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Abstract

This paper considers asset price as a random variable during the averaging interval *Δ* and introduces the *market-based* price probability. We substitute the problem of guessing the "correct" form of the asset price probability by approximate description of random price properties by the finite number of price statistical moments determined by statistical moments of the market trade value and volume during *Δ*. We extend the well-known definition of the volume weighted average price (VWAP) presented more than 30 years ago and introduce market price *n-th* statistical moments as *n-th* power volume weighted average. That results in zero correlations between time-series of *n-th* power of the trade volume and price during *Δ,* but doesn't cause statistical independence between trade volume and price. As example we derive expression for correlation between time-series of the market price and squares of the trade volume. The market trades records during *Δ* allow assess only finite number of statistical moments. Approximations of the price characteristic function that reproduce first *m* price statistical moments generate approximations of the *market-based* price probability. That approach unifies description of the asset price, price indices, returns, inflation and their volatilities. *Market-based* approach impacts the asset pricing models and uncovers hidden vital fault of the widely used hedging tool – Value-at-Risk. Our approach doesn't simplify the random price puzzle but establishes direct economic ties and relations between asset pricing, market stochasticity and economic theory. Description of the *market-based* price and returns volatility, Skewness and Kurtosis requires development of the economic theories those model relations between the second, third and forth order macroeconomic variables. Development of these theories will take a lot of efforts and years.

Keywords : asset price, price probability, returns, inflation, market trades

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1. Introduction

Aspirations to have price predictions for the next day, month or year are as old as market trade. Merchants and pawnbrokers, investors and traders, bankers and stockbrokers, academics and households for years, decades and centuries seek for reliable, secure and precise price forecasts. However, little by little it became clear that exact price guesses as well as tomorrow fortune forecasts are too fickle and variable. Thus, exact predictions of a price for the next time term were replaced by variety predictions of probable set of the price values. Ambiguity of the future was projected into uncertainty of the price prognosis. Centuries of asset pricing studies (Dimson and Mussavian, 1999) track price probability up to Bernoulli's studies 1738, but probably, Bachelier (1900) was one of the most influential paper that outlines probabilistic character of the price behavior and forecasting. “The probabilistic description of financial prices, pioneered by Bachelier”(Mandelbrot, et.al., 1997). “in fact the first author to put forward the idea to use a random walk to describe the evolution of prices was Bachelier.” (Shiryaev, 1999). During last century the endless number of studies discussed asset pricing models and described different hypothesis, laws and properties of random asset price (Kendall and Hill, 1953; Muth, 1961; Sharpe, 1964; Fama, 1965; Stigler and Kindahl, 1970; Black and Scholes, 1973; Merton, 1973; Tauchen and Pitts, 1983; Mackey, 1989; Friedman, 1990; Cochrane and Hansen, 1992; Campbell, 2000; Heaton and Lucas, 2000; Cochrane, 2001; Poon and Granger, 2003; Andersen et.al., 2005a; 2005b; 2006; Cochrane, 2005; Wolfers and Zitzewitz, 2006; DeFusco et.al., 2017; Weyl, 2019; Cochrane, 2022). Rigorous mathematical treatments of stochastic description and probabilistic modelling of asset price are given in (Shiryaev, 1999; Shreve, 2004). It is obvious, that we referred only a negligible part of endless studies on asset pricing.

Asset price dynamics is under action of numerous factors of different nature that result irregular or random price change during almost any time interval. A century ago Fetter (1912) mentioned 117 price definitions, and that for sure increases varieties of asset price considerations, treatments and forecasting. That generates enormous range of price studies that outline price variations and dependence on market (Fama, 1965; Tauchen and Pitts, 1983; Odean, 1998; Poon and Granger, 2003; Andersen et.al., 2005b; DeFusco et.al., 2017..), macroeconomic (Cochrane and Hansen, 1992; Heaton and Lucas, 2000; Diebold and Yilmaz, 2008), business cycles (Mills, 1946; Campbell, 1998), expectations (Muth, 1961; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014), trading volumes (Karpoff, 1987; Campbell et.al., 1993; Gallant et.al., 1992; Brock and LeBaron, 1995;

Llorente et.al., 2001) and many other factors that for sure impact price trends and fluctuations. The line of factors and references can be continued forever (Goldsmith and Lipsey, 1963; Andersen et.al., 2001; Hördahl and Packer, 2007; Fama and French, 2015). However, we have no idea to review the asset pricing studies, but chose one "simple" problem of the asset price universe. That problem concerns the notion and definition of the asset price probability density function (PDF). It seems to be one of the most common and well-studied issues of financial economics. Almost all standard probability distributions (Forbes, et.al., 1992; Walck, 2011) were tested to check how they can model, describe and predict price PDF and random price properties. A lot was done but the asset price PDF puzzle seems to be still inconceivable. Actually, it is twice interesting to have a fresh look at the traditional matter. Indeed, since Bachelier (1900) the joint efforts of economists and statisticians were directed to uncover the "correct" model of the random price change and its PDF. May be credibility and domination of Bachelier and his famous followers put studies of the price probability not that side? We don't critique any notable studies but remind that the asset price is not a single, main and independent issue of economics and finance. Asset pricing is woven deeply into relations, laws and properties of the economy and finance. We consider the asset price PDF problem as a puzzle of the economic and financial relations, as a result of market evolution and not as a standing separately question. As we mentioned above, there are a lot of different definitions of price. Fetter (1912) presents 117 price definitions and hence there are probably a lot of different treatments and approaches to price probability modelling. We don't review that variety of price definitions but consider single, simple and usual market-based price notion that is trivially determined by each market trade. Indeed, each particular market trade at time t_i can be determined by the trade value $C(t_i)$, trade volume $U(t_i)$ and trade price $p(t_i)$ those match simple relations:

$$C(t_i) = p(t_i)U(t_i) \tag{1.1}$$

It is well known that the time-series records of the values, volumes and prices of the market trades are very irregular and usually assumed as random. Trivial equation (1.1) establishes important requirement on the probabilities of time-series that match (1.1). Indeed, PDF of the time-series of market trades performed at moments t_i of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ those match (1.1) cannot be determined independently. Given probabilities of the trade value and volume (1.1) determine the asset price probability. That approach to the market-based asset price probability doesn't simplify the price probability puzzle, but establishes direct links, ties and relations between stochasticity of the market trade value and volume on one hand and randomness of the market price on the other hand. Actually, we replace the classical problem: what is the "correct" price probability? - by a different one. We consider how approximate description of the market trade value and volume stochasticity can approximate random properties of the market asset price, but don't study the specific properties of the market trade value and volume probabilities.

