The stochastic dynamics of business evaluations using Markov models

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Abstract

Current assessments of credit and financial risk based on deterministic analyses provide only a limited understanding of current and future solvency rates. This paper offers an alternate model using two-state Markov chains that produces a more comprehensive and accurate system and allows for broader and more complex analyses of present and future situations.

Building off findings made in the development of the Altman Z-score, this proposed model applies stochastic processes and probability spaces to multivariate normal populations to account for the uncertainty of market conditions. Where one-step Markov chains demonstrate the relevance of this model for finite and infinite variables, the player's downfall theorem indicates that the *n*th value is only dependent on the value before it. Using the Chapman-Kolmogorov equation, multi-step transition probabilities then lead to the final two-state Markov chain.

Keywords

Business evaluations; Markov chain, stochastic processes

1. Introduction: credit and financial risk assessment

Representing complex, non-linear and open systems that are characterized by chaotic dynamics, mathematical models are applied to interpret and simulate financial markets and personal finances. This includes business evaluations produced by banks and other institutions, which are based on a compilation of individual analyses of budget indicators.

Although this information is coalesced and organized in a comprehensive system that incorporates sector data, historical series and qualitative factors, the projections resulting from such a process, especially for insolvency forecasting, are limited by univariate procedures.

It should be said that some studies (e.g. Beaver [3]) have demonstrated the discrete predictive abilities of these assessments for certain companies, especially in relation to the use of the cash flow ratio on total debts—when properly weighted, the Probability of Default of up to 80% of the companies assessed was correctly classified. At $t - 1$, where t is the default year, the accuracy rose to 90%.

Generally, however, a purely deterministic analysis provides only a vague snapshot of market conditions that assumes time to be *periodal* and therefore discounts other decay variables. Alongside a Bayesian basic facility, the disparate treatment of the various assessment criteria, including profitability, financial structure and available liquidity, renders this model ineffectual for more demanding assessments involving longer periods or more complex portfolios.

This procedure further warrants revision following recent guidelines introduced by the European Banking Authority regarding the renewal of certain international accounting standards. This includes IFRS 9, which contains the first mention of expected credit losses. Specific technical provisions must now be met to verify the conditions of probable default.

A mathematical model that simultaneously accounts for all related variables and provides more predictive instruments is therefore crucial. Where the former refers broadly to interchangeable components, it includes the cost of provisions, variables for risk segments and possible loss prevention and mitigation measures.

The latter focuses on the technical requirements of the system and enabling any company to generate results. These may be from forbearance procedures or stochastic evaluations of the impact of a new investment, financial or otherwise, on settlement, development and outlook rating.

A simplified version of the model we aim to construct was formed during the development of the Altman Z-score [1] in which a single matrix of variance and covariance represents the relevant populations. Applied a year in advance, this model predicted unfavourable events or default with 95% accuracy (error $\alpha = 6\%$; $\beta = 3\%$).

As our model involves multivariate normal populations, we take up the linear discriminating analysis proposed by Fisher [4], with classification score:

$$S_j = a_1 X_{1j} + a_2 X_{2j} + \ldots + a_i X_{ij} + \ldots + a_n X_{nj}$$

where:

S_j = the score of the *j*-th enterprise;

X_{ij} = the descriptive variable of the *i*-th feature of the *j*-th enterprise, with x_i representing the column vector of those variables; and

a_j = the coefficient of variable X_{ij}.

If two (or more) known populations A and B, respectively "non-performing" and "in Bonis," are present within time *t*, the allocation of company *j* to either population depends on the distance of S_j from the average scores of A and B:

$$S_j = (\overline{x}_A - \overline{x}_B)\, V^{-1} x_j$$

Applied to samples of populations A and B, column vectors $\overline{x}_A$ and $\overline{x}_B$ correlate with the averages of the selected variables, while *V* refers to an *n* by *n* matrix of variances and covariances, as derived from the union of the two samples relative to the average $\overline{x}$. This configuration will allow multinormal populations to produce quadratic discriminating functions.

2. Markov chains: the one-step process

Whether discrete or in steps, the stochastic process that a countable set may undergo depends on the number of values assumed by its causal variables. If set T in a family of random variables $X = \{X_t : t \in \mathrm{T}\}$ is discrete, the discrete-time process is given as $\{X_n : n \in \mathrm{N}\}$. If set T is continuous, however, its intersection with R, R^+ or any subset of R would result in a continuous-time process.

Here, we focus on discrete-time process $\{X_n\}$. Where each variable X_n can assume only a countable value within the space of *S*, the nature of *S* as either infinite set S^∞ = {0, 1, 2, ...} or finished set S^d = {0, 1, 2, ..., d} determines whether the variables are finite or infinite (cf. Jarrow et al. [6]).

