Impacts of boycotts concerning the Shapley value and extensions

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Abstract

If a player boycotts another player, it means that the cooperation gains of all coalitions containing both players vanish. In the associated coalition function, both players are now disjointly productive with respect to each other. The disjointly productive players property states that a player's payoff does not change when another player who is disjointly productive to that player is removed from the game. We show that the Shapley value is the only TU-value that satisfies efficiency and the disjointly productive players property and for which the impact of a boycott is the same for the boycotting and the boycotted player. Analogous considerations are made for the proportional Shapley value and the class of (positively) weighted Shapley values.

Keywords Cooperative game · (Weighted/proportional) Shapley value · Disjointly productive players · (Weighted/proportional) impacts of boycotts

1 Introduction

Boycotting a player should be well thought out because it not only punishes the boycotted player but also has an enormous impact on the boycotting player. If one player boycotts another, it means that there is no cooperation in any coalition that contains both players. That is, for these two players, the marginal contributions do not depend on the inclusion or exclusion, respectively, of the other player in any considered coalition. In Besner (2022) such players are called *disjointly productive* which may be considered being a special case within the "interaction of cooperation" in Grabisch and Roubens (1999), lacking any interaction. Casajus (2021) describes this "interaction of cooperation" as "second-order marginal contributions" or "second-order productivity" which equals zero for the boycotting and the boycotted player.

The focus of this paper is on the impact of a boycott on both the boycotting and the boycotted player. We call a game where two players i and j of the player set are disjointly productive, and all coalitions without both players have the same worth as in the original

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game an (i, j)-boycott game. However, it is not possible to tell from the coalition function who is the boycotting player and who is the boycotted player.

Since the Shapley value (Shapley, 1953b) only takes into account the coalition function for the payoff calculation, it is not particularly surprising that we have *balanced impacts of boycotts* for the two players who are disjointly productive in the boycott games.

Besides efficiency, an essential axiom that enables our axiomatizations is the *disjointly productive players* property (Besner, 2022), which states that a player's payoff does not change when another player who is disjointly productive to that player is removed from the game. It turns out that the Shapley value is the unique TU-value that satisfies efficiency, the disjointly productive players, and the balanced impacts of boycotts property.

In the next two sections, we transfer this result to the proportional Shapley value (Besner, 2016; Béal et al., 2018) and the class of the weighted Shapley values (Shapley, 1953a).

In our proofs, the *balanced contributions* property (Myerson, 1980) and its proportional and weighted variants play an important role.

This paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we recall some results related to the concept of disjointly productive players, introduce boycott games and the balanced impacts of boycotts property and give a new axiomatization of the Shapley value. Section 4 applies our results to the proportional Shapley value. In Section 5, we first replace the weights of the stand-alone worths with exogenously given weights in the results from the previous section. Thereafter, using the equal proportions of impacts of boycotts property, we give an axiomatization of the class of the weighted Shapley values. Section 6 provides a short conclusion and the Appendix (Section 7) shows the logical independence of the axioms in the axiomatizations.

2 Preliminaries

Let $\mathbb{R}$ be the real numbers, $\mathbb{R}_{++}$ the set of all positive real numbers, $\mathfrak{U}$ be the universe of all players and $\mathcal{N}$ be the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (TU-game) is a pair (N, v) such that $N \in \mathcal{N}$ is a player set and $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is a **coalition function**. The subsets $S \subseteq N$ are called **coalitions** and $v(S)$ is the **worth** of the coalition S. (S, v) is the **restriction** of (N, v) to the player set $S \subseteq N$, $S \neq \emptyset$.

Let $N \in \mathcal{N}$. The set of all TU-games (N, v) is denoted by $\mathbb{V}(N)$. If the stand-alone worths of the players must all be positive real numbers or must all be negative real numbers, this set is denoted by $\mathbb{V}_0(N) := \{(N, v) \in \mathbb{V}(N) : v(i) > 0 \text{ for all } i \in N \text{ or } v(i) < 0 \text{ for all } i \in N\}$. A TU-game $(N, u_S) \in \mathbb{V}(N)$, $S \subseteq N$, $S \neq \emptyset$, defined for all $T \subseteq N$ by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise, is called a **unanimity game**.