In Sec.2 we describe how stochasticity of the trade value and volume can determine random properties of the asset price. Further, in Sec. 3 and 4 we briefly consider consequences of our results on description of random properties of returns and inflation, asset pricing models and Value-at-Risk (VaR) as most conventional risk management tool, and argue some other issues of financial economics. Sec. 5 – Conclusion. We assume that our readers are familiar with standard issues of probability theory as statistical moments, characteristic functions and etc. This paper for sure, is not for novices and we propose that readers already know or able find on their own definitions and explanations of the notions, terms and models that are nor given in the text.

2. Market-based price probability

To start with, let us consider the time-series of the market transactions with selected asset at moments t_i, $i=1,\dots$. Economic analysis of time-series has a long history and references (Davis, 1941; Anderson, 1971; Cochrane, 2005; Diebold, 2019) indicate author's preferences only. We take that time-series records describe the value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ of transaction at time t_i. The times t_i of the time-series records introduce the initial time division of the time axis. We study random properties of market trade using time-series records of performed transactions only. Thus, any considerations of possible impact of agents' price expectations, price forecasts, economic or financial factors and any possible influence on the market price can be omitted. Actually, all possible factors those impact asset pricing are already collected and imprinted into the time-series records of the market trade value $C(t_i)$ and volume $U(t_i)$. Hence, one can state that the randomness of the price time-series is completely determined by stochasticity of time-series of the trade value and volume (1.1).

For simplicity we take that transactions are performed at times t_i multiply of small interval ε:

$$t_i = \varepsilon \cdot i \;;\; i = 0, 1, 2, \dots \tag{2.1}$$

Time-series (2.1) establish time axis division multiple of ε. Let us take that we study the market trade time-series during time horizon T and assume that initial time division $\varepsilon \ll T$. High frequency trading can deliver market trades records with ε as fraction of second. Such precise time division generates high irregular time-series and of little help for modelling price at long time horizon T. Description of market price at horizon T that equals weeks, months or years requires aggregation of the initial market time-series during some reasonable time interval Δ, that takes intermediate value

$$\varepsilon \ll \Delta \ll T \tag{2.2}$$

Market time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ aggregated or averaged during the interval Δ result time-series with time axis division multiple of Δ. For simplicity let take the interval Δ multiple of ε for some n as:

$$\Delta = 2n \cdot \varepsilon \quad ; \quad N = 2n + 1 \ ; \quad \varepsilon \ll \Delta \ll T \tag{2.3}$$

Aggregation of time-series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ during the interval Δ generates corresponding time-series at moments t_k and results time axis division multiple of Δ

$$t_k = \Delta \cdot k \quad ; \quad \Delta_k = \left[t_k - \frac{\Delta}{2}; t_k + \frac{\Delta}{2}\right] \quad ; \quad k = 0, 1, 2, \ldots \tag{2.4}$$

Let us take that each member of time-series of the trade value $C(t_k)$ at time t_k (2.4) is determined by aggregation or averaging of time-series $C(t_i)$ during Δ_k (2.4). Total trade value $C(t_k)$ and total trade volume $U(t_k)$ during the interval Δ_k are determined as

$$C(t_k) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U(t_k) = \sum_{i=1}^{N} U(t_i) \tag{2.5}$$

$$t_k - \frac{\Delta}{2} \leq t_i \leq t_k + \frac{\Delta}{2} \tag{2.6}$$

Due to our assumption (2.3) there are $N=2n+1$ members of time-series $C(t_i)$ or $U(t_i)$ in each time interval Δ_k. We consider time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ as random variables during Δ_k (2.4) and determine the mean market trade value $C(t_k;1)$ and the mean trade volume $U(t_k;1)$ at time t_k averaged during Δ_k as

$$C(t_k; 1) = E[C(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \quad ; \quad U(t_k; 1) = E[U(t_i)] = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{2.7}$$

We use $E[..]$ to denote mathematical expectation. We underline that mean values of market trade (2.7) are determined during the interval Δ_k (2.2-2.4). Different choice of the interval Δ (2.2) results in different average trade value and volume (2.7).

For the given averaging interval Δ_k (2.2-2.4) we consider time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.6) as random variables. Probabilities of the trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.6) are determined in a conventional way (Shiryaev, 1999; Shreve, 2004). Probability of trade value $P(C)$ is proportional to frequency of trades at value C. If during the time interval Δ_k (2.6) there are m_c trades at value C and m_u trades at volume U then, due to (2.3) probabilities $P(C)$ and $P(U)$ are assessed as:

$$P(C) \sim \frac{m_c}{N} \quad ; \quad P(U) \sim \frac{m_u}{N} \tag{2.8}$$

Further we note conventional approach to probability (2.8) as the *frequency-based* in contrary to the *market-based* definition of the price probability below. Statistical moments of the market trade value $C(t_i)$ and volume $U(t_i)$ for t_i inside Δ_k (2.4; 2.6) are assessed as usual:

$$C(t_k;n) = E[C^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(t_k;n) = \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \quad ; \; n = 1, \ldots \tag{2.9}$$

For $n=1,2,\ldots$ statistical moments (2.9) completely determine properties of the trade value $C(t_i)$ and volume $U(t_i)$ treated as random variables during Δ_k (2.4).

