Munich Personal RePEc Archive

Optimal growth with labour market frictions

Guerrazzi, Marco

Department of Economics - DIEC, University of Genoa

2 September 2022

Online at https://mpra.ub.uni-muenchen.de/114422/
MPRA Paper No. 114422, posted 07 Sep 2022 04:05 UTC

Optimal Growth with Labour Market Frictions*

Marco Guerrazzi†

September 2, 2022

Abstract

In this paper, I develop an optimal growth model with labour market frictions in which recruiting efforts are measured in terms of labour instead of output. Specifically, I build an intertemporal framework à la Ramsey in which labour has to be alternatively employed in the production of goods or in the recruitment of workers. Within this setting, assuming that capital is paid according to its marginal productivity, I show that (*i*) capital measured along its intensive margin may converge towards its stationary value in a non-monotonic manner; (*ii*) Pareto optimal allocations typical of a centralized economy can also be achieved in a decentralized environment in which the prevailing wage is indexed to the labour market tightness indicator; (*iii*) the consistency of the wage that implements efficient allocations with the competitiveness of the market for goods relies on vanishing values of the discount rate.

JEL Classification: E22; E24; J64.

Keywords: Capital accumulation; Searching-and-matching frictions; Efficiency; Capitalization effects; Zero discounting.

1 Introduction

The modern theory of optimal growth with endogenous saving took its first steps in a full-employment environment (cf. Koopmans, 1965; Cass, 1965, 1966). After the acknowledgment of the theory of equilibrium unemployment embodied in the Diamond (1982), Mortensen (1982), and Pissarides (1985) model, however, several scholars introduced labour market frictions in the form of searching-and-matching externalities into 'classical' models of capital accumulation by aiming at replicating some crucial business cycles regularities such as real-wage stickiness that frictionless settings where unable to explain in a satisfactory manner (cf. Merz, 1995; Andolfatto, 1996).

*Preliminary draft. Comments are welcome.

†Author contacts: Department of Economics – DIEC, University of Genoa, via F. Vivaldi, n. 5, 16126 Genoa (Italy), Phone (+39) 010 2095702, Fax (+39) 010 2095269, E-mail: guerrazzi@economia.unige.it.

Both from a theoretical and an empirical point of view, the introduction of searching-and-matching frictions within models of capital accumulation and growth raises a number of intriguing issues (cf. Eriksson, 1997; Pissarides, 2000, Chapter 3; Hornstein et al., 2007; Chen et al. 2011). For instance, whenever employment and its dynamics are determined by combining the searching efforts of unemployed workers and the recruiting efforts of entrepreneurs, the actual behaviour of the capital-labour ratio – as well as the one of other economic variables measured along their intensive margin – is affected not only by saving and consumption decisions, but also by the effectiveness of the mentioned labour market activities (cf. Janiak and Wasmer, 2014). Moreover, as it usually happens in non-Walrasian environments, whenever produced output is the result of a combination of different production factors, it may be interesting to assess how are determined the respective remunerations that in a decentralized setting may implement the the Pareto optimal allocations that would be chosen by a hypothetical social planner (cf. Masters, 1998).

In this paper, I aim at exploring these issues by developing an analytically-tractable optimal growth model with capital accumulation and labour market frictions supplemented by some numerical simulations. Specifically, drawing on Farmer (2013) and Guerrazzi (2015), I augment the standard setting à la Ramsey with a searching-and-matching mechanism that conveys the dynamics of the employed labour force by assuming that the wasteful recruiting efforts that move jobless workers from home towards production sites are measured in terms of labour instead of produced output (cf. Farmer, 2010). In other words, I assume that there are no vacant jobs. Nevertheless, a fraction of the employed workers has to be optimally allocated in recruiting activities – such as applications' screening and jobs advertising – that do not contribute to output production.[1] Within such an analytical proposal, the whole labour input enters the model economy as an additional state variable vis-à-vis productive capital, whereas its fraction employed in recruiting activities is modeled as a further control variable that can be set at a centralized or decentralized level just like households' consumption. Therefore, the resulting theoretical framework represents a straightforward reference to model capital accumulation and growth in a searching-and-matching environment with equilibrium unemployment.

Within this dynamic setting, I show that productive capital measured along its intensive margin may converge towards its stationary value in a non-monotonic manner by showing an initial phase of take-off then followed by a subsequent phase of contraction (cf. Fiaschi and Lavezzi, 2007; O'Neill, 2012). In addition, assuming that employed capital is conventionally paid according to its net marginal productivity, I show that the Pareto optimal allocations typical of a centralized economy can also be replicated in a decentralized framework in which the current wage is indexed to the prevailing labour market tightness indicator (cf. Chen et al. 2011; Duval et al. 2022). Furthermore, I show that the wage that implements efficient allocations is consistent with the long-run requirements of perfect competition in the market for goods, only when the discount rate of households and firms takes vanishing values as advocated

[1] This hypothesis can be rationalized on the ground that hiring is a labour-intensive activity (cf. Eriksson, 1997; Pissarides, 2000).

by growth contributions with an environmental flavour (cf. Cline, 1992; Stern, 2007). To the best of my knowledge, these findings are new in the reference literature and contribute to a deeper understanding of growth models with labour market frictions.

The paper is arranged as follows. Section 2 describes the building blocks of the model economy. Section 3 develops the social planner problem. Section 4 offers the derivation of the centralized steady-state solution. Section 5 analyses the local dynamics of the model economy around its first-best equilibrium allocation. Section 6 develops a decentralized version and shows under which conditions it may replicate Pareto optimal allocations and meet the conditions for perfect competition in the market for goods. Section 7 explores some numerical properties of the theoretical framework. Finally, Section 8 concludes.

2 The model

Drawing on Farmer (2013) and Guerrazzi (2015), I consider a closed model economy without a public sector in which time is continuous and supplied labour can be allocated in two alternative and essential economic activities, that is, recruiting of unemployed workers and production of homogenous goods that can be consumed by households or invested in additional productive capacity. On the one hand, in each instant, the recruiting of unemployed workers occurs by matching the fraction of employed workers which are not allocated in production activities with the current fraction of jobless workers. On the other hand, the production of goods is obtained by combining the existing stock of productive capital with the fraction of employed workers which are not allocated in recruiting activities. In the remaining of this section, I enucleate the building blocks of the theoretical framework under scrutiny by starting from its production side.

Suppose that in each instant – say $t \in \mathbb{R}_+$ – there are $L(t)$ workers which are available to work. Such a workforce supplied by households can be directed into two distinct alternative activities by splitting $L(t)$ in two different groups of workers, that is, the ones allocated in recruiting activities – denoted by $V(t)$ – and the ones allocated in production activities – denoted instead by $X(t)$. Consequently, it will hold true that

$$L(t) = V(t) + X(t) \tag{1}$$

According to the available production technology, the flow of output $Y(t)$ can be obtained by combining the existing stock of capital – denoted by $K(t)$ – with the fraction of workers allocated in production activities. Therefore, taking into account that workers allocated in recruiting activities are essential to hire labour but they do not contribute at all to output, the Cobb-Douglas production function that encapsulates the technological possibilities of the model economy can be written as

$$Y(t) = S(\Phi(t))^{\alpha}(L(t) - V(t)) \tag{2}$$

where $S > 0$ is an index for the efficiency of production, $\alpha \in (0, 1)$ is the elasticity of output with respect to employed capital, whereas $\Phi(t) \equiv K(t)/(L(t) - V(t))$ is the stock of capital over the fraction of productive labour allocated in output production and it represents the measure of capital in effective units of labour prevailing in the present theoretical framework.

In each instant, the flow of produced output can be alternatively consumed by households or invested in additional capital goods in order to increase the productivity of workers allocated in production activities. Consequently, considering the expression in eq. (2), the law of capital accumulation over time will be given by

$$\dot{K}(t) = S(\Phi(t))^{\alpha}(L(t) - V(t)) - C(t) - \delta K(t) \tag{3}$$

where $C(t)$ is households' consumption, whereas $\delta > 0$ is the rate of capital depreciation.[2]

I turn now to the behaviour of households and to dynamics of the employed labour force. On the one hand, leisure is assumed to be worthless for the households that populate the model economy, so that they will inelastically supply their own endowment of labour which is normalized to 1. Therefore, the unemployment rate can be written as

$$U(t) = 1 - L(t) \tag{4}$$

In addition, recalling that they do not value leisure – so that they do not dislike working – and assuming that their instantaneous utility is logarithmic, the utility function of the representative household is assumed to be given by the following integral:

$$\mathcal{U} \equiv \int_{t=0}^{\infty} \exp(-\rho t)(\ln C(t))\, dt \tag{5}$$

where $\rho > 0$ is the discount rate.

On the other hand, consistently with the searching-and-matching framework popularized by Pissarides (2000) as then modified by Farmer (2010, 2013), I assume that – in each instant – there is an inflow into employment fuelled by the Cobb-Douglas matching between the fraction of workers allocated in recruiting activities and the fraction of unemployed workers. In parallel, I posit that there is also a simultaneous outflow from employment driven a constant share of the employed workers that lose their positions for exogenous redundancy. Consequently, the evolution of employment over time is conveyed by

$$\dot{L}(t) = B(\Psi(t))^{\theta}(1 - L(t)) - \sigma L(t) \tag{6}$$

where $B > 0$ is an index for the efficiency of matching, $\Psi(t) \equiv V(t)/(1 - L(t))$ is a measure of the labour market tightness, $\theta \in (0, 1)$ is the elasticity of matching with respect to the fraction

[2] In a conventional searching-and-matching model with capital accumulation in which recruiting efforts are measured in terms of output instead of labour, savings are used to finance additions to the stock of capital and pay for the cost of vacancies. Consequently, in eq. (3) should appear and additional term with a negative sign that conveys recruiting costs (cf. Pissarides, 2000, Chapter 3).

of workers allocated in recruiting activities, whereas $\sigma > 0$ is the instantaneous job destruction rate.