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$. For all $S \subseteq N$, the **Harsanyi dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) = \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } |S| \geq 1, \text{ and} \\ 0 & \text{if } S = \emptyset. \end{cases} \tag{1}$$

The **marginal contribution** $MC_i^v(S)$ of a player $i \in N$ to a coalition $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **null player** in (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$.

We define $\mathrm{W} := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$ with $w_i := w(i)$ for all $w \in \mathrm{W}$ and $i \in \mathfrak{U}$ as the set of all positive weight systems on the universe of all players.

For all $N \in \mathcal{N}$, a **TU-value** φ on $\mathbb{V}(N)$ (or, respectively, on $\mathbb{V}_0(N)$) is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ (or, respectively, to any $(N, v) \in \mathbb{V}_0(N)$) a payoff vector $\varphi(N, v) \in \mathbb{R}^N$.

The (positively) **weighted Shapley values** Sh^w (Shapley, 1953a) are defined by

$$Sh_i^w(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S) \text{ for all } i \in N \text{ and } w \in \mathrm{W}. \tag{2}$$

As a special case of this class of TU-values, all weights are equal, the **Shapley value** Sh (Shapley, 1953b) is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \text{ for all } i \in N. \tag{3}$$

Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}_0(N)$. The **proportional Shapley Value** Sh^p (Besner, 2016; Béal et al., 2018) is given by

$$Sh_i^p(v) = \sum_{S \subseteq N,\, S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) \text{ for all } i \in N. \tag{4}$$

We make use of the following axioms for TU-values which hold for all $N \in \mathcal{N}$.

Efficiency, E. For all $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Monotonicity, Mon. For all $(N, v), (N, v') \in \mathbb{V}(N)$ such that $v'(N) > v(N)$ and $v'(S) = v(S)$ for all $S \subsetneq N$, we have $\varphi_i(N, v') > \varphi_i(N, v)$ for all $i \in N$.

Balanced contributions, BC (Myerson, 1980)**.** For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, we have $\varphi_i(N, v) - \varphi_i(N \backslash \{j\}, v) = \varphi_j(N, v) - \varphi_j(N \backslash \{i\}, v)$.

w-Weighted balanced contributions, WBC^w (Myerson, 1980)**.** For all $(N, v) \in \mathbb{V}(N)$, all $i, j \in N$, $i \neq j$, and $w \in \mathrm{W}$, we have

$$\frac{\varphi_i(N, v) - \varphi_i(N \backslash \{j\}, v)}{w_i} = \frac{\varphi_j(N, v) - \varphi_j(N \backslash \{i\}, v)}{w_j}.$$

Proportional balanced contributions, PBC (Besner, 2016; Béal et al., 2018)**.** For all $(N, v) \in \mathbb{V}_0(N)$ and $i, j \in N$, $i \neq j$, we have

$$\frac{\varphi_i(N, v) - \varphi_i(N \backslash \{j\}, v)}{v(\{i\})} = \frac{\varphi_j(N, v) - \varphi_j(N \backslash \{i\}, v)}{v(\{j\})}.$$

We refer to the following axiomatizations.

Theorem 2.1 (Myerson, 1980)**.** *Sh is the unique TU-value that satisfies **E** and **BC**.*

Theorem 2.2 (Myerson, 1980; Hart and Mas-Colell, 1989)**.** *Let $w \in \mathrm{W}$. Sh^w is the unique TU-value that satisfies **E** and **WBC^w**.*

Theorem 2.3 (Besner, 2016; Béal et al., 2018)**.** *Let $N \in \mathcal{N}, (N, v) \in \mathbb{V}_0(N)$. Sh^p is the unique TU-Value that satisfies **E** and **PBC**.*

3 Balanced impacts of boycotts and the Shapley value

The concept of disjointly productive players is the common thread in this study. Two agents are disjointly productive if the marginal contribution of one agent is unchanged by the presence or absence of the other one in the coalition to which it contributes.

Definition 3.1 (Besner, 2022)**.** *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, two players $i,j \in N$, $i \neq j$, are called **disjointly productive** in (N,v) if, for all $S \subseteq N\backslash\{i,j\}$, we have $MC_i^v(S \cup \{j\}) = MC_i^v(S)$ which is the same as*

$$v(S \cup \{i,j\}) - v(S \cup \{j\}) = v(S \cup \{i\}) - v(S). \tag{5}$$

All coalitions containing two disjointly productive players have no cooperation benefit, which is the content of the following lemma in Besner (2022).

