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Abstract

This paper analyses a two-period model in which a consumer faces a future income risk but is uncertain about its probability distribution. We derive three sets of sufficient conditions under which a consumer with generalised recursive smooth ambiguity (GRSA) preferences will save more under ambiguity than in a deterministic environment. Our results show how precautionary saving is jointly determined by attitudes toward atemporal risk, ambiguity and intertemporal substitution. We also find a close connection between risk prudence under non-expected utility and precautionary saving under GRSA preferences.

Keywords: Precautionary Saving; Risk Aversion; Intertemporal Substitution; Smooth Ambiguity Preferences.

JEL classification: D15, D81, E21

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1 Introduction

Precautionary saving refers to the tendency to save more due to the presence of risk or ambiguity.[1] Since the early contributions of Leland (1968), Sandmo (1970) and Kimball (1990), there has been an extensive literature that studies consumers' precautionary saving behaviour under pure risks. Such behaviour has become known as *risk prudence*.[2] By comparison, research on precautionary saving under ambiguity is still in its burgeoning stage.[3] The present study contributes to this literature by offering new insights into four fundamental questions: First, under what conditions on preferences will a consumer facing an ambiguous risk (i.e., a random variable with an unknown probability distribution) choose to save more than in a deterministic environment? For reasons explained below, we refer to this type of saving behaviour as *mixed prudence.* Second, how does ambiguity in itself affect saving decisions? More specifically, will (or when will) the introduction of ambiguity induce a consumer to save more than when she faces a pure risk? Such behaviour is referred to as *ambiguity prudence.* Third, what is the relation (if any) among mixed prudence, ambiguity prudence and risk prudence? Finally, what are the separate importance of risk attitudes, ambiguity attitudes and preferences for intertemporal substitution in generating precautionary saving under ambiguity?

To address these questions, we focus on the consumption-saving problem faced by a single consumer in a two-period setup that is commonly used in the precautionary saving literature. The consumer receives a random income in the second (or future) period, which is the only source of risk and ambiguity. There is a single risk-free asset that can be used to smooth consumption over time. Due to imperfect knowledge, the consumer is uncertain about the true probability distribution of future income. Hence, she has to rely on her own subjective beliefs when making the saving decision. As in Klibanoff *et al.* (2005), these beliefs are captured by a set of plausible, first-order probability distributions of future income, and a second-order probability distribution which describes how likely that a given first-order distribution is the true one. At the centre of our analysis are the assumptions on consumer preferences. Following Hayashi and Miao (2011)

[1] Risk (or pure risk) refers to random variables with a known probability distribution. Ambiguity arises when there is insufficient kowledge to determine the probability distribution of the random variables in question. In this study, we will use the terms "ambiguity" and "uncertainty" interchangeably.

[2] Baiardi *et al.* (2020) provide a selective survey of the theoretical literature on risk prudence. In particular, they focus on studies that adopt a two-period model with a single expected-utility maximiser. A textbook treatment of risk prudence can be found in Gollier (2001a, Chapters 16 and 20). There is also a separate but related literature that studies precautionary saving behaviour in the Bewley-Aiyagari-Huggett model, which involves a longer planning horizon, incomplete markets and general equilibrium analysis. We do not consider this type of model in this study.

[3] See for instance Osaki and Schlesinger (2014), Berger (2014), Baillon (2016), Peter (2019) and André *et al.* (2021).

and Ju and Miao (2012), we assume that the consumer has generalised recursive smooth ambiguity (GRSA) preferences. One major advantage of GRSA preferences is that it allows for the separation of attitudes toward risk, ambiguity and intertemporal substitution. Similar to the Selden/Kreps-Porteus (SKP) utility function, atemporal risk preferences are represented by a von Neumann-Morgenstern (vNM) utility function $u(\cdot)$, while preferences regarding intertemporal substitution are represented by a time aggregator function. In addition to these, the GRSA preferences also feature a second vNM utility function $v(\cdot)$. This function is central to our results, hence it deserves a more detailed explanation. When deciding how much to save in the presence of risk and/or ambiguity, the consumer needs to form *ex ante* evaluation about future consumption or future utility in order to assess the benefits of saving. In the current setting, these evaluations are formed in the following manner: For any given level of saving and for any first-order distribution of future income, the consumer can compute the certainty equivalent of future consumption using $u(\cdot)$,[4] and the expected future value of $u(\cdot)$. Both measures are contingent on a particular first-order distribution, hence they are *ex ante* random in the presence of ambiguity. The function $v(\cdot)$ captures the consumer's attitudes toward the randomness in the certainty equivalents. In particular, a concave $v(\cdot)$ means that the consumer dislikes any mean-preserving spread in the distribution of these certainty equivalents.[5] On the other hand, the composite function $\phi(\cdot) \equiv v \circ u^{-1}(\cdot)$ captures the consumer's attitudes toward the uncertainty in the expected future value of $u(\cdot)$ [Klibanoff *et al.* (2005, Proposition 1)]. Following the convention in this literature, we refer to a consumer with a concave $\phi(\cdot)$ as *ambiguity-averse*.

A second major advantage of GRSA preferences is that it admits several forms of preferences as special cases, including expected-utility (EU) preferences, Selden/Kreps-Porteus (SKP) preferences and the recursive smooth ambiguity preferences of Klibanoff *et al.* (2009).[6] Depending on the specification of the time aggregator function, GRSA preferences can be either time-separable or non-time-separable. This generality makes it possible to unify and compare several groups of precautionary saving models (both with and without ambiguity) under one single framework. This in turn allows us to draw upon and extend the insights gained from the risk prudence literature.

We begin with a detailed analysis of mixed prudence using the time-separable version of GRSA

[4] This certainty equivalent is referred to as a *second-order act* in Klibanoff *et al.* (2005, p.1857).

[5] Throughout this paper, all mentions regarding curvature (concave, convex), quantity comparison (more, less, greater than, less than, positive, negative, etc.) and monotonicity (increasing, decreasing) should be understood as weak inequalities. Similarly, "risk-averse" and "ambiguity-averse" have the same meaning as "non-risk-loving" and "non-ambiguity-loving," respectively. Strict inequalities will be stated explicitly.

[6] As mentioned in Hayashi and Miao (2011, p.425-426), the GRSA preferences also encompass the risk-sensitivity model of Hansen and Sargent (2001) and the recursive multiple-prior model of Epstein and Schneider (2003) as special cases. We do not consider these cases in this paper.

preferences, which involves a per-period felicity function $V(\cdot)$ and a constant subjective discount factor.[7] The consumer's willingness to substitute intertemporally is now captured by the curvature of $V(\cdot)$. Using the time-separable formulation, we are able to identify three different sets of conditions under which optimal saving under ambiguity is higher than that in a deterministic environment. These results are formally stated in Theorems 1-3. The first two are closely related to the risk prudence results in Gollier (2001a, p.300-302) and Kimball and Weil (2009), hence a quick review of these results is warranted. Both of them concern a single risk-averse consumer with SKP preferences who faces a pure income risk in the future period. SKP preferences correspond to the special case of GRSA preferences in which either the consumer is ambiguity-neutral [i.e., $u(\cdot)$ and $v(\cdot)$ are identical up to a positive affine transformation] or ambiguity is absent. Risk prudence arises under two different sets of conditions. First, Gollier (2001a) shows that it suffice if (i) the felicity function $V(\cdot)$ is more concave than the vNM utility function $u(\cdot)$, and (ii) the marginal utility function $u'(\cdot)$ is a convex function.[8] Second, Kimball and Weil (2009, Proposition 1) show that risk-prudent behaviour emerges if $u(\cdot)$ exhibits decreasing absolute risk aversion (DARA), provided that the consumer's future felicity is concave in the choice variable (i.e., savings).[9] Our Theorems 1 and 2 generalise these results to an environment in which both ambiguity and ambiguity aversion matter for the saving problem.

Our first theorem states that a risk-averse and ambiguity-averse consumer is mixed-prudent if two additional conditions are satisfied: (i) $V(\cdot)$ is more concave than $v(\cdot)$ and (ii) **both** $u'(\cdot)$ and $v'(\cdot)$ are convex functions. Our second theorem states that a risk-averse (but not necessarily ambiguity-averse) consumer is mixed-prudent if **both** $u(\cdot)$ and $v(\cdot)$ exhibit DARA, provided that the consumer's future felicity is concave in savings.[10] These results are best explained by starting with two special cases. First, if the consumer is ambiguity-neutral (or if there is no ambiguity), then optimal saving is unaffected by the uncertainty regarding the true distribution of future income. Thus, any precautionary savings (if exist) must be driven by risk alone. In this case, mixed prudence is equivalent to risk prudence and our results coincide with those of Gollier (2001a) and Kimball and Weil (2009). Second, if the consumer is risk-neutral or if the first-order distributions

[7] In Section 4.2, we show that some of our results obtained under the time-separable version of GRSA preferences can be easily extended to a non-time-separable version.

[8] In the absence of ambiguity, condition (i) implies that the consumer has a preference for late resolution of uncertainty, whereas condition (ii) implies that higher savings will create a larger gain in (expected) future utility in a stochastic environment than in a deterministic one. If $V(\cdot)$ is a positive affine transformation of $u(\cdot)$, then Gollier's result is identical to the risk prudence result of Kimball (1990).

[9] Both sets of conditions are summarised in Gollier (2001a, p.302, Proposition 78). The second set of conditions, however, first appears in the 1992 working paper version of Kimball and Weil (2009). Hence, we attribute this to Kimball and Weil (2009).

[10] Our Theorem 2 specifies the conditions under which future felicity is concave in savings.

of future income are all degenerate, then the consumer's saving decision is only affected by the first moment of the income risk. But since the true distribution of this risk is unknown, its first moment remains *ex ante* random. Any precautionary savings that emerge in this case is entirely driven by ambiguity, i.e., ambiguity prudence. Most importantly, this type of ambiguity prudence (that emerges under risk neutrality) can be characterised in the same way that Gollier (2001a) and Kimball and Weil (2009) did for risk prudence, except with $u(\cdot)$ replaced by $v(\cdot)$. The intuition is that a risk-neutral consumer with GRSA preferences who is subject to an ambiguous risk will behave in exactly the same way as a risk-averse consumer with SKP preferences who faces a pure risk. The main difference is that the former's risk preferences are captured by a linear vNM utility function $u(\cdot)$, while the latter's are captured by a concave function $v(\cdot)$. Thus, according to our first two theorems, a consumer is mixed-prudent in general if she is risk-prudent under these two special cases.

More generally, the conditions in Theorem 1 have two effects on the consumer's saving decision: The first one works as if it raises the consumer's subjective discount factor, while the second one ensures that an increase in savings will create a larger gain in (expected) future utility under ambiguity than in a deterministic environment. Both contribute to a higher marginal benefit of saving under ambiguity, which encourages savings. The conditions in Theorem 2 have the same effect on the marginal benefit of saving but work through a different mechanism. Specifically, this result is obtained by comparing the certainty equivalent of future consumption under ambiguity to the future consumption in a deterministic environment. The concavity of $u(\cdot)$ and $v(\cdot)$ imply that the certainty equivalent is lower in *level* than its deterministic counterpart under any given level of savings. The DARA assumptions imply that the same increase in savings will lead to a larger increase in the certainty equivalent than in its deterministic counterpart. These together contribute to a higher marginal benefit of saving under ambiguity.

When comparing between Theorems 1 and 2, one major difference is that ambiguity aversion is not required in the second one. This shows that ambiguity aversion is not a necessary condition for mixed prudence. As noted above, precautionary saving can arise under ambiguity even if the consumer is risk-neutral. This means risk aversion, or any conditions on the higher-order derivative of $u(\cdot)$, are not necessary for mixed prudence. Furthermore, the conditions in Theorems 1 and 2 are weaker than the convexity of $\phi'(\cdot)$ [or a positive third-order derivative of $\phi(\cdot)$] and decreasing absolute ambiguity aversion (DAAA).[11] Thus, neither of these properties is necessary for mixed

[11] Absolute ambiguity aversion $\mathcal{A}_\phi(\cdot)$ is defined in a similar fashion as absolute risk aversion, i.e., $\mathcal{A}_\phi(x) \equiv -\phi''(x)/\phi'(x)$. The function $\phi(\cdot)$ is said to exhibit DAAA if $\mathcal{A}_\phi(\cdot)$ is a decreasing function.

prudence. Our results show that precautionary saving behaviour under ambiguity is a complex issue that cannot be easily determined by a single condition. Instead, it is determined by a number of substitutable factors related to risk preferences, ambiguity preferences and attitudes toward intertemporal substitution. Our results also highlight the often-overlooked importance of $v(\cdot)$ in generating precautionary saving under ambiguity.

Our third theorem delves deeper into the conditions for ambiguity prudence and its relation with mixed prudence. Although the two coincide under risk neutrality and degenerate first-order probability distributions, they are not equivalent in more general cases. If the consumer is *both* strictly risk-averse and strictly ambiguity-averse, then the conditions in Theorems 1 and 2 are in general insufficient to ensure ambiguity prudence. To explain the situation more precisely, let s_A^*, s_R^* and s_D^* denote, respectively, optimal savings under **A**mbiguity, pure **R**isk and **D**eterministic future income. Risk prudence, ambiguity prudence and mixed prudence are, respectively, defined as $s_R^* \geq s_D^*$, $s_A^* \geq s_R^*$ and $s_A^* \geq s_D^*$. The conditions in Theorems 1 and 2 ensure that $s_R^* \geq s_D^*$ and $s_A^* \geq s_D^*$, but do not rule out the possibility of $s_R^* > s_A^* \geq s_D^*$. The strict inequality means that the consumer cuts back her savings when the distribution of future income becomes uncertain. Thus, the precautionary savings characterised in our first two theorems, i.e., $(s_A^* - s_D^*)$, can be possibly due to a mixture of risk prudence and ambiguity "imprudence". For this reason, we call it "mixed" prudence.

Our Theorem 3 provides a set of sufficient conditions under which a strictly risk-averse and strictly ambiguity-averse consumer is *both* risk-prudent and ambiguity-prudent, i.e., $s_A^* \geq s_R^* \geq s_D^*$ (which immediately implies mixed prudence). The risk prudence result is similar to the one in Gollier (2001a). The ambiguity prudence result, on the other hand, requires two additional conditions: first, the first-order probability distributions of future income can be ranked according to first-order stochastic dominance (FOSD); and second, the function $\phi(\cdot)$ exhibits decreasing absolute ambiguity aversion. The first assumption allows us to compare the marginal benefit of saving under different plausible distributions. If one distribution is ranked lower than another according to FOSD, then it is more likely to draw low levels of future income from the former. Since savings have a greater impact on future consumption when future income is low, the marginal benefit of saving is higher under those distributions that are deemed less favourable (ranked lower) by FOSD. In the presence of ambiguity, the consumer will weigh the marginal benefit of saving under different plausible distributions according to her ambiguity preferences. In particular, an ambiguity-averse consumer will put a higher importance on the less favourable distributions.[12] This, together with DAAA, guarantees that the marginal benefit of saving under ambiguity are higher than that under a pure risk.[13]

We conclude this section by mentioning some related studies. Our paper is most closely related to Osaki and Schlesinger (2014) and Berger (2014). Both studies analyse precautionary saving behaviour under ambiguity in a two-period model that is similar to ours, but with recursive smooth ambiguity preferences [Klibanoff *et al.* (2009)]. Osaki and Schlesinger (2014) focus on the conditions for mixed prudence. Their main result (Proposition 2) relies on the assumption that the first-order distributions can be ranked according to some stochastic dominance criterion, hence it is similar to our Theorem 3. They, however, do not distinguish between risk prudence and ambiguity prudence. Berger (2014), on the other hand, focus on ambiguity prudence alone. Our Theorem 4 generalises Berger's Proposition 1 to GRSA preferences and shows that DAAA can be replaced by a weaker condition. Baillon (2016) proposes a model-free definition of ambiguity prudence which is different from Berger's and ours. Baillon's approach is similar in spirit to the risk apportionment approach of Eeckhoudt and Schlesinger (2006) in characterising higher order derivatives of utility function. Whereas, our definition of ambiguity prudence is based on the behavioural prediction of a specific decision problem. Two recent studies, Wang and Li (2020) and André *et al.* (2021), have adopted the GRSA preferences to study consumer behaviour. The former examines how changes in ambiguity aversion will affect different types of intertemporal decisions, one of which is precautionary saving. Some of the assumptions in their Proposition 1 [in particular, A1, A2, A3, (i) and (iv)] are also used in our Theorem 3. André *et al.* (2021) analyse the demand for annuity and saving in a two-period model with longevity uncertainty. They focus on a special case of GRSA preferences in which both $u(\cdot)$ and $v(\cdot)$ are exponential utility functions but with different parameters.

The rest of the paper is organised as follows: Section 2 describes the model environment and the generalised recursive smooth ambiguity preferences. Section 3 presents the main results on mixed prudence (Theorems 1-3). Section 4 presents some further results. Section 5 concludes.

[12] This is consistent with the conventional wisdom that ambiguity-averse consumers act as if they are more pessimistic [Ilut and Schneider (2022)], or pay more attention to the worst-case scenario. This mechanism, however, is not needed in our Theorems 1 and 2.

[13] We also attempt to establish the ordering $s_A^* \geq s_R^* \geq s_D^*$ without imposing any restrictions on the first-order distributions. This proves to be challenging and we are only able to establish this when $u(\cdot)$ displays a constant absolute risk aversion (i.e., exponential utility). This result is stated in Theorem 4.