The matching function on the right-hand-side of eq. (6) implies a straightforward trade-off between the production and matching technologies available in the model economy driven by the alternative uses of labour; indeed, according to eq.s (1) and (2), allocating more (less) workers in production activities boosts (reduces) output production but – at the same time – reduces (boosts) the inflows of new employment. Nevertheless, eq. (6) still mirrors the standard trading externalities that characterize a typical searching-and-matching economy; indeed, it implies that the instantaneous probability to find a job – namely, $\left(\dot{L}(t) + \sigma L(t)\right)/U(t) = B(\Psi(t))^{\theta}$ – and the recruiting effectiveness of labour – namely, $\left(\dot{L}(t) + \sigma L(t)\right)/V(t) = B(\Psi(t))^{-(1-\theta)}$ – are, respectively, an increasing and a decreasing function of the prevailing labour market tightness indicator (cf. Diamond, 1982; Pissarides, 2000).

3 The social planner problem

In the model economy described above, a benevolent and well-informed social planner will choose the level of consumption of the representative household and the share of workers allocated in recruiting activities with the aim of maximizing social welfare – which is assumed to coincide with $\mathcal{U}$ – by considering the accumulation of productive capital, and the evolution of employment over time. Such a social planner will take its decisions in a centralized manner by considering the impact of its choices on the prevailing labour market tightness indicator and solving in an optimal manner the labour allocation trade-off described above. Therefore, considering the expressions in eq.s (1), and (3) – (6), the intertemporal problem of the social planner can be written as

$$\begin{gathered} \max_{\mathcal{C}_{\mathrm{S}} \in \mathcal{A}_0^{\mathrm{S}}} \int_{t=0}^{\infty} \exp(-\rho t)(\ln C(t))\, dt \\ \text{s.to} \\ \dot{K}(t) = S(\Phi(t))^{\alpha} X(t) - C(t) - \delta K(t) \\ \dot{L}(t) = B(\Psi(t))^{\theta} U(t) - \sigma L(t) \\ K(0) = \overline{K}, \quad L(0) = \overline{L} \end{gathered} \tag{7}$$

where $\mathcal{C}_{\mathrm{S}} \equiv \begin{pmatrix} C(\cdot) & V(\cdot) \end{pmatrix}$ is the set of its control functions, $\mathcal{A}_0^{\mathrm{S}}$ is the set of its admissible control strategies, whereas $\overline{K} > 0$ and $\overline{L} > 0$ are, respectively, the initial value of the stock of capital and the initial value of employment.[3]

[3]In an unpublished note, Farmer (2012) shows that controlling for labour supply does not significantly alter the results achieved in this simplest context. Analytical frameworks in which the social planner controls for the extensive measure of labour supply are developed by Merz (2005), Andolfatto (2006) and Chen at al. (2011).

In order to have economically meaningful trajectories, the set of all admissible control strategies $\mathcal{C}_{\mathrm{S}}$ starting from the initial tern $\left\{0, \overline{K}, \overline{L}\right\}$ is defined as

$$\mathcal{A}_0^{\mathrm{S}} := \left\{ \mathcal{C}_{\mathrm{S}} \in \mathbb{L}^1_{\mathbf{loc}} \left(\mathbb{R}_+ ; \mathbb{R}^2_+ \right) : \begin{pmatrix} K(t) \\ L(t) \end{pmatrix} \in \mathbb{R}^2_+ \quad \forall t \in \mathbb{R}_+ \right\} \tag{8}$$

According to the definition in (8), the components of $\mathcal{C}_{\mathrm{S}}$ belongs to the set of locally integrable (or summable) functions such that household's consumption, the fraction of workers allocated in recruiting activities, the stock of capital and the employment rate are non-negative all over the relevant time horizon.

The first-order conditions (FOCs) for the problem in (7) are given by

$$\frac{1}{C(t)} - q(t) = 0 \tag{9}$$

$$-(1-\alpha) S q(t) (\Phi(t))^{\alpha} + \theta B \frac{w(t)}{(\Psi(t))^{1-\theta}} = 0 \tag{10}$$

$$\dot{q}(t) = q(t) \left(\rho - \frac{\alpha S}{(\Phi(t))^{1-\alpha}} + \delta \right) \tag{11}$$

$$\dot{w}(t) = w(t) \left(\rho + \sigma + (1-\theta) B (\Psi(t))^{\theta} \right) - (1-\alpha) S q(t) (\Phi(t))^{\alpha} \tag{12}$$

$$\lim_{t \to \infty} \exp(-\rho t) q(t) K(t) = 0 \tag{13}$$

$$\lim_{t \to \infty} \exp(-\rho t) w(t) L(t) = 0 \tag{14}$$

where $q(t)$ and $w(t)$ are the costate variables associated, respectively, to the capital accumulation constraint and to the dynamics of employment.

The infratemporal relationships in eq.s (8) and (9) hold in each instant and they represent, respectively, the FOCs with respect to consumption and the fraction of employed workers allocated in recruiting. In detail, the former states that the marginal utility of consumption must be equal to the marginal contribution of capital to households' utility, whereas the latter implies that the marginal contribution of labour to output production must be equal to its marginal contribution to the matching process (cf. Chen et al. 2011). Moreover, the intertemporal relationships in eq.s (11) and (12) convey the optimal trajectories of the two costate variables. Furthermore, the two endpoint limits on the value of the two state variables in (13) and (14) are the required transversality conditions.

The expressions in eq.s (9) – (12) can be exploited to derive the optimal dynamics of the two control variables chosen by the social planner. Formally speaking, differentiating eq. (9) with respect to time and exploiting the differential equation in (11), it is possible to obtain the Euler equation for households' consumption – or the Ramsey (1928) rule – whose actual expression is given by

$$\dot{C}(t) = C(t)\left(\frac{\alpha S}{(\Phi(t))^{1-\alpha}} - \delta - \rho\right) \tag{15}$$

In a rather conventional way, the differential equation in (15) implies that the growth rate of households' consumption is positive (negative), whenever the marginal productivity of capital adjusted for its depreciation rate is higher (lower) than the discount rate (cf. Cass, 1965, 1966).

Following a similar strategy, it also possible to find the implied Euler equation for $V(t)$. First, differentiating eq. (10) with respect to time, leads to the following expression:

$$(1-\alpha) S q(t) (\Phi(t))^{\alpha}\left(\frac{\dot{q}(t)}{q(t)} + \alpha\frac{\dot{\Phi}(t)}{\Phi(t)}\right) = \frac{\theta B w(t)}{(\Psi(t))^{1-\theta}}\left(\frac{\dot{w}(t)}{w(t)} - (1-\theta)\frac{\dot{\Psi}(t)}{\Psi(t)}\right) \tag{16}$$

Second, according to the results in eq.s (9) – (12), eq. (16) reduces to

$$\alpha\frac{\dot{\Phi}(t)}{\Phi(t)} + (1-\theta)\frac{\dot{\Psi}(t)}{\Psi(t)} = \sigma + \frac{\alpha S}{(\Phi(t))^{1-\alpha}} - \delta + \frac{B((1-\theta)\Psi(t)-\theta)}{(\Psi(t))^{1-\theta}} \tag{17}$$

Third, relying on eq.s (3) and (6), the growth rates of $\Phi(t)$ and $\Psi(t)$ can be written, respectively, as

$$\frac{\dot{\Phi}(t)}{\Phi(t)} = \frac{S}{(\Phi(t))^{1-\alpha}} - \delta - \frac{C(t)}{K(t)} - \frac{B(\Psi(t))^{\theta} U(t) - \sigma L(t)}{X(t)} + \frac{\dot{V}(t)}{X(t)} \tag{18}$$

$$\frac{\dot{\Psi}(t)}{\Psi(t)} = \frac{\dot{V}(t)}{V(t)} + B(\Psi(t))^{\theta} - \sigma\frac{L(t)}{U(t)} \tag{19}$$

Thereafter, plugging the results in eq.s (18) and (19) into eq. (17), implies that the differential equation for the fraction of workers allocated in recruiting activities is given by

$$\dot{V}(t) = \Lambda(t)\left(\alpha\left(\frac{C(t)}{K(t)} + \frac{B(\Psi(t))^{\theta}(U(t)) - \sigma L(t)}{X(t)}\right) + \frac{(1-\theta)\sigma L(t)}{U(t)} - \frac{\theta B}{(\Psi(t))^{1-\theta}} + \Omega_0\right) \tag{20}$$

where $\Lambda(t) \equiv V(t)X(t)/(\alpha V(t) + (1-\theta)X(t))$ and $\Omega_0 \equiv \sigma - \delta(1-\alpha) \lesseqgtr 0$.

Intuitively, considering the evolution of the two state variables, the Euler equation for $V(t)$ in (20) optimally counterbalances the contribution that employed labour gives to output production and to workforce recruitment. Obviously, whenever $C(t)$, $V(t)$, $K(t)$ and $L(t)$ move over time according to, respectively, eq.s (3), (6), (15) and (20), by complying to the transversality conditions in (13) and (14), the implemented allocations are Pareto optimal.

4 Steady state

In the model economy described in Section 2, steady-state allocations are defined as the set of quadruplets $\mathcal{S} := \{C^*, V^*, K^*, L^*\} \in \mathbb{R}^4_+$ such that $\dot{C}(C^*, V^*, K^*, L^*) = \dot{V}(C^*, V^*, K^*, L^*) =$

$\dot{K}(C^*, V^*, K^*, L^*) = \dot{L}(C^*, V^*, K^*, L^*) = 0$. In case of asymptotic stability, some elements of that set will be also characterized by the fact that $\lim_{t\to\infty} C(t) = C^* \wedge \lim_{t\to\infty} V(t) = V^* \wedge \lim_{t\to\infty} K(t) = K^* \wedge \lim_{t\to\infty} L(t) = L^*$. The unique component of $\mathcal{S}$ can be easily retrieved by finding the steady-state value of the stock of capital over the fraction of labour allocated in production activities – namely, Φ^* – and the steady-state value of the labour market tightness indicator – namely, Ψ^*.

On the one hand, setting $\dot{C}(t) = 0$ in eq. (15), allows us to immediately find that

$$\Phi^* = \left(\frac{\alpha S}{\rho + \delta}\right)^{\frac{1}{1-\alpha}} \tag{21}$$

where $\Phi^* \equiv K^*/(L^* - V^*)$.