Within either of these parameters, probability can be calculated for the instant when *n* is in the state *i*, or $X_n = i$:

$$P_{ij}^n = P(X_{n+1} = j | X_n = i)$$

Reaching the state *j* at the instant *n* + 1, the Markov chain for $\{X_n\}$, *n* = 1, 2, …, is then defined as:

$$P(X_{n+1} = j | X_{n-1} = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j | X_n = i) = P_{ij}^n$$

For all significant probabilities of each state $i_0, i_1, \dots, i_{n-1}, i, j$ where $n \geq 0$, P_{ij}^n represents the probability of a one-step transition for a Markov chain (cf. Israel et al. [5]).

If we take X_n to be the present state of the process, with past states $X_{n-1}, \dots, X_{1,} X_0$, then the future state X_{n+1} depends only on the present state for every instant *n*.

As such, to determine the probability of transition from state *i* to *j*, we take up a homogenous Markov chain, for which:

$$\mathrm{P}(X_{n+1} = j | X_n = i) = P_{ij}$$

And, therefore:

$$P_{ij} = \mathrm{P}(X_1 = j | X_0 = i)$$

Where this law is independent of the time and place of the transition, the conditions to satisfy it are:

$$P_{ij} \geq 0, i, j \geq 0, \sum_{j \epsilon S} P_{ij} = 1$$

Extrapolating this equation, the one-step transition matrix *P* presents the evolution of this particular Markov chain for every instant that $n \geq 1$:

$$P^{\infty} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1j} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots & P_{2j} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}$$

Should the space of the states be finite, the matrix would take the form $d \times d$:

$$P^{d} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1d} \\ P_{10} & P_{11} & P_{12} & \cdots & P_{2d} \\ P_{20} & P_{21} & P_{22} & \cdots & P_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{d1} & P_{d2} & P_{d3} & \cdots & P_{dd} \end{bmatrix}$$

In both cases, matrix *P* remains stochastic as each element never falls below 0, and the sum of each row vector is 1.

Here, we develop the player's downfall theorem to better illustrate the process. Assuming the role of an entrepreneur, the player makes a series of investments that carry an inherent level of risk. Where a positive return refers to full remuneration of the input capital, the probability of achieving this outcome for each investment is $p = a$. The probability of loss is $q = 1 - p =$ b (b > a); investments are assumed to stop when the player's equity reaches value *N* or 0.

After *n* investments, the entrepreneur amasses a wealth of X_n and the preceding family of values forms a Markov chain. Therefore, for all previous instants when $X_n = i$ (for $0 < i < N$):

$$P(X_{n+1} = i + 1 | X_n = i,\ X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \dots,\ X_1 = i_1,\ X_0 = i_0) = a$$

At instant $n + 1$, the player's wealth depends only on the wealth possessed in the previous instant. In other words, the transition probabilities at any point in time are $P_{i,i+1} = a$; $P_{i,i-1} =$ b for $0 < i < N$; $P_{0,0} = 1$; and $P_{N,N} = 1$:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{b} & 0 & \mathrm{a} & 0 & 0 & 0 \\ 0 & \mathrm{b} & 0 & \mathrm{a} & 0 & 0 \\ 0 & 0 & \mathrm{b} & 0 & \mathrm{a} & 0 \\ 0 & 0 & 0 & \mathrm{b} & 0 & \mathrm{a} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In the case that *N* = 5, for example, the space of states involves 5 + 1 elements. In a one-step transition matrix, the elements $P_{0,0} = 1$ e $P_{N,N} = 1$ indicate that at the instant $n^* \geq 0$, the chain leads to state 0 or state *N*. However, as $\mathrm{P}(X_{n+1} = 0|X_n = 0) = 1$ and $\mathrm{P}(X_{n+1} = N|X_n = N) = 1$ for every $n \geq n^*$, these states of loss are temporary.

3. Multi-step transition probabilities

A variation of the Chapman-Kolmogorov equation allows the further calculation of the probability of transition in *n*-steps (cf. Negri [7]).