2 The Model

Consider the consumption-saving problem faced by a consumer in a two-period model. The consumer starts with a known initial wealth $w > 0$ in the first (or current) period and receives a random income $\widetilde{y}$ in the second (or future) period. The true distribution of $\widetilde{y}$ is unknown to the consumer. Thus, she relies on her own subjective beliefs when making the saving decision. These beliefs are formulated as follows: Let Θ be a subset of $\mathbb{R}$ and $\mathcal{F} \equiv \{F(\widetilde{y} \mid \theta) : \theta \in \Theta\}$ be a collection of probability distributions defined on a common support $\Omega = [\underline{y}, \overline{y}]$, with $0 < \underline{y} < \overline{y} < \infty$. The collection $\mathcal{F}$ contains all the distributions of future income that are deemed plausible by the consumer. We refer to these as first-order probability distributions. The perceived likelihood of these distributions is represented by a non-degenerate, second-order probability distribution function $G(\cdot)$ defined on Θ. Ambiguity is absent is $G(\cdot)$ is degenerate at some $\theta^{\dagger}$ in Θ, so that $G(\theta) = 1$ if $\theta \geq \theta^{\dagger}$ and zero otherwise.

There is a single risk-free asset which offers a known gross return $R > 0$. The consumer can save or borrow using this asset. An *ad hoc* borrowing constraint is in place to limit the amount of debt that the consumer may incur. Let (c_1, c_2, s) denote, respectively, current consumption, future consumption and savings in the current period. These choice variables are subject to non-negativity constraints: $c_1 \geq 0$, $c_2 \geq 0$; budget constraints: $c_1 + s = w$ and $c_2 = \widetilde{y} + Rs$; and an *ad hoc* borrowing constraint: $s \geq -\underline{b}$, where $\underline{b} > 0$ is the borrowing limit.[14]

As mentioned in the Introduction, the consumer is assumed to have generalised recursive smooth ambiguity (GRSA) preferences. Lifetime utility under this type of preferences is defined in three stages: First, for each plausible distribution in $\mathcal{F}$ and for each $(s;\theta) \in [-\underline{b}, w] \times \Theta$, a certainty equivalent of future consumption is computed according to

$$\mathbb{M}_u(s;\theta) \equiv u^{-1}\left[\int_{\Omega} u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)\right]. \tag{1}$$

In the above equation, $u : \mathbb{R}_+ \to \mathbb{R}$ is a von Neumann-Morgenstern (vNM) utility function that captures the consumer's atemporal risk preferences. At this stage, we only require $u(\cdot)$ to fulfill some basic properties which are listed below.

[14] If $\underline{y} \geq R\underline{b}$, then future consumption is non-negative even when the consumer owes the maximum amount of debt $(-\underline{b})$ and receives the lowest amount of future income $(\underline{y})$. If $\underline{y} < R\underline{b}$, then the *ad hoc* borrowing constraint will never bind, because otherwise future consumption will be negative in some future income states. Our main results are valid in both cases.

Assumption A1 The function $u : \mathbb{R}_+ \to \mathbb{R}$ is at least twice continuously differentiable, strictly increasing and concave.

Since $u(\cdot)$ is continuous and future consumption is of finite value in all possible states, the expected utility inside the square brackets in (1) is well-defined. A strictly increasing $u(\cdot)$ implies that the inverse function $u^{-1}(\cdot)$ is single-valued and strictly increasing. Hence, the certainty equivalent $\mathbb{M}_u(s;\theta)$ is well-defined for all $(s;\theta) \in [-\underline{b}, w] \times \Theta$.[15] Note that we only require $u(\cdot)$ to be (weakly) concave, hence Assumption A1 includes risk neutrality as a special case.

If $\widetilde{y}$ is drawn from the distribution $F(\widetilde{y} \mid \theta)$, then $\mathbb{M}_u(s;\theta)$ is the quantity of "risk-free" future consumption that yields the same utility as the random consumption profile $c_2 = \widetilde{y} + Rs$. Since the true distribution of $\widetilde{y}$ is unknown, $\mathbb{M}_u(s;\theta)$ is itself a random variable. Thus, in the second stage, the consumer forms a subjective expected utility over the random profile $\{\mathbb{M}_u(s;\theta) \mid \theta \in \Theta\}$ using the second-order probability distribution $G(\cdot)$. The subjective expected utility in the second stage is given by

$$\int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta), \tag{2}$$

where $v(\cdot)$ is another vNM utility function that captures the consumer's attitudes toward the randomness in $\{\mathbb{M}_u(s;\theta) \mid \theta \in \Theta\}$. The basic properties of $v(\cdot)$ are listed in Assumption A2.

Assumption A2 The function $v : \mathbb{R}_+ \to \mathbb{R}$ is at least twice continuously differentiable, strictly increasing and concave.

Define $\phi(\cdot) \equiv v \circ u^{-1}(\cdot)$ and $U(s;\theta) \equiv \int_\Omega u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)$. Then (2) can be rewritten as

$$\int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta) = \int_\Theta v \circ u^{-1}[U(s;\theta)]\, dG(\theta) = \int_\Theta \phi[U(s;\theta)]\, dG(\theta). \tag{3}$$

We refer to $U(s;\theta)$ as first-order expected utility and $\int_\Theta \phi[U(s;\theta)]\, dG(\theta)$ as second-order expected utility. The main difference between $v(\cdot)$ and $\phi(\cdot)$ can be explained as follows: $\mathbb{M}_u(s;\theta)$ and $U(s;\theta)$ can be viewed as *ex ante* evaluation about future consumption and future utility under a given value of savings and a given first-order distribution. Both are *ex ante* random in the presence of ambiguity. A concave $v(\cdot)$ means that the consumer dislikes any mean-preserving spread in the distribution of $\mathbb{M}_u(s;\theta)$, while a concave $\phi(\cdot)$ means that the consumer dislikes the same kind of spread in the distribution of $U(s;\theta)$. Following Klibanoff *et al.* (2005), we refer to a consumer with a concave $\phi(\cdot)$ as ambiguity-averse and one with a linear $\phi(\cdot)$ as ambiguity-neutral.

[15] Under Assumption A1, $\mathbb{M}_u(s;\theta)$ is also differentiable with respect to s for any given $\theta \in \Theta$. The details are shown in Appendix A.1.

Assumptions A1 and A2 together imply that $\phi(\cdot)$ is at least twice continuously differentiable and strictly increasing. But the concavity of $u(\cdot)$ and $v(\cdot)$ does not necessarily imply that of $\phi(\cdot)$. To see this, differentiate the identity $\phi[u(c)] \equiv v(c)$ twice to obtain

$$\phi'(x) = \frac{v'(c)}{u'(c)}, \tag{4}$$

$$\phi''(x) = \frac{v''(c) - \phi'(x)\, u''(c)}{[u'(c)]^2}, \tag{5}$$

where $x = u(c)$ and $c \geq 0$. Let $\mathcal{A}_u(c) \equiv -u''(c)/u'(c)$ and $\mathcal{A}_v(c) \equiv -v''(c)/v'(c)$ be the Arrow-Pratt coefficient of absolute risk aversion for $u(\cdot)$ and $v(\cdot)$, respectively. Then (4) and (5) imply that[16]

$$\phi''(x) \lesseqqgtr 0 \qquad \text{iff} \qquad \mathcal{A}_v(c) \gtreqqless \mathcal{A}_u(c), \tag{6}$$

for $x = u(c)$. Hence, the consumer is ambiguity-averse [i.e., $\phi''(\cdot) \leq 0$] if and only if $v(\cdot)$ is more concave than $u(\cdot)$. This has three immediate implications: First, risk aversion alone does not imply ambiguity aversion because the latter requires another (more) concave function $v(\cdot)$. Second, a risk-neutral (or even risk-loving) consumer can also be ambiguity-averse, provided that $v(\cdot)$ is sufficiently concave. Hence, risk aversion is neither necessary nor sufficient for ambiguity aversion. Third, if the consumer is both risk-averse and ambiguity-averse, then $v(\cdot)$ must be a concave function.

The last component of GRSA preferences is a time aggregator function $W : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, which captures the consumer's attitudes toward intertemporal substitution. Since $W(\cdot)$ is defined over deterministic consumption paths, we first compute the certainty equivalent of $\{\mathbb{M}_u(s;\theta) \mid \theta \in \Theta\}$ using the utility function $v(\cdot)$, i.e.,

$$\mathbb{M}_v(s) \equiv v^{-1}\left\{\int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta)\right\}, \tag{7}$$

for any given $s \in [-\underline{b}, w]$.[17] We refer to $\mathbb{M}_v(s)$ as the second-order certainty equivalent of future consumption obtained from s. A deterministic consumption path in this setting is given by $\{c_1, \mathbb{M}_v(s)\}$, which yields a lifetime utility of $W[c_1, \mathbb{M}_v(s)]$.

[16] This result has been shown in Klibanoff *et al.* (2005, Proposition 1). These details, however, are useful in understanding some of our main results, hence they are mentioned here.

[17] By Assumption A1, $\mathbb{M}_u(s;\theta)$ is of finite value for all $(s;\theta) \in [-\underline{b}, w] \times \Theta$. The continuity of $v(\cdot)$ then ensures that $v[\mathbb{M}_u(s;\theta)]$ is also of finite value for all $(s;\theta)$. Hence, the expectation $\int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta)$ exists. By Assumption A2, $v^{-1}(\cdot)$ is single-valued and strictly increasing. Hence, $\mathbb{M}_v(s)$ is well-defined for all $s \in [-\underline{b}, w]$. In Appendix A.1, it is shown that $\mathbb{M}_v(s)$ is differentiably and strictly increasing under Assumptions A1 and A2.

We start by considering an additively separable aggregator function $W(\cdot)$, which yields the familiar time-additively-separable lifetime utility function,

$$W[c_1, \mathbb{M}_v(s)] \equiv V(c_1) + \beta V[\mathbb{M}_v(s)]. \tag{8}$$

In the above equation, $\beta \in (0,1)$ is the subjective discount factor and $V : \mathbb{R}_+ \to \mathbb{R}$ is the (per-period) felicity function. We choose to start with this specification for two reasons. First, the time-separable specification has been used by the vast majority of existing studies on precautionary saving. Using the same specification will facilitate comparison with this literature. Second, the time-separable specification is easier to grasp, which allows for a clear development of the intuition behind our main results. In Section 4.2, we show that some of the results obtained under (8) can be readily extended to a non-separable aggregator function.

The basic properties of $V(\cdot)$ are summarised in Assumption A3. The limit condition (only one is necessary) is intended to rule out the uninteresting case where $c_1 = 0$ (or equivalently, $s = w$).

Assumption A3 The function $V : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and *strictly* concave. It also satisfies either $\lim_{c \to 0} V(c) = -\infty$ or $\lim_{c \to 0} V'(c) = +\infty$.

We now exploit the intricate structure of GRSA preferences to derive two alternative but equivalent expressions of future felicity, $V[\mathbb{M}_v(s)]$. The first one uses $\phi(\cdot) \equiv v \circ u^{-1}(\cdot)$ and a new composite function $\Psi(\cdot) \equiv V \circ v^{-1}(\cdot)$ to rewrite future felicity as follows:

$$\begin{aligned} V[\mathbb{M}_v(s)] &= V \circ v^{-1} \left\{ \int_\Theta v \circ u^{-1} \left[\int_S u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta) \right] dG(\theta) \right\} \\ &\equiv \Psi \left\{ \int_\Theta \phi[U(s;\theta)]\, dG(\theta) \right\}. \end{aligned} \tag{9}$$

Equation (9) expresses future felicity as a transformation of the second-order expected future utility $\int_\Theta \phi[U(s;\theta)]\, dG(\theta)$. Under this formulation, the consumer's lifetime utility is a nonlinear aggregate of current-period felicity $V(c_1)$ and $\int_\Theta \phi[U(s;\theta)]\, dG(\theta)$. This is similar in spirit to the Kreps-Porteus representation in Kimball and Weil (2009, p.248).[18]

The second alternative expression of $V[\mathbb{M}_v(s)]$ involves a new variable:

$$\mathbb{M}_\phi(s) \equiv \phi^{-1} \left\{ \int_\Theta \phi[U(s;\theta)]\, dG(\theta) \right\}, \tag{10}$$

[18] The original expression $V[\mathbb{M}_v(s)]$, on the other hand, is in the same spirit as Selden's representation [see Kimball and Weil (2009, p.248)].

which is the certainty equivalent of the first-order expected utility profile $\{U(s;\theta)\,|\theta\in\Theta\}$ defined using the composite function $\phi(\cdot)$.[19] Note that $\mathbb{M}_\phi(s)$ is measured in utility units, while $\mathbb{M}_v(s)$ is measured in units of consumption. Using (3), (7) and $\phi^{-1}(\cdot)=u\circ v^{-1}(\cdot)$, we can get $u[\mathbb{M}_v(s)]=\mathbb{M}_\phi(s)$. Define $\Gamma(\cdot)\equiv V\circ u^{-1}(\cdot)$. Then we can rewrite $V[\mathbb{M}_v(s)]$ as follows:

$$V[\mathbb{M}_v(s)]=V\circ u^{-1}\left[u\circ v^{-1}\left\{\int_\Theta \phi[U(s;\theta)]\,dG(\theta)\right\}\right]\equiv\Gamma[\mathbb{M}_\phi(s)]. \tag{11}$$

This alternative formulation expresses future felicity as a transformation of the certainty equivalent $\mathbb{M}_\phi(s)$. We now have three different but equivalent ways of expressing future felicity and also lifetime utility function.[20] These form the basis of three different approaches in characterising precautionary saving behaviour, which are detailed in Section 3.

The lifetime utility function in (8) can help unify three groups of studies in the precautionary saving literature. The first group of studies [e.g., Osaki and Schlesinger (2014, Section 4) and Berger (2014)] adopt the recursive smooth ambiguity preferences advanced by Klibanoff *et al.* (2009). This specification is able to separate risk attitudes from ambiguity attitudes, but attitudes toward risk and intertemporal substitution are confounded. The recursive smooth ambiguity preferences can be recovered from (8) by setting $V(\cdot)\equiv u(\cdot)$, so that

$$\begin{aligned}W[c_1,\mathbb{M}_v(s)] &= u(c_1)+\beta u\circ v^{-1}\left\{\int_\Theta v[\mathbb{M}_u(s;\theta)]\,dG(\theta)\right\}\\ &= u(c_1)+\beta\phi^{-1}\left\{\int_\Theta \phi\left[\int_\Omega u(\widetilde{y}+Rs)\,dF(\widetilde{y}\mid\theta)\right]dG(\theta)\right\}. \end{aligned}\tag{12}$$

The second group of studies [e.g., Gollier (2001a, Section 20.3) and Kimball and Weil (2009)] use the Selden/Kreps-Porteus (SKP) preferences which disentangle risk attitudes from attitudes toward intertemporal substitution, but assume that the consumer is ambiguity-neutral. The SKP preferences can be recovered from (8) by setting $v(\cdot)\equiv u(\cdot)$, i.e.,

$$W[c_1,\mathbb{M}_u(s)]=V(c_1)+\beta V\circ\underbrace{u^{-1}\left[\int_\Theta\int_\Omega u(\widetilde{y}+Rs)\,dF(\widetilde{y}\mid\theta)\,dG(\theta)\right]}_{\mathbb{M}_u(s)}. \tag{13}$$

[19] By Assumptions A1 and A2, $\phi(\cdot)$ is continuous and strictly increasing. The continuity of $\phi(\cdot)$ and the boundedness of $U(s;\theta)$ for all $(s;\theta)\in[-\underline{b},w]\times\Theta$ ensures the existence of the expectation $\int_\Theta\phi[U(s;\theta)]\,dG(\theta)$. A strictly increasing $\phi(\cdot)$ means that the inverse function $\phi^{-1}(\cdot)$ is single-valued and strictly increasing. Hence, $\mathbb{M}_\phi(s)$ is well-defined for all $s\in[-\underline{b},w]$. In Appendix A.1, it is shown that the derivative of $\mathbb{M}_\phi(s)$ is well-defined and is strictly positive under Assumptions A1 and A2.

[20] While $\mathbb{M}_u(s,\theta)$ and $U(s,\theta)$ are *ex ante* evaluation of future consumption and future utility under a given first-order condition, $\mathbb{M}_v(s)$, $\mathbb{M}_\phi(s)$ and $\int_\Theta\phi[U(s;\theta)]\,dG(\theta)$ are *ex ante* evaluations that take into account the consumer's entire belief system, i.e., all the first-order plausible distributions and their subjective likelihood as represented by $G(\cdot)$.

In the above equation, $\mathbb{M}_u(s) \equiv \mathbb{M}_v(s)$ is the certainty equivalent of risky future consumption associated with the compound probability distribution

$$H(\widetilde{y}) \equiv \int_{\Theta} F(\widetilde{y} \mid \theta)\, dG(\theta), \qquad \text{for all } \widetilde{y} \in \Omega.$$

The last group of studies [most notably, Kimball (1990)] use the expected-utility (EU) preferences, which assume that the consumer is ambiguity-neutral and do not separate risk attitudes from attitudes toward intertemporal substitution. The EU preferences correspond to the special case when $V(\cdot) \equiv u(\cdot) \equiv v(\cdot)$ in (8). The consumer's lifetime utility function then becomes

$$u(c_1) + \beta \int_{\Omega} u(\widetilde{y} + Rs)\, dH(\widetilde{y}).$$

In the absence of ambiguity, i.e., when $G(\cdot)$ is degenerate at some $\theta^{\dagger} \in \Theta$, the lifetime utility in (8) becomes

$$W[c_1, \mathbb{M}_u(s)] \equiv V(c_1) + \beta V \circ u^{-1} \left[\int_{\Omega} u(\widetilde{y} + Rs)\, dF\left(\widetilde{y} \mid \theta^{\dagger}\right) \right]. \tag{14}$$

Equation (14) is observationally equivalent to the lifetime utility of a consumer who has SKP preferences and faces a pure future income risk drawn from the distribution $F\left(\widetilde{y} \mid \theta^{\dagger}\right)$. There is, however, a subtle difference between (13) and (14). If the consumer is ambiguity-neutral, then $\mathbb{M}_u(s)$ is computed using the compound probability distribution $H(\widetilde{y})$. If there is no ambiguity, then $\mathbb{M}_u(s)$ is computed using the distribution $F\left(\widetilde{y} \mid \theta^{\dagger}\right)$, which needs not be the same as $H(\widetilde{y})$. Finally, if ambiguity is absent, then the recursive smooth ambiguity preferences in (12) becomes

$$W[c_1, \mathbb{M}_v(s)] = u(c_1) + \beta \int_{\Omega} u(\widetilde{y} + Rs)\, dF\left(\widetilde{y} \mid \theta^{\dagger}\right). \tag{15}$$

This is observationally equivalent to the lifetime utility of an expected-utility maximiser who faces a pure future income risk drawn according to $F\left(\widetilde{y} \mid \theta^{\dagger}\right)$.[21]

[21] The GRSA preferences considered here, however, do not cover the preferences considered in Leland (1968), Sandmo (1970) and Peter (2019). The first two studies consider a consumer with expected utility preferences defined over a non-separable vNM utility function of current and future consumption. Peter (2019) extends this type of preferences to an environment with ambiguity.