The expression in eq. (21) is the modified golden-rule that holds in the model economy under scrutiny; indeed, according to eq. (3), the long-run equilibrium level of the stock of capital over the fraction of workers allocated in production activities that maximizes the corresponding intensive measure of consumption would be given by $\Phi^*_{\text{GR}} \equiv (\alpha S/\delta)^{1/(1-\alpha)}$ which – as long as we assume a positive discounting – is strictly higher than Φ^*.[4] See the diagram in Figure 1.

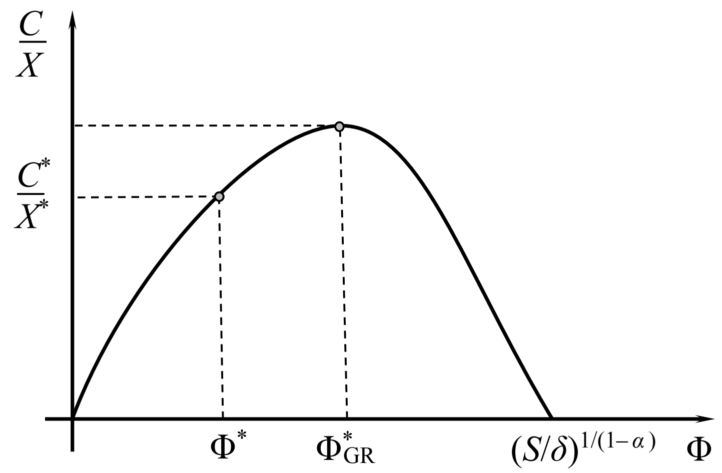

Figure 1: The modified golden-rule

On the other hand, in a steady-state allocation, the expressions for q^*/w^* implied, respectively by the FOC with respect to $V(t)$ in eq. (10) and the optimal dynamics of $w(t)$ conveyed by eq. (12), lead to the following expression:

$$\rho + \sigma + (1-\theta)\, B\, (\Psi^*)^{\theta} - \frac{\theta B}{(\Psi^*)^{1-\theta}} = 0 \tag{22}$$

where $\Psi^* \equiv V^*/(1^* - L^*)$.

[4] As I will show below, the actual distance between Φ^* and Φ^*_{GR} plays a crucial role in determining the long-run behaviour of firms' profits in a decentralized economy.

According to the hypotheses made above about the eligible values of the parameters ρ, σ, B and θ, the expression in eq. (22) is a hyperbolic continuous function of Ψ that tends to $-\infty$ ($+\infty$) as Ψ tends to 0 ($+\infty$).[5] Consequently, as illustrated in the diagram of Figure 2, there will be a unique value of Ψ – denoted by Ψ^* – that represents the steady-state value of the labour market tightness indicator.

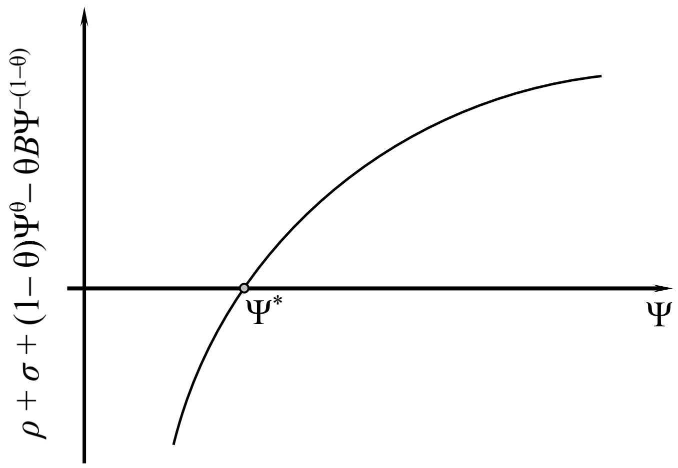

Figure 2: Steady-state determination

It worth noticing that the unique root of eq. (22) is an increasing (decreasing) function of B and θ (ρ and σ). Consequently, a more (less) impatient social planner will achieve a lower (higher) labour market tightness in the steady-state equilibrium.

Given the long-run values of Φ and Ψ, the unique quadruplet of $\mathcal{S}$ can be written straightforwardly. First, setting $\dot{L}(t) = 0$ in eq. (6) by considering the unique positive root of eq. (22), implies that

$$L^* = \frac{B\left(\Psi^*\right)^{\theta}}{\sigma + B\left(\Psi^*\right)^{\theta}} \tag{23}$$

According to eq. (4), the expression in eq. (23) implies that $U^* = \sigma/\left(\sigma + B\left(\Psi^*\right)^{\theta}\right)$. Consequently, in the long-run first-best allocation of the model economy described in Section 2, equilibrium (un)employment is not affected neither by the parameters of the production function nor by the rules of capital accumulation (cf. Layard et al. 1991).

Second, setting $\dot{L}(t) = 0$ in eq. (6) by taking into account the expression in eq. (23), leads to

$$V^* = \frac{\sigma\Psi^*}{\sigma + B\left(\Psi^*\right)^{\theta}} \tag{24}$$

[5] An alternative – but equivalent – way to derive the expression in eq. (22) is the one to consider the steady-state version of eq. (17) by taking into account the equilibrium value of Φ^* conveyed by eq. (21).

Third, according to the definition of Φ, eq.s (21), (23) and (24) imply that the steady-state level of the capital stock can be written as

$$K^* = \frac{\Phi^* \left(B \left(\Psi^* \right)^{\theta} - \sigma \Psi^* \right)}{\sigma + B \left(\Psi \right)^{\theta}} \tag{25}$$

Straightforward algebra reveals that in a companion model economy without the labour market frictions implied by eq. (6) in which all the labour force supplied by households is actually used to produce commodities, so that $L(t) = 1$ for all t, the steady-state value of the capital stock achieved by a social planner endowed with the preferences in eq. (5) would be simply equal to Φ^*.[6] Consequently, in our model economy, the fraction of workers allocated in production activities – whose analytical expression according to eq.s (23) and (24) is given by $X^* = \left(B \left(\Psi^* \right)^{\theta} - \sigma \Psi^* \right) / (\sigma + B \left(\Psi^* \right)^{\theta}) > 0$, provides a measure of the dead-weight loss suffered by the society for the mere presence of attrition in the labour market.

Thereafter, setting $\dot{K}(t) = 0$ in eq. (3) by considering the expressions in eq.s (21), (23) and (24), leads to

$$C^* = \frac{S \left(\Phi^* \right)^{\alpha} \left(B \left(\Psi^* \right)^{\theta} - \sigma \Psi^* \right) \Omega_1}{\left(\rho + \delta \right) \left(\sigma + B \left(\Psi^* \right)^{\theta} \right)} \tag{26}$$

where $\Omega_1 \equiv \rho + \sigma - \Omega_0 > 0$.

5 Local dynamics

Exploiting the expressions in eq.s (3), (6), (15) and (20), the local dynamics of $C(t)$, $V(t)$, $K(t)$ and $L(t)$ around the unique stationary solution defined by eq.s (23) – (26) is conveyed by the following 4×4 linear system:

$$\begin{pmatrix} \dot{C}(t) \\ \dot{V}(t) \\ \dot{K}(t) \\ \dot{L}(t) \end{pmatrix} = \begin{bmatrix} 0 & -\left(1-\alpha\right) S \left(\Phi^*\right)^{\alpha} \Omega_1 & -\frac{(1-\alpha)(\rho+\delta)\Omega_1}{\alpha} & \left(1-\alpha\right) S \left(\Phi^*\right)^{\alpha} \Omega_1 \\ \frac{\alpha}{\Phi^* \Gamma(\Psi^*)} & \frac{\theta B}{(\Psi^*)^{1-\theta}} & -\frac{\Omega_1}{\Phi^* \Gamma(\Psi^*)} & j_{2,4} \\ -1 & -\left(1-\alpha\right) S \left(\Phi^*\right)^{\alpha} & \rho & \left(1-\alpha\right) S \left(\Phi^*\right)^{\alpha} \\ 0 & \frac{\theta B}{(\Psi^*)^{1-\theta}} & 0 & -\left(1-\theta\right) B \left(\Psi^*\right)^{\theta} - \sigma \end{bmatrix} \begin{pmatrix} C(t) - C^* \\ V(t) - V^* \\ K(t) - K^* \\ L(t) - L^* \end{pmatrix} \tag{27}$$

where $\Gamma\left(\Psi^*\right) \equiv \alpha - \left(1-\theta\right) \left(\left(\sigma - B \left(\Psi^* \right)^{-(1-\theta)} \right) / \sigma \right) > 0$.

The non-explicit element on the second row of the Jacobian matrix in eq. (27) can be written as follows

[6] Formal details are available from the author upon request.

$$j_{2,4} \equiv \frac{\theta B \left(B \left(\Psi^*\right)^{-(1-2\theta)} - \sigma \left(\Psi^*\right)^{\theta} \left(1 + \alpha \left(1-\theta\right)\right) \right) + \left(1-\theta\right) \left(B \left(\Psi^*\right)^{\theta} - \sigma \Psi^* \right) \left(\sigma + B \left(\Psi^*\right)^{\theta} \right)}{\sigma \Gamma \left(\Psi^*\right)} \tag{28}$$

Taking a look at the expressions in (27) and (28), the dynamic properties of the unique component of $\mathcal{S}$ found in Section 3 are difficult to assess analytically. The local existence and convergence of dynamic paths, however, are a direct consequence of the turnpike property of optimal growth models; indeed, given the initial conditions for K and L, the social planner problem in (7) consists in discounting at a positive rate a concave instantaneous utility function over an infinite horizon by complying with two convex dynamic constraints. Therefore, such a problem must have a unique meaningful saddle-path stationary solution towards which all the endogenous variables asymptotically tend to converge in order to verify the transversality conditions in (13) and (14) (cf. Cass, 1966). Formally speaking, this means that the Jacobian matrix in (27) need to have two positive and diverging roots associated to households' consumption and to the fraction of employed workers allocated in recruiting activities, as well as two negative and converging roots – say λ_1 and λ_2 – associated, respectively, to the stock of capital and to the whole labour input.