Now let us consider random properties of the market price. As the mean price $p(t_k;1)$ or price 1-st statistical moment we take volume weighted average price (VWAP) that was introduced more than 30 years ago and is widely used now (Berkowitz et.al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; CME Group, 2020; Duffie and Dworczak, 2021). During the time interval Δ_k (2.6) VWAP $p(t_k;1)$ takes form:

$$p(t_k;1) = E[p(t_i)] = \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i)U(t_i) \tag{2.10}$$

Using (1.1) obtain equivalent form of VWAP $p(t_k;1)$:

$$p(t_k;1) = E[p(t_i)] = \frac{\sum_{i=1}^{N} p(t_i)U(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{\sum_{i=1}^{N} C(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{C(t_k;1)}{U(t_k;1)} \tag{2.11}$$

Actually, just one VWAP price $p(t_k;1)$ (2.10; 2.11) is not sufficient to define price properties as a random variable during the interval Δ_k (2.4; 2.6). To define price as random variable one should determine price probability density function (PDF) or price characteristic function or introduce all price *n-th* statistical moments. All three methods give equal description of random variable (Shephard, 1991; Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005). To define price as random variable we introduce all price *n-th* statistical moments $p(t_k;n)$ for $n=1,2,\ldots$:

$$p(t_k;n) = E[p^n(t_i)] = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \tag{2.12}$$

Taking *n-th* power of (1.1) obtain:

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \; n = 1,2,3, \ldots \tag{2.13}$$

Hence, due to (2.5; 2.9; 2.13) obtain:

$$p(t_k;n) = E[p^n(t_i)] = \frac{\sum_{i=1}^{N} p^n(t_i)U^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{\sum_{i=1}^{N} C^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{C(t_k;n)}{U(t_k;n)} \tag{2.14}$$

Relations (2.12; 2.14) for all n determine all price statistical moments $p(t_k;n)$ and hence determine all properties of price as random variable during the interval Δ_k (2.6). Let us discuss the reasons in the foundation of our definition of the market price statistical moments $p(t_k;n)$ (2.12; 2.14). For all $n=1,2,3,\ldots$ relations (2.12; 2.14) introduce average of *n-th* power of price $E[p^n(t_i)]$ as weighted by *n-th* power of the trade volume $U^n(t_i)$ during Δ_k (2.6). Actually, (2.14) defines average *n-th* power of price $E[p^n(t_i)]$ as ratio of sum of *n-th* power of market trade values $C^n(t_i)$ to sum of *n-th* power of market trade volumes $U^n(t_i)$ those match (2.13) during Δ_k (2.4). For $n=1$ we obtain VWAP as ratio of sum of trade values to sum of trade volumes. Relations (2.9; 2.14) demonstrate that *n-th* statistical moments $p(t_k;n) = E[p^n(t_i)]$ are determined as ratio of *n-th* statistical moments of the market trade value $C(t_k;n)$ to volume $U(t_k;n)$. That establishes relations between statistical moments those determine random properties of the market trade and random properties of the market price.

Let us underline that definition of the VWAP $p(t_k;1)=E[p(t_i)]$ (2.11) hides important consequences. Actually, VWAP relations (2.11) result in zero correlations between time-series of the price $p(t_i)$ and volume $U(t_i)$ during Δ_k (2.4). Indeed, from (1.1; 2.7; 2.11) obtain:

$$E[p(t_i)U(t_i)] = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) \equiv \frac{1}{\sum_{i=1}^{N} U(t_i)}\sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U(t_i) \equiv E[p(t_i)]E[U(t_i)] \quad (2.15)$$

Hence (2.15) causes no correlations between time-series of market price and trade volume:

$$corr\{pU\} = E[p(t_i)U(t_i)] - E[p(t_i)]E[U(t_i)] = 0 \quad (2.16)$$

No correlations (2.15; 2.16) between VWAP and trade volume is a result of the price averaging procedure (2.7; 2.10; 2.11). Actually, numerous studies describe "observed" correlations between price and trading volume (Tauchen and Pitts, 1983; Karpoff, 1987; Gallant et.al., 1992; Campbell et.al., 1993; Llorente et.al., 2001; DeFusco et.al., 2017). That underlines the different approaches to definition of price averaging procedure. Researchers often neglect the trivial equation (1.1) that prohibits independent definitions of the trade value, volume and price probabilities. That results in "observation" of correlations between trade volume and price. Usage of VWAP states no correlations between volume and price.

It is obvious that definitions of the average *n-th* power of price $p(t_k;n)=E[p^n(t_i)]$ (2.12; 2.14) as weighed during Δ_k by *n-th* power of the trade volume $U^n(t_i)$ result in zero correlations between time-series of the *n-th* power of the price $p^n(t_i)$ and trade volume $U^n(t_i)$. Using (2.9; 2.12; 2.13) one can easy obtain for all $n=1,2,3,\ldots$

$$E[C^n(t_i)] = E[p^n(t_i)U^n(t_i)] = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) \equiv \frac{1}{\sum_{i=1}^{N} U^n(t_i)}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) \cdot$$

$$\frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \equiv E[p^n(t_i)]E[U^n(t_i)] = p(t_k;n)U(t_k;n) \quad (2.17)$$

Hence, for all n correlations $corr\{p^nU^n\}$ between time-series of *n-th* power of price $p^n(t_i)$ and trade volume $U^n(t_i)$ during Δ_k (2.6) equal zero:

$$corr\{p^nU^n\} = E[p^n(t_i)U^n(t_i)] - E[p^n(t_i)]E[U^n(t_i)] = 0 \quad (2.18)$$

Zero correlations (2.18) provide a different way for derivation of price statistical moments $p(t_k;n)$. Indeed, from (2.14) follows:

$$C(t_k;n) = p(t_k;n)U(t_k;n) \quad (2.19)$$

Relations (2.9; 2.19) define price *n-th* statistical moments $p(t_k;n)$ during Δ_k (2.6) that can be derived directly form (2.13) and zero correlations (2.18). Taking math expectation of (2.13):

$$C(t_k;n) = E[C^n(t_i)] = E[p(t_k;n)U(t_k;n)] = E[p^n(t_i)]E[U^n(t_i)] = p(t_k;n)U(t_k;n) \quad (2.20)$$

Thus zero correlations (2.18) can be treated as consequences of the definition (2.14) and as assumptions for deriving (2.19; 2.20) as mathematical expectation of (2.13).

