments of price *p(t;n)* and price mean *n-th* root $p(t;1/n)$

$$p(t;1/n) = [p(t;n)]^{1/n} \tag{5.22}$$

that reflect increasing impact of large value and volume transactions on market trade price (5.22). Any forecast of price probability should assess the finite set of price statistical moments *p(t;n)* (5.8) it predicts. Thus any price probability forecast should estimate the proportion of impact of large value and volume market trades described by high *n-th* statistical moments and minor market trades with small volume. The choice of proportion of large and minor market trades determines economic meaning of any asset pricing model and economic sense of any price probability forecast.

Random variable can be equally described by probability measure, characteristic function and by set of n-th statistical moments. The set of price *n-th* statistical moments *p(t;n)* (5.10; 5.12) for all n=1,2,3,… determines Taylor series of the price characteristic function *F(t;x)* (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005; 2015):

$$F(t;x) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} p(t;n)\, x^n \tag{5.23}$$

However, any econometric records of the market trade during *Δ* (5.3) allow assess only finite number of statistical moments of the trade value *C(t;n)* (5.7) and volume *U(t;n)* (5.8) Hence, one can assess only finite number of price statistical moments *p(t;n)* (5.10; 5.12). In App.B we consider simple successive approximations of the price characteristic function $F_k(t;x)$ that takes into account finite number *k* members of the Taylor series (5.23) and corresponding approximations of the price probability measures $\eta_k(t;p)$ derived as Fourier transforms of characteristic functions:

$$\eta_k(t;p) = \int dx\, F_k(t;x) \exp(-ixp) \tag{5.24}$$

Relations (5.24) define successive approximations of the price probability measures $\eta_k(t;p)$. In (5.24) we omit factor (2π) for brevity. Observation of finite number of the market trade and price statistical moments cause that one can forecast only approximations of price characteristic function or price probability measure those match finite number *k* of price statistical moments *p(t;n)* (5.10; 5.12).

$$p(t;n) = \frac{d^n}{(i)^n dx^n} F_k(t;x)|_{x=0} = \int dp\, \eta_k(t;p) p^n \tag{5.25}$$

Any hypothesis on the form of the price probability measure $\eta_k(t;p)$ during Δ (5.3) and any predictions of the price probability measure at horizon T should match relations (5.10; 5.12; 5.25) at $t+T$ during Δ. Thus one should predict k statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8) at $t+T$ during Δ for $n \leq k$. That equals prediction of the k-approximations of the market trade probability measures at horizon T during Δ. For example, consider the market-based price volatility $\sigma^2(t;p)$ (Olkhov, 2020):

$$\sigma^2(t;p) = E_t\left[\left(p(t_i) - p(t;1)\right)^2\right] = p(t;2) - p^2(t;1) = \frac{C(t;2)}{U(t;2)} - \frac{C^2(t;1)}{U^2(t;1)} \qquad (5.26)$$

From (5.7; 5.8; 5.11) one can express based price volatility $\sigma^2(t;p)$ as:

$$\sigma^2(t;p) = \frac{C(t;2)}{U(t;2)} - \frac{C^2(t;1)}{U^2(t;1)} = \frac{C_\Sigma(t;2)}{U_\Sigma(t;2)} - \frac{C_\Sigma^2(t;1)}{U_\Sigma^2(t;1)} \qquad (5.27)$$

Prediction of the price volatility $\sigma^2(t;p)$ at horizon T during Δ requires forecasts of the market trade statistical moments $C(t;1)$, $C(t;2)$ (5,7) and $U(t;1)$, $U(t;2)$ (5.8) at the same horizon T. Accuracy of the price probability forecast determined by accuracy of the market trade probabilities forecasts. In simple words: to predict price probability one should predict market trade probabilities. Dependence of the market price probability measure on the market trade probabilities expresses the famous phrase: “You can’t beat the market”.