Suppose that $\mathbb{V}_1(\lambda_1) \in \mathbb{R}^4$ and $\mathbb{V}_2(\lambda_2) \in \mathbb{R}^4$ are, respectively, the eigenvectors associated to λ_1 and λ_2. Thereafter, the evolution of $C(t)$, $V(t)$, $K(t)$ and $L(t)$ over time is given by

$$\begin{pmatrix} C(t) \\ V(t) \\ K(t) \\ L(t) \end{pmatrix} = \begin{pmatrix} C^* \\ V^* \\ K^* \\ L^* \end{pmatrix} + \begin{bmatrix} \frac{v_{1,1}}{v_{1,3}} & \frac{v_{2,1}}{v_{2,4}} \\ \frac{v_{1,2}}{v_{1,3}} & \frac{v_{2,2}}{v_{2,4}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \exp\left(\lambda_1 t\right) \left(\overline{K} - K^*\right) \\ \exp\left(\lambda_2 t\right) \left(\overline{L} - L^*\right) \end{pmatrix} \tag{29}$$

where $v_{i,j}$ is the j-th element of $\mathbb{V}_i(\lambda_i)$.

The linear expression in (29) will be the analytical device for the numerical experiments carried out in Section 7.

6 A decentralized version

The existence of a centralized Pareto-optimal solution raises the issue of assessing what may happen in a decentralized setting, in which some atomistic agents take their decisions on the base of market signals by ignoring the impact of their maximizing choices on aggregate variables. Relying on the building blocks laid down in Section 2, a decentralized version of our capital accumulation model with labour market frictions can be framed by assuming the contemporaneous presence of two distinct players that take their decisions in a simultaneous manner by taking as given market prices and matching ratios. On the one side, I will assume that there is a finite number of identical households endowed with the preferences implied by eq. (5) that choose their flow of consumption by complying to a wealth-accumulation constraint and observing – just like the realization of an exogenous shock – the evolution over time of their employed members. On the other side, I will consider a finite number of identical firms endowed with the production technology described in eq. (2) that choose the fraction of workers allocated in recruiting activities and the amount of capital to employ with the aim of maximizing the discounted flow their profits under the intertemporal constraint implied by the law of employment dynamics. In the remainder of this section, I will develop the household's and the firm's problem, and I will discuss under which conditions such a decentralized economy delivers Pareto optimal trajectories which are consistent with perfect competition in the market for goods.

On the consumption side, denoting its wealth by $A(t)$, the representative household's problem can be written as

$$\begin{gathered} \max_{\mathcal{C}_{\mathrm{H}}\in\mathcal{A}_0^{\mathrm{H}}}\int_{t=0}^{\infty}\exp\left(-\rho t\right)\left(\ln C\left(t\right)\right)dt \\ \text{s.to} \\ \dot{A}(t)=A\left(t\right)R\left(t\right)+W(t)L(t)+\Pi(t)-C(t) \\ \dot{L}\left(t\right)=(1-L(t))\Gamma\left(\Psi(t)\right)-\sigma L\left(t\right) \\ A\left(0\right)=\overline{A},\quad L(0)=\overline{L} \end{gathered} \tag{30}$$

where $\mathcal{C}_{\mathrm{H}}{\equiv}C\left(\cdot\right)$ is its set of control functions, $\mathcal{A}_0^{\mathrm{H}}$ is the set of its admissible control strategies, $R\left(t\right)$ is the instantaneous real return on wealth, $W(t)$ and $\Pi(t)$ are, respectively, the real wage rate and the profit paid by the representative firm, $\Gamma\left(\Psi(t)\right)$ is the probability to find a job for a jobless worker belonging to the household itself, whereas $\overline{A}>0$ the initial value of wealth.

Similarly to $\mathcal{A}_0$, the set of all admissible control strategies for the household starting from the initial pair $\left\{0,\overline{A}\right\}$ is defined as

$$\mathcal{A}_0^{\mathrm{H}}:=\left\{\mathcal{C}_{\mathrm{H}}\in\mathbb{L}^1_{\mathbf{loc}}\left(\mathbb{R}_+;\mathbb{R}_+\right):A(t)>0\quad\forall t\in\mathbb{R}_+\right\} \tag{31}$$

Considering that the household takes as given $R(t)$, $W(t)$ and the whole employment dynamics, the FOCs for the problem in (30) are simply the following:

$$\frac{1}{C\left(t\right)}-q_{\mathrm{H}}\left(t\right)=0 \tag{32}$$

$$\dot{q}_{\mathrm{H}}\left(t\right)=q_{\mathrm{H}}\left(t\right)\left(\rho-R(t)\right) \tag{33}$$

$$\lim_{t\to\infty}\exp\left(-\rho t\right)q_{\mathrm{H}}\left(t\right)A\left(t\right)=0 \tag{34}$$

where $q_{\mathrm{H}}\left(t\right)$ is the costate variable associated to the wealth accumulation constraint.

The infratemporal relationship in eq. (32) is the FOC with respect to consumption and it is qualitatively identical to the one found in the centralized economy. Moreover, the intertemporal relationship in eq. (33) conveys the optimal trajectory of the costate variable and – as I will maintain below – its adherence to the social planner counterpart is strictly conditioned by the determinants of the return on wealth. In addition, (34) is the transversality condition for the household's problem.

Differentiating eq. (32) with respect to time and exploiting the differential equation in (33), it is possible to obtain the Euler equation for the household' consumption that holds in the decentralized economy whose analytical expression given by

$$\dot{C}(t) = C(t)(R(t) - \rho) \tag{35}$$

On the production side, the problem of the representative firm that rents the existing stock of capital from the household can be written as

$$\begin{gathered}\max_{\mathcal{C}_{\mathrm{F}} \in \mathcal{A}_0^{\mathrm{F}}} \int_{t=0}^{\infty} \exp\left(-\rho t\right) \Pi\left(t\right) dt \\ \text{s.to} \\ \dot{L}\left(t\right) = V(t)\Delta\left(\Psi(t)\right) - \sigma L\left(t\right) \\ L(0) = \overline{L}\end{gathered} \tag{36}$$

where $\mathcal{C}_{\mathrm{F}} \equiv \begin{pmatrix} K\left(\cdot\right) & V\left(\cdot\right)\end{pmatrix}$ is its set of control functions, $\Pi\left(t\right) \equiv Y\left(t\right) - W(t)L(t) - \left(R\left(t\right) + \delta\right) K(t)$ is its instantaneous profit, whereas $\Delta\left(\Psi(t)\right)$ is the recruiting effectiveness of workers not engaged in production activities (cf. Eriksson, 1997; Pissarides, 2000).

Obviously, the set of all admissible control strategies for the firm starting from the initial pair $\left\{0, \overline{L}\right\}$ is defined as

$$\mathcal{A}_0^{\mathrm{F}} := \left\{\mathcal{C}_{\mathrm{F}} \in \mathbb{L}^1_{\mathbf{loc}}\left(\mathbb{R}_+; \mathbb{R}^2_+\right) : L(t) > 0 \quad \forall t \in \mathbb{R}_+\right\} \tag{37}$$

Considering that the firm takes as given $W(t)$ and $\Delta\left(\Psi(t)\right)$, the FOCs for the problem in (36) are given by

$$\frac{\alpha S}{\left(\Phi\left(t\right)\right)^{1-\alpha}} - R\left(t\right) - \delta = 0 \tag{38}$$

$$-\left(1-\alpha\right) S\left(\Phi\left(t\right)\right)^{\alpha} + w_{\mathrm{F}}\left(t\right)\Delta\left(\Psi(t)\right) = 0 \tag{39}$$

$$\dot{w}_{\mathrm{F}}\left(t\right) = w_{\mathrm{F}}\left(t\right)\left(\rho + \sigma\right) - \left(1-\alpha\right) S\left(\Phi\left(t\right)\right)^{\alpha} + W(t) \tag{40}$$

$$\lim_{t\to\infty} \exp\left(-\rho t\right) w_{\mathrm{F}}\left(t\right) L\left(t\right) = 0 \tag{41}$$

The relationship in eq. (38) is the FOC with respect to the employed capital, and it trivially states that the marginal productivity of employed capital must be equal to its user cost, whereas eq. (39) is the FOC with respect to fraction of workers allocated in recruiting activities. That latter equation is quite different from the corresponding one – namely, eq. (10) – that holds in the social planner problem. Specifically, according to eq. (39) – in each instant – the firm sets the value of $V(t)$ by omitting to consider the contribution of employed capital to household's utility, as well as the congestion effect driven by putting additional workers in its recruiting department. Similar arguments hold true also for the intertemporal relationship in eq. (40) that instead conveys the optimal trajectory of the costate variable. Such an equation reveals that the determinants of the real wage rate are essential to replicate the Pareto optimal trajectories generated in the centralized model economy. Furthermore, (41) is the transversality condition for the firm's problem.

As I argued above, in a non-centralized setting agents take their optimal decisions on the account of market signals. Consequently, at first, it may be sensible to assume that in the background of the decentralized economy there is an asset market in which the supply of wealth from the household meets the demand for productive capital from the firm. In each instant, the equilibrium condition for such a market will be given by

$$A(t) = K(t) \quad \text{for all } t \tag{42}$$

Everything else being equal, given the expressions in eq.s (3), (15) and (35), the market-clearing condition in eq. (42) implies that whenever the return on wealth is equal to the marginal productivity of capital net of its depreciation rate, the stock of wealth and households' consumption move over time, respectively, as the stock of capital and the flow of consumption generated by the solution of the social planner problem. In other words, whenever it holds true that

$$R(t) \equiv \frac{\alpha S}{(\Phi\,(t))^{1-\alpha}} - \delta \tag{43}$$

the differential equations for $A(t)$ and $C(t)$ in the decentralized economy may replicate the Pareto optimal dynamics of the stock of capital and consumption derived in Section 3.