3 Analysis

Taking the risk-free return $R > 0$ as given, the consumer's problem is to choose a feasible allocation (c_1, c_2, s) so as to maximise his lifetime utility in (8). This problem can be succinctly expressed as

$$\max_{s \in [-\underline{b}, w]} \{V(w - s) + \beta V[\mathbb{M}_v(s)]\}, \tag{P1}$$

where $\mathbb{M}_v(s)$ is defined in (7). Since the choice set is compact and the objective function is continuous under Assumptions A1-A3, (P1) has at least one solution. These assumptions, however, do not guarantee that the objective function is concave in the choice variable. A concave objective function is desirable because it ensures that the first-order condition is sufficient to identify the solution(s).[22] If the objective function is strictly concave, then a unique solution exists. Uniqueness of solution is useful in simplifying the subsequent comparative statics analysis.

With these considerations in mind, our task in this section is twofold: The first one is to derive conditions under which the objective function in (P1) is strictly concave. The second task is to provide conditions under which precautionary saving exists. The definition of precautionary saving is made precise in the next subsection.

3.1 Precautionary Saving Motives

Suppose (P1) has a unique solution denoted by s_A^* (this will be verified later). Precautionary saving is defined by comparing this to the optimal savings in two other environments. In the first one, future income is known with certainty. The consumer's problem in this case is given by

$$\max_{s \in [-\underline{b}, w]} \{V(w - s) + \beta V(\mu + Rs)\}, \tag{P2}$$

where $\mu > 0$ is the mean of the compound distribution $H(\widetilde{y})$. Since $V(\cdot)$ is continuous and strictly concave under Assumption A3, (P2) has a unique solution denoted by s_D^*. In the second environment, future income is drawn from the compound distribution $H(\widetilde{y})$ without any ambiguity. The consumption-saving problem now takes the form

$$\max_{s \in [-\underline{b}, w]} \left\{ V(w - s) + \beta V \circ u^{-1} \left[\int_\Omega u(\widetilde{y} + Rs)\, dH(\widetilde{y}) \right] \right\}. \tag{P3}$$

[22] Most, if not all, of the existing studies on precautionary saving (including this one) focus on the first-order necessary condition of the consumption-saving problem. This approach is valid only if the first-order condition is sufficient.

This problem has a unique solution s_R^*, which will be verified later.

Previous studies, such as Kimball (1990) and Kimball and Weil (2009), focus on the conditions under which $s_R^* \geq s_D^*$. This means the consumer chooses to save more when facing a pure income risk in the future period. This type of saving behaviour is referred to as *risk prudence.* Precautionary saving can also arise due to the introduction of ambiguity, i.e., $s_A^* \geq s_R^*$. This type of saving behaviour has become known as *ambiguity prudence.* Formally, ambiguity prudence refers to the tendency to save more due to a mean-preserving spread in the distribution of $U(s;\theta)$. Note that the mean value of $U(s;\theta)$ satisfies

$$\int_\Theta U(s;\theta)\, dG(\theta) = \int_\Omega u(\widetilde{y} + Rs)\, dH(\widetilde{y}).$$

Hence, when defining ambiguity prudence, we compare s_A^* to the optimal saving when the pure income risk is drawn from the compound distribution $H(\widetilde{y})$. It is also important to note that risk prudence does not necessarily imply ambiguity prudence, and vice versa.

In the current section, we focus on the conditions under which $s_A^* \geq s_D^*$.[23] We refer to this as *mixed prudence.* In all of our results, a mixed-prudent consumer is also risk-prudent, but is not necessarily ambiguity-prudent. If the consumer is both risk-prudent and ambiguity-prudent, i.e., $s_A^* \geq s_R^* \geq s_D^*$, then mixed prudence is immediately implied. In this case, the additional savings induced by mixed prudence can be expressed as a sum of those induced by risk prudence and ambiguity prudence, i.e.,

$$s_A^* - s_D^* = \underbrace{(s_A^* - s_R^*)}_{(+)} + \underbrace{(s_R^* - s_D^*)}_{(+)}. \tag{16}$$

Our Theorem 3 provides a set of conditions under which (16) is valid. Further discussions about ambiguity prudence can be found in Section 4.1.

3.2 Main Results

In this subsection, we present three different approaches to establish the uniqueness of s_A^* and to identify conditions under which $s_A^* \geq s_D^*$. These approaches are based on the three equivalent formulations of future felicity mentioned in Section 2. We begin with an outline of our strategy.

First note that the consumer's felicity in the current period, $V(w-s)$, is identical in (P1)-(P3). This means the marginal cost of saving, $V'(w-s)$, under any given value of s is identical

[23] Obviously, there is an important difference between $s_A^* > s_D^*$ and $s_A^* = s_D^*$. In the following analysis, we will also mention the conditions under which precautionary saving is strictly positive.

in all three problems. Hence, when comparing $\{s_A^*, s_R^*, s_D^*\}$, it suffice to focus on future felicity alone. In particular, $s_A^* \geq s_D^*$ if the marginal benefit of saving in (P1) exceed that in (P2).

In all three approaches, future felicity under ambiguity is expressed as the composition of two suitably chosen functions: $\Phi(\cdot)$ and $g_A(\cdot)$, so that

$$V[\mathbb{M}_v(s)] \equiv \Phi[g_A(s)], \qquad \text{for all } s \in [-\underline{b}, w]. \tag{17}$$

Likewise, future felicity in (P2) is expressed as the composition of $\Phi(\cdot)$ and another function $g_D(\cdot)$, so that

$$V(\mu + Rs) \equiv \Phi[g_D(s)], \qquad \text{for all } s \in [-\underline{b}, w]. \tag{18}$$

The choices of $\Phi(\cdot)$, $g_A(\cdot)$ and $g_D(\cdot)$ are summarised in Table 1.[24] The first approach is motivated by the formulation in (9), hence $\Phi(\cdot)$ corresponds to the composite function $\Psi(\cdot) \equiv V \circ v^{-1}(\cdot)$ and $g_A(\cdot)$ corresponds to the mapping $s \mapsto \int_\Theta \phi[U(s;\theta)]\, dG(\theta)$. The latter captures the effect of savings on second-order expected future utility. The second approach uses the original expression for future felicity, $V[\mathbb{M}_v(s)]$. Hence, $\Phi(\cdot)$ in (17) now corresponds to $V(\cdot)$, and $g_A(\cdot)$ is defined as the mapping $s \mapsto \mathbb{M}_v(s)$, which captures the effect of savings on the second-order certainty equivalent of future consumption. The third approach is motivated by the formulation in (11).

In all three cases, $g_A(\cdot)$ corresponds to an *ex ante* evaluation of future consumption or future utility under ambiguity, $g_D(\cdot)$ is its deterministic counterpart,[25] and $\Phi(\cdot)$ captures the consumer's preferences on $g_A(\cdot)$. Assumptions A1-A3 ensure that $\{\Phi(\cdot), g_A(\cdot), g_D(\cdot)\}$ are all continuously differentiable, strictly increasing functions. In each approach, we will impose additional conditions to ensure that $\Phi(\cdot)$ is also a concave function.

Table 1: Main Ingredients of the Three Approaches

	$\Phi(\cdot)$	$g_A(s)$	$g_D(s)$
Approach #1	$\Psi(\cdot) \equiv V \circ v^{-1}(\cdot)$	$\int_\Theta \phi[U(s;\theta)]\, dG(\theta)$	$\phi \circ u(\mu + Rs)$
Approach #2	$V(\cdot)$	$\mathbb{M}_v(s)$	$\mu + Rs$
Approach #3	$\Gamma(\cdot) \equiv V \circ u^{-1}(\cdot)$	$\mathbb{M}_\phi(s)$	$u(\mu + Rs)$

The main advantage of this strategy is that when comparing s_A^* to s_D^*, it suffice to focus on

[24] The same strategy is used in Theorem 3 and Theorem 4 to establish ambiguity prudence, i.e., $s_A^* \geq s_R^*$.

[25] In all three approaches, $g_D(\cdot)$ can be derived from $g_A(\cdot)$ in either one of the following two ways: (i) By assuming that $G(\theta)$ is degenerate at some $\theta^\dagger \in \Theta$ and $F(\widetilde{y} \mid \theta^\dagger)$ is degenerate at μ. (ii) By assuming that all the first-order distributions in $\mathcal{F}$ are identical and degenerate at μ.

$g_A(\cdot)$ and $g_D(\cdot)$. The rationale is as follows: Based on (17) and (18), the marginal benefit of saving in (P1) and (P2) are given by $\Phi'[g_A(s)]\,g'_A(s)$ and $\Phi'[g_D(s)]\,g'_D(s)$, respectively. Hence, $s^*_A \geq s^*_D$ if

$$\Phi'[g_A(s)]\,g'_A(s) \geq \Phi'[g_D(s)]\,g'_D(s), \qquad \text{for all } s \in [-\underline{b}, w]. \tag{19}$$

Since $\{\Phi(\cdot), g_A(\cdot), g_D(\cdot)\}$ are all strictly increasing functions, the derivatives in (19) are all strictly positive. If, in addition, $\Phi'(\cdot)$ is a decreasing function, then condition (19) is satisfied when

$$g_A(s) \leq g_D(s) \qquad \text{and} \qquad g'_A(s) \geq g'_D(s), \tag{20}$$

for all $s \in [-\underline{b}, w]$. The conditions in (20) states that the consumer is mixed prudent if $g_A(s)$ is lower in level but more responsive to s (i.e., steeper as a function in s) than its deterministic counterpart. This will then contribute to a higher marginal benefit of saving under ambiguity and promote savings. We now discuss each of these three approaches in detail.

Approach #1

Under the first approach, the objective function in (P1) is rewritten as

$$\Pi_A(s) \equiv V(w-s) + \beta\Psi\left\{\int_\Theta \phi[U(s;\theta)]\,dG(\theta)\right\}.$$

We first examine the conditions under which $\Pi_A(\cdot)$ is strictly concave. Since $V(\cdot)$ is strictly concave by Assumption A3, it suffice to consider the consumer's future felicity. The mapping $s \mapsto \Psi\left\{\int_\Theta \phi[U(s;\theta)]\,dG(\theta)\right\}$ is concave if $u(\cdot)$, $\phi(\cdot)$ and $\Psi(\cdot)$ are all increasing concave functions. This follows from the facts that (i) monotonicity and concavity are preserved by integration, and (ii) the composition of two increasing concave functions is again increasing concave.[26] Under Assumptions A1-A3, $u(\cdot)$ is strictly increasing and concave, and both $\phi(\cdot)$ and $\Psi(\cdot)$ are strictly increasing. Hence, additional conditions are needed to ensure the concavity of $\phi(\cdot)$ and $\Psi(\cdot)$. As shown in (6), $\phi(\cdot)$ is concave if and only if $v(\cdot)$ is more concave than $u(\cdot)$. By the same token, $\Psi(\cdot) \equiv V \circ v^{-1}(\cdot)$ is concave if and only if $V(\cdot)$ is more concave than $v(\cdot)$. Define $\mathcal{A}_V(c) \equiv -V''(c)/V'(c)$. Then $u(\cdot)$, $\phi(\cdot)$ and $\Psi(\cdot)$ are all concave functions if and only if

$$\mathcal{A}_V(c) \geq \mathcal{A}_v(c) \geq \mathcal{A}_u(c) \geq 0, \qquad \text{for } c \geq 0. \tag{21}$$

[26] The full argument is developed in the proof of Theorem 1.

Taken together, Assumptions A1-A3 and (21) are sufficient to guarantee that $\Pi_A(\cdot)$ is strictly concave, which then ensures the uniqueness of s_A^*. The conditions $\mathcal{A}_u(\cdot) \geq 0$ and $\mathcal{A}_v(\cdot) \geq \mathcal{A}_u(\cdot)$ correspond to risk aversion and ambiguity aversion, respectively. If ambiguity is absent (or if the consumer is ambiguity-neutral), then $\mathcal{A}_V(\cdot) \geq \mathcal{A}_u(\cdot)$ means that the consumer has a preference for late resolution of intertemporal risk.[27]

Under Approach #1, the first-order condition for (P1) is given by

$$V'(w-s) \geq \beta \underbrace{\Psi' \left\{ \int_\Theta \phi\left[U(s;\theta)\right] dG(\theta) \right\}}_{\Phi'[g_A(s)]} \underbrace{\int_\Theta R\phi'\left[U(s;\theta)\right] U_s(s;\theta)\, dG(\theta)}_{g_A'(s)}, \tag{22}$$

where $U_s(s;\theta)$ is the derivative of $U(s;\theta)$ with respect to s. Condition (22) will hold with strict equality if s_A^* is an interior solution, i.e., $s_A^* > -\underline{b}$. The right side of this condition reflects the marginal benefit of saving. If this is outweighed by the marginal cost under every feasible value of s, then the consumer will choose to exhaust the borrowing limit, i.e., $s_A^* = -\underline{b}$. The marginal benefit of saving can be decomposed into two parts. The first part shows how a change in second-order expected utility will affect future felicity. This corresponds to $\Phi'[g_A(s)]$ in (19). The second part shows how a change in s will affect the second-order expected utility, which corresponds to $g_A'(s)$ in (19).

On a similar vein, the deterministic saving problem (P2) can be rewritten as

$$\max_{s \in [-\underline{b}, z]} \left\{ V(w-s) + \beta \Psi\left[\phi \circ u(\mu + Rs)\right] \right\},$$

and the first-order condition is given by

$$V'(w-s) \geq \beta \underbrace{\Psi'\left[\phi \circ u(\mu + Rs)\right]}_{\Phi'[g_D(s)]} \underbrace{R\phi'\left[u(\mu + Rs)\right] u'(\mu + Rs)}_{g_D'(s)}. \tag{23}$$

The consumer is mixed-prudent (i.e., $s_A^* \geq s_D^*$) if the marginal benefit of saving in (P1) is no less than that in (P2), i.e.,

$$\begin{aligned} & \Psi' \left\{ \int_\Theta \phi\left[U(s;\theta)\right] dG(\theta) \right\} \int_\Theta \phi'\left[U(s;\theta)\right] U_s(s;\theta)\, dG(\theta) \\ \geq \;& \Psi'\left[\phi \circ u(\mu + Rs)\right] \phi'\left[u(\mu + Rs)\right] u'(\mu + Rs). \end{aligned} \tag{24}$$

Our first theorem provides a set of conditions under which (24) is valid. Unless otherwise stated,

[27] This implication may not be true in the presence of ambiguity. See Strzalecki (2013) for details.

all proofs can be found in Appendix A.

Theorem 1 *Suppose Assumptions A1-A3 are satisfied.*

(i) *If (21) is satisfied, then (P1) has a unique solution denoted by* s_A^*.

(ii) *If, in addition, both* $u'(\cdot)$ *and* $v'(\cdot)$ *are convex functions, then* $s_A^* \geq s_D^*$.

Theorem 1 identifies five conditions [not counting the differentiability and monotonicity of $u(\cdot)$, $v(\cdot)$ and $V(\cdot)$] that are sufficient to establish mixed prudence. These are (I) risk aversion, i.e., $\mathcal{A}_u(\cdot) \geq 0$; (II) ambiguity aversion, i.e., $\mathcal{A}_v(\cdot) \geq \mathcal{A}_u(\cdot)$; (III) the concavity of $\Psi(\cdot)$, i.e., $\mathcal{A}_V(\cdot) \geq \mathcal{A}_v(\cdot)$; (IV) risk prudence as in Kimball (1990), i.e., the convexity of $u'(\cdot)$; and (V) the convexity of $v'(\cdot)$. Conditions I and II together imply that the second-order expected utility is no greater than its deterministic counterpart, i.e.,

$$\underbrace{\int_\Theta \phi\left[U(s;\theta)\right] dG(\theta)}_{g_A(s)} \leq \underbrace{\phi \circ u(\mu + Rs)}_{g_D(s)}. \tag{25}$$

This follows immediately from Jensen's inequality. Conditions I, II, IV and V together ensure that the *ex ante* evaluation of future utility is more responsive to changes in s than its deterministic counterpart, i.e.,

$$\underbrace{\int_\Theta \phi'\left[U(s;\theta)\right] U_s(s;\theta)\, dG(\theta)}_{g'_A(s)} \geq \underbrace{R\phi'\left[u(\mu + Rs)\right] u'(\mu + Rs)}_{g'_D(s)}. \tag{26}$$

As noted before, given the concavity of $\Psi(\cdot)$ (condition III), (25) and (26) are sufficient to deliver (24).