Lemma 3.2. *Let $N \in \mathcal{N}, (N,v) \in \mathbb{V}(N)$. Two players $i,j \in N$, $i \neq j$, are disjointly productive in (N,v) if and only if for all $S \subseteq N$, we have*

$$\Delta_v(S) = 0 \text{ if } \{i,j\} \subseteq S. \tag{6}$$

In our axiomatizations, the following axiom plays an important role: if there are two disjointly productive players, one player's payoff is not affected by the other player's exit from the game.

Disjointly productive players, DP (Besner, 2022)**.** For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i,j \in N$ such that i and j are disjointly productive players in (N,v), we have

$$\varphi_i(N,v) = \varphi_i(N\backslash\{j\},v).$$

We introduce games in which the behavior of two players toward each other changes from an original game such that one player boycotts cooperation with another in all coalitions that contain both players.

Definition 3.3. *Let $N \in \mathcal{N}, (N,v) \in \mathbb{V}(N)$, and $i,j \in N$, $i \neq j$. A TU-game (N,v^{ij}) such that i is disjointly productive in relation to j is called the (i,j)**-boycott game** corresponding to (N,v) if*

$$v^{ij}(S) := v(S) \text{ for all } S \subseteq N,\ \{i,j\} \nsubseteq S. \tag{7}$$

Thus, for the coalition function, it does not matter whether player i boycotts player j or vice versa. Accordingly, the impact on the payoff should be the same for both players.

Balanced impacts of boycotts, BIB. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and the (i,j)-boycott games (N,v^{ij}) corresponding to (N,v), we have

$$\varphi_i(N,v) - \varphi_i(N,v^{ij}) = \varphi_j(N,v) - \varphi_j(N,v^{ij}).$$

We introduce our first main result.

Theorem 3.4. *Sh is the unique TU-value that satisfies **E**, **DP**, and **BIB**.*

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$.
I. Existence: By Besner (2022), Sh satisfies **E** and **DP**. By (3) and Lemma 3.2, we have, for $i,j \in N$,

$$Sh_i(N,v) - Sh_i(N,v^{ij}) = \sum_{S\subseteq N,\, \{i,j\}\subseteq S} \frac{\Delta_v(S)}{|S|} = Sh_j(N,v) - Sh_j(N,v^{ij}),$$

and **BIB** is satisfied.
II. Uniqueness: Let φ be a TU-value that satisfies all axioms of Theorem 3.4. If $|N| = 1$, φ is unique by **E**. Let now $|N| \geq 2$ and (N,v^{ij}) be the (i,j)-boycott game corresponding to (N,v). By **DP**, we have $\varphi_i(N,v^{ij}) = \varphi_i(N\backslash\{j\},v^{ij}) = \varphi_i(N\backslash\{j\},v)$ and, analogously, $\varphi_j(N,v^{ij}) = \varphi_j(N\backslash\{i\},v)$. Since $i,j \in N$, $i \neq j$, are arbitrary, φ satisfies **BC** and, therefore, by **E** and Theorem 2.1, uniqueness is shown. □

4 Proportional balanced impacts of boycotts and the proportional Shapley value

Sometimes a player's stand-alone worth significantly influences the cooperation gain of the larger coalitions, for example, when the stand-alone worth reflects the player's capital strength, military strength, or if costs of coalitions depend on the costs of the singletons. In this context, compelling characterizations of the proportional Shapley value can be found in Béal et al. (2018) and Besner (2019). Should we have such conditions for the coalition function, the next axiom, according to which for a boycotting and a boycotted player, the impacts of boycotts are proportional to the stand-alone worths, seems to be more accurate than the balanced impacts of boycotts property.

Proportional balanced impacts of boycotts, PBIB. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_0(N)$, and the (i,j)-boycott games (N,v^{ij}) corresponding to (N,v), we have

$$\frac{\varphi_i(N,v) - \varphi_i(N,v^{ij})}{v(\{i\})} = \frac{\varphi_j(N,v) - \varphi_j(N,v^{ij})}{v(\{j\})}.$$

The following is a characterization analogous to Theorem 3.4.