The choice of price averaging procedure between the *frequency-based* and the *market-based* approaches determines different random properties of price. However, zero correlations (2.18) don't cause statistical independence between the trade volume and price random variables. As example, we derive correlation $corr\{pU^2\}$ between time-series of market price $p(t_i)$ and square of trade volume $U^2(t_i)$ during Δ_k (2.6) (see A.2):

$$corr\{pU^2\} = corr\{CU\} - p(t_k;1)\sigma^2(U)$$

Thus correlations between time-series of price $p(t_i)$ and square of trade volume $U^2(t_i)$ can be positive only in the case of high positive correlations $corr\{CU\}$ between trade value $C(t_i)$ and volume $U(t_i)$:

$$corr\{pU^2\} > 0 \;\; \Leftrightarrow \;\; corr\{CU\} > p(t_k;1)\sigma^2(U) > 0$$

Otherwise correlations are always negative: $corr\{pU^2\} <0$.

It is well known, that the set of *n-th* statistical moments for all $n=1,2,\ldots$ of random variable determines its characteristic function as Taylor series (Shephard, 1991; Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005). Price characteristic function $F(t_k;x)$ as Taylor series at moment t_k for price treated as random variable during Δ_k (2.4) takes form:

$$F(t_k;x) = 1 + \sum_{i=1}^{\infty} \frac{i^n}{n!} p(t_k;n)\, x^n \quad (2.21)$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \frac{d^n}{(i)^n dx^n} F(t_k;x)|_{x=0} \quad (2.22)$$

The most important result of our derivation of the *market-based* asset price characteristic function is follows: price characteristic function $F(t_k;x)$ (2.21) depends on set of *n-th* statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ and hence the price PDF also depends on market trade statistical moments (2.9). Any predictions of the *market-based* asset price PDF at horizon *T* should march forecasts of PDF or set of *n-th* statistical moments of the market trade value and volume at same horizon *T*.

However, exact expressions of the market trade value and volume probability density functions (PDF) are unknown. Any market trade time-series records for any given time interval Δ_k (2.4) permit observe and assess only finite number of the market trade statistical moments $C(t_k;n)$ and $U(t_k;n)$. Hence, one can operate by finite number of price statistical moments $p(t_k;n)$ only. Finite number *m* of price statistical moments describes *approximation* of the price characteristic function $F_m(t_k;x)$. Let us take m-approximations of the price characteristic function $F_m(t_k;x)$ (2.24; 2.25) that generate m-approximations of the price probability measure $\eta_m(t_k;p)$ (2.26; 2.27) (Olkhov, 2021d):

$$F_m(t_k;x) = \exp\left\{\sum_{j=1}^{m} \frac{i^j}{j!}\, a_j\, x^j - b\, x^{2n}\right\} \quad ; \quad m = 1,2,..; \; b > 0; \; 2n > m \qquad (2.24)$$

For each approximation $F_m(t_k;x)$ terms a_j , $j=1,..m$ in (2.24) depend on price statistical moments $p(t_k;j)$, $j\leq m$ and match relations (2.25). The term bx^{2n}, $b>0$, $2n>m$ doesn't impact relations (2.25). However, coefficient $b>0$ ensures integrability of approximation $F_m(t_k;x)$ of characteristic function and existence of the approximation of the price probability measures $\eta_m(t_k;p)$ as Fourier transforms (5.26). Uncertainty of the coefficient $b>0$ and power $2n$ of x^{2n}, $2n>m$ in (2.24) illustrates well-known fact that m statistical moments can't exactly define characteristic function and probability measure of a random variable. Expressions (2.24) present the set of approximations $F_m(t_k;x)$ of the price characteristic functions with different $b>0$ and $2n>m$ and corresponding set of approximations of the probability measures $\eta_k(t;p)$ those match (2.25; 2.26).

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \frac{d^n}{(i)^n dx^n} F_m(t_k;x)|_{x=0} \quad ; \quad n \leq m \qquad (2.25)$$

Such m-approximation $F_m(t_k;x)$ of the characteristic function reproduces first m price statistical moments (2.22) and generates m-approximation of the PDF $\eta_m(t_k;p)$ at moment t_k for price treated as random variable during the interval Δ_k (2.4):

$$\eta_m(t_k;p) = \frac{1}{\sqrt{2\pi}} \int dx\, F_m(t_k;x) \exp(-ixp) \qquad (2.26)$$

$$p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} = \int dp \;\, p^n \eta_m(t_k;p) \quad ; \quad n \leq m$$

For $n=2$ approximation of the price characteristic function $F_2(t_k;x)$ takes simple form (2.27).

$$F_2(t_k;x) = \exp\left\{i\, p(t_k;1)x - \frac{\sigma_p^2(t_k)}{2} x^2\right\} \qquad (2.27)$$

Fourier transform (2.26) of $F_2(t_k;x)$ generates simple Gaussian distribution $\eta_2(t_k;p)$ with the *market-based* asset price volatility $\sigma_p^2(t_k)$:

$$\eta_2(t_k;p) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_p(t_k)} \exp\left\{-\frac{\left(p - p(t_k;1)\right)^2}{2\sigma_p^2(t_k)}\right\} \qquad (2.28)$$

$$\sigma_p^2(t_k) = E[(p(t_i) - p(t_k;1))^2] = p(t_k;2) - p^2(t_k;1) = \frac{C(t_k;2)}{U(t_k;2)} - \frac{C^2(t_k;1)}{U^2(t_k;1)} \qquad (2.29)$$

We underline that simple Gaussian approximation of the asset price probability measure (2.28) depends on second statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$. Prediction of Gaussian price probability $\eta_2(t_k;p)$ (2.28) at horizon T requires forecasts of the second statistical moments of the market trade value and volume (2.9) at the same horizon *T*.