Current economic theory model evolution of macroeconomic variables determined by sums of the *1-st* degree variables like sums of the trade value $C_\Sigma(t;1)$ and volume $U_\Sigma(t;1)$. However, price volatility $\sigma^2(p)$ (5.26; 5.27) is a sample of the *2-d* degree variable, because it depends on sums of squares of trade values value $C_\Sigma(t;2)$ and volume $U_\Sigma(t;2)$ during Δ. Description of the *2-d* degree macro variables as well as description of $C_\Sigma(t;2)$ and $U_\Sigma(t;2)$ requires development of the *2-d* order economic theory (Olkhov, 2021a). Moreover, description of the price skewness $Sk(t;p)$ (B.9) requires *3-d* statistical moments of price $p(t;3)$ determined by 3- statistical moments on trade value $C(t;3)$ and volume $U(t;3)$

$$C(t;3) = p(t;3)U(t;3) \qquad ; \qquad C_\Sigma(t;3) = p(t;3)U_\Sigma(t;3)$$

Hence, predictions of the price skewness $Sk(p)$ requires forecasts of $C(t;3)$ and $U(t;3)$. That need development of the *3-d* order economic theory that models sums of the *3-d* power of the market trade values $C_\Sigma(t;3)$ and volumes $U_\Sigma(t;3)$. Forecasts of price kurtosis (B.11) require development of the 4-th order economic theory and so on.

However, above considerations don’t determine the choice between correct and incorrect treatment of the price probability measure. Economics is a social science and investors are free in their trade decisions, expectations, habits, beliefs, financial and social “myths & legends”. Investors are free to choose any definition of the price probability they prefer.

Brief resume 3. Approximations of the market-based price probability measure are determined by finite number of *n-th* statistical moments of the market trade value *C(t;n)* and volume *U(t;n)* during the interval *Δ*. Any forecasts of the price probability at horizon *T* should match predictions at the same horizon *T* of the finite number of *n-th* statistical moments of the market trade value and volume averaged during *Δ*.

6. Conclusion

1. We derive modification of the basic pricing equation (3.3) that takes into account averaging of investor's utility during *Δ* at moment *t* and *t+1*. The choice of *Δ* and description of the dependence of the mean price, payoff, volatilities and etc., on duration of *Δ* are important for any asset-pricing model.

2. Taylor series expansions of the common (2.6) and modified (3.3) basic pricing equations permit derive successive approximations of the mean price, payoff, their volatilities, skewness and etc. As example of linear Taylor series expansion of (3.3) we mention (4.12)

$$p_0 = m_0 x_0 + \xi_{max}[m_1\sigma^2(x) + m_2\sigma^2(p)]$$

that describes the mean price p_0 at t as function of the mean payoff x_0 and payoff volatility $\sigma^2(x)$ at $t+1$, price volatility $\sigma^2(p)$ at t and the amount of assets ξ_{max} that delivers max to investor's utility and equals the root of the equation (3.3). Other new results present Taylor series expansions generated by the averaging interval *Δ*. Similar relations can be considered for any asset pricing models, economic and financial models those describe relations between averaged variables. Taylor series expansions can be applied to any financial models those consider mean price, payoff or other averaged variables. In particular, for the consumption-based asset pricing model Taylor series help derive relations on price and payoff autocorrelations (Olkhov, 2022a). Linear, quadratic or higher expansions of Taylor series give successive approximations of the mean variables, their volatilities and etc.