Theorem 4.1. *Let $N \in \mathcal{N}, (N,v) \in \mathbb{V}_0(N)$. Sh^p is the unique TU-Value that satisfies **E**, **DP**, and **PBIB**.*

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}_0(N)$.
I. Existence: It is well-known that Sh^p satisfies **E**. By (4) and Lemma 3.2, it is obvious that Sh^p satisfies **DP**. By (4) and Lemma 3.2, we have, for $i,j \in N$,

$$\frac{Sh_i(N,v) - Sh_i(N,v^{ij})}{v(\{i\})} = \sum_{S\subseteq N,\, \{i,j\}\subseteq S} \frac{\Delta_v(S)}{\sum_{k\in S} v(\{k\})} = \frac{Sh_j(N,v) - Sh_j(N,v^{ij})}{v(\{j\})},$$

and **PBIB** is satisfied.
II. The uniqueness part follows completely analogous to the uniqueness part of the proof of Theorem 3.4, replacing **BC** by **PBC** and Theorem 2.1 by Theorem 2.3. □

5 Equal proportions of impacts of boycotts and the weigthed Shapley values

There are many conceivable situations in which, apart from the coalition function, the players should not be treated in a symmetrical manner. These include, for example, that the players are representatives of groups of different sizes or that the players show different levels of effort in cooperating. For further illustrations in this context, we refer the reader to Shapley (1953a) and Kalai and Samet (1987). Similar to the previous section, the impact of boycotts here should not be balanced but should be in proportion to weights of the affected actors, according to factors just mentioned.

w-balanced impacts of boycotts, BIB^w. For all $(N,v) \in \mathbb{V}(N)$, all $i,j \in N$, $i \neq j$, the (i,j)-boycott games (N, v^{ij}) corresponding to (N,v), and $w \in \mathrm{W}$, we have,

$$\frac{\varphi_i(N,v) - \varphi_i(N,v^{ij})}{w_i} = \frac{\varphi_j(N,v) - \varphi_j(N,v^{ij})}{w_j}.$$

As an intermediate step, we present an axiomatization that uses exogenously given weights.

Proposition 5.1. *Let $w \in \mathrm{W}$. Sh^w is the unique TU-value that satisfies **E**, **DP**, and **BIB**^{w}.*

*P*roof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $w \in \mathrm{W}$.
I. Existence: It is well-known that Sh^w satisfies **E**. By (2) and Lemma 3.2, it is obvious that Sh^w satisfies **DP**. By (2) and Lemma 3.2, we have, for $i,j \in N$,

$$\frac{Sh_i^w(N,v) - Sh_i^w(N,v^{ij})}{w_i} = \sum_{S \subseteq N,\, \{i,j\} \subseteq S} \frac{\Delta_v(S)}{\sum_{k \in S} w_k} = \frac{Sh_j^w(N,v) - Sh_j^w(N,v^{ij})}{w_j},$$

and **BIB**^{w} is satisfied.
II. The uniqueness part follows completely analogous to the uniqueness part of the proof of Theorem 3.4, replacing **BC** by **BC**^{w} and Theorem 2.1 by Theorem 2.2. □

We now want to get rid of the dependence on exogenously given weights and regard boycott games corresponding to two different TU-games.

Equal proportions of impacts of boycotts, EPIB.[1] For all $N \in \mathcal{N}$, $(N,v),(N,v') \in \mathbb{V}(N)$, and the (i,j)-boycott games $(N,v^{ij}),(N,v'^{ij})$ corresponding to $(N,v),(N,v')$, respectively, we have

$$\begin{aligned}\left[\varphi_i(N,v) - \varphi_i(N,v^{ij})\right]&\left[\varphi_j(N,v') - \varphi_j(N,v'^{ij})\right] \\ &= \left[\varphi_j(N,v) - \varphi_j(N,v^{ij})\right]\left[\varphi_i(N,v') - \varphi_i(N,v'^{ij})\right].\end{aligned}$$

By this axiom, if for two players and two games their impacts of boycotts are not zero, then the impacts of boycotts for both players are in the same proportion. Again, of course, it does not matter who is doing the boycotting and who is supposed to suffer from the boycott. It follows an axiomatization of the class of weighted Shapley values.