We now explore further the economic meaning of the conditions in Theorem 1. Set $g_A(s) = \int_\Theta \phi\left[U(s;\theta)\right] dG(\theta)$ and $g_D(s) = \phi \circ u(\mu + Rs)$, and rewrite the first-order condition in (22) as

$$\begin{aligned} V'(w - s) \quad \geq \quad & \underbrace{\beta \frac{\Psi'\left[g_A(s)\right]}{\Psi'\left[g_D(s)\right]}}_{\text{Discount Factor Effect}} \times \underbrace{\frac{\int_\Theta \phi'\left[U(s;\theta)\right] U_s(s;\theta)\, dG(\theta)}{R\phi'\left[u(\mu + Rs)\right] u'(\mu + Rs)}}_{\text{Generalised Risk Prudence Effect}} \\ & \times \underbrace{R\Psi'\left[g_D(s)\right] \phi'\left[u(\mu + Rs)\right] u'(\mu + Rs)}_{\text{MB of saving under certainty}}. \end{aligned} \tag{27}$$

Equation (27) breaks down the marginal benefit of saving under ambiguity into that under certainty and two additional factors. The first one is labelled as "discount factor effect."[28] To explain this formally, define

$$\widetilde{\beta}(s) \equiv \beta \frac{\Psi'[g_A(s)]}{\Psi'[g_D(s)]}.$$

If $\Psi(\cdot)$ is a linear function, or if the consumer is ambiguity-neutral *and* risk-neutral so that $g_A(\cdot) \equiv g_D(\cdot)$, then the discount factor effect is absent, i.e., $\widetilde{\beta}(\cdot) = \beta$. But for any risk-averse and ambiguity-averse consumer with a concave $\Psi(\cdot)$ [i.e., under conditions I, II and III in Theorem 1], $\widetilde{\beta}(s) \geq \beta$ for all $s \in [-\underline{b}, w]$. This works as if the consumer has become more patient under ambiguity, which then induces the consumer to save more.

The second factor in (27) is essentially the ratio between $g'_A(s)$ and $g'_D(s)$. If the consumer is ambiguity-neutral, so that $\phi'(\cdot)$ is a constant function, then this ratio will be reduced to

$$\frac{g'_A(s)}{g'_D(s)} = \frac{\int_\Omega u'(\widetilde{y} + Rs)\, dH(\widetilde{y})}{u'(\mu + Rs)}. \tag{28}$$

If $u'(\cdot)$ is convex, then any mean-preserving spread in the distribution of $\widetilde{y}$ will generate a higher (expected) marginal utility of future consumption and promote saving. This is the same mechanism behind the risk prudence result of Kimball (1990). Now consider the case when the consumer is *not* ambiguity-neutral. Risk aversion (condition I) implies that for any $s \in [-\underline{b}, w]$ and for any $\theta \in \Theta$,

$$U(s;\theta) = \int_\Omega u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta) \leq u(\widetilde{\mu}(\theta) + Rs),$$

where $\widetilde{\mu}(\theta)$ is the mean of $F(\widetilde{y} \mid \theta)$. Risk prudence (condition IV), on the other hand, implies that

$$U_s(s;\theta) = \int_\Omega Ru'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta) \geq Ru'[\widetilde{\mu}(\theta) + Rs].$$

Combining these and ambiguity aversion (condition II) gives

$$\phi'[U(s;\theta)]\, U_s(s;\theta) \geq R\phi'[u(\widetilde{\mu}(\theta) + Rs)]\, u'(\widetilde{\mu}(\theta) + Rs),$$

which is equivalent to

$$\frac{d}{ds}\phi[U(s;\theta)] \geq \frac{d}{ds}\{\phi \circ u(\widetilde{\mu}(\theta) + Rs)\}, \qquad \text{for any } \theta \in \Theta. \tag{29}$$

[28] This is similar in spirit to the "timing of uncertainty effect" in Osaki and Schlesinger (2014, p.16).

Equation (29) states that a mean-preserving spread in any first-order distribution $F(\widetilde{y} \mid \theta)$ will make $\phi[U(s;\theta)]$ more sensitive to changes in s.[29] This essentially generalises Kimball's risk prudence result to a concave transformation of (first-order) expected future utility. Note that $\phi \circ u(\widetilde{\mu}(\theta) + Rs) = v(\widetilde{\mu}(\theta) + Rs)$, which means

$$\frac{d}{ds}\{\phi \circ u(\widetilde{\mu}(\theta) + Rs)\} = v'[\widetilde{\mu}(\theta) + Rs].$$

Thus, by the convexity of $v'(\cdot)$ (condition V),

$$\begin{aligned} \int_\Theta \frac{d}{ds}\{\phi \circ u(\widetilde{\mu}(\theta) + Rs)\} dG(\theta) &= \int_\Theta v'[\widetilde{\mu}(\theta) + Rs] dG(\theta) \\ &\geq v'(\mu + Rs) = \phi'[u(\mu + Rs)] u'(\mu + Rs). \end{aligned} \tag{30}$$

The inequality in (30) is again the risk prudence effect, but now applied on a consumer with vNM utility function $v(\cdot)$ who faces a pure risk in $\widetilde{\mu}(\theta)$. Equation (26) can then be obtained by combining (29) and (30). We use the term "generalised risk prudence effect" to reflect the interplay between the convexity of $u'(\cdot)$ and that of $v'(\cdot)$. This, alongside with the discount factor effect, encourages the consumer to save more in (P1) than in (P2).

Our Theorem 1 encompasses at least four important special cases. The first one is the expected-utility model which corresponds to the special case of $V(\cdot) \equiv v(\cdot) \equiv u(\cdot)$ in (8). Under these restrictions, conditions I, II and III in Theorem 1 are equivalent, while IV and V coincide. Uniqueness of solution is guaranteed by the strict concavity of $u(\cdot)$, and precautionary saving exists if $u'(\cdot)$ is weakly convex (condition IV), which is the well-known result in Kimball (1990). In terms of (27), the discount factor effect is absent (reduced to one) and the generalised risk prudence effect is reduced to (28).

The second special case is the SKP preferences, which can be obtained by imposing ambiguity neutrality, i.e., $v(\cdot) \equiv u(\cdot)$. In this case, conditions I and IV in Theorem 1 are equivalent to II and V, respectively. Conditions I, III and IV are then sufficient to ensure the existence of precautionary saving. This is the risk prudence result of Gollier (2001a, p.300-302). Since the consumer is ambiguity neutral, the generalised risk prudence effect is again reduced to (28). When viewed through this lens, Gollier's risk prudence result can be explained by a combination of discount factor effect and Kimball's risk prudence effect.

[29] The same argument remains valid if we compare $F(\widetilde{y} \mid \theta)$ to another non-degenerate distribution, say $M(\widetilde{y} \mid \theta)$, where $F(\widetilde{y} \mid \theta)$ is a mean-preserving spread of $M(\widetilde{y} \mid \theta)$.

We now present two new special cases in which precautionary saving is strictly positive even though $u'''(\cdot) = 0$. Suppose $u(\cdot)$ is quadratic so that

$$u(c) = \alpha_0 + \alpha_1 c - \frac{\alpha_2}{2} c^2, \tag{31}$$

for some real numbers α_0, $\alpha_1 > 0$ and $\alpha_2 > 0$. The main finding here is that if the consumer is *strictly* risk-averse and *strictly* ambiguity-averse, then precautionary saving is *strictly* positive even if $u'''(\cdot) \equiv 0$. This result is formally stated in Corollary 1. An additional restriction on the parameters $\{\alpha_1, \alpha_2, \overline{y}, w\}$ is introduced to ensure $u'(c_1) > 0$ and $u'(c_2) > 0$ under all feasible values of s in (P1).

Corollary 1 *Suppose Assumptions A2-A3 and* $\alpha_1 > \alpha_2 \max\{\overline{y} + Rw, w + \underline{b}\}$ *are satisfied. Suppose* $u(\cdot)$ *takes the quadratic form in (31),* $\mathcal{A}_V(\cdot) \geq \mathcal{A}_v(\cdot) > \mathcal{A}_u(\cdot)$ *and* $v'(\cdot)$ *is convex. Then* $s_A^* > s_D^*$.

One implication of Theorem 1 and Corollary 1 is that a strictly positive third derivative of $\phi(\cdot)$ is sufficient but *not* necessary for mixed prudence. To see this, suppose both $u(\cdot)$ and $v(\cdot)$ are thrice differentiable and have non-negative third-order derivative, i.e., $u'''(\cdot) \geq 0$ and $v'''(\cdot) \geq 0$. Then $\phi(\cdot)$ is also thrice differentiable but $\phi'''(\cdot)$ can be either positive-valued or negative-valued (see Appendix A.1 for details). In other words, the conditions in Theorem 1 and Corollary 1 do not imply $\phi'''(\cdot) \geq 0$. But, on the contrary, if $\phi'''(\cdot) \geq 0$ and $u'''(\cdot) \geq 0$, then $v'''(\cdot)$ must be strictly positive.

Finally, we revisit one of the special cases mentioned in the Introduction. The result is summarised in Corollary 2. It states that if $v(\cdot)$ is strictly concave and $v'(\cdot)$ is strictly convex, then precautionary saving is strictly positive even if the consumer is risk-neutral. In other words, risk aversion is also not necessary for mixed prudence. By setting $u(c) = c$, the lifetime utility function in (8) becomes

$$W[c_1, \mathbb{M}_v(s)] \equiv V(c_1) + \beta V\left[v^{-1}\left\{\int_{\Theta} v[\widetilde{\mu}(\theta) + Rs]\, dG(\theta)\right\}\right].$$

This shows that an ambiguity-averse but risk-neutral consumer who faces an ambiguous risk is observationally equivalent to a risk-averse consumer with SKP preferences who faces a pure risk in $\widetilde{\mu}(\theta)$. Thus, mixed prudence under risk neutrality is observationally equivalent to risk prudence under SKP preferences. This explains the result in Corollary 2, which is essentially the risk prudence result in Gollier (2001a, p.300-302) but with $u(\cdot)$ replaced by $v(\cdot)$.

Corollary 2 *Suppose Assumptions A2 and A3 are satisfied. Suppose* $u(c) = c$, $\mathcal{A}_V(\cdot) \geq \mathcal{A}_v(\cdot) > 0$, *and* $v'(\cdot)$ *is strictly convex. Then* $s_A^* > s_D^*$.

Through Approach #1, we not only identify a set of sufficient conditions for mixed prudence, we also show that neither risk aversion nor risk prudence are necessary for this type of precautionary saving behaviour. A caveat of this approach is that the conditions in (21) are incompatible with the recursive smooth ambiguity preferences put forward by Klibanoff *et al.* (2009) [which requires $V(\cdot) \equiv u(\cdot) \neq v(\cdot)$]. This prompts us to explore a different approach that can be applied to such preferences, which is Approach #2.

Approach #2

Consider the objective function of (P1) in its original form, i.e.,

$$\Xi_A(s) \equiv V(w - s) + \beta V[\mathbb{M}_v(s)].$$

Under Assumption A3, $\Xi_A(s)$ is a concave function in s if the mapping $s \mapsto \mathbb{M}_v(s)$ exhibits concavity, i.e., for any s_1, $s_2 \in [-\underline{b}, w]$ and for any $\alpha \in [0, 1]$,

$$\mathbb{M}_v(s_\alpha) \geq \alpha \mathbb{M}_v(s_1) + (1 - \alpha) \mathbb{M}_v(s_2), \tag{32}$$

where $s_\alpha \equiv \alpha s_1 + (1 - \alpha) s_2$. Since $\mathbb{M}_v(s)$ is itself a certainty equivalent of $\mathbb{M}_u(s; \theta)$, this will also require the concavity of $\mathbb{M}_u(s; \theta)$, i.e.,

$$\mathbb{M}_u(s_\alpha; \theta) \geq \alpha \mathbb{M}_u(s_1; \theta) + (1 - \alpha) \mathbb{M}_u(s_2; \theta), \qquad \text{for any } \theta \in \Theta. \tag{33}$$

The notion of concave certainty equivalent has been previously considered in Gollier (2001a, p.322), Gollier (2001b, Lemma 5), , Hennessy and Lapan (2006, Proposition A) and Kimball and Weil (2009, p.268-269). The necessary and sufficient condition for this property can be traced back to Hardy *et al.* (1934, Theorem 106). In order to state this condition, we need to introduce two additional notations. Define $\mathcal{T}_u(c) \equiv -u'(c)/u''(c)$ as the coefficient of risk tolerance of $u(\cdot)$, which is the reciprocal of $\mathcal{A}_u(c)$. Similarly, define $\mathcal{T}_v(c) \equiv -v'(c)/v''(c)$. To ensure that $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are well-defined and continuously differentiable, we adopt a stronger version of Assumptions A1 and A2 in this part of the analysis.

Assumption A1′ The function $u : \mathbb{R}_+ \to \mathbb{R}$ is *thrice* continuously differentiable, strictly increasing and *strictly* concave.

Assumption A2′ The function $v : \mathbb{R}_+ \to \mathbb{R}$ is *thrice* continuously differentiable, strictly increasing and *strictly* concave.

Lemma 1 states the conditions under which (32) and (33) are valid. The first part states that the concavity of $s \mapsto \mathbb{M}_u(s;\theta)$ is equivalent to the concavity of $\mathcal{T}_u(\cdot)$. This is essentially the same result that appears in the aforementioned studies. The second part states that if both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are concave functions, then the concavity of $\mathbb{M}_v(\cdot)$ can be ensured. This is an extension of the result in part (i).[30] A complete detailed proof of Lemma 1 can be found in Appendix B.

Lemma 1 *Suppose Assumptions A1′ and A2′ are satisfied.*

(i) *For any $\theta \in \Theta$, $\mathbb{M}_u(s;\theta)$ is a concave function in s if and only if $\mathcal{T}_u(\cdot)$ is a concave function.*

(ii) *If both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are concave functions, then $\mathbb{M}_v(\cdot)$ is a concave function.*

The assumption of a concave risk tolerance may sound restrictive at first, but it is satisfied by the commonly used hyperbolic-absolute-risk-aversion (HARA) class of utility functions [e.g., quadratic utility functions, constant-absolute-risk-aversion (CARA) utility functions and constant-relative-risk-aversion (CRRA) utility functions]. Recall that a defining feature of HARA utility functions is a linear (hence weakly concave) risk tolerance. Thus, according to Lemma 1, if both $u(\cdot)$ and $v(\cdot)$ belong to the HARA class, then $\mathbb{M}_v(\cdot)$ is a concave function. Our next result provides a second set of sufficient conditions under which $s_A^* \geq s_D^*$.

Theorem 2 *Suppose Assumptions A1′, A2′ and A3 are satisfied.*

(i) *If both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are concave, then (P1) has a unique solution denoted by s_A^*.*

(ii) *If, in addition, both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are increasing, then $s_A^* \geq s_D^*$.*

There are two fundamental differences between the conditions in Theorem 1 and those in Theorem 2. On the one hand, Theorem 2 does not require any specific ranking of $\mathcal{A}_V(\cdot)$, $\mathcal{A}_v(\cdot)$ and $\mathcal{A}_u(\cdot)$. This means there is no restriction on the consumer's ambiguity attitudes, and it does

[30] The concavity of $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ here means that both $\mathcal{T}_u'(\cdot)$ and $\mathcal{T}_v'(\cdot)$ are decreasing functions. In particular, $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ need not be twice differentiable globally. Hence, we only need to consider up to the third-order derviative of $u(\cdot)$ and $v(\cdot)$.

not require a preference for late resolution of temporal risk under SKP preferences. On the other hand, Theorem 2 requires $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ to be increasing concave functions, which are stronger than the assumption of convex $u'(\cdot)$ and $v'(\cdot)$ in Theorem 1.

We now explain the role played by each of the conditions in Theorem 2. First of all, the concavity of $\mathcal{T}_u(\cdot)$ *and* $\mathcal{T}_v(\cdot)$ are only required to prove the uniqueness of s_A^*. These conditions are not used in the proof of part (ii). Under Approach #2, we set $\Phi(\cdot) = V(\cdot)$, $g_A(\cdot) = \mathbb{M}_v(\cdot)$ and $g_D(s) = (\mu + Rs)$ in (19). Since $V(\cdot)$ is strictly increasing and strictly concave by Assumption A3, $s_A^* \geq s_D^*$ is true if

$$\mathbb{M}_v(s) \leq \mu + Rs \qquad \text{and} \qquad \mathbb{M}_v'(s) \geq R,$$

for any $s \in [-\underline{b}, w]$, where $\mathbb{M}_v'(s)$ is the derivative of $\mathbb{M}_v(s)$ with respect to s. The first inequality follows immediately from the concavity of $u(\cdot)$ and $v(\cdot)$. The second inequality is valid if both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are increasing, i.e., both $u(\cdot)$ and $v(\cdot)$ exhibit decreasing absolute risk aversion (DARA).

Under this approach, the first-order condition of (P1) is given by

$$\begin{aligned} V'(w-s) &\geq \beta V'[\mathbb{M}_v(s)]\,\mathbb{M}_v'(s) \\ &= \underbrace{\beta\frac{V'[\mathbb{M}_v(s)]}{V'(\mu+Rs)}}_{\text{Discount Factor Effect}} \times \underbrace{\frac{\mathbb{M}_v'(s)}{R}}_{\text{Combined DARA Effect}} \times \underbrace{RV'(\mu+Rs)}_{\text{MB of saving under certainty}}. \end{aligned} \tag{34}$$

Similar to (27), the right side of (34) expresses the marginal benefit of saving under ambiguity as the product of the marginal benefit under certainty and two additional factors. The first one is again labelled as "discount factor effect," albeit it is defined differently from the one in Approach #1. For any $s \in [-\underline{b}, w]$, define $\widehat{\beta}(s)$ according to

$$\widehat{\beta}(s) \equiv \beta\frac{V'[\mathbb{M}_v(s)]}{V'(\mu+Rs)}.$$

If $V(\cdot)$ is a linear function, or if the consumer is ambiguity-neutral and risk-neutral so that $\mathbb{M}_v(s) = \mu + Rs$, then $\widehat{\beta}(\cdot) \equiv \beta$. But if $u(\cdot)$, $v(\cdot)$ and $V(\cdot)$ are all strictly concave functions as required in Theorem 2, then $\widehat{\beta}(s) > \beta$.