Where $\{X_n\}$ is a Markov chain with the space of state *S*, $P_{ij}^{(n)}$ is the probability that the chain has transitioned from state *i* to state *j* *n* instants before for every instance when $n \geq 1$:

$$P_{ij}^{(n)} = \mathrm{P}(X_{n+m} = j|X_m = i)$$

A homogenous chain, $P_{ij}^{(n)}$ does not depend on *m*:

$$P_{ij}^{(n)} = \mathrm{P}(X_n = j|X_0 = i)$$

As $P_{ij}^{(1)} = P_{ij}$:

$$P_{ij}^{(n)} = \sum_{h \epsilon S} P_{ih}^{n-1} P_{hj}$$

therefore:

$$P_{ij}^{(n)} = \frac{\mathrm{P}(X_n = j, X_0 = i)}{\mathrm{P}(X_0 = i)} = \sum_{h \epsilon S} \frac{\mathrm{P}(X_n = j, X_{n-1} = h, X_0 = i)}{\mathrm{P}(X_0 = i)}$$

$$= \sum_{h \epsilon S} \mathrm{P}(X_n = j|X_{n-1} = h, X_0 = i)\mathrm{P}(X_{n-1} = h|X_0 = i)$$

$$= \sum_{h \epsilon S} \mathrm{P}(X_n = j|X_{n-1} = h)P_{ih}^{n-1} = \sum_{h \epsilon S} \mathrm{P}(X_1 = j|X_0 = h)P_{ih}^{n-1}$$

from which we return to:

$$P_{ij}^{(n)} = \sum_{h \epsilon S} P_{ih}^{n-1} P_{hj}$$

A $P^{(n)}$ matrix can thus be constructed using the probabilities $P_{ij}^{(n)}$ to reach $P^{(n)} = P^{(n-1)}$. P is then deduced as the product of matrices for infinite values, and $P^{(2)} = P \cdot P = P^2$ and $P^{(n)} = P^n$. We also determine that $P^{(m+n)} = P^{(m)} \cdot P^{(n)}$ as:

$$P_{ij}^{(m+n)} = \sum_{h \epsilon S} P_{ih}^{(m)} P_{hj}^{(n)}$$

The probability π^n is of note as it assumes the chain is in a certain state *i* at the instant *n*:

$$\pi_k^n = \mathrm{P}(X_n = k), \nabla k \in S$$

4. Two-state Markov chain

We propose a two-state Markov chain as a more comprehensive model for effectual assessments of companies and their solvency prospects (cf. Desogus and Casu [3]).

As earlier mentioned, the state of a company can be classified as either Non-performing or in Bonis. The Non-performing state corresponds with 0 and in Bonis with 1. Regardless of its state prior to day *n*, if a company is in state 0 on day *n*, the probability that it will be in state 1 on day *n* + 1 is α. Conversely, if the same company is in state 1 on day *n*, the probability that it will be in state 0 on day *n* +1 is β. This is again independent of its status in the days before *n*. The evolution of the company's state during the days *n* = 1, 2, …, *n* is indicated by the chain $\{X_n\}$.

To avoid a static system of 0 or 1, let us assume that α and β are never 0 in the same instance. Likewise, let us also assume that neither α nor β are 1 at the same time to prevent a deterministic system. The probabilities of a company's state should therefore be 0 < α + β < 2.

The transition matrix of the resulting Markov chain is:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Resolving the equations, the *n*-step transition matrix is:

$$P^{(n)} = \frac{(1-\alpha-\beta)^n}{\alpha+\beta} \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix} + \frac{1}{\alpha+\beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

The distribution of X_n variables is then calculated to determine the probability of the company being in state 0 or 1 at the instant *n*:

$$P(X_n = 0) = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^n (\pi_0^0 - \frac{\beta}{\alpha+\beta})$$

and:

$$P(X_n = 1) = \frac{\beta}{\alpha + \beta} + (1 - \alpha - \beta)^n (\pi_1^0 - \frac{\alpha}{\alpha + \beta})$$

Representing state 0 when n = 0 in the first equation, $\pi_0^0 = \mathrm{P}(X_0 = 0)$. In the second equation, π_1^0 reflects the state 1 when n = 1, or $\pi_1^0 = 1 - \pi_0^0 = \mathrm{P}(X_0 = 1)$.

For the hypothesis $| 1 - \alpha - \beta | < 1$:

$$\lim_{n \to +\infty} \mathrm{P}(X_n = 0) = \frac{\beta}{\alpha + \beta}$$

and:

$$\lim_{n \to +\infty} \mathrm{P}(X_n = 0) = \frac{\alpha}{\alpha + \beta}$$

5. Conclusions

The proposed model is positioned within the new needs expressed by the regulatory system, specifically on credit risk assessments.

An alternate series of mathematical-financial calculations must be applied to properly assess the inherent uncertainty of dynamic and complex systems, such as markets and expected returns on investments, and to produce more accurate predictions that inform crucial decisions for portfolio strategies.

We believe that a two-state Markov chain model can conduce to this achievement.

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