Theorem 5.2. *A TU-value φ satisfies **E**, **DP**, **EPIB**, and, in two-player games, **Mon** if and only if there exists a $w \in \mathrm{W}$ such that $\varphi = Sh^w$.*

[1]This axiom has some closeness to the mutual dependence property in Nowak and Radzik (1995).

Proof. Let $N \in \mathcal{N}$, $(N,v),(N,v') \in \mathbb{V}(N)$.
I. Let $w \in \mathrm{W}$. By Proposition 5.1, Sh^w satisfies **E** and **DP**, and, by Hart and Mas-Colell (1989), **Mon**. By (2) and Lemma 3.2, we have, for $i,j \in N$,

$$\begin{aligned}
&\left[Sh_i^w(N,v) - Sh_i^w(N,v^{ij})\right]\left[Sh_j^w(N,v') - Sh_j^w(N,v'^{ij})\right] \\
&\qquad = \left[\sum_{S\subseteq N,\,\{i,j\}\subseteq S} \frac{w_i}{\sum_{k\in S} w_k}\Delta_v(S)\right]\left[\sum_{S\subseteq N,\,\{i,j\}\subseteq S} \frac{w_j}{\sum_{k\in S} w_k}\Delta_{v'}(S)\right] \\
&\qquad = \left[\sum_{S\subseteq N,\,\{i,j\}\subseteq S} \frac{w_j}{\sum_{k\in S} w_k}\Delta_v(S)\right]\left[\sum_{S\subseteq N,\,\{i,j\}\subseteq S} \frac{w_i}{\sum_{k\in S} w_k}\Delta_{v'}(S)\right] \\
&\qquad = \left[Sh_j^w(N,v) - Sh_j^w(N,v^{ij})\right]\left[Sh_i^w(N,v') - Sh_i^w(N,v'^{ij})\right],
\end{aligned}$$

and **EPIB** is satisfied.
II. Let φ be a TU-value that satisfies all axioms of Theorem 5.2. We show that $\varphi = Sh^w$ for some $w \in \mathrm{W}$.

If $|N| = 1$, we have, by **E**, $\varphi = Sh^w$ for all $w \in \mathrm{W}$.

Let now $|N| \geq 2$ and $\{i,j\} \subseteq N$. By Lemma 3.2, we have $(N, u_{\{i,j\}}^{i,j})$ is a null game which means $u_{\{i,j\}}^{i,j}(S) = 0$ for all $S \subseteq N$ and all players $k \in N$ are mutually disjointly productive in $(N, u_{\{i,j\}}^{i,j})$. By using **DP** successively and **E**, we have $\varphi_i(N, u_{\{i,j\}}^{i,j}) = \varphi_j(N, u_{\{i,j\}}^{i,j}) = 0$.

All players $k \in N \backslash \{i,j\}$ are disjointly productive in relation to player i and j in $(N, u_{\{i,j\}})$. Using **DP** successively, we have $\varphi_i(N, u_{\{i,j\}}) = \varphi_i(\{i,j\}, u_{\{i,j\}})$ By **Mon** in two-player games, it follows $\varphi_i(N, u_{\{i,j\}}) - \varphi_i(N, u_{\{i,j\}}^{i,j}) = c_{\{i,j\},i}$, $c_{\{i,j\},i} \in \mathbb{R}_{++}$ and $\varphi_j(N, u_N^{i,j}) = c_{\{i,j\},j}$, $c_{\{i,j\},j} \in \mathbb{R}_{++}$. Thus exists a $w \in \mathrm{W}$ such that $w_i = c_{\{i,j\},i}$ and $w_j = c_{\{i,j\},j}$. Therefore, by **EPIB**, also $\mathbf{BIB}^w$ is satisfied and the claim follows by Proposition 5.1. □

6 Conclusion

Many participants in a boycott situation may not initially be aware of the extent to which they are affecting themselves by boycotting a former partner. If we consider the boycotted state between two actors, i.e., the state in which there is no cooperation in all coalitions in which both actors are represented, as the initial state, the impact of the boycott is simply the sum of the cooperation gains that can be achieved through cooperation in all coalitions in which both actors are members. This means that the impact of a boycott hits the actor most, who also benefited most from the cooperation. If the cooperation gain was 'fairly' distributed in the sense of the Shapley value, i.e. equally, the impacts of the boycott are also equal.