Approximations $F_4(t_k;x)$ (2.30-2.33) match first four price statistical moments for different $b>0$, $2n>4$, (Olkhov, 2021d). Approximations $F_4(t_k;x)$ depend on the price volatility $\sigma_p^2(t_k)$ (2.29), price skewness $Sk(p)$ (2.31) and price kurtosis $Ku(p)$ (2.32)

$$F_4(t;x) = \exp\left\{i\, p(t;1)x - \frac{\sigma_p^2(t_k)}{2}x^2 - i\,\frac{a_3}{6}x^3 + \frac{a_4}{24}x^4 - bx^{2n}\right\}\ ;\ 2n > 4 \qquad (2.30)$$

$$a_3 = E\left[\left(p - p(t;1)\right)^3\right] = Sk(p)\sigma_p^3(t_k) \qquad (2.31)$$

$$Kr(p)\sigma_p^4(t_k) = E\left[\left(p(t_i) - p(t;1)\right)^4\right] \qquad (2.32)$$

$$a_4 = [Kr(p) - 3]\sigma_p^4(t_k) \qquad (2.33)$$

It is important to underline that the market-based price volatility $\sigma_p^2(t_k)$ (2.29) depends on second statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$ (2.9) (Olkhov, 2020). Description of the market trade *2-d* statistical moments requires description of second statistical moments of all macroeconomic variables those depend on market trading. That needs development of the second-order economic theory that describes relations between *2-d* statistical moments of market trades and second-order macroeconomic variables (Olkhov, 2020; 2021a; 2021e). Usage of the *market-based* approach to price probability opens the way for description of market price autocorrelations (Olkhov, 2022a; 2022b).

Our consideration of the price statistical moments $p(t_k;n)$ through statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ only complement well-known properties of the random variable determined as difference of two random variables Indeed, taking logarithm of (1.1) one easy obtains that logarithm of price *ln(p)* equals logarithm of the trade value *ln(C)* minus logarithm of the trade volume *ln(U).* That case is described in many probability introductory notes (Papoulis and Pillai, 2002; p.181) and we consider it briefly in App. B. However we repeat that exact expressions of market trade PDF are unknown and econometric records of market trade allows assess only finite number of statistical moments. The market trade time series records on their own don't identify the exact form of the trade value PDF, trade volume PDF and their joint PDF that are required to derive log price PDF (App.B). At that point usage of our approximate approach to definition of the market-based price statistical moments (2.12; 2.14; 2.19) is more preferable.

3. Returns and inflation

The market-based asset price probability permit describe statistical properties of returns via statistical moments of the market trade value and volume. Actually, returns $r(t_1,t_2)$ are determined as

$$r(t_1,t_2) = \frac{p(t_2)-p(t_1)}{p(t_1)} = \frac{p(t_2)}{p(t_1)} - 1 \tag{3.1}$$

Let's take price index $a(t_1,t_2)$ (3.2) as:

$$p(t_2) = p(t_1)a(t_1,t_2) \tag{3.2}$$

$$r(t_1,t_2) = a(t_1,t_2) - 1 \tag{3.3}$$

In Sec.2 we already derived the *market-based* asset price probability statistical moments $p(t_k;n)$ (2.19), price characteristic function $F(t_k;x)$ (2.21; 2.22) and approximations of the *market-based* price probability measure $\eta_m(t_k;p)$ (2.26-2.29). Let's use the same approach to describe the *market-based* probability of the price index $a(t_1,t_2)$ (3.2). We shall consider two cases. First, we assume that the price index $a(t_1,t_2)$ (3.2) is determined with respect to the fixed price $p(t_1)$ and consider statistical properties of the price index by time t_2 averaged during the interval Δ_k (2.4). In the second case we consider random properties of the price index $a(t_k,t_2)$ with respect to market price averaged during Δ_k taking price $p(t_2)$ as random variable during interval Δ_{k+m} (2.4).

1-st case - Returns.

Relations (3.1) define returns $r(t_1,t_2)$ at moment t_2 with price $p(t_2)$ with respect of previous moment t_1 with price $p(t_2)$. As usual price $p(t_2)$ is unpredictable and one assesses average returns $r(t_1,t_2)$ or its volatility taking price $p(t_2)$ as random variable during the averaging interval Δ_k . As usual one considers returns irregular time-series as initial data to assess statistical moment of the returns, using conventional *frequency-based* probability of returns. We propose *market-based* assessment of returns statistical moments based on the proposed in Sec.2 market price statistical moments (2.12; 2.14). We state that statistical properties of returns are determined by statistical moments of the market price (2.12; 2.14) and derive statistical moments of returns as functions of statistical moments of market price and market trade value and volume.

Let us take into account (2.9; 2.19; 3.2) and for the *n-th* statistical moments of the price index $a(t_1,t_k;n)$ averaged by t_2 during the interval Δ_k (2.4) obtain:

$$a(t_1,t_k;n) = E[a^n(t_1,t_2)] \tag{3.4}$$

$$a(t_1,t_k;n)p^n(t_1) = p(t_k;n) = \frac{C(t_k;n)}{U(t_k;n)} \tag{3.5}$$

$E[...]$ – math expectation during the interval Δ_k (2.4). From (3.4; 3.5) obtain expressions for *n-th* statistical moments of returns $r(t_1,t_k;n)$:

$$r(t_1,t_k;n) = E[r^n(t_1,t_2)] \tag{3.6}$$

$$r(t_1,t_k;n) = E[(a(t_1,t_2)-1)^n] \tag{3.7}$$

Due to (3.4; 3.5; 3.7) *n-th* returns statistical moment $r(t_1,t_k;n)$ is a simple sum of *m-th* statistical moments of the price index $a(t_1,t_k;m)$, $m \le n$:

$$r(t_1,t_k;n) = \sum_{m=0}^{n}(-1)^{(n-m)}\frac{n!}{m!(n-m)!}a(t_1,t_k;m) \;\; ; \;\; a(t_1,t_k;0) = 1 \tag{3.8}$$

Due to (3.5) returns *n-th* statistical moments $r(t_1,t_k;n)$ can be presented through the market trade value and volume statistical moments (3.9):

$$r(t_1,t_k;n) = \sum_{m=0}^{n}(-1)^{(n-m)}\frac{n!}{m!(n-m)!}p^{-m}(t_1)\,\frac{C(t_k;m)}{U(t_k;m)} \;\; ; \;\; \frac{C(t_k;0)}{U(t_k;0)} = 1 \tag{3.9}$$

or equally through the *n-th* price statistical moment (see (3.5)).