Unfortunately, the market-clearing argument exploited for the asset market cannot be replicated for labour demand and labour supply; indeed, the presence of frictions summarized by eq. (6) rules out the possibility to assume the presence of a labour market through which the household and the firm may coordinate their actions by observing a price. Nevertheless, it is worth noticing that if the matching probabilities for households and firms are defined as in Section 2, namely, if $\Gamma\,(\Psi(t)) \equiv B\,(\Psi\,(t))^{\theta}$ and $\Delta\,(\Psi(t)) \equiv B\,(\Psi\,(t))^{-(1-\theta)}$, then the differential equation for total employment is the same in the decentralized as well as in the centralized economy. Consequently, assuming that in each instant the asset market is in equilibrium as stated by eq. (42) and that it prices $R(t)$ according to eq. (43), the decentralized economy exactly retraces the same trajectories chosen by the social planner in a centralized setting whenever the expression for the real wage plugged into the individual problems in (30) and (36) implies a differential equation for the share of workers allocated in recruiting activities equivalent to the expression in eq. (20). In this way, the prevailing real wage rate will internalize all the external effects that the individual firm does not consider when it sets the fraction of workers allocated in recruiting activities (cf. Diamond, 1982; Hosios, 1990).

In a non-stationary environment, the actual expression for $W(t)$ that leads the firm to choose instant-by-instant the first-best value of the fraction of workers allocated in recruiting activities can be retrieved by following a procedure similar to the one implemented in Section 3 to derive $\dot{V}(t)$. First, differentiating eq. (39) with respect to time leads to the following expression:

$$\alpha\left(1-\alpha\right)S\left(\Phi\left(t\right)\right)^{\alpha}\frac{\dot{\Phi}\left(t\right)}{\Phi\left(t\right)}=B\frac{w_{\mathrm{F}}\left(t\right)}{\left(\Psi\left(t\right)\right)^{1-\theta}}\left(\frac{\dot{w}_{\mathrm{F}}\left(t\right)}{w_{\mathrm{F}}\left(t\right)}-\frac{\dot{\Psi}\left(t\right)}{\Psi\left(t\right)}\right) \tag{44}$$

Second, according to the results in eq.s (39) and (40), eq. (44) reduces to

$$\alpha\frac{\dot{\Phi}(t)}{\Phi(t)}+(1-\theta)\frac{\dot{\Psi}(t)}{\Psi(t)}=\rho+\sigma+B\frac{W\left(t\right)-\left(1-\alpha\right)S\left(\Phi\left(t\right)\right)^{\alpha}}{\left(1-\alpha\right)S\left(\Phi\left(t\right)\right)^{\alpha}\left(\Psi\left(t\right)\right)^{1-\theta}} \tag{45}$$

Thereafter, exploiting the definitions of the growth rates of $\Phi(t)$ and $\Psi(t)$ in eq.s (18) and (19), the expression in eq. (45) delivers the same differential equation for $V(t)$ implied by the solution of the social planner problem if and only if the real wage paid to employed workers by the representative firm is equal to

$$W(t)=\left(1-\alpha\right)S\left(\Phi\left(t\right)\right)^{\alpha}\left(\left(1-\theta\right)\left(1+\Psi\left(t\right)\right)+\frac{\left(\Psi\left(t\right)\right)^{1-\theta}\left(R(t)-\rho\right)}{B}\right) \tag{46}$$

Although derived in a different manner, the expression in eq. (46) corresponds to the outcome of an efficient bargaining process in which the representative firm takes a fraction θ of the total surplus that – in present setting, where the representative household is assumed to take the remaining $1-\theta$ – amounts to $C\left(t\right)w(t)$ units of output in each instant (cf. Chen et al. 2011).[7] Whenever capital is remunerated at is net marginal productivity, such a wage support reveals that the Pareto efficiency of firm's choices requires a real wage rate which is directly proportional to the elasticity of the labour input in the production technology, to the overall efficiency of production, to the stock of capital over the fraction of labour allocated in production activities but – at the same time – positively indexed to the labour market tightness indicator and to the return on wealth.[8]

The wage support in eq. (46) allows the representative firm to obtain non-negative profits all over its optimization horizon. If we assume that the market for goods is competitive, however, then a positive flow of profits is possible only in the short run before the achievement of the steady-state solution, whereas in the long-run the value of $\Pi\left(t\right)$ needs to vanish to prevent the entry of additional firms. In order to show under which conditions the market for goods meets the mentioned requirement for perfect competition, it is worth noticing that under the pricing assumption of eq. (43), the instantaneous profit of the representative firm can be written as

[7] The splitting-the-surplus condition is derived in Appendix.

[8] A positive relationship between wages and labour market tightness seems to hold true in many advanced economies (cf. Duval, et. al. 2022).

$$\Pi(t) = \left(\overline{W}(t) - W(t)\right) L(t) \tag{47}$$

where $\overline{W}(t) \equiv ((1-\alpha) S\Phi(t)(L(t) - V(t)))/L(t)$ can be dubbed as the competitive real wage rate (cf. Chen et al. 2011).

According to the expression in eq. (47), in the long run the flow of profits of the representative firm tend to vanish whenever $\lim_{t\to\infty} \overline{W}(t) - W(t) = 0$. Given the results in eq.s (23), (24) and (35), the asymptotic behaviours of the competitive wage and the one of the wage support in eq. (46) tend to be the same whenever it holds

$$\sigma + (1-\theta) B (\Psi^*)^{\theta} - \frac{\theta B}{(\Psi^*)^{1-\theta}} = 0 \tag{48}$$

A straightforward comparison of eq.s (48) and (22) reveals that the long-run condition for a competitive market for goods is met whenever the discount rate ρ tends to zero. In other words, recalling the diagram in Figure 1, the steady-state values of the competitive wage and the one of the wage that implements efficient allocations tend asymptotically to coincide whenever the model economy tends to approach Φ^*_{GR}, that is, the stock of capital over the fraction of workers allocated in production activities that maximizes the corresponding intensive measure of consumption. Under these circumstances, remunerating capital according to its net marginal productivity and using the remaining produced output to pay all the employed workers without distinguishing the activity in which they are allocated, provides exactly the same outcome of the efficient bargaining underlying the wage support in eq. (46). By contrast, whenever the household and the firm discount their respective future streams arising from their objective functions at a common positive rate, the real wage rate that implements efficient allocations is always below the competitive wage defined in eq. (47). In this case, the flow of profits of the representative firm is persistently positive so that the non-Walrasian features of labour market spill over also in the market for goods.

From an economic point of view, the analytical result conveyed by eq. (48) can be rationalized as follows. In a competitive model economy in which capital is remunerated according to its net marginal productivity and it is possible to pay workers according to the actual activity in which they are allocated, the real wage rate received by the ones employed in recruiting would be equal to zero no matter the value of the discount rate, because they do not directly contribute to produced output.[9] Such a discrimination among employed workers, however, may be unfeasible because trade unions may dictate a uniform wage treatment among workers randomly allocated in different activities exactly as assumed in the household's and the firm's problem pinned down in (30) and (36) (cf. Card, 2001). In this case, paying capital at its net marginal productivity, the representative firm will find profitable to replicate the same trajectories for the fraction of workers allocated in recruiting activities and investment chosen by the social planner only when the discount rate tends to vanish because this generates the required capitalization effect on its forward-looking strategy for the employed levels of labour and cap-

[9] A formal proof is sketched in Appendix.

ital (cf. Aghion and Howitt, 1994; Hall, 2017). In other words, since the share of the wage bill paid for these unproductive workers and the user cost of capital are borne now, whereas the profits from their employment accrue in the future, a discount rate that tends to zero is exactly the requirement for targeting a vanishing flow of profits typical of a long-run equilibrium in a competitive market for goods.[10] Consequently, recalling the arguments in Section 3 on the long-run equilibrium intensive measure of capital, in our model economy closing the gap between Φ^* and Φ^*_{GR} allows the conventional pricing rules grounded on marginal productivity to implement Pareto optimal allocations even a decentralized setting without efficient wage bargaining.

7 Numerical properties

Here I explore the numerical properties of the theoretical framework outlined above.[11] In that direction, the model economy is calibrated by taking as reference the US economy (cf. Chen et al. 2011). Specifically, the elasticity of produced output with respect to capital (α) entering the production function in eq. (2) and the depreciation rate of capital (δ) collected in eq. (3) are set at the same values chosen by Kydland and Prescott (1982). Moreover, the values of the elasticity of matching with respect to recruiting efforts (θ) and the job destruction rate (σ) in eq. (6) are fixed according to the estimations retrieved by Shimer (2005). Furthermore, the value of the discount rate entering in the utility function (ρ) in eq. (5) is set at the point value suggested by Itskhoki and Moll (2019) and Nordhaus (1994). In addition, the productivity index entering the production function (S) is normalized to 1 whereas the corresponding index entering the matching function (B) is set in order to convey an equilibrium unemployment rate equal to 5%, a figure that is consistent with the long-run US unemployment rate (cf. Guerrazzi, 2015). The description of the model parameters and their baseline values are collected in Table 1.

Parameter	Description	Value
S	Index of production efficiency	1
B	Index of matching efficiency	2.54
α	Output elasticity with respect to capital	0.36
θ	Matching elasticity with respect to recruiting	0.28
δ	Depreciation rate of capital	0.025
ρ	Discount rate	0.03
σ	Job destruction rate	0.10

Table 1: Baseline calibration

[10]Within such a long-run competitive equilibrium even the return on wealth tends to vanish, whereas the user cost of capital converges towards its depreciation rate.

[11]The MATLAB codes exploited to derive the numerical findings of this section are available from the author upon request.