The second factor in (34) is due to the DARA property of $u(\cdot)$ and $v(\cdot)$. If both $u(\cdot)$ and $v(\cdot)$ are CARA utility functions, then this term will be reduced to one, i.e., $\mathbb{M}_v'(s) = R$. But if both $u(\cdot)$ and $v(\cdot)$ are DARA utility functions, then $\mathbb{M}_v'(s) \geq R$ for all $s \in [-\underline{b}, w]$. The reason is as follows: By saving more in the current period, the consumer can expect to have higher future consumption in all possible states under all plausible first-order distributions of future income. If $u(\cdot)$ exhibits DARA, then such an increase will lower the consumer's risk aversion in the future period and raise the certainty equivalent $\mathbb{M}_u(s;\theta)$. Saving more also means that risky income $(\widetilde{y})$ will become less important than accumulated wealth (Rs) in the future period. Thus, when s is sufficiently large, $\mathbb{M}_u(s;\theta)$ will catch up with its deterministic counterpart $\widetilde{\mu}(\theta)+Rs$ from below [since risk aversion implies $\mathbb{M}_u(s;\theta)\leq\widetilde{\mu}(\theta)+Rs$]. This is possible only if $\mathbb{M}_u(s;\theta)$ is increasing at a faster rate than $\widetilde{\mu}(\theta)+Rs$ when s increases, i.e.,

$$\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta)\geq R, \qquad \text{for any given } \theta\in\Theta. \tag{35}$$

By the same token, if $v(\cdot)$ exhibits DARA, then as each $\mathbb{M}_u(s;\theta)$ increases the consumer will become less averse to the randomness in $\{\mathbb{M}_u(s;\theta)\mid\theta\in\Theta\}$. As a result, $\mathbb{M}_v(s)$ will catch up with its deterministic counterpart $(\mu+Rs)$ from below by increasing at a faster rate, i.e., $\mathbb{M}_v'(s)\geq R$, for any $s\in[-\underline{b},w]$. Intuitively, $\mathbb{M}_v'(s)$ can be interpreted as the marginal gain in risk-free future consumption due to an increase in s, while R is the counterpart in the deterministic environment. The condition $\mathbb{M}_v'(s)\geq R$ thus implies that it is more rewarding to save under ambiguity.

Theorem 2 encompasses at least three special cases which are of interest. First, as shown in (13), if the consumer is ambiguity-neutral, then the lifetime utility function in (8) is observationally equivalent to SKP preferences under a pure future income risk that is drawn according to $H(\widetilde{y})$. In this case, a unique solution of (P1) exists if both $V(\cdot)$ and $\mathcal{T}_u(\cdot)$ exhibits concavity; and precautionary saving exists if $u(\cdot)$ exhibits DARA. These are the same conditions stated in Kimball and Weil (2009, Propositions 1 and Proposition A3). Second, as shown in (12), if $V(\cdot)\equiv u(\cdot)$ then the lifetime utility function in (8) becomes the recursive smooth ambiguity aversion preferences developed by Klibanoff *et al.* (2009). Thus, according to Theorem 2, if both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are increasing concave functions then precautionary saving exists under this type of preferences. Third, since Theorem 2 does not require any ranking of $\mathcal{A}_v(c)$ and $\mathcal{A}_u(c)$, precautionary saving can exist even if the consumer is ambiguity-loving, i.e., $\mathcal{A}_v(\cdot)<\mathcal{A}_u(\cdot)$. This shows that ambiguity aversion is not necessary for $s_A^*\geq s_D^*$.

Approach #3

We now present the third approach in characterising precautionary saving behaviour under ambiguity. Using this approach, we are able to derive conditions under which the consumer is both risk-prudent and ambiguity-prudent, i.e., $s_A^*\geq s_R^*\geq s_D^*$.

Under Approach #3, the objective function in (P1) is expressed as

$$\Delta_A(s) \equiv V(w-s) + \beta\Gamma[\mathbb{M}_\phi(s)], \tag{36}$$

where $\Gamma(\cdot) \equiv V \circ u^{-1}(\cdot)$ and $\mathbb{M}_\phi(\cdot)$ is as defined in (10). Given Assumption A3, $\Delta_A(\cdot)$ is strictly concave if (i) $\Gamma(\cdot)$ is increasing concave and (ii) the mapping $s \mapsto \mathbb{M}_\phi(s)$ exhibits concavity. Assumptions A1 and A3 ensure that $\Gamma(\cdot)$ is strictly increasing. It is concave if and only if $V(\cdot)$ is more concave than $u(\cdot)$, i.e., $\mathcal{A}_V(\cdot) \geq \mathcal{A}_u(\cdot)$. Using the same line of argument as in Lemma 1, $\mathbb{M}_\phi(\cdot)$ is concave if the reciprocal of absolute ambiguity aversion $\mathcal{T}_\phi(\cdot) \equiv -\phi'(\cdot)/\phi''(\cdot)$ is a concave function. This result is formally stated in Lemma 2, the proof of which can be found in Appendix B.

Lemma 2 *Suppose Assumptions A1, A2′ and $\mathcal{A}_v(\cdot) > \mathcal{A}_u(\cdot)$ are satisfied so that $\phi''(\cdot) < 0$. Then $\mathbb{M}_\phi(\cdot)$ is a concave function if $\mathcal{T}_\phi(\cdot)$ is concave.*

The objective function in (P2) and (P3) can be similarly rewritten as

$$\Delta_D(s) \equiv V(w-s) + \beta\Gamma[u(\mu + Rs)],$$

$$\Delta_R(s) \equiv V(w-s) + \beta\Gamma\left[\int_\Omega u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right].$$

Assumptions A1 and A3, together with a concave $\Gamma(\cdot)$, are enough to ensure that $\Delta_D(\cdot)$ and $\Delta_R(\cdot)$ are strictly concave.

Our next result provides a set of sufficient conditions under which $s_A^* \geq s_R^* \geq s_D^*$. Unlike our first two theorems, Theorem 3 requires an additional condition on the set of prior distributions, which is stated in Assumption A4.

Assumption A4 For any continuous, increasing function $\eta : \Omega \to \mathbb{R}$, the expected value $\int_\Omega \eta(\widetilde{y})\, dF(\widetilde{y} \mid \theta)$ is increasing in θ on Θ.

Assumption A4 is equivalent to saying that $F(\widetilde{y} \mid \theta_1)$ is first-order stochastically dominated by $F(\widetilde{y} \mid \theta_2)$ for any $\theta_1 < \theta_2$ in Θ. To use the terminology of Topkis (1998, Section 3.9.2), $F(\widetilde{y} \mid \theta)$ is stochastically increasing in θ on Θ and $\mathcal{F} \equiv \{F(\widetilde{y} \mid \theta) : \theta \in \Theta\}$ is a collection of stochastically increasing distribution functions. Similar assumptions have been used by Osaki and Schlesinger (2014, Section 5), Berger (2014), Peter (2019) and Wang and Li (2020). Assumption A4 has two immediate implications: Fix $s \in [-\underline{b}, w]$. Since $u(\widetilde{y} + Rs)$ is increasing in $\widetilde{y}$, this assumption ensures that $U(s;\theta) \equiv \int_S u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)$ is an increasing function in θ. By a similar token, since $u'(\widetilde{y} + Rs)$ is decreasing in $\widetilde{y}$, $U_s(s;\theta) \equiv \int_S Ru'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)$ is a decreasing function in θ. The relevance of these properties will be explained later.

Theorem 3 *Suppose Assumptions A1, A2′and A3 are satisfied.*

(i) *If $\mathcal{A}_V(\cdot) \geq \mathcal{A}_u(\cdot)$ and $\mathcal{T}_\phi(\cdot)$ is a concave function, then (P1) has a unique solution denoted by s_A^*.*

(ii) *If, in addition, $\mathcal{A}_v(\cdot) > \mathcal{A}_u(\cdot)$, $\mathcal{T}_\phi(\cdot)$ is an increasing function, $u'(\cdot)$ is convex, and Assumption A4 is satisfied, then $s_A^* \geq s_R^* \geq s_D^*$.*

Similar to the concavity of $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ in Theorem 2, the concavity of $\mathcal{T}_\phi(\cdot)$ is only required in the proof of part (i). Theorem 3 identifies six conditions that are sufficient to establish both risk prudence and ambiguity prudence. This includes (A) the concavity of $\Gamma(\cdot)$, i.e., $\mathcal{A}_V(\cdot) \geq \mathcal{A}_u(\cdot)$; (B) $u(\cdot)$ is strictly increasing and concave; (C) strict ambiguity aversion, i.e., $\mathcal{A}_v(\cdot) > \mathcal{A}_u(\cdot)$;[31] (D) decreasing absolute ambiguity aversion, i.e., $\mathcal{T}_\phi(\cdot)$ is increasing; (E) risk prudence as in Kimball (1990), i.e., $u'(\cdot)$ is convex; and (F) Assumption A4. We now explain the role played by each of these conditions.

To start, the risk prudence result (i.e., $s_R^* \geq s_D^*$) holds if the marginal benefit of saving in (P3) exceed that in (P2), i.e.,

$$\Gamma'\left[\int_\Omega u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right] \int_\Omega u'(\widetilde{y} + Rs)\, dH(\widetilde{y}) \geq \Gamma'[u(\mu + Rs)]\, u'(\mu + Rs).$$

Provided that $\Gamma(\cdot)$ is concave (condition A), the above inequality is valid if

$$\int_\Omega u(\widetilde{y} + Rs)\, dH(\widetilde{y}) \leq u(\mu + Rs) \quad \text{and} \quad \int_\Omega u'(\widetilde{y} + Rs)\, dH(\widetilde{y}) \geq u'(\mu + Rs).$$

The first inequality is valid if and only if $u(\cdot)$ exhibits risk aversion (condition B). The second inequality holds if and only if $u'(\cdot)$ is convex (condition E). Thus, conditions A, B and E in Theorem 3 are enough to establish $s_R^* \geq s_D^*$. This is essentially the risk prudence result of Gollier (2001a).

[31] *Strict* ambiguity aversion is required only to ensure that $\phi''(\cdot) < 0$ so that $\mathcal{T}_\phi(\cdot)$ is well-defined.

The ambiguity prudence result can be explained in a similar manner. Specifically, $s_A^* \geq s_R^*$ holds if the marginal benefit of saving in (P1) exceed that in (P3), i.e.,

$$\Gamma' \left[\mathbb{M}_\phi (s) \right] \mathbb{M}'_\phi (s) \geq \Gamma' \left[\int_\Omega u \left(\widetilde{y} + Rs \right) dH \left(\widetilde{y} \right) \right] \int_\Omega Ru' \left(\widetilde{y} + Rs \right) dH \left(\widetilde{y} \right).$$

Given that $\Gamma(\cdot)$ is strictly increasing and concave, this condition is satisfied if $\mathbb{M}_\phi (s)$ is lower in level but more responsive to changes in s than its counterpart in (P3), which is $\int_\Omega u \left(\widetilde{y} + Rs \right) dH \left(\widetilde{y} \right)$. Formally, $s_A^* \geq s_R^*$ holds if

$$\mathbb{M}_\phi (s) \leq \int_\Omega u \left(\widetilde{y} + Rs \right) dH \left(\widetilde{y} \right) \qquad \text{and} \qquad \mathbb{M}'_\phi (s) \geq \int_\Omega Ru' \left(\widetilde{y} + Rs \right) dH \left(\widetilde{y} \right). \tag{37}$$

The first inequality follows immediately from ambiguity aversion (condition C). Since we are assuming strict ambiguity aversion, this condition will hold with strict inequality. The second inequality is the one that requires some elaboration. By saving more in the current period, the consumer can expect a higher value $U(s;\theta)$ under all plausible first-order distributions. This will in turn raise the value of $\mathbb{M}_\phi (s)$. Formally, the derivative $\mathbb{M}'_\phi (s)$ is given by

$$\mathbb{M}'_\phi (s) = \frac{\int_\Theta \phi' \left[U(s;\theta) \right] U_s (s;\theta) \, dG(\theta)}{\int_\Theta \phi' \left[U(s;\theta) \right] dG(\theta)} \times \underbrace{\frac{\int_\Theta \phi' \left[U(s;\theta) \right] dG(\theta)}{\phi' \left[\mathbb{M}_\phi (s) \right]}}_{\text{DAAA Effect}}. \tag{38}$$

The first term on the right side of (38) captures the effect of s on a weighted average of $\{U(s;\theta) \mid \theta \in \Theta\}$, where the weights are determined by the consumer's ambiguity preferences $\phi(\cdot)$ and second-order beliefs $G(\cdot)$. The second term is related to the shape of the absolute ambiguity aversion coefficient $\mathcal{A}_\phi (\cdot)$. We start with the meaning of the second term. Consider a hypothetical scenario in which each $U(s,\theta)$ is increased by the *same* infinitesimal amount $\varepsilon > 0$. This will raise the certainty equivalent of $\{U(s;\theta) \mid \theta \in \Theta\}$ derived under $\phi(\cdot)$ by

$$\frac{d}{d\varepsilon} \phi^{-1} \left\{ \int_\Theta \phi \left[U(s;\theta) + \varepsilon \right] dG(\theta) \right\} \bigg|_{\varepsilon=0} = \frac{\int_\Theta \phi' \left[U(s;\theta) \right] dG(\theta)}{\phi' \left[\mathbb{M}_\phi (s) \right]}.$$

If $\phi(\cdot)$ exhibits DAAA (condition D), then such an increase will make the consumer less ambiguity-averse. This will narrow the gap between $\phi^{-1} \left\{ \int_\Theta \phi \left[U(s;\theta) \right] dG(\theta) \right\}$ and its counterpart in (P3), i.e., $\int_\Omega u(\widetilde{y}+Rs)\,dH(\widetilde{y})$. In other words, for any $\varepsilon > 0$,

$$\begin{aligned} & \int_\Omega u(\widetilde{y}+Rs)\,dH(\widetilde{y}) - \phi^{-1}\left\{\int_\Theta \phi\left[U(s;\theta)\right]dG(\theta)\right\} \\ \geq & \int_\Omega u(\widetilde{y}+Rs)\,dH(\widetilde{y}) + \varepsilon - \phi^{-1}\left\{\int_\Theta \phi\left[U(s;\theta)+\varepsilon\right]dG(\theta)\right\}. \end{aligned}$$

Rearranging terms and dividing both sides by $\varepsilon > 0$ gives

$$\frac{1}{\varepsilon}\left[\phi^{-1}\left\{\int_\Theta \phi\left[U(s;\theta)+\varepsilon\right]dG(\theta)\right\} - \phi^{-1}\left\{\int_\Theta \phi\left[U(s;\theta)\right]dG(\theta)\right\}\right] \geq 1.$$

By taking the limit $\varepsilon \to 0$, this becomes

$$\frac{\int_\Theta \phi'\left[U(s;\theta)\right]dG(\theta)}{\phi'\left[\mathbb{M}_\phi(s)\right]} \geq 1. \tag{39}$$

Thus, the DAAA effect in (38) tells us how much $\mathbb{M}_\phi(s)$ will increase when all $U(s;\theta)$ increase by the same amount.

We now explain the first term on the right side of (38), which captures the differential effect of s across different $U(s;\theta)$. We first rewrite it as

$$\frac{\int_\Theta \phi'\left[U(s;\theta)\right]U_s(s;\theta)\,dG(\theta)}{\int_\Theta \phi'\left[U(s;\theta)\right]dG(\theta)} = \int_\Theta U_s(s;\theta)\,\sigma(s;\theta)\,dG(\theta),$$

where $\sigma(s;\theta)$ is a Radon-Nikodym derivative defined by

$$\sigma(s;\theta) \equiv \frac{\phi'\left[U(s;\theta)\right]}{\int_\Theta \phi'\left[U(s;\theta)\right]dG(\theta)}, \quad \text{with} \quad \int_\Theta \sigma(s;\theta)\,dG(\theta) = 1.$$

If the consumer is ambiguity-neutral, then $\phi'(\cdot)$ is a positive constant and $\sigma(s;\theta) = 1$ for all $(s;\theta) \in [-\underline{b}, w] \times \Theta$. In this case, the first term on the right side of (38) is simply the expected value of $U_s(s;\theta)$ under the second-order distribution $G(\cdot)$. But if the consumer is strictly ambiguity-averse (condition C), then $\sigma(s;\theta)$ is not a constant in general. Holding s fixed, a higher value of $\sigma(s;\theta)$ means that the corresponding first-order distribution $F(\widetilde{y} \mid \theta)$ is more important in the consumer's decision process.[32] To explain this more precisely, pick any θ_1 and θ_2 in Θ so that $\theta_2 > \theta_1$. Assumption A4 implies that $F(\widetilde{y} \mid \theta_1)$ is less desirable than $F(\widetilde{y} \mid \theta_2)$ under the first-order stochastic dominance (FOSD) criterion. In other words, it is less likely to draw a high value of

[32] A similar Radon-Nikodym derivative also appears in Gollier [2011, Equation (8)]. Gollier interprets this derivative as a distortion to the consumer's second-order beliefs.

future income under $F(\widetilde{y} \mid \theta_1)$ than under $F(\widetilde{y} \mid \theta_2)$. Hence, the first-order expected utility $U(s;\theta)$ is lower under θ_1 than under θ_2. This, together with ambiguity aversion, implies $\phi'[U(s;\theta_1)] \geq \phi'[U(s;\theta_2)]$ and $\sigma(s;\theta_1) \geq \sigma(s;\theta_2)$. At the same time, since savings are more valued when future income is low, the marginal benefit of saving is greater under $F(\widetilde{y} \mid \theta_1)$ than under $F(\widetilde{y} \mid \theta_2)$, i.e., $U_s(s;\theta_1) \geq U_s(s;\theta_2)$. Thus, when the consumer is deciding how much to save, she will put greater importance on those first-order distributions that yield a higher marginal benefit of saving. Consequently, the weighted average $\int_\Theta U_s(s;\theta)\,\sigma(s;\theta)\,dG(\theta)$ will be greater than the expected value $\int_\Theta U_s(s;\theta)\,dG(\theta)$, i.e.,

$$\int_\Theta U_s(s;\theta)\,\sigma(s;\theta)\,dG(\theta) \geq \int_\Theta U_s(s;\theta)\,dG(\theta) = \int_\Omega Ru'(\widetilde{y}+Rs)\,dH(\widetilde{y}). \tag{40}$$

Another way to derive (40) is by considering the covariance between $\phi'[U(s;\theta)]$ and $U_s(s;\theta)$. Since both of them are decreasing functions in θ under Assumption A4, they are comonotone and have a positive covariance, i.e.,

$$\int_\Theta \phi'[U(s;\theta)]\,U_s(s;\theta)\,dG(\theta) \geq \left[\int_\Theta \phi'[U(s;\theta)]\,dG(\theta)\right]\left[\int_\Theta U_s(s;\theta)\,dG(\theta)\right].$$

The condition in (40) can be obtained by rearranging terms.