3. We introduce new market-based price probability measure through definition of all price *n-th* statistical moments *p(t;n)* (5.10; 5.12). Price statistical moments *p(t;n)* are determined by *n-th* statistical moments of the trade value *C(t;n)* (5.7) and volume *U(t;n)* (5.8). Market trade records permit assess only finite number *k* of trade statistical moments. First $n \leq k$ *n-th* statistical moments of trade value *C(t;n)* (5.7) and volume *U(t;n)* (5.8) determine *k*-approximations of market price probability measures. Any predictions of price probability at horizon *T* should match forecasts of $n \leq k$ *n-th* trade statistical moments at the same horizon *T*. Our definition of price statistical moments *p(t;n)* (5.10; 5.12) extends definition of VWAP (5.1). We outline important consequences of VWAP (5.1) and definitions (5.10; 5.12): usage

of price statistical moments $p(t;n)$ (5.1; 5.10; 5.12) results in zero correlations $corr\{p^n(t_i)U^n(t_i)\}=0$ (5.14; 5.16) between time series of *n-th* power of price $p^n(t_i)$ and trade volume $U^n(t_i)$ averaged during Δ (5.3). In particular, use of VWAP causes zero correlations (5.14) between time-series of price $p(t_i)$ and volume. That impact numerous studies on price-volume correlations those use other definition of the mean price. Most results should be reconsidered. Zero correlations (5.16) between *n-th* power of price $p^n(t_i)$ and trade volume $U^n(t_i)$ don't cause statistical independence between price and volume random variables during Δ (5.3). We derive expression for correlation corr$\{p(t_i)U^2(t_i)\}$ (5.17) between price and squares of volume during Δ (5.3).

New market-based price probability uncovers tough problems for the effective usage of the well known Value-at-Risk (Olkhov, 2021b); reassessment of the classical Black-Scholes option pricing shows the need of two-dimensional space (Olkhov, 2021c). Taylor series expansions help derive new representations of the market-based price autocorrelation (Olkhov, 2022a; 2022b).

The price probability depends upon the set of *n-th* statistical moments of trade value and volume. For different *n* the market price *n-th* statistical moments $p(t;n)$ (5.10; 5.12) describe impact of different proportions of major deals at high trade values and volumes and minor deals at low trade volumes. With growing *n* the impact of large transactions on market price grows up. Development of economic models with different proportion between of high and low trade volumes is a necessary condition for correct prediction of the price probability.

The trinity – explicit usage of the averaging interval, Taylor series and similar assessment of the probability measures can provide successive approximations for other versions of asset pricing, financial and economic models.

Appendix A

Max of Utility

$$p^2 > -\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [{x_0}^2 + \sigma^2(x)] - \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max} [2x_0\sigma^2(x) + \gamma^3(x)] \qquad \text{(A.1)}$$

If the right side is negative then it is valid always. If the right side is positive – then there exist a lower limit on the price *p*. For simplicity, neglect term $\gamma^3(x)$ to compare with $2x_0\sigma^2(x)$ and take the conventional power utility $u(c)$ (Cochrane, 2001) as:

$$u(c) = \frac{1}{1-\alpha} c^{1-\alpha} \qquad \text{(A.2)}$$

Let us consider the case with negative right side for (A.1). Simple but long calculations give:

$$-\beta \frac{u''(c_{t+1;0})}{u''(c_t)} [{x_0}^2 + \sigma^2(x)] < \beta \frac{u'''(c_{t+1;0})}{u''(c_t)} \xi_{max}\, 2x_0\sigma^2(x)$$

$$\xi_{max}\, 2x_0\sigma^2(x) < -\frac{u'''(c_{t+1;0})}{u''(c_{t+1;0})} [{x_0}^2 + \sigma^2(x)] \qquad \text{(A.3)}$$

Let us take into account (A.2) and for (A.3) obtain:

$$\frac{u''(c)}{u'''(c)} = \frac{-\alpha c^{-\alpha-1}}{\alpha(1+\alpha)c^{-\alpha-2}} = -\frac{c}{1+\alpha} \quad ; \quad \xi_{max}\, 2x_0\sigma^2(x) < \frac{e_{t+1} + x_0\xi_{max}}{1+\alpha} [{x_0}^2 + \sigma^2(x)]$$

$$\xi_{max} x_0 \,[(1+2\alpha)\sigma^2(x) - {x_0}^2] < e_{t+1}[{x_0}^2 + \sigma^2(x)] \qquad \text{(A.4)}$$

Inequality (A.4) determines that the right side (A.1) is negative in two cases.