As shown in the figures of Table 2, the baseline calibration reported in Table 1 has many interesting implications for the optimal growth model with labour market frictions under examination. First, in the steady-state equilibrium, 1.87% of the available labour force is allocated in recruiting activities, whereas 16.36% of produced output is saved and invested in new productive capital. Second, recalling that in a frictionless economy the equilibrium stock of capital would be equal to $\Phi^* = 18.8324$, the steady-state values of capital, output and consumption are 7.26% lower in our model economy with labour market frictions and equilibrium unemployment. Third, while the equilibrium capital share coincides with the elasticity of output with respect to capital, the equilibrium labour share is equal to 0.6363 which is lower than $1 - \alpha$ to accommodate firm's profit. Moreover, the eigenvalue associated to capital adjustments is much lower, in modulus, than the one associated to labour adjustments. Consequently, the out-of-equilibrium adjustments of capital/wealth and consumption are much slower than the ones involving employment and the fraction of employed workers allocated in recruiting activities.[12]

Variable	Description	Value
Y^*	*Output*	2.6823
C^*	*Consumption*	2.2434
$K^* = A^*$	*Capital/Wealth*	17.5569
L^*	*Employment*	0.9500
X^*	*Workers allocated in production activities*	0.9323
V^*	*Workers allocated in recruiting activities*	0.0178
W^*	*Wage*	1.7967
R^*	*Return on wealth*	0.0300
λ_1	*Eigenvalue associated to capital adjustments*	-0.0537
λ_2	*Eigenvalue associated to labour adjustments*	-2.0707

Table 2: Steady-state values and convergent eigenvalues

Taking values of $\overline{K}$ and $\overline{L}$ one percent above their steady-state references, the saddle-path trajectories of the four quantities the model economy are illustrated in the two panels of Figure 3.

[12]The value retrieved for λ_1 is of the same order of magnitude of the convergent root usually retrieved in the standard Ramsey model (e.g. Barro and Sala-i-Martin, 2004).

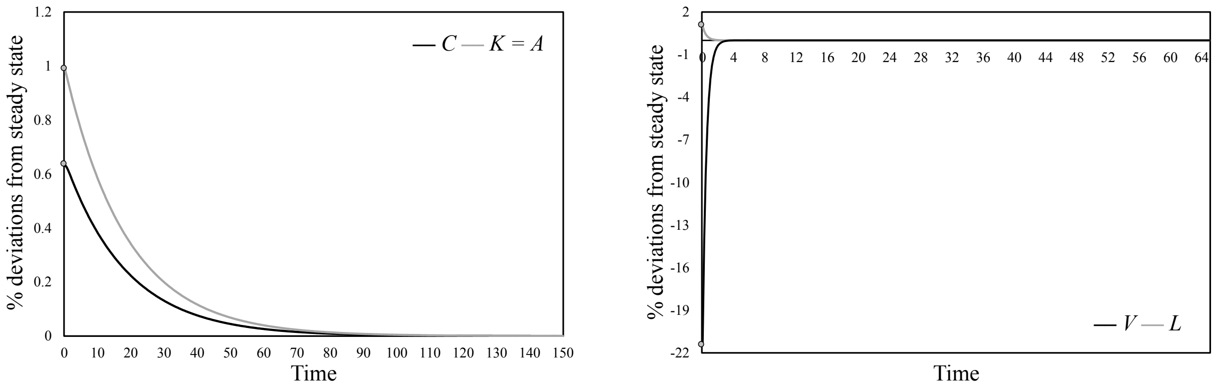

Figure 3: Saddle-path dynamics of quantities

On the one hand, the diagram on the left-hand-side of Figure 3 shows that when capital overshoots its steady-state value by 1%, households' consumption jumps only 0.62% above its equilibrium value and then the two tend to converge towards their long-run references by moving in the same direction. Such a pro-cyclical pattern in which the deviations of C from its steady state are always lower than the corresponding deviations of capital/wealth mirrors the risk aversion underlying the households' preferences in eq. (5) and it is in line with the textbook Ramsey model (e.g. Barro and Sala-i-Martin, 2004).

On the other hand, the diagram on the right-hand-side shows that when the whole labour input overshoots its steady-state value by 1%, the fraction of employed workers allocated in recruiting activities jumps 21.48% below its equilibrium value and then the two tend to converge towards their long-run references in a shorter time with respect to capital and consumption. Considering the definition of unemployment given in eq. (4), this means that the fraction of employed workers allocated in recruiting activities and the fraction of jobless workers tend to converge towards their long-run references by moving in the same direction. This pattern replicates the overshooting – or forward-looking – feature of vacancies displayed by the textbook matching model. In other words, if unemployment is expected to rise (fall) from its initial value, the return form allocating workers in recruiting activities is lower (higher) than the anticipated return during the adjustment. This is because at lower (higher) unemployment rates, the recruiting effectiveness of labour is lower (higher) as well. Therefore, as illustrated in the right-hand-side panel of Figure 3, in the starting period of time, there will be the tendency to allocate a lower (higher) fraction of workers in recruiting activities with respect to the share expected in equilibrium (cf. Pissarides, 2000, Chapter 1).

The dynamic patterns described above reveal that the optimal growth model with labour market frictions developed in this paper merges the out-of-equilibrium adjustments of conventional models with capital accumulation and search-and-matching externalities. As far the link with production and the matching technologies is concerned, however, some intriguing differences can be retrieved in the out-of-equilibrium adjustments of some critical ratios. Specifically, while the plots on the right-hand-side panel of Figure 3 imply monotonic adjustments for the labour market tightness indicator – denoted by $\Psi(t)$ – as it happens in the standard searching-and-matching model, the same does not hold true for the intensive measure of productive capital – denoted instead by $\Phi(t)$. In other words, the overshooting undertaken by the fraction of employed workers allocated in recruiting activities implies that the stock of productive capital over the fraction of employed workers allocated in output production adjusts towards its long-run value in a non-monotonic manner. Taking the same initial conditions used to track the diagrams in Figure 3, such a conjecture is tested in the two panels of Figure 4 where are plotted the implied trajectory of the labour market tightness indicator and the one of the intensive measure of productive capital.

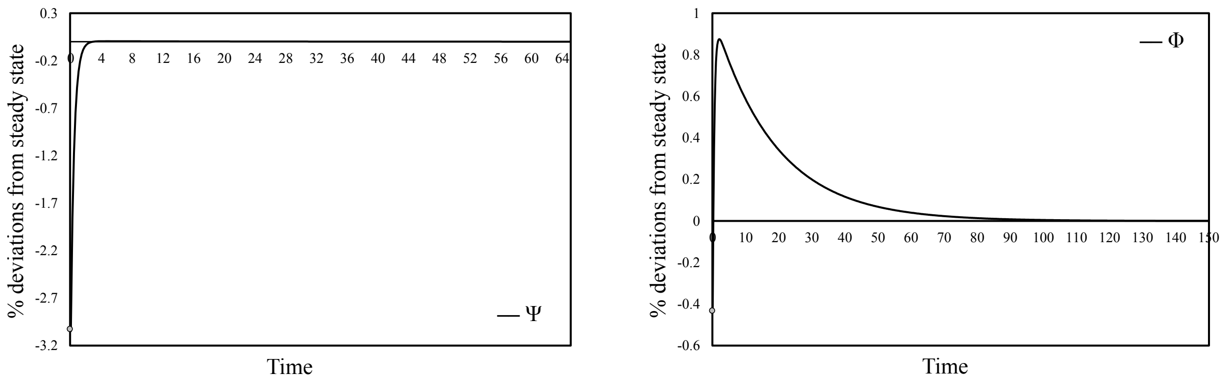

Figure 4: The implied dynamics of Ψ and Φ

Recalling that $K(t)$ and $L(t)$ are assumed to start 1% above their steady-state values, the plot in the left-hand-side of Figure 4 shows that at the beginning of its adjustment process $\Psi(t)$ undershoots its long-run value by 3.01% and then it monotonically converges to its steady-state reference (cf. Pissarides, 2000). By contrast, the plot on the right-hand-side shows that $\Phi(t)$ follows a non-monotonic adjustment process which is at odds with respect to the monotonic path tracked by the stock of capital in unit of effective labour within the textbook Ramsey model (cf. Barro and Sala-i-Martin, 2004). Specifically, at the beginning, the stock of capital over the fraction of workers allocated in production activities undershoots its long-run equilibrium value by 0.42%, it quickly goes up until it overshoots its state-state reference by 0.87%, and then it monotonically converges to its stationary level. This kind of dynamic behaviour is strictly related to the overshooting of the fraction of employed workers allocated in recruiting activities documented above; indeed, at the beginning of the adjustment process, the reduction of $V(t)$ is so strong that – given the prevailing values of $L(t)$ – the reduction of the fraction of employed workers allocated in production activities – labelled with $X(t)$ – is higher than the reduction of the overall capital stock, and this obviously pushes $\Phi(t)$ upwards. Thereafter, the monotonic adjustment of the stock of capital in units of effective labour towards its steady-state equilibrium begins when $X(t)$ starts to approach its long-run value.

The overshooting of the long-run value of Φ tracked in the right-hand-side panel of Figure 4 is consistent with recent findings on the non-linearity of the growth process observed in many countries (cf. Fiaschi and Lavezzi, 2007). Specifically, as illustrated in the discretized phase diagram of Figure 5, our optimal growth model with labour market frictions implies that in the region from 0 to Φ^* the intensive measure of the capital stock tends to increase at sustained rates that lead the actual value of Φ – and the value of output per worker – to exceed for a while their long-run references. Thereafter, this tendency is reversed and there is a phase of monotonic contraction that leads the intensive measure of the stock of capital to converge towards Φ^*. This kind of dynamic pattern that merely follows from the optimal adjustments of the state and control variables of our optimal growth model with searching-and-matching frictions is also consistent with the degrowth transition to a steady-state economy often claimed by environmental economists; indeed, within this literature, the long-run equilibrium may be achieved after a period of contraction that follows a phase of sustained expansion (cf. O'Neill, 2012).