The inequalities in (39) and (40) together establish the second inequality in (37). From this discussion, it is clear that the only reason of having Assumption A4 is to generate a positive covariance between $\phi'[U(s;\theta)]$ and $U_s(s;\theta)$. Berger (2014, Proposition 2) and Osaki and Schlesinger (2014, p.14-15) provide other assumptions can also achieve the same effect.

Our Theorem 3 is similar to the first part of Proposition 2 in Osaki and Schlesinger (2014), but there are three non-trivial differences: First, Osaki and Schlesinger do not specify the conditions under which the first-order condition of the consumer's problem is sufficient to identify the optimal level of savings. These conditions are explicitly stated in our Theorem 3. Second, they focus on the recursive smooth ambiguity preferences developed by Klibanoff *et al.* (2009), which corresponds to the case when $V(\cdot) \equiv u(\cdot)$. Our Theorem 3 extends this to the more general case in which $V(\cdot)$ is an increasing concave transformation of $u(\cdot)$. Finally, Osaki and Schlesinger do not explore the connection among risk prudence, ambiguity prudence and mixed prudence.

4 Further Results

4.1 More on Ambiguity Prudence

A natural follow-up question to Theorem 3 is whether we can establish the ambiguity prudence result without imposing any restrictions on the set of first-order distributions, as in our first two approaches. In this subsection, we show that this is possible but at the expense of the generality of $u(\cdot)$. The result is formally stated in Theorem 4, which shares a resemblance with Proposition 1 in Berger (2014). Using the recursive smooth ambiguity preferences of Klibanoff *et al.* (2009) [i.e., when $V(\cdot) \equiv u(\cdot)$ in (8)], Berger shows that the consumer is ambiguity-prudent if (i) $\phi(\cdot)$ is increasing and concave, (ii) $\phi(\cdot)$ exhibits DAAA, and (iii) $u(\cdot)$ is either a linear function or an exponential function (i.e., CARA). Using a modified version of Approach #2 (which now requires ambiguity aversion), our Theorem 4 generalises Berger's result in two ways: First, we generalise his result to the case when $V(\cdot) \neq u(\cdot)$. Second, we show that DAAA is overly sufficient for ambiguity prudence. Instead, this can be replaced by a weaker condition, which is DARA of $v(\cdot)$. To see why this is true, differentiate $\mathcal{A}_\phi(u(c))$ with respect to c to get

$$\mathcal{A}'_\phi(x) = \frac{1}{u'(c)}\left[\mathcal{A}'_v(c) - \mathcal{A}'_u(c)\right] + \mathcal{A}_u(c)\,\mathcal{A}_\phi(x),$$

where $x = u(c)$. Suppose $\phi(\cdot)$ exhibits ambiguity aversion so that $\mathcal{A}_\phi(\cdot) \geq 0$. Then $\mathcal{A}'_v(\cdot) \leq 0$ and $\mathcal{A}'_u(\cdot) = 0$ (CARA utility) does not necessarily imply $\mathcal{A}'_\phi(\cdot) \leq 0$. But if $\mathcal{A}'_\phi(\cdot) \leq 0$ and $\mathcal{A}'_u(\cdot) = 0$, then $\mathcal{A}'_v(\cdot) \leq 0$ must be true.

Theorem 4 *Suppose $u(\cdot)$ exhibits constant absolute risk aversion so that $\mathcal{T}_u(c) = \overline{T}_u > 0$ for all $c \geq 0$, and Assumptions A2′ and A3 are satisfied.*

(i) *If $\mathcal{T}_v(\cdot)$ is concave, then both (P1) and (P3) have a unique solution denoted by s^*_A and s^*_R, respectively.*

(ii) *If, in addition, $\mathcal{T}_v(\cdot)$ is an increasing function and $\phi(\cdot)$ is concave, then $s^*_A \geq s^*_R$.*

The CARA assumption is useful because under this type of preferences, $\mathbb{M}_u(s;\theta)$ and the certainty equivalent under $H(\cdot)$, i.e., $u^{-1}\left[\int u(\widetilde{y} + Rs)\,dH(\widetilde{y})\right]$, are both linear functions in s. Given that $V(\cdot)$ is strictly increasing and strictly concave, $s^*_A \geq s^*_R$ holds if

$$\mathbb{M}_v(s) \leq u^{-1}\left[\int u(\widetilde{y} + Rs)\,dH(\widetilde{y})\right] \qquad \text{and} \qquad \mathbb{M}'_v(s) \geq 1.$$

The first inequality is valid under ambiguity aversion. The second inequality can be established using the DARA property of $v(\cdot)$ alone.

Finally, since $V(\cdot)$ is strictly increasing and strictly concave, and $u(\cdot)$ exhibits CARA, the risk prudence result of Kimball and Weil (2009) remains valid here. Thus, the conditions in Theorem 4 also guarantee that $s_R^* \geq s_D^*$.

4.2 Non-Time-Separable Utility

In this subsection, we show that the results in Theorem 2 can be readily extended to a non-time-separable version of GRSA preferences. Suppose now the consumer's attitudes toward intertemporal substitution is captured by a general aggregator function $W : \mathbb{R}_+^2 \to \mathbb{R}$. Let $W_i(c_1, c_2)$ denote the partial derivative of $W(\cdot)$ with respect to the ith argument, for $i \in \{1, 2\}$, and $W_{ij}(c_1, c_2)$ be the partial derivative of $W_i(\cdot)$ with respect to the jth argument, for $i, j \in \{1, 2\}$. The properties of W are stated in Assumption A5, which will replace Assumption A3.

Assumption A5 The function $W : \mathbb{R}_+^2 \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and jointly strictly concave in both arguments. In addition, $W_1(c_1, c_2)/W_2(c_1.c_2)$ is increasing in c_2 for any $c_1 \geq 0$, and $\lim_{c_1 \to 0} W_1(c_1, c_2) = \infty$ for all $c_2 \geq 0$.

The main assumption here is that the marginal rate of substitution $W_1(c_1, c_2)/W_2(c_1.c_2)$ is increasing in c_2 for all $c_1 \geq 0$. This is true if and only if

$$\frac{W_{12}(c_1, c_2)}{W_2(c_1, c_2)} - \frac{W_1(c_1, c_2) W_{22}(c_1, c_2)}{[W_2(c_1, c_2)]^2} \geq 0, \tag{41}$$

for all $(c_1, c_2) \in \mathbb{R}_+^2$. Intuitively, this means any increase in c_2 will increase the marginal benefit of c_1 relative to that of c_2. Holding other things constant, this will encourage the consumer to substitute c_2 for c_1. A sufficient condition for (41) is $W_{12}(c_1, c_2) \geq 0$, for all $(c_1, c_2) \in \mathbb{R}_+^2$. The limit condition in Assumption A5 ensures that it is never optimal to choose $c_1 = 0$.

The consumer's lifetime utility is now given by $W[c_1, \mathbb{M}_v(s)]$ and the consumption-saving problem under ambiguity can be expressed as

$$\max_{s \in [-\underline{b}, w]} \{W[w - s, \mathbb{M}_v(s)]\}. \tag{P4}$$

The deterministic version of (P4) is given by

$$\max_{s\in[-\underline{b},w]}\left\{W\left(w-s,\mu+Rs\right)\right\}. \tag{P5}$$

Since $W(\cdot)$ is continuous and jointly strictly concave in its arguments, if $\mathbb{M}_v(s)$ satisfies the concavity condition in (32), then the objective function in (P4) is continuous and strictly concave in s. This ensures the existence of a unique solution for (P4), denoted by $\widetilde{s}_A^*$. The first-order condition of (P4) can be expressed as

$$\frac{W_1\left[w-s,\mathbb{M}_v(s)\right]}{W_2\left[w-s,\mathbb{M}_v(s)\right]}\geq\mathbb{M}_v'(s),$$

which holds with equality if $\widetilde{s}_A^*>-\underline{b}$. As for (P5), Assumption A5 alone is enough to ensure the existence of a unique solution, denoted by $\widetilde{s}_D^*$. The first-order condition of (P5) implies

$$\frac{W_1\left(w-s,\mu+Rs\right)}{W_2\left(w-s,\mu+Rs\right)}\geq R,$$

which holds with equality if $\widetilde{s}_D^*>-\underline{b}$.

The consumer is mixed-prudent, i.e., $\widetilde{s}_A^*\geq\widetilde{s}_D^*$, if

$$\frac{W_1\left(w-s,\mu+Rs\right)}{W_2\left(w-s,\mu+Rs\right)}\geq\frac{W_1\left[w-s,\mathbb{M}_v(s)\right]}{W_2\left[w-s,\mathbb{M}_v(s)\right]}\geq\mathbb{M}_v'(s)\geq R, \tag{42}$$

for all $s\in[-\underline{b},w]$. Since $W_1(c_1,c_2)/W_2(c_1.c_2)$ is increasing in c_2, the first inequality in (42) is equivalent to

$$\mu+Rs\geq\mathbb{M}_v(s).$$

The intuition is as follows: Under Assumption A5, a higher future consumption in the deterministic environment will encourage the consumer to substitute future consumption (c_2) for more current consumption (c_1). This discourages saving in the deterministic environment. On the other hand, $\mathbb{M}_v'(s)\geq R$ means that it is more rewarding to save under ambiguity. These two forces together ensure $\widetilde{s}_A^*\geq\widetilde{s}_D^*$.

Thus, similar to Approach #2 in Section 3, the consumer is mixed-prudent if $\mathbb{M}_v(s)$ is lower in level but more sensitive to changes in s than its deterministic counterpart. From this point onward, the proof of $\widetilde{s}_A^*\geq\widetilde{s}_D^*$ and its interpretation are exactly the same as in Theorem 2. Thus, by replacing Assumption A3 with A5, we can generalise Theorem 2 to a non-time-separable lifetime utility function. This is formally stated as Theorem 5 (the proof is omitted).

Theorem 5 *Suppose Assumptions A1′, A2′* and *A5 are satisfied.*

(i) *If both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are concave, then both (P4) and (P5) have a unique solution, denoted by $\widetilde{s}_A^*$ and $\widetilde{s}_D^*$, respectively.*

(ii) *If, in addition, both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are increasing, then $\widetilde{s}_A^* \geq \widetilde{s}_D^*$.*

5 Conclusions

In this paper we adopt a two-period model to analyse precautionary saving behaviour under ambiguity. Our goal is to better understand the conditions that lead to precautionary saving in this setting. To this end, we adopt the generalised recursive smooth ambiguity (GRSA) preferences of Hayashi and Miao (2011) and distinguish between two types of precautionary saving motives under ambiguity, namely mixed prudence and ambiguity prudence. Our first two major results show a close connection between risk-prudence under Selden/Kreps-Porteus preferences and precautionary saving under GRSA preferences. In particular, these results do not require any stochastic ordering on the first-order probability distributions of future income. This type of ordering, however, is needed in the analysis of ambiguity prudence.

We believe the methodology developed in this paper can also be useful in two other directions of research. The first one is comparative statics analysis. For instance, under what conditions will a more ambiguity-averse consumer save more in the presence of ambiguity? The second direction is to analyse precautionary saving behaviour under other types of ambiguous risks, e.g., interest rate risk. We plan to pursue these in future work.

Appendix A

A1. Preliminaries

This subsection serves two purposes. First, it collects some preliminary known results that are useful for subsequent proofs. Second, it verifies certain claims that we have made in the main text.

Properties of $\mathbb{M}_u(s;\theta)$**.** For any $(s;\theta) \in [-\underline{b}, w] \times \Theta$, $\mathbb{M}_u(s;\theta)$ is implicitly defined by

$$u[\mathbb{M}_u(s;\theta)] \equiv \int_{\Omega} u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta). \tag{A.1}$$

If $u(\cdot)$ is concave, then

$$u[\mathbb{M}_u(s;\theta)] \equiv \int_{\Omega} u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta) \leq u[\widetilde{\mu}(\theta) + Rs],$$

where $\widetilde{\mu}(\theta)$ is the expected value of $\widetilde{y}$ under $F(\widetilde{y} \mid \theta)$. Hence, we have $\mathbb{M}_u(s;\theta) \leq \widetilde{\mu}(\theta) + Rs$. Differentiating both sides of (A.1) with respect to s and rearranging terms give

$$\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta) = \frac{R}{u'[\mathbb{M}_u(s;\theta)]} \int_{\Omega} u'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta) > 0. \tag{A.2}$$

The denominator on the right side is non-zero as $u(\cdot)$ is strictly increasing, hence the partial derivative is well-defined.

If $u(\cdot)$ exhibits decreasing absolute risk aversion (DARA), then $-u'(\cdot)$ is a concave transformation of $u(\cdot)$, or in other words, $-u' \circ u^{-1}(\cdot)$ is a concave function (Gollier, 2001a, p.25). It follows that

$$-u'[\mathbb{M}_u(s;\theta)] = -u' \circ u^{-1}\left[\int_{\Omega} u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)\right] \geq -\int_{\Omega} u'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)$$

$$\Rightarrow u'[\mathbb{M}_u(s;\theta)] \leq \int_{\Omega} u'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta). \tag{A.3}$$

Equations (A.2) and (A.3) together imply that, if $u(\cdot)$ is a DARA utility function, then $\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta) \geq R$, for all $(s,\theta) \in [-\underline{b}, w] \times \Theta$. By the same argument, if $u(\cdot)$ exhibits increasing absolute risk aversion [*resp.*, constant absolute risk aversion (CARA)], then $\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta) \leq R$ [*resp.*, $\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta) = R$].

Properties of $\mathbb{M}_v(s)$ For any $s \in [-\underline{b}, w]$, the second-order certainty equivalent $\mathbb{M}_v(s)$ is implicitly defined by

$$v[\mathbb{M}_v(s)] \equiv \int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta). \tag{A.4}$$

If both $u(\cdot)$ and $v(\cdot)$ are concave, then by Jensen's inequality

$$\begin{aligned} v[\mathbb{M}_v(s)] &\equiv \int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta) \leq v\left[\int_\Theta \mathbb{M}_u(s;\theta)\, dG(\theta)\right] \\ &\leq v\left\{\int_\Theta [\widetilde{\mu}(\theta) + Rs]\, dG(\theta)\right\}. \end{aligned}$$

The second line uses the fact that $\mathbb{M}_u(s;\theta) \leq \widetilde{\mu}(\theta) + Rs$ for all (s,θ). Hence, we have

$$\mathbb{M}_v(s) \leq \int_\Theta [\widetilde{\mu}(\theta) + Rs]\, dG(\theta) \equiv \mu + Rs. \tag{A.5}$$

If both $u(\cdot)$ and $v(\cdot)$ are strictly concave, then (A.5) will hold with strict inequality. Differentiating both sides of (A.4) with respect to s and rearranging terms give

$$\mathbb{M}_v'(s) = \frac{1}{v'[\mathbb{M}_v(s)]} \int_\Theta v'[\mathbb{M}_u(s;\theta)] \left[\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta)\right] dG(\theta).$$

The expression on the right is well-defined as $v'[\mathbb{M}_v(s)] \neq 0$. Substituting (A.2) into the above equation gives

$$\mathbb{M}_v'(s) = \frac{R}{v'[\mathbb{M}_v(s)]} \int_\Theta \frac{v'[\mathbb{M}_u(s;\theta)]}{u'[\mathbb{M}_u(s;\theta)]} \left[\int_\Omega u'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)\right] dG(\theta) > 0, \tag{A.6}$$

which is strictly positive if $u(\cdot)$ and $v(\cdot)$ are strictly increasing. The same expression can be obtained if we write (A.4) as

$$v[\mathbb{M}_v(s)] = \int_\Theta \phi[U(s;\theta)]\, dG(\theta)$$

and differentiate both sides with respect to s. In particular, if we rewrite (4) as

$$\phi'(x) = \frac{v'\left[u^{-1}(x)\right]}{u'\left[u^{-1}(x)\right]},$$

for any x in the domain of $u(\cdot)$, then we can get

$$\phi'[U(s;\theta)] = \frac{v'[\mathbb{M}_u(s;\theta)]}{u'[\mathbb{M}_u(s;\theta)]}. \tag{A.7}$$

If $v(\cdot)$ exhibits decreasing absolute risk aversion, then

$$-v'[\mathbb{M}_v(s)] = -v' \circ v^{-1}\left[\int_\Theta v[\mathbb{M}_u(s;\theta)]\, dG(\theta)\right] \geq -\int_\Theta v'[\mathbb{M}_u(s;\theta)]\, dG(\theta)$$

$$\Rightarrow v'[\mathbb{M}_v(s)] \leq \int_\Theta v'[\mathbb{M}_u(s;\theta)]\, dG(\theta). \tag{A.8}$$

Properties of $\mathbb{M}_\phi(s)$. For any $s \in [-\underline{b}, w]$, $\mathbb{M}_\phi(s)$ is implicitly defined by

$$\phi[\mathbb{M}_\phi(s)] \equiv \int_\Theta \phi[U(s;\theta)]\, dG(\theta). \tag{A.9}$$

If $\phi(\cdot)$ is increasing and concave, then by Jensen's inequality

$$\phi[\mathbb{M}_\phi(s)] \leq \phi\left[\int_\Theta U(s;\theta)\, dG(\theta)\right]$$

$$\Rightarrow \mathbb{M}_\phi(s) \leq \int_\Theta U(s;\theta)\, dG(\theta) = \int_\Omega u(\widetilde{y} + Rs)\, dH(\widetilde{y}) \leq u(\mu + Rs). \tag{A.10}$$

The last inequality follows from the concavity of $u(\cdot)$. Differentiating both sides of (A.9) with respect to s and rearranging terms give

$$\mathbb{M}'_\phi(s) = \frac{1}{\phi'[\mathbb{M}_\phi(s)]}\int_\Theta \phi'[U(s;\theta)]\, U_s(s;\theta)\, dG(\theta).$$

This derivative is well-defined and strictly positive as $\phi(\cdot)$ is strictly increasing. Using the same line of argument as in (A.3) and (A.8), one can show that if $\phi(\cdot)$ exhibits decreasing absolute ambiguity aversion (DAAA), then

$$\frac{\int_\Theta \phi'[U(s;\theta)]\, dG(\theta)}{\phi'[\mathbb{M}_\phi(s)]} \geq 1, \qquad \text{for all } s \in [-\underline{b}, w].$$

This, however, is different from $\mathbb{M}'_\phi(s) \geq 1$.