In particular, one can easy derive relations (3.11) between the *market-based* price volatility $\sigma_p^2(t_k)$ (2.29), price $p(t_1)$ at moment t_1 and *market-based* volatility of returns $\sigma_r^2(t_1,t_k)$ (3.10)

$$\sigma_r^2(t_1,t_k) = r(t_1,t_k;2) - r^2(t_1,t_k;1) = E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^2\right] \tag{3.10}$$

$$\sigma_p^2(t_k) = p^2(t_1)\;\sigma_r^2(t_1,t_k) \tag{3.11}$$

Using (3.4-3.9) one can derive that return's Skewness $Sk_r(t_1,t_k)$ (3.13) that describe asymmetry of the market-based return's distribution equals (3.14) the *market-based* asset price Skewness $Sk_p(t_k)$ (3.12):

$$Sk_p(t_k)\sigma_p{}^3(t_k) = E\left[\left(p(t_2) - p(t_k;1)\right)^3\right] \tag{3.12}$$

$$Sk_r(t_1,t_k)\sigma_r{}^3(t_1,t_k) = E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^3\right] \tag{3.13}$$

$$Sk_p(t_k) = \; Sk_r(t_1,t_k) \tag{3.14}$$

One can easy obtain that the *market-based* price Kurtosis $Ku_p(t_k)$ (3.15) equals (3.17) returns Kurtosis $Ku_r(t_1,t_k)$ (3.16):

$$Ku_p(t_k)\sigma_p{}^4(t_k) = E\left[\left(p(t_2) - p(t_k;1)\right)^4\right] \tag{3.15}$$

$$Ku_r(t_1,t_k)\sigma_r{}^4(t_1,t_k) = E\left[\left(r(t_1,t_2) - r(t_1,t_k;1)\right)^4\right] \tag{3.16}$$

$$Ku_p(t_k) = \; Ku_r(t_1,t_k) \tag{3.17}$$

2-d case - Inflation.

Inflation has almost the same meaning as returns (3.1). Contrary to returns, inflation is considered as ratio of collective price level of goods and services averaged during chronological time interval $\Delta_{k+\mathrm{m}}$ with respect to price level averaged during earlier interval Δ_k. The interval Δ_k can be equal week, month, quarter or year. Assessment of inflation's price level takes into account collective values and volumes of market trade of goods, services etc. (Fox, et al., 2017). Let us consider collective market trade value and volume that model the price level in a way similar to Sec.2. Let us define "instantaneous" inflation $In(t_i;t_k)$ of price level $p(t_i)$ during averaging interval Δ_{k+m} as:

$$In(t_i, t_k) = \frac{p(t_i)}{p(t_k;1)} - 1 \tag{3.18}$$

Then inflation *n-th* statistical moments $In(t_{k+m},t_k;n)$ averaged during interval Δ_{k+m} take form:

$$In(t_{k+m}, t_k; n) = E[In^n(t_i, t_k)] = E[\left(\frac{p(t_i)}{p(t_k;1)} - 1\right)^n] \tag{3.19}$$

From (2.14) obtain:

$$\frac{p(t_{k+m}; n)}{p^n(t_k; 1)} = \frac{C(t_{k+m}; n)}{C^n(t_k; 1)} \frac{U^n(t_k; 1)}{U(t_{k+m}; n)}$$

Hence from (3.19) obtain *n-th* statistical moment of inflation $In(t_{k+m},t_k;n)$:

$$In(t_{k+m}, t_k; n) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!(n-j)!} \frac{p(t_{k+m};n-j)}{p^{n-j}(t_k;1)} \tag{3.20}$$

$$In(t_{k+m}, t_k; n) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!\,(n-j)!} \frac{C(t_{k+m}; n-j)}{C^{n-j}(t_k; 1)} \frac{U^{n-j}(t_k; 1)}{U(t_{k+m}; n-j)}$$

Let us introduce trade value index $c(t_{k+m};n|t_k,1)$ (3.21) as ratio of *n-th* trade value statistical moment $C(t_{k+m};n)$ (2.9) averaged during the interval Δ_{k+m} to *n-th* power of mean trade value $C(t_k;1)$ (2.7) averaged during the earlier interval Δ_k. The similar meaning has trade volume index $u(t_{k+m};n|t_k,1)$ (3.21):

$$c(t_{k+m}; n|t_k; 1) = \frac{C(t_{k+m};n)}{C^n(t_k;1)} \quad ; \quad u(t_{k+m}; n|t_k; 1) = \frac{U(t_{k+m};n)}{U^n(t_k;1)} \tag{3.21}$$

Using (3.21) inflation *n-th* statistical moments $In(t_{k+m},t_k;n)$ take form:

$$In(t_{k+m}, t_k; n) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!(n-j)!} \frac{c(t_{k+m};n-j|t_k;1)}{u(t_{k+m};n-j|t_k;1)} \tag{3.22}$$

Mean inflation $In(t_k,t_{k+m};1)$ during Δ_{k+m} with respect to time term Δ_k takes form:

$$In(t_{k+m}, t_k; 1) = \frac{p(t_{k+m};1)}{p(t_k;1)} - 1 = \frac{C(t_{k+m};1)}{C(t_k;1)} \frac{U(t_k;1)}{U(t_{k+m};1)} - 1 \tag{3.23}$$