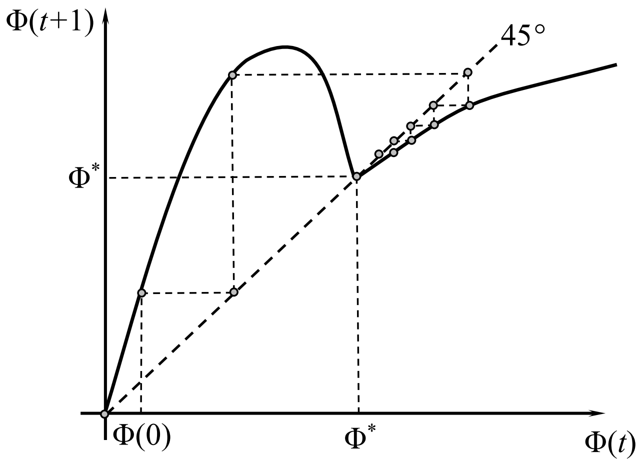

Figure 5: The discretized dynamics of Φ

In the same way as it happens on right-hand-side panel of Figure 4, the stylized phase portrait in Figure 5 plastically shows that whenever Φ starts below Φ^*, it tends to increase by overshooting its long-run reference before starting its convergence process. By contrast, whenever the initial value of Φ is above Φ^* the intensive measure of the productive capital stock monotonically converges towards its long-run value. Consequently, measuring time along the natural scale, that is, taking $t \in \mathbb{N}$, the optimal adjustments of the endogenous variables is able to generate a kink in the relationship between Φ and its lagged value just in correspondence of Φ^*.

The baseline calibration in Table 1 collects a positive discount rate of 3 percent which is mirrored by in the equilibrium return on wealth in Table 2. Consequently, according to the analytical results presented in Section 7, in this case we should observe persistently positive profits which are inconsistent with perfect competition in the market for goods. In order to verify the reliability of that pattern, the two panels of Figure 6 plot the trajectories of the net return on wealth conveyed by eq. (43), the one of the wage support that implements efficient allocations in eq. (46) and the one of the firm's profits implied by eq. (47).

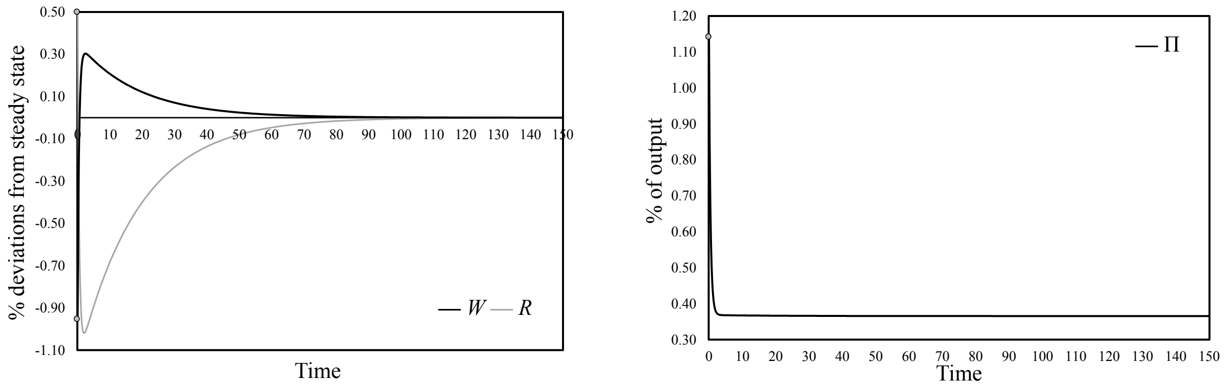

Figure 6: The implied dynamics of market prices and profits

The diagram on the left-hand-side panel of Figure 6 shows that the return on wealth and the wage tend to move in opposite directions during their adjustment process exactly as it happens in the text-book Ramsey model (cf. Barro and Sala-i-Martin, 2004). Specifically, such a diagram reveals that at the beginning of the adjustment process R (W) overshoots (undershoots) its long-run value by 0.49% (0.94%), it quickly goes below (above) its equilibrium reference by 1.01% (0.30), and then it converges towards its steady-state. Taking a look at the right-hand-side panel of Figure 3, it is worth noticing that for most of its transitional path, the real wage rate that implement efficient allocations moves in the same direction of total employment by displaying lower deviations from its long-run mean. Obviously, this is consistent with the available empirical business cycle evidence according to which real wages are mildly pro-cyclical and less volatile than (un)employment (cf. Merz, 1995; Andolfatto, 1996; Shimer, 2005). Moreover, the diagram on the right-side panel of Figure 6 confirms the analytical results derived above; indeed, the profits of the representative firm are persistently positive by converging to a value of 0.36% of produced output.

I close my computational experiments by showing what happens when the baseline calibration of Table 1 is altered by considering a vanishing value of the discount rate. In this direction, aiming at evaluating the welfare of future generations in the context of climate change, Cline (1992) and Stern (2008) suggest values of ρ fairly close to zero in order to overcome the ethical concerns raised by Ramsey (1928) in his seminal contribution on optimal saving (cf. Nordhaus, 1994). In the theoretical framework under scrutiny, vanishing values of the discount rate should lead the decentralized economy with the pricing rules conveyed by eq.s (43) and (46) to fulfil the long-run features of a competitive market for goods in which profits tend to vanish by discouraging the entrance of firms. Consequently, considering a point value of ρ equal to 0.0001 by leaving unaltered the values of all the remaining parameters and the initial conditions exploited to plot Figures 3 and 4, Figure 7 illustrates the path of the firm's profits.

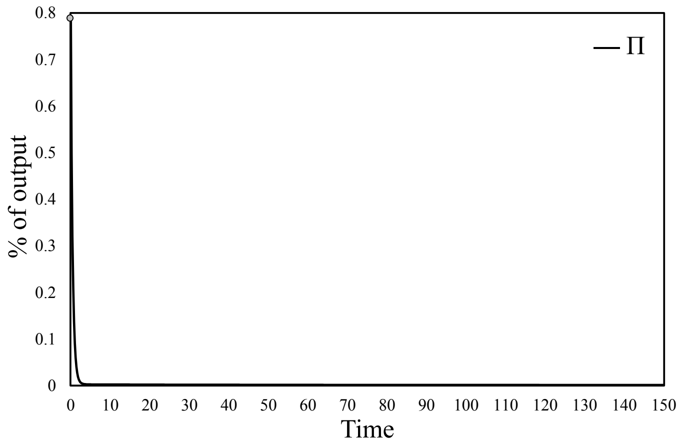

Figure 7: The dynamics of firm's profits with $\rho = 0.0001$

The plot of Figure 7 shows that at the beginning of the adjustment process firm's profits amount about 0.78% of produced output. Obviously, according to eq. (47), this means that the wage that implements efficient allocations is initially lower than the competitive wage so that the representative firm starts with positive profits. Just after a handful of instants, however, profits tend to vanish because $\overline{W}$ and W tend to coincide each other in a long run perspective. Such a dynamic pattern corroborates the analytical finding derived in Section 6 according to which vanishing values of the discount rate are actually consistent with perfect competition in the market for goods.

8 Concluding remarks

In this paper, I developed an optimal growth model with capital accumulation and labour market frictions. Specifically, assuming that vacancies are posted by means of labour instead of output, I augmented the traditional setting à la Ramsey with an additional intertemporal equation that describes the law of movement of employment (cf. Farmer, 2013; Guerrazzi, 2015). Relying on that framework, I shown that the forward-looking behaviour of recruiting efforts lead the capital stock per efficiency unit of productive labour to converge towards its stationary value in a non-monotonic manner by mirroring some recent findings on non-linear economic growth (cf. Fiaschi and Lavezzi; 2007; O'Neill, 2012). Moreover, assuming that productive capital is paid according to its marginal productivity, I shown that Pareto optimal allocations typical of a centralized economy can also be replicated in a decentralized environment in which the prevailing wage is indexed to the current value of the labour market tightness indicator (cf. Chen et al. 2011). Furthermore, I shown that the wage that allows to implement efficient allocations is less volatile than (un)employment and it is consistent with perfect competition in the markets for goods only with vanishing values of the discount rate (cf. Aghion and Howitt, 1994; Hall, 2017).

The analysis presented in this paper could be extended in many different directions. For instance, it could be interesting to address the consequences of some imperfections in the capital market vis-à-vis labour market frictions that move away the return on wealth from the marginal productivity of capital (cf. Lee, 2021). Furthermore, another prominent extension could be the exploration of a two-sector economy in which there are distinct production and matching technologies for consumption and capital goods (cf. Uzawa, 1961). Within such an economy, the available stock of capital should be optimally allocated in the two productive sectors. At the same time, within each sector, the available labour force should be optimally allocated in production and recruiting activities. Consequently, the dynamic patterns followed by this model economy are likely to be influenced not only by the relative capitalization of each sector – as it happens in standard two-sector models (cf. Galor, 1992) – but also by the corresponding degree of labour market tightness. All the mentioned extensions are left, however, to further developments.

Appendix A: The efficiency of decentralized allocations

In the stationary equilibrium of the decentralized economy described in Section 6, the Bellman equation for the value of an unmatched firm – say π_U^* – can be written as

$$\rho\pi_{\mathrm{U}}^* = -VC + \eta\left(\pi_{\mathrm{M}}^* - \pi_{\mathrm{U}}^*\right) \tag{A1}$$

where $VC \equiv -\partial Y(t)/\partial V(t)|_{\mathcal{S}} = (1-\alpha)\, S\left(\Phi^*\right)^{\alpha}$ and $\eta \equiv B\left(\Psi^*\right)^{-(1-\theta)}$.