Third-Order Derivative of $\phi(\cdot)$ Suppose $u(\cdot)$ and $v(\cdot)$ [and hence $\phi(\cdot)$] are thrice differentiable. We now show that $u'''(\cdot) \geq 0$ and $v'''(\cdot) \geq 0$ do not necessarily imply $\phi'''(\cdot) \geq 0$. Recall the expression in (5), which is

$$\phi''(x) = \frac{v''(c) - \phi'(x)\, u''(c)}{[u'(c)]^2},$$

for any $x = u(c)$ and $c \geq 0$. Differentiating both sides with respect to c and rearranging terms give

$$\phi'''(x) = \frac{1}{[u'(c)]^3}\left[v'''(c) - \underbrace{\phi''(x)\, u'(c)\, u''(c)}_{(+)} - \underbrace{\phi'(x)\, u'''(c)}_{(+)}\right] - \underbrace{2\frac{u''(c)\,\phi''(x)}{[u'(c)]^2}}_{(+)}.$$

Suppose $u(\cdot)$, $v(\cdot)$ and $\phi(\cdot)$ are all strictly increasing and concave functions. Then $v'''(\cdot) = 0$ and $u'''(\cdot) \geq 0$ imply $\phi'''(\cdot) < 0$. If $v'''(\cdot) > 0$ and $u'''(\cdot) \geq 0$, then $\phi'''(\cdot)$ can be either positive-valued or negative-valued. But on the contrary, if $\phi'''(\cdot) \geq 0$ and $u'''(\cdot) \geq 0$, then $v'''(\cdot)$ must be strictly positive.

A.2 Proof of Theorem 1

Part (i) As mentioned in the main text, it suffice to show that future felicity $\Psi\left\{\int_\Theta \phi\left[U(s;\theta)\right] dG(\theta)\right\}$ is a concave function in s. Pick any s_1 and s_2 from $[-\underline{b}, w]$. For any $\alpha \in [0,1]$, define $s_\alpha \equiv \alpha s_1 + (1-\alpha)s_2$. Since monotonicity and concavity are preserved by integration, it follows from Assumption A1 that $U(s;\theta) \equiv \int_S u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)$ is strictly increasing and concave in s, for any $\theta \in \Theta$. Hence,

$$U(s_\alpha;\theta) \geq \alpha U(s_1;\theta) + (1-\alpha)U(s_2;\theta). \tag{A.11}$$

Since $\phi(\cdot)$ is strictly increasing and concave,

$$\begin{aligned}\int_\Theta \phi\left[U(s_\alpha;\theta)\right] dG(\theta) &\geq \int_\Theta \phi\left[\alpha U(s_1;\theta) + (1-\alpha)U(s_2;\theta)\right] dG(\theta) \\ &\geq \alpha \int_\Theta \phi\left[U(s_1;\theta)\right] dG(\theta) + (1-\alpha)\int_\Theta \phi\left[U(s_2;\theta)\right] dG(\theta).\end{aligned} \tag{A.12}$$

By the same token, since $\Psi(\cdot)$ is strictly increasing and concave,

$$\begin{aligned}&\Psi\left\{\int_\Theta \phi\left[U(s_\alpha;\theta)\right] dG(\theta)\right\} \\ \geq\ &\Psi\left\{\alpha\int_\Theta \phi\left[U(s_1;\theta)\right] dG(\theta) + (1-\alpha)\int_\Theta \phi\left[U(s_2;\theta)\right] dG(\theta)\right\} \\ \geq\ &\alpha\Psi\left\{\int_\Theta \phi\left[U(s_1;\theta)\right] dG(\theta)\right\} + (1-\alpha)\Psi\left\{\int_\Theta \phi\left[U(s_2;\theta)\right] dG(\theta)\right\}.\end{aligned} \tag{A.13}$$

This proves that future felicity is concave in s.

Part (ii) Set $g_A(s) = \int_\Theta \phi[U(s;\theta)]\,dG(\theta)$ and $g_D(s) = \phi \circ u(\mu + Rs)$, for all $s \in [-\underline{b}, w]$. By the concavity of $\phi(\cdot)$ and $u(\cdot)$,

$$\begin{aligned} g_A(s) &\leq \phi\left[\int_\Theta U(s;\theta)\,dG(\theta)\right] = \phi\left[\int_\Theta u(\widetilde{y} + Rs)\,dH(\widetilde{y})\right] \\ &\leq \phi \circ u\left[\int_\Theta (\widetilde{y} + Rs)\,dH(\widetilde{y})\right] = \phi \circ u(\mu + Rs). \end{aligned}$$

This proves $g_A(s) \leq g_D(s)$ for all $s \in [-\underline{b}, w]$. The derivative of $g_A(\cdot)$ is given by

$$\begin{aligned} g_A'(s) &= \int_\Theta \phi'[U(s;\theta)]\,U_s(s;\theta)\,dG(\theta) \\ &= R\int_\Theta \frac{v'[\mathbb{M}_u(s;\theta)]}{u'[\mathbb{M}_u(s;\theta)]}\int_\Omega u'(\widetilde{y} + Rs)\,dF(\widetilde{y}|\theta)\,dG(\theta). \end{aligned} \tag{A.14}$$

The second line uses (A.7) and the definition of $U(s;\theta)$. Since $g_D(s) = \phi \circ u(\mu + Rs) = v(\mu + Rs)$,

$$g_D'(s) = Rv'(\mu + Rs). \tag{A.15}$$

As shown in Section A.1, the concavity of $u(\cdot)$ implies $\mathbb{M}_u(s;\theta) \leq \widetilde{\mu}(\theta) + Rs$, for all $(s,\theta) \in [-\underline{b}, z] \times \Theta$. Since $v(\cdot)$ is more concave than $u(\cdot)$ under ambiguity aversion, the ratio $v'(c)/u'(c)$ is a decreasing function in c. Combining these two observations gives

$$\phi'[U(s;\theta)] = \frac{v'[\mathbb{M}_u(s;\theta)]}{u'[\mathbb{M}_u(s;\theta)]} \geq \frac{v'[\widetilde{\mu}(\theta) + Rs]}{u'[\widetilde{\mu}(\theta) + Rs]} > 0. \tag{A.16}$$

By the convexity of $u'(\cdot)$,

$$\int_S u'(\widetilde{y} + Rs)\,dF(\widetilde{y}|\theta) \geq u'[\widetilde{\mu}(\theta) + Rs] > 0. \tag{A.17}$$

Substituting (A.16) and (A.17) into (A.14) yields

$$g_A'(s) \geq R\int_\Theta v'[\widetilde{\mu}(\theta) + Rs]\,dG(\theta).$$

Finally, the convexity of $v'(\cdot)$ implies

$$R\int_\Theta v'[\widetilde{\mu}(\theta) + Rs]\,dG(\theta) \geq v'(\mu + Rs) = g_D'(s). \tag{A.18}$$

This establishes $g_A'(s) \geq g_D'(s)$ and completes the proof of Theorem 1.

A.3 Proof of Corollary 1

Given the quadratic utility function in (31), the marginal utility of consumption is non-negative in both time periods and in all possible states if

$$\frac{\alpha_1}{\alpha_2} \geq w + \underline{b} \geq w - s = c_1 \geq 0, \tag{A.19}$$

and

$$\frac{\alpha_1}{\alpha_2} \geq \overline{y} + Rw \geq \widetilde{y} + Rs = \widetilde{c}_2 \geq 0,$$

for all $s \in [-\underline{b}, z]$, where $\overline{y}$ is the highest possible level of future income. This explains the additional condition $\alpha_1 > \alpha_2 \max\{\overline{y} + Rw, w + \underline{b}\}$. The inequality in (A.19) implies $s \geq w - \alpha_1/\alpha_2$. Note that $\alpha_1 > \alpha_2 (w + \underline{b})$ can be rewritten as

$$-\underline{b} > w - \frac{\alpha_1}{\alpha_2}.$$

Hence, the feasible set of s remains the same as $[-\underline{b}, w]$. There are now two possible scenarios for s_A^* : either $s_A^* = w$ or $w > s_A^* \geq -\underline{b}$. In the first scenario, the desired result $s_A^* \geq s_D^*$ is trivially true. In the second scenario, the first-order condition in (22) is again valid. The rest of the proof focuses on this scenario.

Set $g_A(s) = \int_\Theta \phi[U(s;\theta)]\, dG(\theta)$ and $g_D(s) = \phi \circ u(\mu + Rs)$. If both $u(\cdot)$ and $\phi(\cdot)$ are strictly concave, then as shown in the proof of Theorem 1 part (ii) $g_A(s) < g_D(s)$ for any $s \in [-\underline{b}, w]$. Strict concavity of $u(\cdot)$ also implies

$$\mathbb{M}_u(s;\theta) < \widetilde{\mu}(\theta) + Rs, \qquad \text{for all } (s,\theta) \in [-\underline{b}, z] \times \Theta.$$

Meanwhile, a strictly concave $\phi(\cdot)$ means that $v'(c)/u'(c)$ is a strictly decreasing function in c. These two observations imply that the inequality in (A.16) will be strict, i.e.,

$$\frac{v'[\mathbb{M}_u(s;\theta)]}{u'[\mathbb{M}_u(s;\theta)]} > \frac{v'[\widetilde{\mu}(\theta) + Rs]}{u'[\widetilde{\mu}(\theta) + Rs]}.$$

Substituting this into (A.14) yields

$$g_A'(s) > R \int_\Theta \frac{v'[\widetilde{\mu}(\theta) + Rs]}{u'[\widetilde{\mu}(\theta) + Rs]} \int_\Omega u'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)\, dG(\theta) = R \int_\Theta v'[\widetilde{\mu}(\theta) + Rs]\, dG(\theta).$$

The equality uses the fact that $u'(\cdot)$ is linear, which means

$$\int_{\Omega} u'(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta) = u'[\widetilde{\mu}(\theta) + Rs].$$

The last step is to apply the convexity of $v'(\cdot)$ as in (A.18). This proves that $g'_A(s) > g'_D(s)$ and establishes the result in Corollary 1.

A.4 Proof of Corollary 2

Set $g_A(s) = \int_{\Theta} \phi[U(s;\theta)]\, dG(\theta)$ and $g_D(s) = \phi \circ u(\mu + Rs)$. If $u(c) = c$ and $v(\cdot)$ is strictly concave, then $\phi(\cdot)$ is also strictly concave. It follows that

$$g_A(s) < \phi\left[\int_{\Theta} U(s;\theta)\, dG(\theta)\right] = \phi\left[\int_{\Theta}\int_{\Omega} (\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta)\, dG(\theta)\right] = g_D(s).$$

Risk neutrality also means that $\mathbb{M}_u(s;\theta) = \widetilde{\mu}(\theta) + Rs$, for all $(s, \theta) \in [-\underline{b}, z] \times \Theta$, and

$$\phi'[U(s;\theta)] = v'[\mathbb{M}_u(s;\theta)] = v'[\widetilde{\mu}(\theta) + Rs].$$

Substituting this into (A.14) gives

$$g'_A(s) = R\int_{\Theta} v'[\widetilde{\mu}(\theta) + Rs]\, dG(\theta) > Rv'\left(\int_{\Theta} [\widetilde{\mu}(\theta) + Rs]\, dG(\theta)\right) = g'_D(s).$$

The inequality follows from the strictly convexity of $v'(\cdot)$. This proves Corollary 2.

A.5 Proof of Theorem 2

Part (i) By Assumption A3, $V(\cdot)$ is a strictly concave function. By Lemma 1 part (ii), $\mathbb{M}_v(s)$ is a concave function in s if both $\mathcal{T}_u(\cdot)$ and $\mathcal{T}_v(\cdot)$ are concave functions. These two observations together implies that $\Xi_A(\cdot)$ is strictly concave.

Part (ii) As shown in (A.5), if both $u(\cdot)$ and $v(\cdot)$ are strictly concave, then $\mathbb{M}_v(s) < \mu + Rs$, for all $s \in [-\underline{b}, w]$. Also, as shown in Section A.1, $\mathbb{M}'_v(s)$ can be expressed as

$$\mathbb{M}'_v(s) = \frac{1}{v'[\mathbb{M}_v(s)]}\int_{\Theta} v'[\mathbb{M}_u(s;\theta)]\left[\frac{\partial}{\partial s}\mathbb{M}_u(s;\theta)\right] dG(\theta).$$

If $u(\cdot)$ exhibits DARA, then $\mathbb{M}_u'(s;\theta) \geq R$ for all $(s,\theta) \in [-\underline{b}, w] \times \Theta$. Combining these two gives

$$\mathbb{M}_v'(s) \geq \frac{R}{v'[\mathbb{M}_v(s)]} \int_\Theta v'[\mathbb{M}_u(s;\theta)]\, dG(\theta) \geq R.$$

The second inequality follows from (A.8), which is based on the DARA property of $v(\cdot)$. This completes the proof of Theorem 2.

A.6 Proof of Theorem 3

The proof of part (ii) has already been explained in the main text. Hence, we will only mention the proof of part (i) in here. If $\mathcal{A}_V(\cdot) \geq \mathcal{A}_u(\cdot)$, then the composite function $\Gamma(\cdot)$ is concave. This, together with Lemma 2, ensures that the consumer's future felicity $\Gamma[\mathbb{M}_\phi(s)]$ is a concave function in s. The strict concavity of $V(\cdot)$ then implies that $\Delta_A(\cdot)$ is strictly concave.

A.7 Proof of Theorem 4

Part (i) The uniqueness proof of s_A^* is the same as in Theorem 2, hence we focus on the uniqueness of s_R^*. Under Approach #2, the objective function in (P3) can be expressed as

$$\Xi_R(s) \equiv V(w-s) + \beta V\left\{u^{-1}\left[\int_S u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right]\right\}.$$

If $u(\cdot)$ exhibits CARA, then $u^{-1}\left[\int_S u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right]$ is a linear function in Rs. This, together the strictly concavity of $V(\cdot)$ under Assumption A3, ensures that $\Xi_R(\cdot)$ is a strictly concave function. Hence, (P3) has a unique solution.

Part (ii) The desired result $s_A^* \geq s_R^*$ holds if and only if the marginal benefits of saving under (P1) is greater than that under (P3), i.e.,

$$V'[\mathbb{M}_v(s)]\, \mathbb{M}_v'(s) \geq V'\left\{u^{-1}\left[\int_S u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right]\right\} \frac{d}{ds}\left\{u^{-1}\left[\int_S u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right]\right\}.$$

Given that $V(\cdot)$ is strictly increasing and concave, this condition holds if

$$\mathbb{M}_v(s) \leq u^{-1}\left[\int_S u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right], \tag{A.20}$$

$$\mathbb{M}_v'(s) \geq \frac{d}{ds}\left\{u^{-1}\left[\int_S u(\widetilde{y} + Rs)\, dH(\widetilde{y})\right]\right\} = R. \tag{A.21}$$

Condition (A.20) follows immediately from ambiguity aversion. To see this, note that

$$\begin{aligned}
\mathbb{M}_v(s) &= v^{-1}\int_\Theta v\circ u^{-1}\left[\int_\Omega u(\widetilde{y}+Rs)\,dF(\widetilde{y}\mid\theta)\right]dG(\theta) \\
&\leq v^{-1}\circ v\circ u^{-1}\left[\int_\Theta\int_\Omega u(\widetilde{y}+Rs)\,dF(\widetilde{y}\mid\theta)\,dG(\theta)\right] \\
&= u^{-1}\left[\int_S u(\widetilde{y}+Rs)\,dH(\widetilde{y})\right].
\end{aligned}$$

The second line uses the concavity of $\phi(\cdot)\equiv v\circ u^{-1}(\cdot)$. The equality in (A.21) follows from the CARA assumption for $u(\cdot)$. As shown in (A.6), the derivative of $\mathbb{M}_v(s)$ is given by

$$\mathbb{M}_v'(s)=\frac{R}{v'[\mathbb{M}_v(s)]}\int_\Theta\frac{v'[\mathbb{M}_u(s;\theta)]}{u'[\mathbb{M}_u(s;\theta)]}\left[\int_\Theta u'(\widetilde{y}+Rs)\,dF(\widetilde{y}\mid\theta)\right]dG(\theta).$$

Again by the CARA assumption for $u(\cdot)$,

$$\frac{1}{u'[\mathbb{M}_u(s;\theta)]}\left[\int_\Theta u'(\widetilde{y}+Rs)\,dF(\widetilde{y}\mid\theta)\right]=1,$$

for all $\theta\in\Theta$. Hence, the derivative of $\mathbb{M}_v(s)$ can be simplified to become

$$\mathbb{M}_v'(s)=\frac{R}{v'[\mathbb{M}_v(s)]}\int_\Theta v'[\mathbb{M}_u(s;\theta)]\,dG(\theta). \tag{A.22}$$

The last step is to invoke the DARA property of $v(\cdot)$. In particular, combining (A.8) and (A.22) gives (A.21). This completes the proof of Theorem 4.