Volatility of inflation $\sigma^2_{In}(t_{k+m},t_k)$ during Δ_{k+m} with respect to time term Δ_k

$$\sigma^2_{In}(t_{k+m}, t_k) = In(t_{k+m}, t_k; 2) - In^2(t_{k+m}, t_k; 1) = b(t_{k+m}, t_k; 2) - b^2(t_{k+m}, t_k; 1)$$

$$\sigma^2_{In}(t_{k+m}, t_k) = \frac{C(t_{k+m}; 2)}{C^2(t_k; 1)} \frac{U^2(t_k; 1)}{U(t_{k+m}; 2)} - \frac{C^2(t_{k+m}; 1)}{C^2(t_k; 1)} \frac{U^2(t_k; 1)}{U^2(t_{k+m}; 1)}$$

$$\sigma^2_{In}(t_{k+m}, t_k) = \frac{\sigma^2_p(t_{k+m})}{p^2(t_k;1)} \tag{3.24}$$

It is reasonable that volatility of inflation $\sigma^2_{In}(t_{k+m},t_k)$ during Δ_{k+m} with respect to Δ_k equals the ratio of price volatility $\sigma^2_p(t_{k+m})$ during Δ_{k+m} to square of mean price $p^2(t_k,1)$ during Δ_k. The trade value index $c(t_{k+m};n|t_k,1)$ (3.21) and trade volume index $u(t_{k+m};n|t_k,1)$ (3.21) describe growth of the market trade value and volume during the interval $\Delta_{k+\mathrm{m}}$ with respect to Δ_k. Market trade is important indicator of the economic growth and development. Relations (3.20-3.24) link description of asset returns and inflations with economic growth during selected time interval Δ_k and Δ_{k+m} (2.4). We leave further investigation of above relations between economic growth and market trade indexes for future.

4. Asset pricing and value-at-risk

Asset-pricing

All asset-pricing models deal with prices averaged by some probability (Cochrane, 2001; Campbell, 2002). Predictions of the price probability at certain time horizon T play the core role for the assessments of price forecasts at horizon T. Introduction of the *market-based* price probability determined by statistical properties of the market trade value and volume (2.19 - 2.22) during the averaging interval Δ_k (2.4) makes predictions of the price PDF one of the core problems of macroeconomics and finance. Indeed, any prediction of the price PDF at time horizon T for the averaging interval Δ_k (2.4) requires forecasting of the market trade value and volume probabilities at the same horizon T and during the same interval Δ_k (2.4). In simple words, to predict price PDF one should forecast the market trade values and volumes probabilities during the interval Δ_k (2.4) at horizon T. That causes forecasting of supply and demand, production function and investment, economic development and growth and etc., and other factors those impact market trade on one hand and depend on results of market trade on the other hand at horizon T. Introduction of the *market-based* price probability ties up the problems of prediction of financial markets price with problems of forecasting of market trade, economic development and growth. It is obvious, that exact prediction of economic future is impossible. However, development of the approximations may give assessments of future market trade and price probability. One should take into account basic relations (2.19-2.22) those determine price statistical moments through statistical moments of the market trade value and volume. Approximations (2.24-2.27) that take into account first 2,3,4 statistical moments should check how approximate price probability forecasts match predictions of the market trade value and volatility statistical moments during selected time interval Δ_k (2.4). We underline that duration of the interval Δ_k is very important for assessments of statistical moments and for sustainability of predictions at the given time horizon T.

Value-at-risk

Predictions of the asset price probability approximations determine accuracy and reliability of Value-at-Risk (VaR) - the most widespread tool for hedging risk of market price change. Economic ground of VaR was developed more than 30 years ago (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Choudhry, 2013). "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon" (Longerstaey and Spencer, 1996). Nevertheless the progress in VaR performance since then, the core features of VaR remain the same. To assess the VaR at horizon T one should calculate integral of the left tail of the returns PDF predicted at horizon T to estimate "potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon". Such assessment limits the possible capital loss due to market price variations for selected time horizon T for given probability. Usage of any VaR version requires predictions of returns PDF at horizon T. As we show above, returns probability and returns statistical moments are completely determined by asset price probability and price statistical moments (3.1-3.14) and by market trade statistical moments (2.9; 3.9). Prediction of returns probability almost equal prediction of the asset price probability. In Sec.2 we show that the *market-based* asset price probability and price statistical moments are determined by random properties of the market trade value and volume. In simple words: VaR assessment as prediction of price probability at horizon T depends on forecasts of the market trade value and volume probabilities at the same horizon T. The problem of VaR assessment is determined by accuracy of the market trade PDF prediction. The more statistical moments of market trade are predicted, the higher accuracy of VaR can be obtained. However, imaginable exact forecast of the market trade PDF at horizon T would provide for that lucky man a unique opportunity to manage and beat the market alone. That is much more profitable then any VaR calculations. One who will succeed in exact prediction of the market PDF will forget about VaR assessments and will enjoy beating the market alone! However, there still remains a small, negligible and trivial problem – how one can *exactly* predict the market trade PDF? It is a good issue for research.

These obstacles arises the problem of accuracy of any price probability predictions in compare with imaginable price probabilities determined by market trade probability forecasts. That problem may help establish economic ground and introduce possible limits on reliability of usage of VaR based on the *market-based* price probability. We believe that is a great problem for future.

5. Conclusion

The asset price probability plays the core role in macroeconomics and finance. Introduction of the *market-based* asset price PDF through the statistical moments of the market trade value and volume ties up the asset pricing theories and financial market studies with description of market trade evolution, economic development and growth. We don't solve the price probability puzzle but substitute the problem of the "correct" description of the price PDF by the problem of approximate description of the market trade and market price PDF. Investigation of the market trade value and volume PDF and prediction of their statistical moments are the problems for future.