1. The left side in (A.4) is negative and

$$(1+2\alpha)\sigma^2(x) < {x_0}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \qquad \text{(A.5)}$$

Inequality (A.5) describes small payoff volatility $\sigma^2(x)$. In this case the right side of (A.1) is negative for all ξ_{max} and all price *p* and hence (4.27) that defines max of utility (2.5) is valid.

2. The left side in (A.4) is positive and

$$(1+2\alpha)\sigma^2(x) > {x_0}^2 \quad ; \quad \xi_{max} < \frac{e_{t+1}[{x_0}^2 + \sigma^2(x)]}{x_0\,[(1+2\alpha)\sigma^2(x) - {x_0}^2]} \qquad \text{(A.6)}$$

This case describes high payoff volatility and the upper limit on ξ_{max} to utility (2.5). Take the positive right side in (A.1). Then (A.4) is replaced by the opposite inequality

$$\xi_{max} x_0 \,[(1+2\alpha)\sigma^2(x) - {x_0}^2] > e_{t+1}[{x_0}^2 + \sigma^2(x)] \qquad \text{(A.7)}$$

It is valid for (A.6) only. (A.7) determines a lower limit on ξ_{max} to utility (2.5):

$$\xi_{max} > \frac{e_{t+1}[{x_0}^2 + \sigma^2(x)]}{x_0\,[(1+2\alpha)\sigma^2(x) - {x_0}^2]}$$

Appendix B

Approximations of the price characteristic function and probability measure

Taylor series expansions of market price characteristic function help derive successive approximations of characteristic function. Derivation of approximations is a self-standing research and here we present simple examples of such approximations only.

We consider the approximations of price characteristic function $F_k(t;x)$ and price probability measure $\eta_k(t;p)$ Δ (5.3) those fit simple condition. As such we require that approximation $F_k(t;x)$ of the price characteristic function $F(t;x)$ (5.23) and probability measure $\eta_k(t;p)$ determined by Fourier transform (5.24) define first k price statistical moments $p(t;n)$ (5.25) during Δ (5.3) those match (B.1):

$$p(t;n) = \frac{C(t;n)}{U(t;n)} = \frac{d^n}{(i)^n dx^n} F_k(t;x)|_{x=0} = \int dp\, \eta_k(t;p) p^n \; ; \; n \le k \qquad \text{(B.1)}$$

Statistical moments determined by $F_k(t;x)$ for $n>k$ will be different from price statistical moments $p(t;n)$ (5.10; 5.12) but first k moments will be equal to $p(t;n)$ (5.10; 5.12; 5.23).

We suggest approximation $F_k(t;x)$ of price characteristic function $F(t;x)$ (5.23) as

$$F_k(t;x) = \exp\left\{\sum_{m=1}^{k} \frac{i^m}{m!}\, a_m\, x^m - b\, x^{2n}\right\} \; ; \; k = 1,2,\ldots; \; k < 2n \; ; \; b > 0 \qquad \text{(B.2)}$$

For each approximation $F_k(t;x)$ terms a_m in (B.2) depend on price statistical moments $p(t;m)$, $m \le k$ and match relations (B.1). The term bx^{2n}, $b>0$, $2n>k$ doesn't impact relations (B.1) but guarantees existence of the price probability measures $\eta_k(t;p)$ as Fourier transforms (5.24). Uncertainty and variability of the coefficient $b>0$ and power $2n>k$ in (B.2) underlines well-known fact that first k statistical moments don't explicitly and exactly determine characteristic function and probability measure of a random variable. Relations (B.2) describe the set of characteristic functions $F_k(t;x)$ with different $b>0$ and $2n>k$ and corresponding set of probability measures $\eta_k(t;p)$ those match (B.1; 5.24).