If there is free-entry in the market for goods $\pi_{\mathrm{U}}^* = 0$. Consequently, the equilibrium value of matched firm is given by

$$\pi_{\mathrm{M}}^* = \frac{(1-\alpha)\, S\left(\Phi^*\right)^{\alpha}}{B\left(\Psi^*\right)^{-(1-\theta)}} \tag{A2}$$

In the present setting, the matching surplus accrued from additional employment is given by individual consumption times the marginal contribution of employment to household's welfare as measured by the costate variable on the constraint for employment dynamics (cf. Chen et al. 2011). Therefore, assuming that $\xi \in (0,1)$ is the surplus's share accruing to the representative firm, whereas $1-\xi$ is the corresponding share accruing to the household, the equilibrium value of matched firm reads also as

$$\pi_{\mathrm{M}}^* = \xi C^* w^* \tag{A3}$$

In a Pareto optimal steady-state equilibrium, eq.s (9), (32), (A2) and (A3) imply that

$$-(1-\alpha)\, Sq^*\left(\Phi^*\right)^{\alpha} + \xi B\frac{w^*}{\left(\Psi^*\right)^{1-\theta}} = 0 \tag{A4}$$

The steady-state solution pinned down by eq. (A4) is equal to the one implied by eq. (10) only when ξ is equal to θ. *Q.E.D.*

Appendix B: The decentralized economy with a differentiated wage treatment

Whenever it possible to pay workers according to the activity in which they are employed by the representative firm the household problem reads as

$$\begin{gathered}\max_{\mathcal{C}_{\mathrm{H}}\in\mathcal{A}_0^{\mathrm{H}}}\int_{t=0}^{\infty}\exp\left(-\rho t\right)\left(\ln C\left(t\right)\right)dt\\ \text{s.to}\\ \dot{A}(t)=A\left(t\right)R\left(t\right)+W_V(t)V(t)+W_X(t)X(t)+\Pi\left(t\right)-C(t)\\ \dot{L}\left(t\right)=(1-L(t))\Gamma\left(\Psi(t)\right)-\sigma L\left(t\right)\\ A\left(0\right)=\overline{A},\quad L(0)=\overline{L}\end{gathered} \tag{B1}$$

where $W_V(t)$ ($W_X(t)$) is the real wage rate paid to workers employed in recruiting (production) activities.

The solution of the problem in (B1) is identical to the solution of the problem in (30). On the other side, when a separating wage strategy is feasible, the problem of the firm becomes the following:

$$\begin{gathered}\max_{\mathcal{C}_{\mathrm{F}}\in\mathcal{A}_0^{\mathrm{F}}}\int_{t=0}^{\infty}\exp\left(-\rho t\right)\left(Y\left(t\right)-W_V(t)V(t)-W_X(t)\left(L(t)-V(t)\right)-\left(R\left(t\right)+\delta\right)K(t)\right)dt\\ \text{s.to}\\ \dot{L}\left(t\right)=V(t)\Delta\left(\Psi(t)\right)-\sigma L\left(t\right)\\ L(0)=\overline{L}\end{gathered} \tag{B2}$$

The FOC with respect to $K(t)$ as well as the transversality condition for the problem in (B2) are identical to the expressions in (38) and (41). By contrast, the FOC with respect to $V(t)$ and the optimal evolution of the costate variable associated to the dynamic constraint for employment are, respectively, given by

$$-\left(1-\alpha\right)S\left(\Phi\left(t\right)\right)^{\alpha}-W_V(t)+W_X(t)+w_{\mathrm{F}}\left(t\right)\Delta\left(\Psi(t)\right)=0 \tag{B3}$$

$$\dot{w}_{\mathrm{F}}\left(t\right)=w_{\mathrm{F}}\left(t\right)\left(\rho+\sigma\right)-\left(1-\alpha\right)S\left(\Phi\left(t\right)\right)^{\alpha}+W_X(t) \tag{B4}$$

On the one hand, in a steady-state equilibrium, eq.s (B3) and (B4) implies that

$$W_X^*=(1-\alpha)S\left(\Phi^*\right)^{\alpha}+W_V^*\frac{\rho+\sigma}{\rho+\sigma-\Delta\left(\Psi^*\right)} \tag{B5}$$

On the other hand, assuming that the asset market clear by pricing capital at its marginal productivity and that the market for goods is competitive allows us to derive the following equilibrium expression:

$$W_V^* = \left(\left(1-\alpha\right) S\left(\Phi^*\right)^\alpha - W_X^*\right) \frac{X^*}{V^*} \tag{B6}$$

Plugging eq. (B6) into eq. (B5) implies that $W_X^* = (1-\alpha) S\left(\Phi^*\right)^\alpha$. Consequently, it necessarily follows that $W_V^* = 0$. *Q.E.D.*

References

[1] Aghion, P., Howitt, P. (1994), Growth and Unemployment, *Review of Economic Studies*, Vol. 61, No. 3, pp. 477-494.

[2] Andolfatto, D. (1996), Business Cycles and Labor-Market Search, *American Economic Review*, Vol. 86, No. 1, pp. 112-132.

[3] Barro, J.R., Sala-i-Martin, X. (2004), *Economic Growth*, Second Edition, MIT Press, Cambridge, Massachusetts.

[4] Card, D. (2001), The Effect of Unions on Wage Inequality in the U.S. Labor Market, *Industrial and Labor Relations Review*, Vol. 54, No. 2, pp. 296-315.

[5] Cass, D. (1966), Optimum Growth in an Aggregative Model of Capital Accumulation: A Turnpike Theorem, *Econometrica*, Vol. 34, No. 4, pp. 833-850.

[6] Cass, D. (1965), Optimum Growth in Aggregative Model of Capital Accumulation, *Review of Economic Studies*, Vol. 32, No. 3, pp. 233-240.

[7] Chen, B-L, Chen. H-J, Wang, P. (2011), Labor-Market Frictions, Human Capital Accumulation, and Long-Run Growth: Positive Analysis and Policy Evaluation, *International Economic Review*, Vol. 52, No. 1, pp. 131-160.

[8] Cline, W.R. (1992), *The Economics of Global Warming*, Peterson Institute for International Economics.

[9] Diamond, P. A. (1982), Wage Determination and Efficiency in Search Equilibrium, *Review of Economic Studies*, Vol. 49, No. 2, pp, 217-227.

[10] Duval, R., Ji, Y., Li, L., Oikonomou. M., Pizzinelli, C., Shibata, I., Sozzi, A., Tavares, M.M. (2022), Labor Market Tightness in Advanced Economies, *IMF Discussion Notes*, No. 1.

[11] Eriksson, C. (1997) Is There a Trade-Off Between Employment and Growth?, Oxford Economic Papers, Vol. 49, No. 1, pp. 77-88.

[12] Farmer, R.E.A. (2013), Animal Spirits, Financial Crises and Persistent Unemployment, *Economic Journal*, Vol. 123, No. 568, pp. 317-340.

[13] Farmer, R.E.A. (2012), Technical Appendix to Animal Spirits, Financial Crises and Persistent Unemployment, *mimeo.*

[14] Farmer, R.E.A. (2010), *Expectations, Employment and Prices*, Oxford University Press.

[15] Fiaschi, D., Lavezzi, A.M. (2007), Nonlinear Economic Growth: Some Theory and Cross-Country Evidence, *Journal of Development Economics*, Vol. 84, No. 1, pp. 271-290.

[16] Galor, O. (1992), A Two-Sector Overlapping-Generations Model: A Global Characterization of the Dynamical System, *Econometrica*, Vol. 60, No. 6, pp. 1351-1386.

[17] Guerrazzi, M. (2015), Animal Spirits, Investment and Unemployment: An Old Keynesian View of the Great Recession, *EconomiA*, Vol. 16, No. 3, pp. 343-358.

[18] Hall, R. (2017), High Discounts and High Unemployment, *American Economic Review*, Vol. 107, No. 2, pp. 305-330.

[19] Hornstein, A., Krussel, P., Violante, G.L. (2007), Modelling Capital in Matching Models: Implications for Unemployment Fluctuations, *Working Papers in Economics of the Princeton University*, No. 2.

[20] Hosios, A.J. (1990), On the Efficiency of Matching and Related Models of Search and Unemployment, *Review of Economic Studies*, Vol. 57, No. 2, pp. 279-298.

[21] Itskhoki, O., Moll, B. (2019), Optimal Development Policies with Financial Frictions, *Econometrica*, Vol 87, No. 1, pp. 139-173.

[22] Janiak, A., Wasmer, E. (2014), Employment Protection and Capital-Labor Ratios, *IZA Discussion Papers*, No. 8362.

[23] Koopmans, T.C. (1965) On the Concept of Optimal Economic Growth, in Johansen, J. (Ed.), *The Econometric Approach to Development Planning*, North Holland, Amsterdam.

[24] Kydland, F.E., Prescott, E.C. (1982), Time to Build and Aggregate Fluctuations, *Econometrica*, Vol. 50, No. 6, pp. 1345-1370.

[25] Layard, R., Nickell, S.J., Jackman, R. (1991), *Unemployment: Macroeconomic Performance and the Labour Market*, Oxford University Press, New York.

[26] Lee, K. (2021), Labor Market Frictions, Capital, Taxes and Employment, *International Tax and Public Finance*,Vol. 28, No. 6, pp. 1329-1359.

[27] Masters, A.M. (1998), Efficiency of Investment in Human and Physical Capital in a Model of Bilateral Search and Bargaining, *International Economic Review*, Vol. 39, No. 2, pp. 477-494.

[28] Merz, M. (1995), Search in the Labor Market and the Real Business Cycle, *Journal of Monetary Economics*, Vol. 36, No. 2, pp. 269-300.

[29] Mortensen, D. T. (1982), The Matching Process as a Noncooperative Bargaining Game, in McCall, J.J. (ed.), *The Economics of Information and Uncertainty*, pp. 233-254, University of Chicago Press, Chicago.

[30] Nordhaus, W.D. (1994), *Managing the Global Commons: The Economics of Climate Change*, MIT Press, Cambridge, Mass.

[31] O'Neill, D.W. (2012), Measuring Progress in the Degrowth Transition to a Steady State Economy, *Ecological Economics*, Vol. 84, No. C, pp. 221-231.

[32] Pissarides, C.A. (2000), *Equilibrium Unemployment Theory*, 2nd Edition, MIT Press, Cambridge, Mass.

[33] Pissarides, C.A. (1985), Short-Run Dynamics of Unemployment, Vacancies, and Real Wages, *American Economic Review*, Vol. 75, No. 4, pp. 676-690.

[34] Ramsey, F.P. (1928), A Mathematical Theory of Saving, *Economic Journal*, Vol. 38, No. 152, pp. 543-559.

[35] Shimer, R. (2005), The Cyclical Behavior of Equilibrium Unemployment and Vacancies, *American Economic Review*, Vol. 95, No. 1, pp. 25-48.

[36] Stern, N.H. (2007), *The Economics of Climate Change: The Stern Review*, Cambridge: Cambridge University Press.

[37] Uzawa, H. (1961), On a Two-Sector Model of Economic Growth, *Review of Economic Studies*, Vol. 29, No. 1, pp. 40-47.