The *market-based* approach to price probability establishes the unified description of the price statistical moments, price indices and returns statistical moments and ties up the predictions of price, price indices, returns probabilities with the ambition of forecasting of the market trade value and volume probabilities and statistical moments. In particular, the market-based price probability implies that the classical Black-Scholes option pricing models should be considered not at one, but at two-dimensional spaces (Olkhov, 2021c). We study bounds of reliability for usage of Value-at-Risk determined by the accuracy of forecasting of the market trade probability (Olkhov, 2021b). We use the *market-based* price PDF to describe dependence of the asset price autocorrelations on the market trade value and volume correlations (Olkhov, 2022a; 2022b). As we show, price volatility (2.29) depends on the second statistical moments of the market trade value and volume and *n-th* price statistical moments are determined by *n-th* statistical moments of market trade (2.9; 2.11; 2.14; 2.19). Description of the asset price volatility as well as description of price Skewness and Kurtosis requires description of the *2-d*, *3-d* and *4-th* market trade statistical moments. As we show (Olkhov, 2021a; 2021e) *n-th* market trade statistical moments depend on and impact at similar statistical moments of macroeconomic variables. Thus, description of the market-based price volatility implies development of the second-order economic theory. Description of price Skewness requires development of the 3-d order economic theory and so on. Studies of theoretical economics (Olkhov, 2022c) have an endless and interesting list of problems for many years ahead.

However, agents and academics are free in their expectations and preferences in choosing conventional *frequency-based* approach to asset price probability. Numerous studies in game theory, expectations, market studies and etc., propose various methods and factors that may impact the price change, define evolution of the asset price, returns, volatility and etc. All such studies investigate complex problems those govern price evolution, market dynamics and economic development. However, all possible factors, expectation, relations and laws, those impact and define price choice and price change, are imprinted in and are recorded by time-series of the performed market trades and are available as market time-series for our analysis (2.1-2.29).

We complement the current studies in asset pricing with a new look and demonstrate hidden complexity, unity and beauty of the *market-based* asset price probability theory. Further development of that theory will take a lot of efforts and years.

Appendix A

For simplicity and brevity we omit here time t_i of random variables and denote trade value $C(t_i)$ as C, trade volume as U and price as p. We denote here statistical moments as $p(1)$, $U(1)$, $U(2)$. Taking into account (2.19) we denote here statistical moments of trade value $C(n)$, volume $U(n)$ and price $p(n)$ as:

$$C(n) = E[C^n] = E[p^n U^n] = E[p^n]E[U^n] = p(n)U(n)$$

Let us take into account (2.15; 2.19) and consider

$$E[pU^2] = E[CU] = E[C]E[U] + corr\{CU\} \quad \text{(A.1)}$$

On the other hand

$$E[pU^2] = E[p]E[U^2] + corr\{pU^2\}$$

Hence obtain:

$$corr\{pU^2\} = p(1)[U^2(1) - U(2)] + corr\{CU\}$$

$$corr\{pU^2\} = corr\{CU\} - p(1)\sigma^2(U) \quad \text{(A.2)}$$

Similar consideration give relations between square of price p^2 and trade volume U:

$$E[p^2U] = E[C^2U^{-1}] \quad \text{(A4)}$$

$$E[p^2U] = p(2)U(1) + corr[p^2U] = [C^2U^{-1}] = p(2)U(2)U(-1) + corr\{C^2U^{-1}\}$$

$$U(-1) = E[U^{-1}]$$

$$corr[p^2U] = corr\{C^2U^{-1}\} + p(2)[U(2)U(-1) - U(1)] \quad \text{(A.5)}$$

The value and sign of correlations depend on the duration of the averaging time interval Δ and that important dependence should be investigated. Let us underline that correlations (A.5) between trade volume U and square of price p^2 determine relations between trade volume and volatility studies (Ito and Lin, 1993; Brock and LeBaron, 1995; Ciner and Sackley, 2007; Bogousslavsky and Collin-Dufresne, 2019), but correlations (A.5) are determined by the market-based price probability (2.12; 2.14; 2.19).

Appendix B

Taking *ln* of equation (1.1) one obtains sums of random variables:

$$\pi(t) = c(t) - u(t) \tag{B.1}$$

$$\pi = \ln p \quad ; \quad c = \ln C \quad u = \ln U \tag{B.2}$$

That well-know problem is described in many introductory notes on probability (Papoulis and Pillai, 2002; p.181). Let function $g(c,u)$ determines the joint probability density function (PDF) of variables c and u (B.2). Probability distribution $Q(\pi)=P\{c\text{-}u< \pi\}$ is determined by conditions

$$Q(\pi) = P\{c < \pi + u\} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\pi+u} dc\, g(c,u)$$

Then PDF $q(\pi)$ is determined as:

$$q(\pi) = \frac{d}{d\pi}Q(\pi) = \int_{-\infty}^{\infty} dx\, g(\pi + x, x) = \int_{-\infty}^{\infty} dx\, g(x, x - \pi) \tag{B.3}$$

To assess volume weighed average price (VWAP) (2.10; 2.11) from (1.1) and (B.1; B.2) obtain

$$E[C] = E[exp(c)] = E[exp(\pi + u)] = E[exp(\pi)]E[exp(u)] = E[p]\,E[U] \tag{B.4}$$

Respectively, relations (2.15-2.20) result:

$$E[C^n] = E[exp(nc)] = E[exp\{n(\pi + u)\}] = E[exp(n\pi)]E[exp(nu)] = E[p^n]\,E[U^n]$$

However, PDF $q(\pi)$ (B.3) is not too useful for calculation of price *n-th* statistical moments $E[p^n(t)]$. Moreover, time-series of the market trade value $C(t_i)$ and volume $U(t_i)$ give the opportunity to assess *n-th* statistical moments of the trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.9) but don't derive sufficient information to approve the exact form of the joint PDF $g(c,u)$ required to calculate (B.3). On the contrary, usage of assessments of first 1,2,3.. statistical moments allow estimate PDF approximations those match first statistical moments (2.9).

However, one should remind that above consideration of market-based asset price probability through assessment of the market trade statistical moments (2.9; 2.12; 2.14; 2.19) derive successive approximations only. Relations (B.1-B.3) present a different way to derive approximations. No "exact" solution of the asset price PDF exists, simply because no exact joint PDF of the market trade value and volume are known. Different averaging intervals Δ, different markets, assets, phases of business cycles and etc., result variations of market trade PDF and corresponding variations of the asset price PDF.

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