For $k=1$ the approximate price characteristic function $F_1(t;x)$ and measure $\eta_1(t;p)$ are trivial:

$$F_1(t;x) = \exp\{i\, a_1 x\} \; ; \; p(t;1) = -i \frac{d}{dx} F_1(t;x)|_{x=0} = a_1 \qquad \text{(B.3)}$$

$$\eta_1(t;p) = \int dx \;\, A_1(xt;)\, exp - ipx = \delta\big(p - \; p(t;1)\big) \qquad \text{(B.4)}$$

For $k=2$ approximation $F_2(t;x)$ describes the Gaussian probability measure $\eta_2(t;p)$:

$$F_2(xt;) = \exp\left\{i\, p(t;1)x - \frac{a_2}{2} x^2\right\} \qquad \text{(B.5)}$$

It is easy to show that

$$p_2(t;2) = -\frac{d^2}{dx^2} F_2(t;x)|_{x=0} = a_2 + p^2(t;1) = p(t;2)$$

Hence:

$$a_2 = p(t;2) - p^2(t;1) = \sigma^2(t;p) \tag{B.6}$$

Coefficient a_2 equals price volatility $\sigma^2(t;p)$ (5.26) and Fourier transform (5.24) for $F_2(t;x)$ gives Gaussian price probability measure $\eta_2(t;p)$:

$$\eta_2(pt;) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(p)} \exp\left\{-\frac{(p-p(t;1))^2}{2\sigma^2(t;p)}\right\} \tag{B.7}$$

For $k=3$ approximation $F_3(t;x)$ has form:

$$F_3(t;x) = \exp\left\{i\, p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\, \frac{a_3}{6}x^3\right\} \tag{B.8}$$

$$p_3(t;3) = i\frac{d^3}{dx^3}F_3(t;x)|_{x=0} = a_3 + 3p(t;1)\sigma^2(t;p) + \; p^3(t;1) = p(t;3)$$

$$a_3 = p(t;3) - 3p(t;2)p(t;1) + 2\,p^3(t;1) = E\left[\left(p - p(t;1)\right)^3\right] = Sk(t;p)\sigma^3(t;p) \tag{B.9}$$

Coefficient a_3 (B.9) depends on price skewness $Sk(t;p)$ that describe asymmetry of the price probability from normal distribution. For $k=4$ approximation $F_4(t;x)$ during Δ (5.3) depends on choice of $b>0$ and degree $2n>4$:

$$F_4(t;x) = \exp\left\{i\, p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\, \frac{a_3}{6}x^3 + \frac{a_4}{24}x^4 - bx^{2n}\right\}\ ; 2n > 4 \tag{B.10}$$

Simple, but long calculations give:

$$a_4 = p(t;4) - 4p(t;3)p(t;1) + 12p(t;2)p^2(t;1) - 6p^4(t;1) - 3p^2(t;2)$$

$$a_4 = E\left[\left(p(t_i) - p(t;1)\right)^4\right] - 3E^2\left[\left(p(t_i) - p(t;1)\right)^2\right]$$

Price kurtosis $Ku(p)$ (B.11) describes how the tails of the price probability measure $\eta_k(t;p)$ differ from the tails of a normal distribution.

$$Ku(p)\sigma_p^4(t;p) = E\left[\left(p(t_i) - p(t;1)\right)^4\right] \tag{B.11}$$

$$a_4 = [Ku(p) - 3]\sigma_p^4(t;p)$$

Even simplest Gaussian approximation $F_2(t;x)$, $\eta_2(t;p)$ (B.5; B.7) uncovers direct dependence of price volatility $\sigma^2(t;p)$ (B.6; 5.26) on *2-d* statistical moments of the trade value $C(t;2)$ and volume $U(t;2)$. Thus, prediction of price volatility $\sigma^2(t;p)$ for Gaussian measure $\eta_2(t;p)$ (B.9) should follow non-trivial forecasting of the statistical moments of the market trade value $C(t;2)$ and volume $U(t;2)$.

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