Appendix B: Concavity of Certainty Equivalent

This appendix provides a detailed and self-contained proof of Lemmas 1 and 2. We begin by establishing an intermediate result, which is a variant of Theorem 106 in Hardy *et al.* (1934).

B.1 Preliminaries

For any positive integer $J > 1$, define a function $\widetilde{M}_u^J : \mathbb{R}_+^J \to \mathbb{R}_+$ according to

$$\widetilde{M}_u^J(\mathbf{x}) \equiv u^{-1}\left[\sum_{j=1}^{J} p_j u(x_j)\right], \tag{B.1}$$

where $\mathbf{x} = (x_1, ..., x_J) \in \mathbb{R}_+^J$, $p_j \in [0,1]$ for each j, and $\sum_{j=1}^{J} p_j = 1$. $\widetilde{M}_u^J(\mathbf{x})$ is said to be concave if for any $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}_+^J$, and for any $\alpha \in [0,1]$,

$$\widetilde{M}_u^J(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \geq \alpha\widetilde{M}_u^J(\mathbf{x}_1) + (1-\alpha)\widetilde{M}_u^J(\mathbf{x}_2). \tag{B.2}$$

Lemma A1 *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a thrice continuously differentiable, strictly increasing and strictly concave function. Then the following statements are equivalent:*

(i) $\mathcal{T}_u(c) \equiv -u'(c)/u''(c)$ *is a concave function.*

(ii) $\widetilde{M}_u^J(\mathbf{x})$ *is concave.*

Proof of Lemma A1 To prove that (i) implies (ii), first rewrite equation (B.1) as

$$u\left[\widetilde{M}_u^J(\mathbf{x})\right] = \sum_{j=1}^{J} p_j u(x_j). \tag{B.3}$$

Since $u(\cdot)$ is at least twice continuously differentiable, so is $\widetilde{M}_u^J(\mathbf{x})$. Let $h_i(\mathbf{x})$ be the derivative of $\widetilde{M}_u^J(\mathbf{x})$ with respect to the ith element of $\mathbf{x}$, and $h_{i,j}(\mathbf{x})$ be the derivative of $h_i(\mathbf{x})$ with respect to the jth element of $\mathbf{x}$. The Hessian matrix of $\widetilde{M}_u^J(\mathbf{x})$ is denoted by $\mathbf{H}(\mathbf{x}) = [h_{i,j}(\mathbf{x})]$ for any $\mathbf{x} \in \mathbb{R}_+^J$. The certainty equivalent $\widetilde{M}_u^J(\mathbf{x})$ is concave if and only if $\mathbf{H}(\mathbf{x})$ is a negative semi-definite matrix. We now show that $\mathbf{H}(\mathbf{x})$ is negative semi-definite if $\mathcal{T}_u(\cdot)$ exhibits concavity.

Straightforward differentiation of (B.3) gives

$$u'\left[\widetilde{M}_u^J(\mathbf{x})\right] h_i(\mathbf{x}) = p_i u'(x_i), \tag{B.4}$$

$$u''\left[\widetilde{M}_u^J(\mathbf{x})\right][h_i(\mathbf{x})]^2 + u'\left[\widetilde{M}_u^J(\mathbf{x})\right]h_{i,i}(\mathbf{x}) = p_i u''(x_i)\,, \tag{B.5}$$

for $i = 1, ..., J$, and

$$u''\left[\widetilde{M}_u^J(\mathbf{x})\right]h_i(\mathbf{x})\,h_j(\mathbf{x}) + u'\left[\widetilde{M}_u^J(\mathbf{x})\right]h_{i,j}(\mathbf{x}) = 0, \qquad \text{if } i \neq j. \tag{B.6}$$

Combining (B.4) and (B.5) gives

$$h_{i,i}(\mathbf{x}) = p_i \frac{u''(x_i)}{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]} - \frac{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]}{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^3}\left[p_i u'(x_i)\right]^2.$$

Similarly, combining (B.4) and (B.6) gives

$$h_{i,j}(\mathbf{x}) = -\frac{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]}{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^3} p_i p_j u'(x_i)\,u'(x_j)\,.$$

Thus, for any $\boldsymbol{\varpi} = (\varpi_1, ..., \varpi_J) \in \mathbb{R}^J$, we can write

$$\begin{aligned}
&\boldsymbol{\varpi}^T \cdot \mathbf{H}(\mathbf{x})\,\boldsymbol{\varpi} \\
= \;& \frac{\sum_{j=1}^J p_j \varpi_j^2 u''(x_j)}{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]} - \frac{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]}{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^3}\left[\sum_{j=1}^J p_j \varpi_j u'(x_j)\right]^2 \\
= \;& \frac{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]}{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^3}\left[\sum_{j=1}^J p_j \varpi_j^2 u''(x_j)\right]\left\{\frac{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^2}{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]} - \frac{\left[\sum_{j=1}^J p_j \varpi_j u'(x_j)\right]^2}{\sum_{j=1}^J p_j \varpi_j^2 u''(x_j)}\right\}.
\end{aligned}$$

Hence, $\boldsymbol{\varpi}^T \cdot \mathbf{H}(\mathbf{x})\,\boldsymbol{\varpi} \leq \mathbf{0}$ if and only if

$$\frac{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^2}{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]} \leq \frac{\left[\sum_{j=1}^J p_j \varpi_j u'(x_j)\right]^2}{\sum_{j=1}^J p_j \varpi_j^2 u''(x_j)}, \tag{B.7}$$

for any $\boldsymbol{\varpi} \in \mathbb{R}^J$. To see the connection between this and the concavity of $\mathcal{T}_u(\cdot)$. First, define an auxiliary function $\Sigma : \mathbb{R} \to \mathbb{R}_+$ according to

$$\Sigma(m) \equiv \frac{\left\{u'\left[u^{-1}(m)\right]\right\}^2}{u''\left[u^{-1}(m)\right]}.$$

Straightforward differentiation gives

$$\Sigma'(m) = 2 - \frac{u'\left[u^{-1}(m)\right]u'''\left[u^{-1}(m)\right]}{\left\{u''\left[u^{-1}(m)\right]\right\}^2} = 1 - \mathcal{T}_u'\left[u^{-1}(m)\right].$$

Since $u^{-1}(\cdot)$ is strictly increasing, it follows that $\mathcal{T}_u(\cdot)$ is weakly concave if and only if $\Sigma(\cdot)$ is weakly convex.

If $\Sigma(\cdot)$ is weakly convex, then

$$\begin{aligned}
\frac{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^2}{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]} &\equiv \Sigma\left[\sum_{j=1}^J p_j u(x_j)\right] \\
&\leq \sum_{j=1}^J p_j \Sigma\left[u(x_j)\right] \\
&= \sum_{j=1}^J p_j \frac{\left[u'(x_j)\right]^2}{u''(x_j)} \leq \frac{\left[\sum_{j=1}^J p_j \varpi_j u'(x_j)\right]^2}{\sum_{j=1}^J p_j \varpi_j^2 u''(x_j)}.
\end{aligned}$$

The second line is obtained by using Jensen's inequality. The third line follows from the definition of $\Sigma(m)$. The last inequality follows from the Cauchy-Schwartz inequality and the fact that $u''(\cdot) < 0$. This proves that if $\mathcal{T}_u(\cdot)$ is concave then the condition in (B.7) is satisfied and $\mathbf{H}(\mathbf{x})$ is negative semi-definte.

To prove the necessity of a concave $\mathcal{T}_u(\cdot)$, suppose $\mathbf{H}(\mathbf{x})$ is negative semi-definite so that (B.7) holds for all $\boldsymbol{\varpi} \in \mathbb{R}^J$. Set $\varpi_j = u'(x_j)/u''(x_j)$ for each j. Then (B.7) becomes

$$\Sigma\left[\sum_{j=1}^J p_j u(x_j)\right] \equiv \frac{\left\{u'\left[\widetilde{M}_u^J(\mathbf{x})\right]\right\}^2}{u''\left[\widetilde{M}_u^J(\mathbf{x})\right]} \leq \sum_{j=1}^J p_j \frac{\left[u'(x_j)\right]^2}{u''(x_j)} \equiv \sum_{j=1}^J p_j \Sigma\left[u(x_j)\right],$$

which proves that $\Sigma(\cdot)$ is convex, and hence $\mathcal{T}_u(\cdot)$ is concave. This completes the proof of Lemma A1. ■

B.2 Proof of Lemma 1

Part (i) Fix $s \in (-\underline{b}, z)$ and $\theta \in \Theta$. Recall the definition of $\mathbb{M}_u(s;\theta)$, i.e.,

$$u\left[\mathbb{M}_u(s;\theta)\right] = \int_\Omega u(\widetilde{y} + Rs)\, dF(\widetilde{y} \mid \theta).$$

In order to apply the result in Lemma A1, we first construct a discrete approximation for the integral on the right side. For any positive integer $J > 1$, let $\{\widehat{y}_0, \widehat{y}_1, ..., \widehat{y}_J\}$ be an arbitrary partition of Ω so that $\underline{y} = \widehat{y}_0 \leq \widehat{y}_1 \leq ... \leq \widehat{y}_J = \overline{y}$. Define a set of probabilities $\{p_1(\theta), ..., p_J(\theta)\}$ according to

$$p_j(\theta) \equiv F(\widehat{y}_j \mid \theta) - F(\widehat{y}_{j-1} \mid \theta), \qquad \text{for each } j \geq 1.$$

The corresponding cumulative distribution function is denoted by $F_J(\widetilde{y} \mid \theta) \equiv \sum_{j=1}^{J} \chi_j(y) F(\widehat{y}_j \mid \theta)$, where $\chi_j(y) = 1$ if $y \in [\widehat{y}_{j-1}, \widehat{y}_j)$ and zero otherwise. Finally, define $\widetilde{M}_u^J(\mathbf{x})$ according to

$$\widetilde{M}_u^J(\mathbf{x}) \equiv u^{-1}\left[\sum_{j=1}^{J} p_j(\theta) u(x_j)\right], \tag{B.8}$$

where $x_j = \widehat{y}_j + Rs$ for all j. Since $F_J(\widetilde{y} \mid \theta)$ converges pointwise to $F(\widetilde{y} \mid \theta)$ as J approaches infinity, we can get

$$\lim_{J \to \infty} \sum_{j=1}^{J} p_j(\theta) u(x_j) = \int_{\Omega} u(y + Rs) \, dF(\widetilde{y} \mid \theta),$$

and by the continuity of $u(\cdot)$, we can get

$$\lim_{J \to \infty} \widetilde{M}_u^J(\mathbf{x}) = \mathbb{M}_u(s; \theta),$$

for each $s \in (-\underline{b}, z)$ and $\theta \in \Theta$.

By Lemma A1, if $\mathcal{T}_u(c) \equiv -u'(c)/u''(c)$ is concave then $\widetilde{M}_u^J(\mathbf{x})$ satisfies the condition in (B.2). Since $\mathbf{x}$ is a linear function in s, it follows that $\widetilde{M}_u^J(\mathbf{x})$ is a concave function in s for each $J > 1$. Hence, $\left\{\widetilde{M}_u^J(\cdot)\right\}$ forms a sequence of concave functions in s that converges pointwise to $\mathbb{M}_u(\cdot; \theta)$ for any given $\theta \in \Theta$. By Theorem 10.8 in Rockfellar (1970), the limiting function $\mathbb{M}_u(\cdot; \theta)$ must be concave as well. This proves the desired result.

Part (ii) Let $g_1 : \Theta \to \mathbb{R}_+$ and $g_2 : \Theta \to \mathbb{R}_+$ be two continuous functions. Suppose $\mathcal{T}_v(c) \equiv -v'(c)/v''(c)$ is a concave function. Using the same line of argument as in part (i), we can prove that

$$\begin{aligned} & v^{-1}\left\{\int_{\Theta} v[\alpha g_1(\theta) + (1-\alpha) g_2(\theta)] \, dG(\theta)\right\} \\ \geq \; & \alpha v^{-1}\left\{\int_{\Theta} v[g_1(\theta)] \, dG(\theta)\right\} + (1-\alpha) v^{-1}\left\{\int_{\Theta} v[g_2(\theta)] \, dG(\theta)\right\}, \end{aligned} \tag{B.9}$$

for any $\alpha \in [0, 1]$. Pick any $s_1 \neq s_2$ from $[-\underline{b}, w]$ and define $s_\alpha = \alpha s_1 + (1-\alpha) s_2$. By the result in part (i),

$$\mathbb{M}_u(s_\alpha; \theta) \geq \alpha \mathbb{M}_u(s_1; \theta) + (1-\alpha) \mathbb{M}_u(s_2; \theta), \qquad \text{for all } \theta \in \Theta.$$

Since both $v(\cdot)$ and $v^{-1}(\cdot)$ are strictly increasing,

$$\begin{aligned}\mathbb{M}_v(s_\alpha) &\equiv v^{-1}\left\{\int_\Theta v\left[\mathbb{M}_u(s_\alpha;\theta)\right]dG(\theta)\right\} \\ &\geq v^{-1}\left\{\int_\Theta v\left[\alpha\mathbb{M}_u(s_1;\theta)+(1-\alpha)\mathbb{M}_u(s_2;\theta)\right]dG(\theta)\right\}. \qquad \text{(B.10)}\end{aligned}$$

Substituting $g_1(\theta)=\mathbb{M}_u(s_1;\theta)$ and $g_2(\theta)=\mathbb{M}_u(s_2;\theta)$ into (B.9) gives

$$\begin{aligned}& v^{-1}\left\{\int_\Theta v\left[\alpha\mathbb{M}_u(s_1;\theta)+(1-\alpha)\mathbb{M}_u(s_2;\theta)\right]dG(\theta)\right\} \\ \geq\; & \alpha v^{-1}\left\{\int_\Theta v\left[\mathbb{M}_u(s_1;\theta)\right]dG(\theta)\right\}+(1-\alpha)v^{-1}\left\{\int_\Theta v\left[\mathbb{M}_u(s_2;\theta)\right]dG(\theta)\right\} \\ =\; & \alpha\mathbb{M}_v(s_1)+(1-\alpha)\mathbb{M}_v(s_2). \qquad \text{(B.11)}\end{aligned}$$

The desired result follows by combining (B.10) and (B.11). This proves Lemma 1.

B.3 Proof of Lemma 2

The proof is similar in spirit to the proof of Lemma 1 part (ii). Let $g_1:\Theta\to\mathbb{R}_+$ and $g_2:\Theta\to\mathbb{R}_+$ be two continuous functions. Suppose $\mathcal{T}_\phi(x)\equiv-\phi'(x)/\phi''(x)$ is a concave function. Using the same line of argument as in part (i), we can prove that

$$\begin{aligned}& \phi^{-1}\left\{\int_\Theta \phi\left[\alpha g_1(\theta)+(1-\alpha)g_2(\theta)\right]dG(\theta)\right\} \\ \geq\; & \alpha\phi^{-1}\left\{\int_\Theta \phi\left[g_1(\theta)\right]dG(\theta)\right\}+(1-\alpha)\phi^{-1}\left\{\int_\Theta \phi\left[g_2(\theta)\right]dG(\theta)\right\}. \qquad \text{(B.12)}\end{aligned}$$

for any $\alpha\in[0,1]$. Pick any $s_1\neq s_2$ from $[-\underline{b},w]$ and define $s_\alpha=\alpha s_1+(1-\alpha)s_2$. By the concavity of $u(\cdot)$,

$$\mathbb{U}(s_\alpha;\theta)\geq\alpha U(s_1;\theta)+(1-\alpha)U(s_2;\theta),\qquad\text{for all }\theta\in\Theta.$$

Since both $\phi(\cdot)$ and $\phi^{-1}(\cdot)$ are strictly increasing,

$$\begin{aligned}\mathbb{M}_\phi(s_\alpha) &\equiv \phi^{-1}\left\{\int_\Theta \phi\left[U(s_\alpha;\theta)\right]dG(\theta)\right\} \\ &\geq \phi^{-1}\left\{\int_\Theta \phi\left[\alpha U(s_1;\theta)+(1-\alpha)U(s_2;\theta)\right]dG(\theta)\right\}. \qquad \text{(B.13)}\end{aligned}$$

Substituting $g_1(\theta)=U(s_1;\theta)$ and $g_2(\theta)=U(s_2;\theta)$ into (B.12) gives

$$
\begin{aligned}
& \phi^{-1}\left\{\int_{\Theta} \phi\left[\alpha U\left(s_1 ; \theta\right)+(1-\alpha) U\left(s_2 ; \theta\right)\right] d G(\theta)\right\} \\
\geq \;& \alpha \phi^{-1}\left\{\int_{\Theta} \phi\left[U\left(s_1 ; \theta\right)\right] d G(\theta)\right\}+(1-\alpha) \phi^{-1}\left\{\int_{\Theta} \phi\left[U\left(s_2 ; \theta\right)\right] d G(\theta)\right\} \\
= \;& \alpha \mathbb{M}_{\phi}\left(s_1\right)+(1-\alpha) \mathbb{M}_{\phi}\left(s_2\right) .
\end{aligned} \tag{B.14}
$$

The desired result can be obtained by combining (B.13) and (B.14). This completes the proof of Lemma 2.

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