When external factors such as economic, political, or military power enter into the distribution, or the distribution depends on one's own performance or the number of participants represented by a proxy, the boycott affects the stronger actor all the more.

This study did not consider the impact of a boycott on those not directly targeted. These players also each lose their share of the cooperation gains of the coalitions that contain the boycotting and the boycotted player. Their losses are just as high as the cooperation gains of these coalitions were before. A boycotting actor should also take these effects into account if it does not want to incur the displeasure of the indirect actors involved.

Also not taken into account were the impacts of multiple players boycotting one or more other players at the same time. This broad area is left for further research in cooperative game theory.

7 Appendix

Remark 7.1. *The axioms in Theorem 3.4 are logically independent:*

- ***E***: *The TU-value* $\varphi := 2Sh$ *satisfies* ***DP*** *and* ***BIB*** *but not* ***E***.
- ***DP***: *The* ***equal division value*** *ED, given by*

$$ED_i(N,v) := \frac{v(N)}{n} \text{ for all } i \in N, \tag{8}$$

satisfies ***E*** *and* ***BIB*** *but not* ***DP***.

- ***BIB***: *The TU-values* ϕ^c*, defined in Besner (2020) for all* $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}^N$*, and all* $c > 0$*, by*

$$\phi_i^c(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{|v(\{i\})| + c}{\sum_{j \in S}(|v(\{j\})| + c)} \Delta_v(S) \text{ for all } i \in N. \tag{9}$$

satisfy ***E*** *and* ***DP*** *but not* ***BIB***.

Remark 7.2. *Let* $N \in \mathcal{N}, (N,v) \in \mathbb{V}_0(N)$*. The axioms in Theorem 4.1 are logically independent:*

- ***E***: *The TU-value* $\varphi := 2Sh^p$ *satisfies* ***DP*** *and* ***PBIB*** *but not* ***E***.
- ***DP***: *The* ***proportional rule*** π *(Moriarity, 1975), given by*

$$\pi_i(N,v) := \frac{v(\{i\})}{\sum_{j \in N} v\{j\}} \text{ for all } i \in N, \tag{10}$$

satisfies ***E*** *and* ***PBIB*** *but not* ***DP***.

- ***PBIB***: *The TU-values* ϕ^c*, defined in (9), satisfy* ***E*** *and* ***DP*** *but not* ***PBIB***.

Remark 7.3. *The axioms in Theorem 5.2 are logically independent:*

- ***E***: *The TU-value* $\varphi := 2Sh$ *satisfies* ***DP***, ***EPIB***, *and* ***Mon*** *but not* ***E***.
- ***DP***: *The* ***equal division value*** *ED satisfies* ***E***, ***EPIB***, *and* ***Mon*** *but not* ***DP***.
- ***EPIB***: *The TU-values* ϕ^c*, defined in (9), satisfy* ***E***, ***DP***, *and* ***Mon*** *but not* ***EPIB***.
- ***Mon***: *We define* $\Lambda := \{f : \mathfrak{U} \to \mathbb{R}\}$ *with* $\lambda_k := \lambda(k)$ *for all* $\lambda \in \Lambda$ *and* $k \in \mathfrak{U}$ *as the set of all weight systems on the real numbers on the universe of all players. Let* $\lambda^i \in \Lambda$ *be given by* $\lambda_i^i = -1$ *and* $\lambda_k^i = 2$ *for all* $k \in \mathfrak{U}\backslash\{i\}$*. The TU-value* ϕ^i*,*[2] *defined for all* $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}^N$*, by*

$$\phi_k^i(N,v) := \sum_{\substack{S \subseteq N,\, S \ni k,\\ S \neq \{k\}}} \frac{\lambda_k^i}{\sum_{j \in S} \lambda_j^i} \Delta_v(S) \text{ for all } k \in N. \tag{11}$$

satisfy ***E***, ***DP***, ***EPIB***, *but not* ***Mon***.

[2]The TU-value ϕ^i is a special case of a multiweighted Shapley value (Dragan, 1992).

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